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Johannes Kolb

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An Analytic Description of Local Intersection Numbers at Non-Archimedian Places for Products of Semi-Stable Curves

Johannes Kolb[(1)]

Résumé. — Nous généralisons une formule de Shou-Wu Zhang [8, Thm 3.4.2], qui donne une description en termes d'analyse élémentaire pour le nombre d'intersections arithmétiques locales de trois diviseurs de Cartier à support dans la fibre spéciale sur l'auto-produit d'une surface arithmétique semi-stable. Par un argument d'approximation, Zhang étend sa formule à une formule pour les nombres d'intersections arithmétiques locaux de trois fibrés en droites avec des métriques adéliques sur l'auto-produit d'une courbe sous condition que le fibré en droites sous-jacent soit trivial. En utilisant les résultats en théorie de l'intersection de [5] nous généralisons les résultats de Zhang aux d-iemes auto-produits pour un nombre naturel arbitraire d. Pour que les approximations convergent, nous devons supposer que le nombre naturel d satisfait une certaine condition d'annulation [5, 4.7]. Cette condition est satisfaite au moins pour $d \in \{2, 3, 4, 5\}$[2]

Abstract. — We generalize a formula of Shou-Wu Zhang [8, Thm 3.4.2], which describes local arithmetic intersection numbers of three Cartier divisors with support in the special fiber on a self-product of a semi-stable arithmetic surface using elementary analysis. By an approximation argument, Zhang extends his formula to a formula for local arithmetic intersection numbers of three adelic metrized line bundles on the self-product of a curve with trivial underlying line bundle. Using the results on intersection theory from [5] we generalize these results to d-fold self-products for arbitrary d. For the approximations to converge, we have to assume that d satisfies the vanishing condition [5, 4.7], which is true at least for $d \in \{2, 3, 4, 5\}$[3]

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(1) Universitaet Regensburg, Mathematik, 93040
johannes.kolb@mathematik.uni-regensburg.de

(2) Récemmment Omid Amini a annoncé à l'auteur qu'il pouvait démontrer que la condition d'annulation [5, 4.7] est satisfaite pour tous les nombres naturels d.

(3) Recently Omid Amini has announced to the author that he can prove the vanishing condition [5, 4.7] for all natural numbers d.

Article proposé par Damien Rössler.

Contents

1. Introduction

Let R be a complete discrete valuation ring with algebraically closed residue field k. We denote the quotient field $\mathrm{Quot}(R)$ by K and a uniformizing element with $\pi \in R$. Furthermore let S denote the scheme $\operatorname{Spec} R$ with generic point η and special point s. Let X be a regular strict semi-stable S-scheme. We denote by $\mathrm{CaDiv}_{X_s}(X)$ the group of Cartier divisors on X with support in the special fiber X_s. In [5] we studied the pairing

$$\begin{aligned} \left(\mathrm{CaDiv}_{X_s}(X)\right)^{d+1} &\to \mathbb{Z}, \\ (C_0, \dots, C_d) &\mapsto \mathrm{ldeg}(C_0 \cdot \cdots \cdot C_d). \end{aligned} \tag{1.1}$$

given by the intersection product and the degree map.

For regular strict semi-stable curves it is easy to give an analytic description of this pairing. Let therefore X be a regular strict semi-stable curve. The group of Cartier divisors $\mathrm{CaDiv}_{X_s}(X)$ with support in the special fiber coincides with the free abelian group generated by the vertices of $\Gamma(X)$, the reduction graph of X. If we endow the graph $\Gamma(X)$ with a metric such that each edge has length 1, we can describe Cartier divisors by continuous functions $f_C : |\Gamma(X)| \to \mathbb{R}$, which are affine on each edge and take only values from $\mathbb{Z}$ on the vertices. We denote the set of functions of this type by $\mathcal{C}^{\mathrm{lin}}_{\vartriangle}(\Gamma(X))$.

Thus the intersection pairing induces a bilinear pairing

$$\langle \cdot, \cdot \rangle : \mathcal{C}^{\mathrm{lin}}_{\vartriangle}(\Gamma(X)) \times \mathcal{C}^{\mathrm{lin}}_{\vartriangle}(\Gamma(X)) \to \mathbb{R},$$

which is uniquely determined by

$$\langle f_{C_1}, f_{C_2}\rangle = \mathrm{ldeg}(C_1 \cdot C_2).$$

For this pairing we can give an elementary analytic description:

FACT 1.1. — *Let X be a proper regular strict semi-stable curve and $f_1, f_2 \in \mathcal{C}^{\mathrm{lin}}_{\vartriangle}(\Gamma, \mathbb{Q})$. Then*

$$\langle f_1, f_2\rangle = -\int_{\Gamma(X)} (D^1 f_1)(D^1 f_2) \tag{1.2}$$

holds. Here we endow the edges of $\Gamma(X)$ with an arbitrary orientation and denote by $D^1 f$ the differential of the function f in the direction of this orientation.

Using approximation arguments we may continue (1.2) to a pairing of piecewise smooth functions given by the same equation (note that the differential D^1 of piecewise smooth functions is defined almost everywhere, therefore we may still integrate). The aim of this paper is to give a generalization of this description for higher-dimensional schemes following an idea of Zhang ([8, §3]).

We restrict ourselves to d-fold self-products of a smooth curve, since we can describe a regular strict semi-stable model explicitly for these K-schemes: If X_η is a smooth curve over K, then the semi-stable reduction theorem yields a semi-stable model of X (possibly after base change). The desingularisation of Gross and Schoen yields a uniquely defined regular strict semi-stable scheme W of X^d (see [5, Thm 3.3]).

We use as generalization of the reduction graph the geometric realization of the simplicial reduction set $|\mathscr{R}(W)| = |\Gamma(X)|^d$; this is a locally affine space, which encodes the incidence relations between the components of W_s. Each Cartier divisor $C \in \mathrm{CaDiv}_{W_s}(W)$ with support in W_s is a model of the trivial line bundle, induces therefore a metric on the trivial line bundle, which corresponds to a piecewise affine function

$$f_C : |\Gamma(X)|^d \to \mathbb{R}.$$

We denote the set of these piecewise affine functions by $\mathcal{C}^{\mathrm{lin}}_{\vartriangle}(\Gamma(X)^d)$. The intersection pairing (1.1) thus induces a multi-linear pairing between piecewise affine functions

$$\langle \cdot, \dots, \cdot\rangle : \left(\mathcal{C}^{\mathrm{lin}}_{\vartriangle}(\mathscr{R}(W))\right)^{d+1} \to \mathbb{R},$$

defined by $\langle f_{C_0}, \dots, f_{C_d}\rangle = \mathrm{ldeg}(C_0 \cdot \dots \cdot C_d)$.

