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Extension of germs of holomorphic foliations

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Résumé. — On considère le problème d'extension de germes de feuilletages holomorphes. On montre qu'un germe régulier après un éclatement, admettant une intégrale première méromorphe, peut être prolongé à une surface algébrique. On montre que les éléments topologiquement les plus simples dans cette classe peuvent être définis par des champs polynomiaux. Par ailleurs, en l'absence d'intégrales premières, on montre qu'il existe une quantité non dénombrable d'élements dans cette classe n'admettant pas de modèles polynomiaux.

Abstract. — We consider the problem of extending germs of plane holomorphic foliations to foliations of compact surfaces. We show that the germs that become regular after a single blow up and admit meromorphic first integrals can be extended, after local changes of coordinates, to foliations of compact surfaces. We also show that the simplest elements in this class can be defined by polynomial equations. On the other hand we prove that, in the absence of meromorphic first integrals there are uncountably many elements without polynomial representations.

1. Introduction

In this paper we treat the following problem: let $\mathcal{F}ol(\mathbb{C}^2, 0)$ be the set of germs of holomorphic foliations defined in a neighborhood of $0 \in \mathbb{C}^2$ which are singular at the origin; we consider two such foliations to be equivalent when they are conjugated by a local holomorphic diffeomorphism of $\mathbb{C}^2$ at

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$0 \in \mathbb{C}^2$. We select some family $L \subset \mathcal{F}ol(\mathbb{C}^2,0)$ and ask if an equivalence class contains a foliation that is defined by polynomial differential equations. Any member of such a class admits an extension to a foliation of the complex projective plane after a suitable local change of coordinates.

A very simple example is given by the set of hyperbolic singularities in the Poincaré domain, namely, foliations defined 1-forms of type

$$(x + A(x,y))\,dy - (\lambda y + B(x,y))\,dx = 0,$$

where $\lambda \notin \mathbb{R}$ and A and B are holomorphic functions such that $A(0,0) = B(0,0) = 0$ and whose derivatives at $(0,0) \in \mathbb{C}^2$ vanish. By the theorem of linearization of Poincaré ([11]), we have a holomorphic equivalence to the linear part

$$x\,dy - \lambda y\,dx = 0.$$

It is interesting to notice that when $\lambda \leqslant 0$ but $\lambda \notin \mathbb{Q}$ it is not known if any equivalence class contains a foliation defined by polynomial equation (see [10]).

There are other instances where we can always find a local model defined by polynomial equations. For example, let $f : (\mathbb{C}^2,0) \to (\mathbb{C},0)$ be a holomorphic germ having an isolated singularity at 0. By a theorem of Mather, f is of finite determinacy: there exists $k \in \mathbb{N}$ such that the k-jet f_k of f at $0 \in \mathbb{C}^2$ is conjugated to f by means of a holomorphic diffeomorphism ϕ, i.e., $f = f_k \circ \phi$. It follows that the foliation defined by $d\,f = 0$ is conjugated by ϕ to the foliation defined by $d\,f_k = 0$. The question of whether a similar statement holds true for foliations defined by meromorphic functions arises: is a foliation in $(\mathbb{C}^2,0)$ defined by a meromorphic function equivalent to a foliation defined by a polynomial differential equation? A theorem by Cerveau and Mattei ([3]) gives sufficient conditions on the function to conclude that it is the case: let f/g be a germ at $0 \in \mathbb{C}^2$ of meromorphic function (f and g are supposed to be relatively prime germs of holomorphic functions) such that the 1-form $f d\,g - g d\,f$ defines a foliation with an isolated singularity at $0 \in \mathbb{C}^2$. Then f/g has finite determinacy, that is, f/g is conjugated to f_k/g_k for some $k \in \mathbb{N}$, where f_k and g_k are the k-jets of f and g at $0 \in \mathbb{C}^2$. One of the simplest examples of a meromorphic function that does not satisfy the hypothesis of Cerveau and Mattei's result is the germ of foliation $\mathcal{F}$ defined by $\frac{y^2 - x^3\phi(x)}{x^2} = const$ at $(0,0)$ (ϕ is a germ of holomorphic function at $0 \in \mathbb{C}$).

Let $\mathcal{D}$ *be the family of germs of foliations that are regular after a single blow-up at the origin*, so that all leaves are transverse to the exceptional divisor, except for a finite number which are tangent to it. Our first result analyzes the question of the existence of algebraic models for the elements of $\mathcal{D}$ which have a meromorphic first integral. In [2] this problem is studied for the case of just one tangent leaf with a simple tangency point. It is proven that the presence of a meromorphic first integral implies holomorphic equivalence with a germ of a foliation which extends to some algebraic surface. Using the results in [1] we are able to extend the conclusion to all foliations in $\mathcal{D}$ which admit a meromorphic first integral (see Theorem 1, Section 3).

We remark that the elements in $\mathcal{D}$ belong to a wider class: the $\mathcal{M}$-simple foliations, those which are topologically equivalent to foliations defined by a meromorphic first integral (but not necessarily admit such an integral); see [8].

A good point in the previous discussion is whether we can replace "algebraic surface" by "projective plane" Let us go back to the situation in [2] and denote by $\mathcal{D}_1$ *the subset of foliations of* $\mathcal{D}$ *which have just one simple point of tangency with the exceptional divisor*. In Theorem 2, Section 3 we improve the result of [2]: any foliation in $\mathcal{D}_1$ which admits a meromorphic first integral is equivalent to a foliation defined in the projective plane (or equivalently, defined by polynomial equations).

The idea behind the proofs of the previous theorems is to transform the problem of extending the germ of foliation to that of extending a germ of curve in a convenient space of rational functions. In the second case the germ of curve is contained in a rational curve, producing a foliation on a rational surface; in the first, after an appropriate local change of coordinates, it is contained in some algebraic curve, whose genus is not known in general.

Using completely different techniques, we prove that a "polynomial-like" statement is not true for all foliations in $\mathcal{D}$: in each topological class (that is, with a given number of tangencies and with a given choice of orders of tangency) there are foliations that are not holomorphically equivalent to foliations defined by polynomial equations.

In order to simplify the exposition, we will carry the proof of this statement (Theorem 3, Section 4) for the class $\mathcal{D}_1$, but the proof extends *quid pro quo* to the other classes. It follows the lines of [5], where a tool that allows to treat the problem from an analytic point of view is introduced. Let us give an outline of the method: given a family $L \subset \mathcal{F}ol(\mathbb{C}^2, 0)$, a surjective map ψ from the set $[L]$ of equivalence clases in L to a space I of invariants is defined; it is assumed that there are appropriate topologies to turn ψ into an "analytic" map. The image of the equivalence classes of polynomially defined foliations is then a countable union of analytic manifolds (of finite dimension!) and cannot be the whole of I provided I is a "huge" space. This is a sort of analytic Baire property. As an application, the authors consider the family L of saddle-node singularities of Milnor number 2 (in fact the choice of the Milnor number is not relevant). Those singularities are defined by forms

$$[y(1+\mu x)+R(x,y)]\,dx-x^2\,dy=0$$

where $\mathrm{ord}_{(0,0)}(R)\geqslant 3$. According to [9], these singularities can be obtained by applying a convenient gluing procedure to normal forms of type

$$y(1+\mu x)\,dx-x^2\,dy=0\,;$$

in this gluing process, non trivial local holomorphic diffeomorphisms $h(z)=z+\dots$ are used and become elements of the space of invariants I. The conclusion is that there are uncountably many saddle-nodes which are not equivalent to saddle-nodes defined by polynomial equations.

In the present paper we apply these ideas to the family $\mathcal{D}_1$. The space of invariants in this situation contains the local germs of finite order at the tangency points defined as follows: if f is a primitive local holomorphic first integral at a tangency point of order n between a regular foliation and a non-invariant regular curve C (in our case it will be the exceptional divisor, or some rational curve), each fiber of the germ $f|_C$ describes the intersection between a local leaf and C. Since $f|_C$ is a holomorphic germ in one variable, it is locally conjugated to $z\mapsto z^{n+1}$, whose fibers are described by the orbits of the rotation of angle $2\pi/(n+1)$. Thus, the fibers of $f|_C$ are described by the orbits of a cyclic group of order $n+1$ generated by a holomorphic germ of diffeomorphism whose linear part is the corresponding rotation (an involution if $n=1$).

This paper is organized as follows: in Section 2 we reproduce a basic construction introduced in [6]; Section 3 is devoted to Theorems 1 and 2, and Section 4 to Theorem 3. Section 4 is completely independent of the rest and can be read separately.

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2. A Model

In this section we will carry our first step towards compactifying a holomorphic germ of foliation. We explain why, up to birational equivalence, a holomorphic germ of foliation in $\mathcal{D}$ (see the Introduction) is equivalent to a holomorphic foliation on $(\mathbb{C},0)\times\mathbb{P}^1$.

Let us consider a holomorphic foliation $\mathcal{G}' \in \mathcal{D}$; We will now conjugate $\mathcal{G}'$ to a special model $\mathcal{G}$.

- **Step 1**: we blow up at $0 \in \mathbb{C}^2$; the exceptional divisor E_1' is not invariant for the blown-up foliation $\tilde{\mathcal{G}}_1'$. We select a point $p \in E_1'$ where $\tilde{\mathcal{G}}_1'$ is transverse to E_1' and take a neighborhood V_1 of this point where $\tilde{\mathcal{G}}_1'$ is trivial. In parallel, we blow up at $0 \in \mathbb{C}^2$ the trivial foliation $d\,y = 0$ to a foliation $\tilde{\mathcal{G}}_2'$ which now has the exceptional divisor E_2' as an invariant set (with one singularity). We take a regular point of $\tilde{\mathcal{G}}_2'$ in E_2' and a neighborhood V_2 of this point where $\tilde{\mathcal{G}}_2'$ is trivial. We then glue $\tilde{\mathcal{G}}_1'$ to $\tilde{\mathcal{G}}_2'$ by a holomorphic diffeomorphism from V_1 to V_2 which sends $\tilde{\mathcal{G}}_1'|_{V_1}$ to $\tilde{\mathcal{G}}_2'|_{V_2}$. We get a surface which contains two divisors, still denoted by E_1' and E_2', with $E_1' \cdot E_1' = E_2' \cdot E_2' = -1$ and $E_1' \cdot E_2' = 1$, and a foliation $\tilde{\mathcal{G}}'$ conjugated to $\tilde{\mathcal{G}}_1'$ and $\tilde{\mathcal{G}}_2'$ in neighborhoods of E_1' and E_2' respectively.

- **Step 2**: we consider now the surface obtained after blowing up $\mathbb{D}\times\mathbb{P}^1$ at some point of $\{0\} \times \mathbb{P}^1$; we have inside it two divisors E_1 and E_2 such that $E_1 \cdot E_1 = E_2 \cdot E_2 = -1$ and $E_1 \cdot E_2 = 1$. Since a neighborhood of $E_1 \cup E_2$ is biholomorphically equivalent to a neighborhood of $E_1' \cup E_2'$ by a diffeomorphism that takes E_1 to E_1' and E_2 to E_2', we may define a foliation $\tilde{\mathcal{G}}$ in a neighborhood of $E_1 \cup E_2$ as the image of $\tilde{\mathcal{G}}'$. We restrict $\tilde{\mathcal{G}}$ to a neighborhood of E_1 and blow-down E_1 to get the model $\mathcal{G}$ we look for. If we blow-down E_2 we get a foliation $\mathcal{G}_1$ defined in a surface diffeomorphic to $\mathbb{D} \times \mathbb{P}^1$.

In other words, modulo holomorphic equivalence, a foliation in $\mathcal{D}$ is obtained by blowing-up a foliation defined in $\mathbb{D}\times\mathbb{P}^1$ at some point of transversality with $\{0\} \times \mathbb{P}^1$, then taking the restriction of this foliation to a neighborhood of the strict transform of $\mathbb{D} \times \mathbb{P}^1$ and finally blowing-down this restriction. There are many choices involved in the construction and the model obtained in the product is not unique. A strategy to find a holomorphic equivalence with an algebraic foliation is to try to "extend" such a foliation in $\mathbb{D} \times \mathbb{P}^1$ to $C \times \mathbb{P}^1$ where C is a compact Riemann surface; we will succeed in the presence of a first integral.

Let us remark that if $\mathcal{G} \in \mathcal{D}$, then $\mathcal{G}_1$ is regular along $\{0\} \times \mathbb{P}^1$ as well. Furthermore, if $\mathcal{G}$ has a meromorphic first integral then the same is true for $\mathcal{G}_1$ (notice that the first integral has no indeterminacy points since $\mathcal{G}_1$ is regular); in particular a first integral $R(x,t)$ can be seen as a holomorphic family of rational functions $x \in \mathbb{D} \mapsto R_x(t) = R(x,t) \in \mathbb{P}^1$ of some degree d. It is not difficult to show that $x \in \mathbb{D} \mapsto R_x$ is locally injective at $x = 0$.

3. Algebraic Case

In order to state our first theorem along the same lines of [2], we use the notion of *algebraic-like foliation*: we say that a germ of holomorphic foliation $\mathcal{G}$ is an *algebraic-like foliation* when there exists a holomorphic foliation of an algebraic surface which is equivalent to $\mathcal{G}$ in a neighborhood of some singularity.