We see this pairing as the local contribution to the intersection product of metrized line bundles with underlying trivial line bundles. We want to extend this pairing to a larger class of metrized line bundles. By approximation we may continue $\langle \cdot, \dots, \cdot \rangle$ on the set of piecewise smooth functions, $\mathcal{C}^\infty_\triangle(\Gamma(X)^d)$ and give an analytical formula for this pairing, if a certain vanishing condition holds. This vanishing condition only depends on the positive integer d and can be verified explicitly in the cases $d = 2, 3$. For a piece-wise smooth function $f \in \mathcal{C}^\infty_\triangle(\Gamma(X)^d)$ let $f^{(1)}, f^{(2)}, \dots$ denote a certain approximation by piece-wise affine functions (see Definition 3.29).

Theorem 1.2. — *If $d \in \mathbb{N}$ satisfies the vanishing condition of Definition 3.31, then for all functions $f_0, \dots, f_d \in \mathcal{C}^\infty_\triangle(\Gamma^d)$, the limit*

$$\langle f_0, \dots, f_d \rangle := \lim_{n \to \infty} \langle f_0^{(n)}, \dots, f_d^{(n)} \rangle_{W,n}$$

exists. It can be calculated by

$$\langle f_0, \dots, f_d \rangle = \sum_{\mathcal{P}\ \mathit{Partition}} \frac{1}{2^{d+|\mathcal{P}|}} \sum_{\substack{v_0,\dots,v_d \in \mathbb{F}_2^d, \\ \sum \alpha(v_i,\mathcal{P}) = d+|\mathcal{P}|}} \operatorname{ldeg}_{I^d}\left(\prod_{i=0}^d F_{v_i}\right) \int_{\mathcal{D}_\mathcal{P}} \prod_{i=0}^d D^{v_i}_{\alpha(v_i,\mathcal{P})}(f_i). \tag{1.3}$$

In this equation the terms $D^{v_i}_{\alpha(v_i,\mathcal{P})}(f_i)$ are elementary analytical expressions in the functions f_i. The coefficients of the integrals $\operatorname{ldeg}_{I^d}\left(\prod_i F_{v_i}\right)$ are independent of X and can be calculated using the simplicial calculus of [5, 4.3].

The case $d = 2$ was already proved by Zhang in [5, Prop 3.3.1, Prop 3.4.1] for a variant of the Gross-Schoen-desingularisation. Our proof follows essentially the proof of Zhang, but adds new ideas to the proof: First of all we use the original method of Gross-Schoen [4] for desingularisation. This requires more technical effort for the description of the special fiber, but gives a model with a simpler structure.

In [5, Def 4.22, Prop 4.23] we developed a localization argument which reduces the computation of intersection to a simple local situation. It reduces in fact the computation of intersection numbers to $\left(\operatorname{Spec} R[x_0, x_1]/(x_0x_1 - \pi)\right)^d$, a simple standard-scheme which is independent of X.

For the derivation of Theorem 3.33 we have to calculate intersection numbers of divisors which correspond to certain vertices of $\Gamma(X)^d$. The vertices serve as nodes for the approximation of piecewise smooth functions $f_0, \dots, f_d \in \mathcal{C}^\infty_\triangle(\Gamma(X)^d)$ from (1.3). To get the limit in Theorem 3.33, Zhang uses a laborious investigation. The resulting formula does not contain the intersection numbers calculated in the first place, which makes the generalization difficult. We are able to simplify the argument by using a Fourier transform. This explains the terms $\mathcal{F}(f_i)$, which appear in the calculation. An elementary argument allows us to show that these Fourier transforms converge to the generalized differentials $D_y^x(f)$.

Contrary to the method of Zhang we are able to explain the coefficients of the integrals as intersection numbers $\operatorname{ldeg}(F_{v_0} \cdot \dots \cdot F_{v_d})$ of certain divisors F_v in the standard situation I^d.

Especially in the case $d = 3$ the intersection numbers of the F_v can be calculated completely, so it is possible to give an explicit description of the pairing in this case. In order to simplify the exposition in the introduction, we restrict to the special case of functions which are smooth on each cube of Γ^3, a set denoted by $\mathcal{C}^\infty_\square(\Gamma^3)$.

Theorem 1.3. — *Let $f_0, \dots, f_3 \in \mathcal{C}^\infty_\square(\Gamma^3)$ be functions smooth on cubes. Then the limit of the quadruple pairing $\langle f_0, f_1, f_2, f_3\rangle$ exists and can be calculated as*

$$\lim_{n\to\infty} \langle f_0^{(n)}, f_1^{(n)}, f_2^{(n)}, f_3^{(n)}\rangle = \int_{\Gamma^3} \sum_{\substack{v_0,v_1,v_2,v_3 \in \mathbb{F}_2 \\ \{v_0,v_1,v_2,v_3\}\in B}} \prod_{i=0}^{3} D^{v_i}_{|v_i|}(f_i),$$

where the set $B \subset \mathcal{P}(\mathbb{F}_2^3)$ is defined as follows

$$\begin{aligned} B := \Big\{ &\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\},\\ &\{(1,0,0),(0,1,0),(1,0,1),(0,1,1)\},\\ &\{(1,0,0),(0,0,1),(1,1,0),(0,1,1)\},\\ &\{(0,1,0),(0,0,1),(1,1,0),(1,0,1)\}\Big\}. \end{aligned}$$

The author has the strong impression, that the main results are true not only for the self-product of a semi-stable curve, but also for products of arbitrary semi-stable curves, i.e. with $W = X_1 \times X_2 \times \dots \times X_d$ where each X_i is a semi-stable model of a smooth proper curve over S. The elaboration of the details is left to the interested reader.

It would be an interesting question to determine the largest class of functions, for which Theorem 3.33 or a variant remains valid, assuming the vanishing conjecture. Also the dependence of (1.3) on the choice of the semi-stable model could be interesting for further research.

This investigation can be seen as a first step to an analytical description of local arithmetic intersection numbers at non-archimedean places in Arakelov theory. For this interpretation of Theorem 3.33 let $(X_\eta)^{\mathrm{an}}$ denote the Berkovich analytification of X_η. By a canonical construction the model X yields a skeleton of $(X_\eta)^{\mathrm{an}}$ which coincides with the geometric realization of $\mathscr{R}(W)$. Thus Theorem 3.33 gives an intersection number for functions on $(X_\eta)^{\mathrm{an}}$, if these functions are induced by metrics on the trivial line bundle on X_η.

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2. Metrized Line Bundles and The Reduction Map

Let R be a complete discrete valuation ring with uniformizer π, whose residue field $k := R/(\pi)$ is algebraically closed. Let $S := \operatorname{Spec} R = \{\eta, s\}$ be the corresponding spectrum with generic point η and special point s. Let X be a proper regular strict semi-stable S-curve, whose generic fiber X_η is smooth and whose special fiber is a reduced divisor with strict normal crossings. After choosing a total ordering $\leqslant$ on $X_s^{(0)}$, the components of X_s we can define a directed reduction graph $\Gamma(X)$ and a well-defined Gross-Schoen desingularization of X^d by the following algorithm:

Algorithm 2.1. — *Let $d \in \mathbb{N}$ and X be a regular strict semi-stable S-curve with total ordering $\leqslant$ on $X_s^{(0)}$ and $\Gamma(X)$ be a simplicial set without multiple simplices. We denote the product by $W_0 := X^d$. Since the components of X_s are geometrically integral, we can describe the irreducible components of $(W_0)_s$ as product*

$$(W_0)_s^{(0)} = X_s^{(0)} \times \cdots \times X_s^{(0)}.$$

We endow this product $(W_0)_s^{(0)}$ with the lexicographical order and denote the elements in ascending order $B_1, \dots B_k$. Now denote by B_1' the irreducible component B_1 endowed with the induced reduced structure and set $W_1 := \mathrm{Bl}_{B_1'}(W_0)$. Inductively let $B_i' \subseteq W_i$ be the strict transform of the irreducible component B_i endowed with the induced reduced structure and set $W_{i+1} := \mathrm{Bl}_{B_i'}(W_i)$. The last scheme in this chain W_k is also denoted by $W(X, \leqslant, d) := W_k$. These blowups introduce no new components in the special fiber $(W_k)_s$, so the lexicographical ordering on $(W_0)_s^{(0)}$ induces also a total ordering on $W(X, \leqslant, d)$.

We proved in [5, Thm 3.3]:

Theorem 2.2. — *The scheme $W := W(X, \leqslant, d)$ is a regular strict semi-stable R-scheme and for the simplicial reduction set the equation*

$$\mathscr{R}(W) = \Gamma(X)^d$$

holds.

This desingularization is denoted by W and is a regular strict semi-stable scheme according to the definition of de Jong [2], which means W_η is smooth, W_s is regular and the components of W_s intersect properly and with multiplicity 1 (for details see [5, Prop 4.8]).

We are about to describe the intersection pairing as paring of functions on the reduction set. The relation between these functions and the Chow group $\mathrm{CH}^1_{W_s}(W)$ is best described using metrics on line bundles.

Thus we repeat the definition of metrized line bundles on a complete discrete valued field according to [7]. In particular we have to deal with metrics induced by models of the trivial line bundle. These metrics are in connection with the irreducible components of the special fiber. Using metrics we find an alternative description of the reduction map $\mathrm{Red} : W(\bar{K}) \to |\mathscr{R}(W)|$. Eventually this allows us to define a bijection between cycles on W and functions on the special fiber.

2.1. Metrics on Line Bundles

Let R be a complete discrete valuation ring with algebraically closed residue field and $|\cdot|$ a norm on its quotient field $\operatorname{Quot} R$. In most cases we normalize $|\cdot|$ by setting $|\pi| = 1/b$ for a fixed basis $b \in \mathbb{R}_{>0}$.

We denote by $\bar{K}$ an algebraic closure of K and continue the norm $|\cdot|$ on $\bar{K}$. Since R is complete the completion is unique. We denote by $R_{\bar{K}}$ the ring of integers $\{a \in \bar{K} \mid |a| \leqslant 1\}$ in $\bar{K}$.

Definition 2.3. — *Let W_K be a K-scheme and L a line bundle on W_K. Let $x \in W_K(\bar{K})$ be a $\bar{K}$-rational point. The global sections of $x^*(L)$ are called* geometric fiber *of L in x and denoted by $L(x) := \Gamma(\operatorname{Spec} \bar{K}, x^*(L))$. A family of morphisms in the geometric fibers $(\|\cdot\|_x : L(x) \to \mathbb{R})_{x \in W_K(\bar{K})}$ is called* metric on L, *if the map $\|\cdot\|_x$ is a $(\bar{K}, |\cdot|)$-norm for each $x \in W_k(\bar{K})$.*

Important metrics are given by models of the line bundle L (see [7, (1.1)]):

Definition 2.4. — *Let W be a proper S-scheme and L a line bundle on W_η. Let $\mathcal{L}$ be a line bundle on W, which is a model of L by the isomorphism $\varphi : \mathcal{L}_\eta \to L$. Then there is a metric $\|\cdot\|_x$ on L defined for a geometric point $x \in W(\bar{K})$ and an element of the geometric fiber $l \in L(x)$ as follows: Let $\tilde{x} : \bar{S} \to W$ be the unique continuation of $x : \bar{K} \to W$ by the valuative criterion of properness and let $\tilde{\varphi}$ be the canonical isomorphism induced by φ:*

$$\tilde{\varphi} : \tilde{x}^*\mathcal{L}(\tilde{S}_\eta) \to x^*L(\operatorname{Spec}\bar{K})$$

Localization gives a canonical injection of the $\bar{R}$-module $\Gamma(\bar{S}, \mathcal{L})$ into the $\bar{K}$-module $\Gamma(\bar{S}_{\bar{\eta}}, \mathcal{L})$, which allows us to identify $\Gamma(\bar{S}, \mathcal{L})$ with a subset of $\Gamma(\bar{S}_{\bar{\eta}}, \mathcal{L})$. We set

$$\|l\|_x := \inf_{a \in \bar{K}^\times} \left(|a| \middle| a^{-1}\tilde{\varphi}^{-1}(l) \in \Gamma(\bar{S}, \mathcal{L})\right).$$

Remark 2.5. — Let W be a proper S-scheme and D a Cartier divisor with support $\operatorname{supp} D \subseteq W_s$ in the special fiber of W. Then the line bundle $\mathcal{O}_W(D)$ can be regarded as model of the trivial line bundle $\mathcal{O}_{W_\eta}$ by the isomorphism

$$\varphi_D : \mathcal{O}_W(D) \mid_{W_\eta} \xrightarrow{\sim} \mathcal{O}_W \mid_{W_\eta},$$

which maps the canonical section $s_D \in \mathcal{O}_W(D)(W_\eta)$ onto $1 \in \mathcal{O}_W \mid_{W_\eta}$.

In this special case the metrics are given by the values of the one-section, thus by a function. We will show, that this function plays the role of a coordinate function:

Definition 2.6. — *Let $|\cdot|$ be a norm on $\bar{K}$, W a proper S-scheme and D a Cartier divisor on W with $\operatorname{supp} D \subseteq W_s$. The model $(\mathcal{O}_W(D), \varphi_D)$ of the trivial line bundle induced by D yields by Definition 2.4 a metric $\|\cdot\|_{\mathcal{O}_W(D)}$ on $\mathcal{O}_{W_\eta}$. We evaluate this metric on the one section $1 \in \Gamma(W_\eta, \mathcal{O}_{W_\eta})$ to get a function*

$$f_D^{|\cdot|} : W_\eta(\bar{K}) \to \mathbb{R}, x \mapsto -\log_b(\|1\|_{\mathcal{O}_W(D),x}).$$

It is called the tropical coordinate function *of the divisor D. If the norm is evident by the context we will denote the coordinate function also by f_D.*

Proposition 2.7. — *Let W be a proper S-scheme and D a Cartier divisor on W with support in W_s. Let $x \in W(\bar{K})$ be a $\bar{K}$-rational point and $U \subseteq W$ an open subset, in which x specializes and the Cartier divisor D is given by a rational function $f \in \mathcal{K}_W(U)$. Then $f \mid_{U_\eta} \in \mathcal{O}_W(U_\eta)$ holds and we have*

$$f_D(x) = -\log_b\left(\left|x^\#\left(f|_{U_\eta}\right)\right|\right),$$

where $x^\#$ denotes the canonical map $x^\# : \mathcal{O}_W(U_\eta) \to \bar{K}$.