THEOREM 1. — *Any foliation in $\mathcal{D}$ admitting a meromorphic first integral is an algebraic-like foliation.*

It generalizes the main Theorem in [2], which refers to foliations in the subset $\mathcal{D}_1 \subset \mathcal{D}$ of germs having only one leaf with a simple tangency with the exceptional divisor. On the other hand in the case of $\mathcal{D}_1$ we can improve the result to prove that the singularity occurs as the germ at a singularity of a foliation of the complex projective plane:

THEOREM 2. — *Any foliation in $\mathcal{D}_1$ admitting a meromorphic first integral is equivalent to a foliation defined by polynomial equations.*

What we are going to do to prove Theorem 1 is to use the model introduced above (we keep the same notations). Starting from a foliation $\mathcal{G}_1$ defined in $\mathbb{D} \times \mathbb{P}^1$ which is regular along $\{0\} \times \mathbb{P}^1$ and admits a first integral $R(x,t)$, we will approximate the corresponding family of rational functions $R_x(t)$ by another one which goes through $R_0(t)$ and whose parameter space is a compact Riemann surface C. The associated foliation on $C \times \mathbb{P}^1$ –an algebraic surface– will be close enough to $\mathcal{G}_1$ along $\{0\} \times \mathbb{P}^1$ to allow the use of the conjugation theorem of [1].

As for Theorem 2, we will show that, up to reparametrizing the x-variable, the (local) family of rational functions can be extended to a family parametrized by $\mathbb{P}^1$. The induced foliation will then define an extension of $\mathcal{G}_1$ to $\mathbb{P}^1 \times \mathbb{P}^1$.

3.1. Critical Points

Let us consider a foliation $\mathcal{G}_1$ defined in $\mathbb{D} \times \mathbb{P}^1$ which is regular along $\{0\} \times \mathbb{P}^1$ and admits a first integral $R(x,t)$.

We will assume that there exists a fixed neighborhood U (independent of x) of $\infty \in \mathbb{P}^1$ such that no critical point of $R_x(t)$ is inside this neighborhood. We may assume also that $R_x(t)$ is holomorphic at $t = \infty$, that its poles are simple (so that they are not critical points) and the lines of poles of R are leaves of $\mathcal{G}_1$ transversal to $\{0\} \times \mathbb{P}^1$, say $A_j(x,t) = t - c_j(x) = 0$ for $1 \leqslant j \leqslant d$. The 1-form dR has its poles along the same lines (with order

2); therefore $A^2\, dR$, where $A(x,t) = A_1(x,t)\dots A_d(x,t)$, is a holomorphic 1-form that defines $\mathcal{G}_1$, possibly with lines of zeroes; these lines are necessarily contained in the curves of critical points of $x \mapsto R_x(t)$.

Let us discuss how to eliminate these zeroes in the expression of the 1-form defining $\mathcal{G}_1$. We start then by analyzing the curves of critical points of $x \mapsto R_x(t)$; the zeroes of $d\,R$ are inside the zeroes of $\frac{\partial R}{\partial t} = 0$. We have the following possibilities:

(i) the leaf of $\mathcal{G}_1$ that passes through a critical point of $R_0(t)$ (of order $m \in \mathbb{N}$) is transversal to $\{0\} \times \mathbb{P}^1$; we parametrise the leaf as $x \mapsto (x, f(x))$. Since the first integral assumes a constant value along each nearby leaf, we see that each point $(x, f(x))$ is also a critical point of order m of $R_x(t)$. Consequently the curve $t - f(x) = 0$ is contained in the singular set of the foliation defined by $d\,R = 0$; we call such a curve of critical points (or singular points) a *level type curve*. We may write locally (assuming $t_0 = 0$ for simplicity) that

$$R(x,t) = a + (t - f(x))^{m+1} h(x,t)$$

where $a \in \mathbb{C}$, $h(0,0) \neq 0$. Therefore

$$d\,R = [(m+1)(t-f(x))^m h + (t-f(x))^{m+1} \frac{\partial h}{\partial x}] d\,x$$

$$+[-(m+1)(t-f(x))^m h f' + (t-f(x))^{m+1} \frac{\partial h}{\partial t}] d\,t$$

The 1-form $d\,R = 0$ has $(t - f(x))^m = 0$ as its equation of zeroes.

(ii) the critical point $(0, t_0)$ is a point of tangency of $\mathcal{G}_1$ with $\{0\} \times \mathbb{P}^1$; it gives rise to a curve of critical points of $R_x(t)$, or points of tangency between $\mathcal{G}_1$ and the vertical lines $x = const$, which crosses $\{0\} \times \mathbb{P}^1$ at the point $(0, t_0)$ (we put again $t_0 = 0$). The foliation $\mathcal{G}_1$ is obtained in a neighborhood of $(0,0)$ once we divide $d\,R = 0$ by the equation of its zeroes. If a component of the curve of critical points is invariant by $\mathcal{F}_1$, it necessarily coincides with the leaf which is tangent to $\{0\} \times \mathbb{P}^1$ at $(0,0)$; we call it also a *level type curve* of critical points (of some order M). It has as equation $x - g(t) = 0$, where $g(t) = t^{l+1}\tilde{g}(t)$ with $l \geqslant 1$ and $\tilde{g}(0) \neq 0$. We apply the same argument as in case (i) to a neighborhood of a point of this curve for which $x \neq 0$ and conclude that $(x - g(t))^M = 0$ is inside the set of zeroes of $d\,R$ (a fortiori in a neighborhood of $(0,0)$ as well).

Now let us analyze the case of a component of a *non-invariant curve* of critical points, that is, one that is not $\mathcal{G}_1$-invariant. We observe that the zeroes of dR are inside the zeroes of $\frac{\partial R}{\partial t} = 0$. Locally *at a point where* $\mathbf{x} \neq \mathbf{0}$ we have

$$R(x,t) = a(x) + (t-u(x))^{l+1}h(x,t)$$

where $a(x)$ is not constant (otherwise we would have case (i)), $h(0,0) \neq 0$, $l \geqslant 1$ and $t - u(x) = 0$ is the local equation of the component. It follows from

$$dR = [a'(x) - (l+1)(t-u(x))^l u'(x)h + (t-u(x))^{l+1}\frac{\partial h}{\partial x}]dx$$

$$+[(l+1)(t-u(x))^l h + (t-u(x))^{l+1}\frac{\partial h}{\partial t}]dt$$

that the coefficients of dx and dt have no common factors; therefore there is no new curve of zeroes arising from the type of curve of critical points under consideration. We conclude that in a neighborhood of $(0,0)$ we only have to take $(x-g(t)^M = 0$ in order to define the zeroes of dR. Of course it may happen that the leaf of $\mathcal{G}_1$ which is tangent to $\{0\} \times \mathbb{P}^1$ is not a level type curve of critical points.