Proof. — Since the support of D is outside of U_η, we have $f\,|_{U_\eta} \in \mathcal{O}_W(U_\eta)$. Furthermore D is a principal divisor on U and therefore allows us to identify $\mathcal{O}_U$ with $\mathcal{O}_U(D)$. Note that this is nevertheless a non-trivial model of $\mathcal{O}_{U_\eta}$, since the model morphism $\varphi : \mathcal{O}_U(D)\,|_{U_\eta} \to \mathcal{O}_{U_\eta}$ maps the global section $f \in \Gamma(U_\eta, \mathcal{O}_U(D))$ onto $1 \in \Gamma(U_\eta, \mathcal{O}_{U_\eta})$.

Denote by $\tilde{x} : \bar{S} \to U$ the continuation of x as before. Then the morphism $\tilde{x}^*\mathcal{O}_U(D)(U) \to \tilde{x}^*\mathcal{O}_U(D)(U_\eta)$ is just the injection $\bar{R} \to \bar{K}$. The section $1 \in \mathcal{O}_{\bar{S}}(\bar{S}_\eta)$ is mapped by the model morphism on $x^\#(f) \in \bar{K}$. We therefore get

$$f_D(x) = -\log_b(\|1\|_{\mathcal{O}_W(D),x}) = -\log_b(\inf_{a \in K^\times} \{|a| \mid a^{-1}x^\#(f) \in R\})$$

$$= -\log_b(\left|x^\#(f)\right|).$$

□

Proposition 2.8. — *The coordinate function of the trivial Cartier divisor is the zero function. Let D_1, D_2 be Cartier divisors on W with* $\operatorname{supp} D_i \subseteq W_s$. *Then $f_{D_1+D_2} = f_{D_1} + f_{D_2}$ holds.*

Proof. — For each point $x \in W(R_{\bar{K}})$ it suffices to examine a neighborhood U of x which trivializes D. In U the claim follows directly from Proposition 2.7. □

Coordinate functions are compatible with base change:

Proposition 2.9. — *Let W, V be proper integral S-schemes, $g : W \to V$ a dominant morphism and D a Cartier divisor on V with support in V_s. According to [3, Prop 11.48] there exists a well-defined Cartier divisor g^*D. Then for each point $x \in W(\bar{K})$ the equation*

$$f_D(g(x)) = f_{\varphi^*(D)}(x).$$

holds.

Proof. — Let $x \in W(\bar{K})$ a $\bar{K}$-rational point and $\tilde{x} : \bar{S} \to W$ its continuation on $\bar{S}$. We denote its image under g by $\tilde{y} := g \circ \tilde{x}$. As above it suffices to prove the claim in an open neighborhood of $\tilde{x}$. Thus we may assume that the Cartier divisor D is given on V by a rational function $f \in \mathcal{K}_V(V)$. By definition g^*D is represented by g^*f and the claim is a consequence of $\tilde{y}^\#(f) = \tilde{x}^\#(g^\#(f))$ and Proposition 2.7. □

Example 2.10. — Let W be a proper S-scheme $U \subseteq W$ an affine open subset with a dominant morphism $f : U \to L = \operatorname{Spec} R[z_0, z_1]/(z_0 z_1 - \pi)$.

Let D be a Cartier divisor on W with $\operatorname{supp}(D) \subseteq W_s$, which coincides on U with $f^*(\operatorname{div}(z_0))$.

We can describe a geometric point $x : \operatorname{Spec} \bar{K} \to U$ of U by its coordinates $(f \circ x)^{\#}(z_0), (f \circ x)^{\#}(z_1)$. If $x \in U(\bar{K})$ specializes into U, we have $(f \circ x)^{\#}(z_0), (f \circ x)^{\#}(z_1) \in R_{\bar{K}}$. Since the divisor D is given on U by $f^{\#}(z_0)$, Proposition 2.7 implies

$$f_D(x) = -\log_b(\big|(f \circ x)^{\#}(z_0))\big|).$$

Thus we get the coordinate function f_D by the valuation of the z_0-component of the point x.

2.2. The Reduction Map

Let us now study the reduction map. First we recall the definition of the reduction map [6, 2.4.2] for curves and define a natural generalization on products of curves. Then we describe this reduction map using tropical coordinates.

Definition 2.11 ([6, 2.4.2]). — *Let X be a regular strict semi-stable S-curve having a reduction set without multiple simplices. Let κ denote the inductive system*

$$\kappa = \{\tilde{K}_n \mid n \in \mathbb{N}, K \subseteq \tilde{K}_n \subseteq \bar{K}, [K_n : K] = n\}.$$

Then there are canonical isomorphisms

$$\varinjlim_{\tilde{K}_n \in \kappa} |\,\mathscr{R}(X_{\mathcal{O}_{\tilde{K}_n}})| \simeq \varinjlim_{\tilde{K}_n \in \kappa} |\mathrm{sd}_n\, \mathscr{R}(X)| \simeq |\,\mathscr{R}(X)|.$$

Denote by $\mathrm{Red}_{\tilde{K}_n} : X(\tilde{K}_n) \to \mathscr{R}(X_{\mathcal{O}_{\tilde{K}_n}})$ the map which sends a point $x \in X(\tilde{K}_n)$ to the component $C \in \mathscr{R}(X_{\mathcal{O}\tilde{K}_n})$, in which x specializes. The limit of these maps induces a map

$$\mathrm{Red} : X(\bar{K}) = \varinjlim X(\tilde{K}_n) \to \varinjlim |\,\mathscr{R}(X_{\mathcal{O}_{\tilde{K}_n}})| \simeq |\,\mathscr{R}(X)|,$$

which is called the reduction map.

We generalize the reduction map to products of curves:

Definition 2.12. — *Let X be a regular strict semi-stable S-curve having a reduction set without multiple simplices, $d \in \mathbb{N}$ and $W = W(X, <, d)$ the* *product model constructed in Algorithm 2.1. Then the map* $\mathrm{Red} : W(\bar{K}) \to |\mathscr{R}(W)|$*, which makes the diagram*

$$\begin{array}{ccc} W(\bar{K}) & \xrightarrow{\mathrm{Red}} & |\mathscr{R}(W)| \\ {\scriptstyle \mathrm{pr}_i}\big\downarrow & & {\scriptstyle \mathrm{pr}_i}\big\downarrow \\ X(\bar{K}) & \xrightarrow{\mathrm{Red}} & |\mathscr{R}(X)| \end{array}$$

for each $i = 1, \ldots d$ *commutative, is called the* reduction map.

We can give an alternative description of the reduction map using coordinate functions. Let X be a regular strict semi-stable curve having a reduction set without multiple simplices, $d \in \mathbb{N}$ and $W = W(X, <, d)$ the product model of X_η.

We identify the vertices $C \in \mathscr{R}(W)_0$ of the reduction set with irreducible components of W_s. Since W is regular strict semi-stable, each $C \in \mathscr{R}(W)_0$ represents a Cartier divisor and by Definition 2.6 we get an associated coordinate function, which we denote by $f_C : X_\eta(\bar{K}) \to \mathbb{R}$.

Theorem 2.13. — *Let X be a regular strict semi-stable curve, $d \in \mathbb{N}$ and $W = W(X, <, d)$ the product model of X_η^n. Let $x \in W(\bar{K})$ be a geometric point. Then the values of the coordinate functions $(f_C(x))_{C \in \mathscr{R}(W)_0}$ yield a probability distribution on $\mathscr{R}(W)_0$ with support in a simplex of $\mathscr{R}(W)$. They determine a point $p \in |\mathscr{R}(W)|$, which coincides with* $\mathrm{Red}(x) \in |\mathscr{R}(W)|$.

We split the proof in three parts. First let us show that $(f_C(x))_{C \in \mathscr{R}(W)}$ gives a probability distribution:

Proposition 2.14. — *Let W be a proper regular strict semi-stable S-scheme having a reduction set without multiple simplices. Let $x \in W(\bar{K})$ be a geometric point. Then $\sum_{C \in \mathscr{R}(W)_0} f_C(x) = 1$ and $f_C(x) \geqslant 0$ for all $C \in \mathscr{R}(W)_0$. For each $C \in \mathscr{R}(W)_0$ the relation $f_C(x) > 0$ holds, iff x specializes into the component C.*

Proof. — The special fiber W_s is given by the principal divisor $D_\pi = \operatorname{div} \pi$. Since X_s is reduced, $\sum_{C \in \mathscr{R}(X)_0} C = D_\pi$ holds and by Proposition 2.8 we get $\sum_{C \in \mathscr{R}(X)_0} f_C(x) = f_{D_\pi}(x)$. By Proposition 2.7 $f_{D_\pi}(p) = 1$ for each p, which implies the first claim.

For the second claim let $\tilde{x} \in X(R_{\bar{K}})$ be the continuation of $x \in X(\bar{K})$. Let $U \subset X$ be an open neighborhood of $\tilde{x}$, in which the effective Cartier divisor C is trivialized by a section $h \in \mathcal{O}_U(U)$. Then Proposition 2.7 implies $f_{D_C}(x) = -\log_b(|x^\# h|) \geqslant 0$.

By definition the point x specializes into the component C, iff $\tilde{x}(\bar{s})$ is in C, this means $h_{\tilde{x}(\bar{s})}$ is in the maximum ideal $\mathfrak{m}_{\tilde{x}(\bar{s})}$ of the local ring $U_{\tilde{x}(\bar{s})}$. Therefore $\tilde{x}^{\#}(h)$ lies in the maximum ideal $\{x \in R_{\bar{K}} \mid |x| < 1\}$ of $R_{\bar{K}}$. According to Proposition 2.7 this is equivalent to $f_C(x) > 0$. $\square$

Using Proposition 2.14 we are able to prove Theorem 2.13 for $d = 1$:

Proposition 2.15. — *Let $W = X$ be a proper regular strict semi-stable S-curve. Let $n \in \mathbb{N}$ be a natural number, K_n/K a finite field extension of degree n and $R_n := \mathcal{O}_{K_n}$ the ring of integers in K. Let $x \in X(K_n)$ be a K_n-rational point. Then* $\operatorname{Red}(x)$ *coincides with the point given by the coordinate functions $(f_C(x))_{C\in\mathscr{R}(X)_0}$.*

Proof. — Denote by X_n the model of Theorem 2.18. By Proposition 2.14 it suffices to consider the components of X_s resp. $(X_n)_s$, in which the point x specializes. Thus we may assume that X has the form $X = L := \operatorname{Spec} R[x_0, x_1]/\ (x_0x_1 - \pi)$. Since $\mathscr{R}(L)$ has only one edge, the reduction set $\mathscr{R}(X_n) = \operatorname{sd}_n(\mathscr{R}(L))$ looks like

$$C_0' - C_1' - \cdots - C_n' \ .$$

Choose $i \in \{0, \dots, n\}$ such that x specializes into C_i'. Then there is a neighborhood of x of the form $U := \operatorname{Spec} R_n[y_0, y_1]$ with structure morphism

$$\begin{aligned} \operatorname{Spec} R[x_0, x_1]/(x_0x_1 - \pi) &\to \operatorname{Spec} R_n[y_0, y_1]/(y_0y_1 - \tilde{\pi}), \\ x_0 \mapsto y_0^i y_1^{i+1}, &\ x_1 \mapsto y_0^{n-i} y_1^{n-i-1}. \end{aligned}$$

We may assume that C_i' is given on U by $\operatorname{div}(y_0)$. By Proposition 2.14 we get $f_{C_i'}(x) = 1$. Using Proposition 2.14 again we deduce $f_{C_0}(x) = \frac{i}{n}$, $f_{C_1}(x) = \frac{n-i}{n}$ and the claim. $\square$

The last part is to consider the product situation $W = W(X, <, d)$:

Proposition 2.16. — *Let X be a proper regular strict semi-stable curve, further $W = W(X, <, d)$ and $x \in W(\bar{K})$. Then the point* $\operatorname{Red}(x) \in |\mathscr{R}(W)| = |\mathscr{R}(X)^d|$ *is given by the probability distribution $(f_C(x))_{C\in\mathscr{R}(W)_0}$.*

Proof. — By Definition 2.12 and Proposition 2.15 it suffices to show that for each $i \in \{1, \dots d\}$ the diagram

$$\begin{array}{ccc} W & \longrightarrow & |\mathscr{R}(W)| \simeq |\mathscr{R}(X)^d| \\ {\scriptstyle \operatorname{pr}_i}\downarrow & & {\scriptstyle \operatorname{pr}_i}\downarrow \\ X & \longrightarrow & |\mathscr{R}(X)| \end{array}$$

commutes. Let $i \in \{1, \dots d\}$. Since pr_i is a dominant morphism between reduced local noetherian schemes, there exists a pull-back of Cartier divisors pr_i^*. As W and X are regular strict semi-stable we may identify each element $C \in \mathscr{R}(X)_0$ with a Cartier divisor and have the equation

$$\mathrm{pr}_i^*(C) = \sum_{C' \in \mathscr{R}(W)_0, \mathrm{pr}_i(C') = C} C'.$$

Then Proposition 2.8 and Proposition 2.9 imply

$$f_C(x) = \sum_{C' \in \mathscr{R}(W)_0, \mathrm{pr}_i(C') = C} f_{C'}(x)$$

which is exactly the description of $\mathrm{pr}_i : \mathscr{R}(X)^d \to \mathscr{R}(X)$ in coordinate functions. □

The reduction map allows us to describe vertical Cartier divisors on X by analytic objects, precisely by functions on $|\mathscr{R}(X)|$, the geometric realization of the reduction set. We formulate this in the following proposition:

Proposition 2.17. — *Let X be a regular strict semi-stable S-curve with total ordering $<$ on $X_s^{(0)}$. Let $d \in \mathbb{N}$ and $W = W(X, \leqslant, d)$ be the product model constructed in Algorithm 2.1 and let $D \in \mathrm{CaDiv}_{W_s} W$ be a Cartier divisor with support in W_s. Then the function f_D factorizes through the reduction map. We denote the induced map by $\tilde{f}_D : |\mathscr{R}(W)| \to \mathbb{R}$. The function $\tilde{f}_D$ is affine in each simplex of $\mathscr{R}(W)$ and therefore uniquely defined by the values on the vertices $(f(C)) |_{C \in \mathscr{R}(W)_0}$. The divisor can be retrieved by*

$$D = \sum_{C \in \mathscr{R}(W)_0} f_D(C)[C].$$

Proof. — The factorization is trivial: If D is the Cartier divisor of one component $C' \in \mathscr{R}(W)_0$, then $f_{C'}$ is itself a coordinate function and therefore affine on each simplex. Since each Cartier divisor is a linear combination of divisors of this form, the claim is implied by Proposition 2.8.

Let $D = \sum_{C \in \mathscr{R}(W)_0} n_C[C]$. For each $C' \in \mathscr{R}(W)_0$ we choose a point $x \in W(\bar{K})$, which specializes only into the component C'. By Proposition 2.8 and Proposition 2.14 we get

$$f_D(x) = \sum_{C \in \mathscr{R}(W)_0} n_C f_C(x) = n_{C'}$$

and the claim is proven. □

2.3. Morphisms Between Models

The coordinate functions f_D are useful to construct morphisms between product models of the type described by Algorithm 2.1 and models arising from ramified base-change in the following way:

THEOREM 2.18. *[5, Theorem 3.1]. — Let $S := \operatorname{Spec} R$ be the spectrum of a complete discrete valuation ring. Let K_n/K be a field extension of degree $n \in \mathbb{N}$ and R_n the ring of integers in K_n. We denote $S_n := \operatorname{Spec} R_n$. Let X be a regular strict semi-stable S-curve with a total ordering on $X^{(0)}$, whose simplicial reduction set $\Gamma(X)$ has no multiple simplices. Let X_n be the scheme obtained by blowing up $X \times_S S_n$ successively in all singular points, blowing up the resulting scheme successively in all singular points, and so on $\lfloor n/2 \rfloor$ times. Then X_n is a regular strict semi-stable S_n curve with $(X_n)_{\eta_n} = (X_\eta) \times_{\operatorname{Spec} K} \operatorname{Spec} K_n$. Furthermore there exists a total ordering of $(X_n)^{(0)}$ such that there is a canonical isomorphism*

$$\Gamma(X_n) \simeq \mathrm{sd}_n(\Gamma(X)),$$

which maps the simplicial reduction set of X_n to the canonical n-fold subdivision $\mathrm{sd}_n(\Gamma(X))$ of the simplicial set $\Gamma(X)$ (see Definition A.7).

The coordinate functions f_D are useful to construct morphisms between models of the type described by Algorithm 2.1 and

THEOREM 2.19. — *Let K_n be an algebraic field extension of degree n, R_n the ring of integers of a finite field extension K_n/K and $S_n := \operatorname{Spec} R_n$. Let X be a regular strict semi-stable S-curve and X_n the model of $X_\eta \times_K K_n$ constructed in Theorem 2.18. Let $W = W(X, <, d)$ denote the product model of X_η^d and $W_n = W(X_n, <, d)$ the analogous model of $(X_n)_{\eta_n}^d$. Then there exists a morphism $\varphi : W_n \to W$.*

The proof uses the universal property of blow-up. Its basic idea is the following:

LEMMA 2.20. — *Let W be a proper regular strict semi-stable scheme and D_1, D_2 two effective Cartier divisors with support in W_s. The scheme theoretic intersection $D_1 \cap D_2$ is a Cartier divisor iff the function $\min(f_{D_1}, f_{D_2})$ is affine on each simplex of $\mathscr{R}(W)$. In this case the Cartier divisor $D := D_1 \cap D_2$ provides*

$$f_D = \min(f_{D_1}, f_{D_2}).$$

Proof. — Assume that the scheme theoretic intersection $D = D_1 \cap D_2$ is a Cartier divisor. Then f_D is affine on each simplex and it suffices to show that $f_D = \min(f_{D_1}, f_{D_2})$. Let $\tilde{x} \in W(R_{\bar{K}})$ be a $R_{\bar{K}}$-valued point of X and $U \subseteq W$ a trivializing neighborhood of the divisors D, D_1, D_2. Since the divisors are effective, they are given by sections $r, r_1, r_2 \in \Gamma(\mathcal{O}_W, U)$. The prerequisite $D = D_1 \cap D_2$ implies $(r) = (r_1, r_2)$ and thus

$$(x^*(r)) = (x^*(r_1), x^*(r_2)).$$

Since $R_{\bar{K}}$ is a principal ideal domain, we have $|x^*(r)| = \min(|x^*(r_1)|, |x^*(r_2)|)$ and the claim results from Proposition 2.7.

For the converse we may restrict ourself to one simplex. There the proposition implies that one function dominates the other; without loss of generality $f_{D_1} \leqslant f_{D_2}$. Using Proposition 2.7 the divisor $D_2 - D_1$ is effective and thus $D = D_1 \cap D_2 = D_1$ is again a Cartier divisor. □

Before we can apply Lemma 2.20 in the setting of Theorem 2.19 we show the following compatibility of reduction sets:

Lemma 2.21. — *Let X, X_n, W, W_n be as in Theorem 2.19. Then the diagram*

$$\begin{tikzcd} W_n(\bar{K}) \arrow[r, "\simeq"] \arrow[d, "\mathrm{Red}"'] & W(\bar{K}) \arrow[d, "\mathrm{Red}"'] \\ |\mathscr{R}(W_n)| \arrow[r, "\tau"] & |\mathscr{R}(W)| \end{tikzcd}$$

commutes, where $\tau : |\mathscr{R}(W_n)| \simeq |\mathrm{sd}_n \mathscr{R}(W)| \to |\mathscr{R}(W)|$ denotes the canonical morphism of the geometric realization of the n-th subdivision (see Proposition A.8). Let $f : |\mathscr{R}(W)| \to \mathbb{R}$ be a function which is affine on each simplex. Then $f \circ \tau$ is affine on each simplex of $\mathscr{R}(W_n)$.

Proof. — According to Definition 2.11 and Definition 2.12 the diagram

$$\begin{tikzcd} W_n(\bar{K}) \arrow[r] \arrow[d] & \prod_i X_n(\bar{K}) \arrow[r] \arrow[d] & \prod_i X(\bar{K}) \arrow[r] \arrow[d] & W(\bar{K}) \arrow[d] \\ |\mathscr{R}(W_n)| \arrow[r] & \prod_i |\mathscr{R}(X_n)| \arrow[r] & \prod_i |\mathscr{R}(X)| \arrow[r] & |\mathscr{R}(W)| \end{tikzcd}$$

commutes and it can be shown that the concatenation of the canonical isomorphisms in the second line equals τ.