We may summarise this information about the zeroes of dR as follows:

- there are curves of level type $x \mapsto f_1(x), \ldots, f_k(x)$ which correspond to critical points of orders $m_1, \ldots, m_k$; these curves are transversal to $\{0\} \times \mathbb{P}^1$, and locally $R(x,t) = a_j + (t - f_j(x))^{m_j+1}h_j(x,t)$. Locally at each of these critical points the zeroes of dR are given by the equation $(t - f_j(x))^{m_j} = 0$.

- there are curves $P_1, \ldots, P_s$ of critical points of orders $M_1, \ldots, M_s$ which are curves of level type (for $x \neq 0$); each curve P_j is tangent to $\{0\} \times \mathbb{P}^1$ in order $l_j \geqslant 1$ at a critical point t_j of R_0, so that it has as equation $x - g_j(t) = 0$ with $g_j(t) = (t - t_j)^{l_j+1}h_j(t)$ and $h_j(t_j) \neq 0$. We have $R \equiv A_j$ along P_j. Each curve P_j has $(l_j + 1)$ points $p_{j,1}(x), \ldots, p_{j,l_j+1}(x)$ corresponding to the coordinate x.

The zeroes of dR are then described by the curve

$$P(x,t) = (\prod_{j=1}^{j=k}(t - f_j(x))^{m_j}.(\prod_{j=1}^{j=s}[(t-p_{j,1}(x))\ldots(t-p_{j,l_j+1})(x)]^{M_j}) = 0$$

and $\mathcal{G}_1$ may be defined by the non-vanishing holomorphic 1-form $A^2P^{-1}\,dR$.

3.2. Proof of Theorem 1

Let us consider, as before, a foliation $\mathcal{G}_1$ defined in $\mathbb{D}\times\mathbb{P}^1$ which is regular along $\{0\}\times\mathbb{P}^1$ and possesses a first integral $R(x,t)$. After blowing-up at some non tangency point of $\{0\}\times\mathbb{P}^1$, we will get a foliation whose restriction to a neighborhood of the strict transform of $\{0\}\times\mathbb{P}^1$ has to be proven to be holomorphically equivalent to the restriction of a foliation of some algebraic surface to a neighborhood of a projective line of selfintersection -1. We take the algebraic variety which is the closure of the space of degree d rational functions of $\mathbb{P}^1$ which have the configuration of critical points we presented, namely:

* the rational function has values $a_1,\dots,a_k$ at critical points which have orders $m_1,\dots,m_k$ respectively.

** the rational function has values $A_1,\dots,A_s$ at $(l_1+1),\dots,(l_s+1)$ critical points which have orders $M_1,\dots,M_s$ respectively.

Let us denote also by R the germ of curve in this variety parametrized as $R(x)=R_x$; it belongs to a smooth stratum B for $x\neq 0$ small and $R(0)$ belongs to $\bar{B}$, which is also an algebraic variety. Let π be a desingularisation of $\bar{B}$ and of R at the point $R(0)$. The strict transform $\tilde{R}$ of R crosses the boundary of $\pi^{-1}(B)$ at a smooth point $r\in\pi^{-1}(\bar{B})$. We have a foliation $\mathcal{R}$ in $\tilde{R}\times\mathbb{P}^1$ given by the level curves of the meromorphic function $(\tilde{p},t)\mapsto R_{\pi(\tilde{p})}(t)$, which is conjugated to the foliation in $\mathbb{D}\times\mathbb{P}^1$ defined by $dR=0$ (because $x\in\mathbb{D}\mapsto R_x$ is injective).

Next we take an algebraic curve $\tilde{S}$ in $\pi^{-1}(\bar{B})$ which passes through the point r smoothly with order of tangency N as big as we wish with $\tilde{R}$; the choice of N will depend on the statements which will follow. Consequently in $\bar{B}$ we have the algebraic family $S=\pi(\tilde{S})$ of rational functions and in $\tilde{S}\times\mathbb{P}^1$ we have the foliation $\mathcal{S}$ given by the level curves of the meromorphic function $(\tilde{q},t)\mapsto S_{\pi(\tilde{q})}(t)$. In the sequel we describe $\mathcal{S}$ and compare it to $\mathcal{R}$.

The first thing to notice is that the curves R and S can be parametrized in a neighborhood of $R(0)$ as

$$R_x(t)=\sum_{i=0}^{i=d} a_i(x)t^i/\sum_{i=0}^{i=d} b_i(x)t^i \text{ and } S_x(t)=\sum_{i=0}^{i=d} \hat{a}_i(x)t^i/\sum_{i=0}^{i=d} \hat{b}_i(x)t^i$$

in such a way that $a_i(x)=\hat{a}_i(x)$ and $b_i(x)=\hat{b}_i(x)$ up to some order as large as we want (depending on N). In the coordinates (x,t) the foliations $\mathcal{R}$ and $\mathcal{S}$ are given as the level curves of $R(x,t)=R_x(t)$ and $S(x,t)=S_x(t)$ respectively.

We have seen before how to eliminate poles and zeroes of dR with the expression $A^2P^{-1}dR = 0$ ($A = 0$ is the set of poles and $P = 0$ is the set of zeroes of dR). We start by eliminating poles of dS multiplying by a holomorphic function $\hat{A}^2$ where $\hat{A} = 0$ defines the set of poles of dS. Writing $A(x,t) = \sum_{j=0}^{j=\infty} c_j(x)t^j$ and $\hat{A}(x,t) = \sum_{j=0}^{j=\infty} \hat{c}_j(x)t^j$, it can be assumed that $c_j(x) = \hat{c}_j(x)$ up to some order as large as we want (depending again on N).

Consequently in $\bar{B}$ we may choose an algebraic family $S = \pi(\tilde{S})$ of rational functions parametrized by a map of $x \in \mathbb{D}$ near the point $S(0) = R(0)$ such that both associated foliations $d\,R = 0$ and $d\,S = 0$ are as close as we wish in $\mathbb{D} \times \mathbb{P}^1$ (in fact, we need to cover $\mathbb{D} \times \mathbb{P}^1$ by two coordinates systems; in the chart that contains $\{0\} \times \{\infty\}$ we use $R = const$ and $S = const$ to define the associated foliations, which are both regular ones; in the chart that contains $\{0\} \times \{0\}$ the foliations $d\,R = 0$ and $d\,S = 0$ are singular).

Next we need to prove that after eliminating the singularities of $\hat{A}^2 d\,S = 0$ we obtain a foliation which is regular and has the same type of tangencies with $\{0\} \times \mathbb{P}^1$ as $\mathcal{G}_1$.