The second claim is an implication of the definition of the canonical morphism τ (see Proposition A.9). □

Proof of Theorem 2.19. — By definition W is constructed as gradual blow-up of W_d:

$$W = W^{[N]} \to \cdots \to W^{[1]} \to W^{[0]}.$$

Obviously there is a morphism $\varphi^{[0]} : W_n \to W^{[0]}$. We verify the universal property of the blow-up to get morphisms $\varphi^{[i]} : W_n \to W^{[i]}$.

Let $\varphi^{[i]} : W_n \to W^{[i]}$ be already constructed. According to [5, Lemma 3.6 (iii)] the center C of the next blow-up $W^{[i+1]} \to W^{[i]}$ is given by an intersection of Cartier divisors $C = D_1 \cap \cdots \cap D_l$. The component C is a Cartier divisor on $W^{[i+1]}$ and on $W = W^{[N]}$, thus the functions $\min(f_{D_1}, \dots f_{D_l})$ are affine on each simplex of $\mathscr{R}(W)$ (Lemma 2.20).

By Lemma 2.21 the pull-backs under $\varphi^{[i]}$ are also affine on each simplex of $\mathscr{R}(W)$ and by Lemma 2.20 the intersection $(\varphi^{[i]})^{-1}(C) = (\varphi^{[i]})^* D_1 \cap \cdots \cap (\varphi^{[i]})^* D_l$ is a Cartier divisor. The universal property of the blow-up gives the postulated morphism $\varphi^{[i+1]} : W_n \to W^{[i+1]}$. $\square$

3. Limits of Intersection Numbers

In this last section we combine the theory of metrics with the results of [5]. This allows us to describe the localized intersection numbers of vertical divisors by methods of analysis on the simplicial reduction set and use this description to approximate hermitean metrics on the trivial bundle.

Let R be as usual a complete discrete valuation ring with algebraically closed residue field and X a proper regular strict semi-stable curve on $S = \operatorname{Spec} R$ with a total ordering $<$ on $X_s^{(0)}$. We want to assume that the reduction set has no multiple simplices. Let $W := W(X, <, d)$ be the regular strict semi-stable model of the product $(X_\eta)^d$ as defined in Algorithm 2.1. By Proposition 2.17 there is a bijection between Cartier divisors on W with support in the special fiber W_s and piecewise affine functions on $|\mathscr{R}(W)|$. Thus we can view the intersection product as $(d+1)$-fold pairing between piecewise affine functions on $|\mathscr{R}(W)|$.

We use a limit argument in the spirit of Zhang ([8, Sec. 3]) to continue this pairing on piecewise smooth functions on $|\mathscr{R}(W)|$ and give an analytic description of this continuation.

3.1. Analysis on Simplicial Reduction Sets

We start with the definition of analytical objects on products of graphs. Let Γ be a finite graph. By functoriality ([5, Prop. 18]) each edge $\gamma \in \Gamma_1$ induces an embedding $i_\gamma : I \to \Gamma$ of the standard graph $I = \Delta[1]$ into Γ. In the product we get for each d tuple $\gamma = (\gamma_1, \dots \gamma_d) \in \Gamma_1^d$ of edges an embedding

$$i_\gamma := i_{\gamma_1} \times \cdots \times i_{\gamma_d} : I^d \to \Gamma^d$$

(compare Remark A.2). By functoriality this induces a morphism $(i_\gamma)_* : |I^d| \to |\Gamma^d|$ of the d-dimensional "standard cube" $|I|^d$ into $|\Gamma^d|$.

We see these embeddings $(i_\gamma)_*$ as charts. To ease the definition of analytical terms we identify $|I|^d$ with the standard cube $[0,1]^d$ in $\mathbb{R}^d$. Let $(C_0, \dots C_k) \in (I^d)_k$ be a k-simplex given by its vertices $C_0, \dots C_k \in (I^d)_0$. Then we denote by $[C_0, \dots C_k]$ the closed convex hull of the points $C_0, \dots C_k$ — seen as points in $[0,1]^d$. A continuous function $f : [C_0, \dots C_k] \to \mathbb{R}$ is called smooth if it can be continued to a smooth function on an open neighborhood $U \supseteq [C_0, \dots C_k]$.

Definition 3.1. —

1. *A continuous function $f : |I^d| \simeq [0,1]^d \to \mathbb{R}$ is called*
   (a) smooth in the cubes *if f is smooth on $[0,1]^d$,*
   (b) smooth in the simplices *if for each $k \in \mathbb{N}$ and each k-simplex $(C_0, \dots C_k) \in (I^d)_k$ the restriction $f\,|_{[C_0,\dots,C_k]}$ is smooth,*
   (c) affine *if for each $k \in \mathbb{N}$ and each k-simplex $(C_0, \dots C_k) \in (I^d)_k$ the restriction $f\,|_{[C_0,\dots C_k]}$ is affine.*

   The set of continuous functions on $|I^d|$ is denoted by $\mathcal{C}^0(I^d)$, the set of functions smooth in the cubes by $\mathcal{C}^\infty_\square(I^d)$, the set of functions smooth in the simplices by $\mathcal{C}^\infty_\triangle(I^d)$ and the set of affine functions by $\mathcal{C}^{\mathrm{lin}}_\triangle(I^d)$.

2. *Let Γ be a graph. The set of continuous functions $f : |\Gamma^d| \to \mathbb{R}$ is denoted by $\mathcal{C}^0(\Gamma^d)$. A function $f \in \mathcal{C}^0(\Gamma^d)$ is called smooth in the cubes (smooth in the simplices, affine), if for each $\gamma \in (\Gamma_1)^d$ the function $(i_\gamma)^* f = f \circ i_\gamma : [0,1]^d \to \mathbb{R}$ is smooth in the cubes (smooth in the simplices, affine).*

   The set of these functions is denoted by $\mathcal{C}^\infty_\square(\Gamma^d)$ ($\mathcal{C}^\infty_\triangle(\Gamma^d)$, $\mathcal{C}^{\mathrm{lin}}_\triangle(\gamma^d)$).