Let us fix a family of disjoint polydiscs, one for each critical point of $R_0 = S_0$. If $(0, t_j)$ is a critical point, we take $\Delta_j = \{(x,t); |x| \leqslant \epsilon, |t - t_j| \leqslant \epsilon\}$. If S_x is sufficiently close to R_x and ϵ is small, the configuration of critical points of S_x in each set $K_j = \{(x,t); \frac{\epsilon}{2} \leqslant |x| \leqslant \epsilon, |t - t_j| \leqslant \epsilon\}$ is the same as the configuration of R_x. This means that for S_x we have in $K_1 \cup \ldots K_j$:

- new connected curves of level type $x \mapsto \hat{f}_1(x), \ldots, \hat{f}_k(x)$ (close to $x \mapsto f_1(x), \ldots, f_k(x)$) which correspond to critical point of orders $m_1, \ldots, m_k$; S takes the values $a_1, \ldots, a_k$ along these curves.
- new connected curves $\hat{P}_1, \ldots \hat{P}_s$ (close to $P_1, \ldots, P_s$) which are curves of level type corresponding to critical points of orders $M_1, \ldots, M_s$; S takes the values $A_1, \ldots, A_s$ along these curves. Above each $x \in \mathbb{D}$, $\frac{\epsilon}{2} \leqslant |x| \leqslant \epsilon$, $\hat{P}_j$ has $l_j + 1$ points $\hat{p}_{j,1}(x), \ldots, \hat{p}_{j,l_j+1}(x)$ in Δ_j.

Since the set of critical points of S_x inside each Δ_j is an analytic curve, we conclude that the critical curve of level type that lies in Δ_j has an extension (still denoted by $x \mapsto \hat{f}_j(x)$ or $\hat{P}_j$) which passes through the point $(0, t_j)$ and reproduces the same type of the corresponding critical curve of R_x. Each pair $f_j(x), \hat{f}_j$ agree up to an order as large as we want (depending on N) . Since $\hat{P}_j$ can be defined by the equation $x - \hat{g}_j(t) = 0$ with $\hat{g}_j(t) = (t - t_j)^{l_j+1}\hat{h}_j(t)$ and $\hat{h}_j(t_j) \neq 0$, we have that $h_j(t)$ and $\hat{h}_j(t)$ agree up to an order as large as we want (depending on N).

In other words, the critical set of the families R_x and S_x are tangent at each point $(0, t_j)$ at an order as large as we want (depending on N). Therefore we conclude that the polynomial equations in t

$$\hat{P}(x,t) = (\prod_{j=1}^{j=k}(t - \hat{f}_j(x))^{m_j}.(\prod_{j=1}^{j=s}[(t - \hat{p}_{j,1}(x))\dots(t - \hat{p}_{j,l_j+1})(x)]^{M_j}) = 0$$

and

$$P(x,t) = (\prod_{j=1}^{j=k}(t - f_j(x))^{m_j}.(\prod_{j=1}^{j=s}[(t - p_{j,1}(x))\dots(t - p_{j,l_j+1})(x)]^{M_j}) = 0$$

have coefficients that agree to an order as large as we want (depending on N).

The singular sets of the 1-forms $Ad\,R$ and $\hat{A}d\,S$ are exactly the curves of critical points of level type because of condition (**), therefore they are also given by the previous equations. We finally conclude that the 1-forms $A^2P^{-1}dR$ and $\hat{A}^2\hat{P}^{-1}dS$ agree along $\{0\} \times \mathbb{P}^1$ at an order as large as we want (depending on N).

Notice that the equality $R_0 = S_0$ implies that the germs of periodic maps associated to the points of tangency of the foliations with $\{0\} \times \mathbb{P}^1$ coincide.

Now we blow-up the point $(r, \infty) \in \{r\} \times \mathbb{P}^1$ first as a point of $\tilde{\mathcal{R}} \times \mathbb{P}^1$ and afterwards as a point of $\tilde{\mathcal{S}} \times \mathbb{P}^1$; we obtain two foliations (one is the blow-up of $\tilde{\mathcal{R}}$ and the other one is the blow-up of $\tilde{\mathcal{S}}$). We claim that they are conjugated in neighborhoods of the strict transforms of $\{r\} \times \mathbb{P}^1$. It is enough to consider both foliations in the coordinates (x, t) with their expressions $A^2P^{-1}dR$ and $\hat{A}^2\hat{P}^{-1}dS$; we blow up at the point $(0, \infty)$. The blown-up foliations have the same germs of periodic maps at the points of tangency with the strict transform of $\{0\} \times \mathbb{P}^1$ since $R(0) = S(0)$. Furthermore, they may be assumed to coincide to an order as large as we want along the strict transform of $\{0\} \times \mathbb{P}^1$. We may then apply [1] to get a conjugation between the foliations in neighborhoods of the strict transforms of $\{0\} \times \mathbb{P}^1$. This ends the proof of Theorem 1.

We point out that the foliations defined by the 1-forms $A^2P^{-1}dR$ and $\hat{A}^2\hat{P}^{-1}dS$ are not necessarily conjugated in $\mathbb{D} \times \mathbb{P}^1$.

3.3. Proof of Theorem 2

The proof of Theorem 2 does not use approximation and can be done after a suitable change of the first integral and of the coordinates on the product $\mathbb{D} \times \mathbb{P}^1$. In this subsection we suppose that $\mathcal{G}$ is a foliation in $\mathcal{D}_1$ admitting a meromorphic first integral and consider its model $\mathcal{G}_1$. The idea of the proof of Theorem 2 is to exploit the equivalence between meromorphic funtions and branched ramified coverings of the sphere onto itself. We will show that by appropriately choosing a meromorphic first integral $R = R(x, t)$ for $\mathcal{G}_1$, the map defined by $x \mapsto R_x$ for $x \neq 0$ close to 0 can be thought as a holomorphic map from $\mathbb{D}^*$ into a suitable Hurwitz space of branched covers over the sphere. To be able to extend this map to a holomorphic map defined on some punctured sphere $\mathbb{P}^1 \setminus \{v_1, \ldots, v_k\}$ we will need to control how the critical fibers of R_x (and not only the critical points!) develop along the parameter x, even when x is far from 0. In particular we will choose the meromorphic first integral to guarantee that every collapse of points in these fibers along the parameter occurs in the domain of the original foliation, and precisely around the tangency point between the foliation and the curve $x = 0$.

Let $R : \mathbb{D} \times \mathbb{P}^1 \to \mathbb{P}^1$ be a meromorphic first integral of $\mathcal{G}_1$. If we post-compose it with a non-constant rational function Q of $\mathbb{P}^1$, the level sets of $Q \circ R$ still define the same foliation. Its fibers are unions of fibers of R, and its critical fibers contain the critical fibers of R and the fibers of R over critical points of Q. By choosing Q and the x coordinate appropriately we claim that we can suppose that the first integral R for $\mathcal{G}_1$ satisfies

1. For any critical *value* $v \neq 0$ of R_0 except possibly for one of them, there is a connected component of $R^{-1}(v)$ that is not critical for R, intersecting $0 \times \mathbb{P}^1$ in two points $q, h(q)$ where h is the involution associated to $\mathcal{G}_1$ at $(0, 0)$.
2. $(x, 0) \in \mathbb{D} \times \mathbb{P}^1$ is a critical point of $R_x(t) = R(x, t)$ with critical value $R_x(0) = x^n$ where $\mathrm{ord}_0(R_0) = 2n$.

Remark that condition 1.. does not make sense for the germ of $\mathcal{G}_1$ around $x = 0$. It is a global condition that tells us that collapses of points in critical fibers occur in the domain of the foliation and precisely around $(0, 0)$. To prove that it can be attained, take a domain D where $h : D \to D$ is conjugated to a rotation and each leaf cutting $D \setminus 0$ is a disc intersecting D on two points. Take a round disc $D_r \subset R_0(D)$ containing 0. By composing R_0 with a Moebius transformation we can suppose that $D_r = \mathbb{H}$, the upper half plane in $\mathbb{C}$, and the critical values $v_1, \ldots, v_k \in \mathbb{C} \setminus R_0(D)$ of R_0 belong to a small neighbourhood of ∞.

Next take a polynomial $Q(z) = z^5 + a_4 z^4 + \ldots + a_1 z + a_0$ with real coefficients $a_i \in \mathbb{R}$ satisfying that its four critical points $c_1 < c_2 < c_3 < c_4$ in $\mathbb{C}$ lie in $\mathbb{R}$, and the equation $Q(z) = Q(c_i)$ has precisely two distinct real roots for each $i = 1, \dots, 4$. By construction the other two roots of each such equation are complex conjugate. In particular all finite critical values of Q are attained at regular points in $\mathbb{H}$. To show that the finite critical values of $Q \circ R_0$ are also attained in D it suffices to remark that in a neighbourhood $U_\rho = \{z \in \mathbb{H} : |z| > \rho\}$ for ρ sufficiently big p acts like $z \mapsto z^5$ and thus $Q(U_\rho)$ covers a pointed neighbourhood of infinity. As $Q(v_i)$ are close to ∞ we have that $Q(v_i) \subset Q(\mathbb{H})$.

Once condition 1. is satisfied, condition 2. can be obtained by a change of variables. Indeed, if R already satisfies 1. then in some connected and simply connected neighbourhood $U \subset 0 \times \mathbb{P}^1$ where the involution associated to $\mathcal{G}_1$ is defined, we can define two branched coverings: on the one hand $R_{|U}$, which is branched at 0 and a $2n : 1$ covering map around 0, and the projection $\pi : U \to V \subset \{t = 0\}$ along the leaves of the foliation from U onto an open set $V \subset \{t = 0\}$. It is branched at 0 and $2 : 1$ around it. By construction $R(x, 0) = R_0 \circ \pi^{-1}(x, 0)$ for any $(x, 0) \in V$ and it is a $n : 1$ branched cover $V \to R_0(U)$. Up to composing R with a Moebius map, we can suppose $R_0(U)$ is the unit disc $\mathbb{D}$. Let $P_n(x) = x^n$ for $x \in \mathbb{D}$ denote the branched cover $\mathbb{D} \to \mathbb{D}$. By construction there exists an injective holomorphic map $\varphi : \mathbb{D} \to V$ such that $P_n(x) = R \circ \varphi(x)$. The map $\hat{R}(x, t) := R_{\varphi(x)}(t)$ for $(x, t) \in \mathbb{D} \times \mathbb{P}^1$ satisfies both conditions 1. and 2.

Let $C = \{v_1, \dots, v_k\} \subset \mathbb{P}^1 \setminus 0$ be the set of critical values of R_0 different from 0 and $\widetilde{C} = P_n^{-1}(C)$ where $P_n : \mathbb{P}^1 \to \mathbb{P}^1$ is defined by $P_n(x) = x^n$. By construction, for each $x \in \mathbb{D} \setminus (\widetilde{C} \cap \mathbb{D})$ the rational function R_x has degree d and has critical values at $\{x^n\} \cup C$. Indeed, since the tangency point between $\mathcal{G}_1$ and $x = 0$ is simple and unique, there is a unique component of the tangency divisor between $\mathcal{G}_1$ and the vertical fibration, and it corresponds to the set $t = 0$ by construction. Each other critical value of R_0 produces a critical value of R_x having a critical point at the point of intersection of the corresponding leaf with the fiber $\{x\} \times \mathbb{P}^1$. The restriction of R_x to $R_x^{-1}(\mathbb{P}^1 \setminus C \cup \{x\})$ defines a topological degree d covering having monodromy in a conjugacy class of a subgroup G_x of the symmetric subgroup in d symbols. By continuity the class of G_x is constant G for all $x \in \mathbb{D} \setminus C$. By connectedness of the covering we know that G acts transitively on each fiber.

Let $\mathcal{H}$ be the Hurwitz space associated to the triple $(d, k+1, G)$, that is, the space of isomorphism classes of topological coverings of the sphere minus $k+1$ points having degree d and monodromy conjugated to G. Two coverings X, X' are isomorphic if there exists a homeomorphism between the covering spaces $H : X \to X'$ such that $\pi = \pi' \circ H$, where π, π' denote the covering projections. In particular for two coverings to be equivalent they need to omit the same set of values on the sphere. Let $\mathcal{V}$ be the set of unordered $(k+1)$-uples of distinct points in $\mathbb{P}^1$. Hurwitz (see [7] or [4]) showed that the projection $P : \mathcal{H} \to \mathcal{V}$, defined by associating to any class of coverings the set of values it omits on the sphere, is itself a topological covering map. We have a natural, continuous, non-constant map $f : \mathbb{D} \setminus \widetilde{C} \to \mathcal{V}$ defined by $f(x) = P([R_x])$. If we take the coordinates in $\mathbb{P}^1$ we took before it can be written as $f(x) = [\{x^n, v_1, \ldots, v_k\}] \in \mathcal{V}$ and it extends naturally to a map $f : \mathbb{P}^1 \setminus \widetilde{C} \to \mathcal{V}$ that is actually holomorphic. To lift f to a map $F : \mathbb{P}^1 \setminus C \to \mathcal{H}$ continuously it suffices to guarantee that at the fundamental group level we have the inclusion $\mathrm{Im} f_* \subset \mathrm{Im} P_*$. This condition is satisfied since we can find generators $\gamma_1, \ldots, \gamma_{k-1}$ of the fundamental group of $\mathbb{P}^1 \setminus \widetilde{C}$ whose images lie in $\mathbb{D} \setminus (\widetilde{C} \cap \mathbb{D})$, and thus the loops $t \mapsto f(\gamma_i(t))$ in $\mathcal{V}$ lift to loops $t \mapsto [R_{\gamma_i(t)}]$ in $\mathcal{H}$. The resulting F has finite fibers and is holomorphic when we consider the unique complex structure on $\mathcal{H}$ for which P is holomorphic (recall that $\mathcal{V}$ already carries a holomorphic structure).

For each $x \in \mathbb{P}^1 \setminus \widetilde{C}$, by pulling back the complex structure from $\mathbb{P}^1$ through the branched covering, we can consider $F(x)$ as a degree d meromorphic function defined on $\mathbb{P}^1$, hence rational of degree d. Since F is holomorphic, we get a new holomorphic map $\mathbb{P}^1 \setminus \widetilde{C} \to \mathrm{Rat}_{\leqslant d}$. By construction it has finite fibers. Hence it has no essential singularity and it extends to a holomorphic map $\mathbf{F} : \mathbb{P}^1 \to \mathrm{Rat}_{\leqslant d}$.

By construction and uniqueness of complex structure on the sphere, there exists for each $x \in \mathbb{D} \setminus \widetilde{C}$ a Moebius transformation H_x such that $R_x \circ H_x = F(x)$. In particular, by pulling R back by the change of coordinates $(x,t) \mapsto (x, H_x(t))$ defined in a neighbourhood of $x = 0$ we have that the the germ of $x \mapsto \mathbf{F}(x)$ at 0 describes the pull back of the foliation $\mathcal{G}_1$. This foliation extends to $\mathbb{P}^1 \times \mathbb{P}^1$ by the level sets of $F(x,t) = \mathbf{F}(x)(t)$. By blowing up a point of transversality of the foliation and the central fibre and contracting the strict transform of the fibre we obtain a foliation in $\mathbb{P}^1 \times \mathbb{P}^1$ having a singularity in $\mathcal{D}_1$ with the same holonomy involution as $\mathcal{G}$ modulo conjugation by the Moebius transforamtion H_0. As will be seen in Section 4.2 two foliations in $\mathcal{D}_1$ having the same involution modulo conjugation by a Moebius transformation are analytically equivalent. Hence we have that the germ $\mathcal{G}$ is equivalent to the germ of that singularity. The obtained foliation is obviously defined by polynomial equations.

This proof cannot be extended to other foliations in $\mathcal{D}$ in general because there appear many components of the tangency divisor between $\mathcal{G}_1$ and the vertical foliation and there is no way of finding a coordinate where all the curves of critical values can be extended in the same parametrization to $\mathbb{P}^1$. Even if the extension existed there would be intersections of the parametrized curves of critical values and we would have no control over the monodromies around those intersection points. It is for this reason that instead of trying to extend the germ of curve $x \mapsto R_x$, we have approximated it in Section 3.2 by another one which has a global extension.

4. The general case

In the previous sections we always worked under the hypothesis of the existence of a first integral for the foliation in $\mathcal{D}$. In the general case we cannot hope to get extensions of Theorem 2. :

THEOREM 3. — *In any topological class in $\mathcal{D}$ there exist uncountably many elements that are not holomorphically equivalent to foliations defined by polynomial equations.*

We will give the proof of Theorem 3 only in the simplest topological class $\mathcal{D}_1$ of foliations with a single simple tangency with the exceptional divisor. The other cases are covered by an equivalent argument, but for simplicity of exposition we restrict ourselves to $\mathcal{D}_1$.

By considering some coordinates $(x, y) \in (\mathbb{C}^2, 0)$, every element in $\mathcal{D}_1$ is equivalent to some germ of holomorphic foliation defined by a germ of differential 1-form of type

$$\sum_{j\geqslant 2} b_j(x,y)dx - \sum_{j\geqslant 2} a_j(x,y)dy = 0 \tag{4.1}$$

where $a_2(x, y) = xy, b_2(x, y) = y^2$ and $xb_3(x, y) - ya_3(x, y) = \beta x^4, \beta \neq 0$.

After one blow-up $(x, t) \mapsto (x, tx)$, the foliation is regular, with only one point of tangency of order 1 with the exceptional divisor (the equation is normalized as to have the tangency point given by $t = 0$).

To each $\mathcal{F} \in \mathcal{D}_1$ we can associate a local involution $i_{\mathcal{F}}(t)$ defined for $t \in \mathbb{C}$ close to $0 \in \mathbb{C}$; moreover, it can be easily seen that for a holomorphic family $\alpha \in U \subset \mathbb{C}^m \mapsto \mathcal{F}_\alpha \in \mathcal{D}_1$, the function $(\alpha, t) \mapsto i_{\mathcal{F}_\alpha}(t)$ is holomorphic.

Let $Inv := \{i(t) = \sum_{j\geqslant 1} a_j t^j \in \mathbb{C}\{t\}, a_1 = -1, i \circ i(t) = t\}$; we consider in $\mathbb{C}\{t\}$ the norm $||\sum_{j\geqslant 0} c_j t^j|| := \sum_{j\geqslant 0} \dfrac{|c_j|}{j!}$, which induces a distance d. Since

$$Inv_k := \{i(t) \in \mathbb{C}\{t\}; i(0) = 0, i'(0) = -1 \text{ and } i \circ i(t) = t \bmod t^{k+1}\} \tag{4.2}$$

is closed in $(\mathbb{C}\{t\}, d)$ for each $k \geqslant 1$, and $Inv = \cap_{k\geqslant 1} Inv_k$, we conclude that Inv is closed in $(\mathbb{C}\{t\}, d)$.

Now we take

$$\mathcal{L}_1\{t\} := \{\sum_{j\geqslant 0} c_j t^j \in \mathbb{C}\{t\};\ \sum_{j\geqslant 0} |c_j| < \infty\} \tag{4.3}$$

Clearly $\mathcal{L}_1\{t\}$ is a vector subspace of $\mathbb{C}\{t\}$; any power series in $\mathcal{L}_1\{t\}$ defines a holomorphic function whose domain of definition contains the unit disc $\mathbb{D} = \{z \in \mathbb{C};\ |z| \leqslant 1\}$. On the other hand, the Taylor series centered at $0 \in \mathbb{C}$ of a holomorphic function defined in a neighborhood of $\bar{\mathbb{D}}$ belongs to $\mathcal{L}_1\{t\}$.

We define $||\sum_{j\geqslant 0} c_j t^j||_1 := \sum_{j\geqslant 0} |c_j|$ for $\sum_{j\geqslant 0} c_j t^j \in \mathcal{L}_1\{t\}$; with this norm $\mathcal{L}_1\{t\}$ becomes a Banach space. Let d_1 be the associated distance.

Lemma 1.— *The inclusion map from $(\mathcal{L}_1\{t\}, d_1)$ to $(\mathbb{C}\{t\}, d)$ is continuous.*

Proof. — It is enough to remark that

$$||\sum_{j\geqslant 0} c_j t^j|| = \sum_{j\geqslant 0} \frac{|c_j|}{j!} \leqslant \sum_{j\geqslant 0} |c_j| = ||\sum_{j\geqslant 0} c_j t^j||_1 \tag{4.4}$$

□

It follows that $Inv \cap \mathcal{L}_1\{t\}$ is closed in $(\mathcal{L}_1\{t\}, d)$. Therefore, $Inv \cap \mathcal{L}_1\{t\}$, endowed with the metric d_1, becomes a complete metric space, in particular a Baire space.

4.1. Realizing Involutions

We introduced in the last section a map i that takes foliations of $\mathcal{D}_1$ to involutions of $\mathbb{C}\{t\}$.

Lemma 2.— *The map $i : \mathcal{D}_1 \longrightarrow Inv$ is surjective.*

Proof. — 1) Given some $i(t) \in Inv$, we construct first a local foliation around the disc $\mathbb{D} \times \{0\}$ which has a tangency point at $(0,0)$ with this disc and whose associated involution is $i(t)$. We start by mapping $\mathbb{D} \times 0$ to $\mathbb{C} \times 0$ via some holomorphic diffeomorphism ϕ which satisfies

- $\phi(0) = 0$.
- ϕ conjugates $t \mapsto -t$ to $i(t)$.

We then extend ϕ to some holomorphic diffeomorphism Φ in a neighborhood of $\mathbb{D} \times \{0\}$ and define the foliation $\mathcal{H}$ as the image by Φ of the foliation defined as $d(x - t^2) = 0$.

2) The next step consists in the following gluing process:

- we take the surface S obtained after blowing-up $\mathbb{C}^2$ at $(0,0)$, foliated by $dt = 0$ ((x,y) are coordinates in $\mathbb{C}$, $(t = \frac{y}{x}, x)$ are coordinates in S). In S we remove a disc $\mathbb{D}_{\frac{1}{2}} \times U$, where U is a small neighborhood of $0 \in \mathbb{C}$.

- the trivial foliation $dt = 0$ in $\overline{(\mathbb{D} \setminus \mathbb{D}_{\frac{1}{2}})} \times U$ is equivalent to the restriction of $\mathcal{H}$ to a region R given as a local saturation (along the leaves of $\mathcal{H}$) of some annulus $\mathbb{A} \times \{0\}$ around $(0,0) \in \mathbb{D} \times \{0\}$. This equivalence is then used to glue $\mathcal{G}|_{S \setminus \{\overline{(\mathbb{D}\setminus\mathbb{D}_{\frac{1}{2}})} \times U\}}$ with $\mathcal{H}$; since it can be taken close to the Identity, the resulting foliation is defined around a (-1)-curve and is thus equivalent to the blow-up of an element of $\mathcal{D}_1$. $\square$

4.2. Adapting Genzmer-Teyssier

Our aim is to show that there are foliations in $\mathcal{D}_1$ which are not holomorphically equivalent to any foliation in $\mathcal{D}_1$ defined by a polynomial equation. In order to do that, we need to change the map i. Let G be the group of Moebius transformations of $\mathbb{P}^1$ which fix $0 \in \mathbb{C}$ (in the t-coordinate associated to the blow up). We consider the map

$$I : G \times \mathcal{D}_1 \longrightarrow Inv, \;\; I(g, \mathcal{F}) = g^{-1} \circ i_{\mathcal{F}} \circ g. \tag{4.5}$$

Remark. — In fact the map of Lemma 2 induces a bijection between $[\mathcal{D}_1]$ and Inv/G (see [1]).

Let $\mathcal{D}_1^{(k)}$ denote the subset of elements of $\mathcal{D}_1$ defined by a polynomial equation of degree k. The goal is therefore to prove that

$$\cup_k I(G \times \mathcal{D}_1^{(k)}) \neq Inv \tag{4.6}$$

We follow the procedure exposed in [5]. We have to prove that the image of an embedding $\xi : \overline{\mathbb{D}}^l \longrightarrow (Inv, d)$ leaves a trace in $Inv \cap \mathcal{L}_1\{t\}$ which has empty interior in the topology defined by d_1.

Let us consider then some $f \in Im(\xi) \cap \mathcal{L}_1\{t\}$ and $0 < \lambda < 1$. Any power series defined as $f_\lambda(t) = \lambda^{-1} f(\lambda t)$ belongs to $\mathcal{L}_1\{t\}$ and $d_1(f_\lambda, f) \to 0$ as $\lambda \to 1$; furthermore, the radius of convergence of f_λ is greater than 1. If for some sequence $\lambda_m \to 1$ it happens that $f_{\lambda_m} \notin Im(\xi)$, we are done; otherwise we replace f by some d_1-close $f_{\bar{\lambda}}$ and we still have $f_{\bar{\lambda}} \in Im(\xi) \cap \mathcal{L}_1\{t\}$. In order to simplify the notation we use f instead of $f_{\bar{\lambda}}$.

We then have $f = -t + \sum c_j t^j \in Im(\xi) \cap \mathcal{L}_1\{t\}$, with radius of convergence greater than 1. The tangent space $T_f Im(\xi)$ has some finite dimension l. Any element in $T_f Im(\xi)$ is a power series $\sum a_j t^j \in \mathbb{C}\{t\}$; after truncating the elements of $T_f Im(\xi)$ up to some sufficiently high order m_0, we still have a linear subspace of dimension l. Therefore, for each $m \geqslant m_0$, a power series in $T_f Im(\xi)$ is completely determined once we know the first m coefficients.

Now we consider the path $\alpha(u) := h_u^{-1} \circ f \circ h_u$, where $h_u(t) = t + ut^m$ for $m \geqslant 0$. Clearly h_u^{-1} is well defined in some disc of radius greater than 1 for $|u|$ small enough. This guarantees that $\alpha(u)$ is inside $Inv \cap \mathcal{L}_1\{t\}$. The tangent vector $\alpha'(0)$ (*which we intend to prove that is transverse to* $T_f Im(\xi)$) has its $(m-1)$-jet equal to zero, therefore $\alpha'(0) = 0$ if it belongs to $T_f Im(\xi)$. But an easy computation shows that

$$\alpha(u)(t) = h_u^{-1} \circ f \circ h_u(t) = -t + \sum_{j=2}^{m-1} c_j t^j + (c_m - 2u)t^m + \cdots \tag{4.7}$$

and then

$$\alpha'(0) = -2t^m + \cdots \tag{4.8}$$

which is a contradiction that proves Theorem 3.

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