To define partial derivatives we have to discuss the points on which the functions from $\mathcal{C}^\infty_\triangle(\Gamma^d)$ and $\mathcal{C}^{\mathrm{lin}}_\triangle(\Gamma^d)$ have singularities. In the standard cube $|I^d|$ these are exactly the points $x = (x_1, \dots x_n) \in [0,1]^d$ where two or more coordinates coincide. We call sets of this type *generalized diagonals*:

Definition 3.2. — *The points in $|I^d| \setminus \partial |I^d|$ are called* inner points *of $|I^d|$. Let Γ be an arbitrary graph. Then $x \in |\Gamma^d|$ is called* inner point, *if there is a tuple $\gamma = (\gamma_1, \dots \gamma_d) \in \Gamma_1^d$ such that $x = (i_\gamma)_*(x')$ holds with $x' \in |I^d|$ an inner point. In this case γ and x' is unique. The set of inner points is denoted by $|\Gamma^d|^{\mathrm{i}}$.*

Definition 3.3. —

1. *Let $x \in |I^d|^{\mathrm{i}}$ be an inner point in $|I^d| \simeq [0,1]^d$, given by its coordinates $(x_1, \ldots, x_d) \in [0,1]^d$. We define a partition $\{1, \ldots, d\} = A_1 \amalg \cdots \amalg A_l$ such that*
$$x_i = x_j \Leftrightarrow \exists h : i, j \in A_h$$
holds. This partition is unique and is denoted by
$$d(x) := \{A_1, \ldots A_l\}.$$

2. *Let $x \in |\Gamma^d|^{\mathrm{i}}$ be an inner point in $|\Gamma^d|$. Then there exists a unique tuple $\gamma \in \Gamma_1^d$ and a unique point $x' \in |I^d|$ such that $(i_\gamma)_*(x') = x$. We set*
$$d(x) := d(x').$$

The partition $d(x)$ of $x \in |I^d|$ holds the information which coordinates of x coincide. For example the points $x \in |I^d|$ with $d(x) = \{\{1, \ldots, d\}\}$ are exactly the points of the usual diagonal $\{(t, \ldots, t) \mid t \in (0,1)\}$ in $[0,1]^d$.

Definition 3.4. — *Let Γ be a graph and $\mathcal{P} = \{A_1, \ldots, A_k\}$ a partition of $\{1, \ldots, d\}$. We call the subset*
$$\mathcal{D}_{\mathcal{P}}(\Gamma^d) := \left\{ x \in |\Gamma^d| \middle| d(x) = \mathcal{P} \right\} \subseteq |\Gamma^d|$$
generalized diagonal *to the partition $\mathcal{P}$.*

Remark 3.5. — *Let $\mathcal{P} = \{A_1, \ldots, A_l\}$ be a partition of $\{1, \ldots, d\}$ and $I = \{i_1, \ldots, i_l\}$ with $i_1 \in A_1, \ldots, i_l \in A_l$. Then*
$$\mathcal{D}_{\mathcal{P}}(I^d) = \left\{ x = (x_1, \ldots, x_d) \in |I^d| \middle| x_{j_1} = x_{j_2} \Longleftrightarrow \exists m \in \{1, \ldots, l\} : j_1, j_2 \in A_m \right\}$$
and by the projection pr_I on the coordinates $i_1, \ldots, i_l$ we get a bijection
$$c_{\mathcal{P}} : \mathcal{D}_{\mathcal{P}}(I^d) \to [0,1]^l, (x_1, \ldots, x_d) \mapsto (x_{i_1}, \ldots, x_{i_l}).$$
The chart $c_{\mathcal{P}}$ does not depend on the choice of $i_1, \ldots, i_l$.

Since continuous functions are integrable, we are able to define an integral on Γ^d and on generalized diagonals $\mathcal{D}(\mathcal{P})$.

Definition 3.6. —

1. *Let $\Gamma = I$ and $f \in \mathcal{C}^0(I^d)$ be a continuous function on the standard cube I^d. Then $|I^d|$ is canonically homeomorphic to $[0,1]^d$ and we may define the* integral *of f by*

$$\int_{I^d} f := \int_{[0,1]^d} f(x) d\mu,$$

where μ denotes the Lebesgue-measure on $[0,1]^d$. Let $\mathcal{P}$ be a partition of the set $\{1, \ldots, d\}$. Then the integral along the diagonal $\mathcal{D}_{\mathcal{P}}$ is defined by

$$\int_{\mathcal{D}_{\mathcal{P}}(I^d)} f := \int_{[0,1]^{|\mathcal{P}|}} f \circ c_{\mathcal{P}}^{-1} d\mu,$$

where $c_{\mathcal{P}}$ denotes the chart from Remark 3.5.

2. *Let Γ be an arbitrary graph, $f \in \mathcal{C}^0(\Gamma^d)$ a continuous function on Γ^d and $\mathcal{P}$ a partition of $\{1, \ldots, d\}$. Then the* integral *of f along Γ^d respectively along $\mathcal{D}_{\mathcal{P}}$ is defined by*

$$\int_{\Gamma^d} f := \sum_{\gamma \in (\Gamma_1)^d} \int_{I^d} (f \circ (i_\gamma)_*),$$

$$\int_{\mathcal{D}_{\mathcal{P}}(\Gamma^d)} f := \sum_{\gamma \in (\Gamma_1)^d} \int_{\mathcal{D}_{\mathcal{P}}(I^d)} (f \circ (i_\gamma)_*).$$

For the definition of generalized differential operators on functions smooth in simplices we use a kind of discrete Fourier transform:

Let $v = (v_1, \ldots, v_d) \in \mathbb{F}_2^d$ be a vector and denote by $|v|$ the number of non-trivial coordinates, i.e. $|v| = \#\{i \mid v_i \neq 0\}$. We use these vectors to index the vertices of I^d. For a real number $h \in \mathbb{R}$ denote by $h^v \in \mathbb{R}^d$ the point

$$h^v := h \cdot ((-1)^{v_1}, \ldots, (-1)^{v_d})$$

in $\mathbb{R}^d$. For each continuous function $f \in \mathcal{C}^0(I^d)$ and each $x \in |I^d|^{\mathrm{i}}$ we study the values of f in a cube surrounding x:

$$f_x^h : \mathbb{F}_2^d \to \mathbb{R}, v \mapsto f(x + h^v).$$

Definition 3.7. —

1. *Let $f \in \mathcal{C}^0(I^d)$ be a continuous function on I^d, $x \in |I^d|^{\mathrm{i}}$, $v \in \mathbb{F}_2^d$. Let $h > 0$ be small enough, such that $x + h^w$ is an inner point for all $w \in \mathbb{F}_2^d$. Then we define a function $\Delta_h^v f$ by the discrete Fourier transform*

$$\begin{aligned}\Delta_h^v f : |I^d|^{\mathrm{i}} &\to \mathbb{R}\\ x &\mapsto \frac{1}{2^d} \sum_{w \in \mathbb{F}_2^d} (-1)^{\langle v,w \rangle} f_x^h(w).\end{aligned}$$

2. *Let Γ be a graph without multiple edges and $f \in \mathcal{C}^0(\Gamma^d)$. Let $x \in |\gamma^d|^{\mathrm{i}}$ an inner point which is given as image of a point $x' \in |I^d|^{\mathrm{i}}$ under $(i_\gamma)_* : |I^d| \to |\Gamma^d|$ for $\gamma \in \Gamma_1^d$. Let $v \in \mathbb{F}_2^d$ and $h > 0$ small enough so that $x' + h^w \in |I^d|^{\mathrm{i}}$ holds for all $w \in \mathbb{F}_2^d$. Then define*

$$\Delta_h^v f(x) := \Delta_h^v f \circ (i_\gamma)_*.$$

Since x is an inner point, γ is unique and thus this notation is well-defined.

Proposition 3.8. — *Let $f \in \mathcal{C}^0(\Gamma^d)$ a continuous function which is smooth in a neighborhood of an inner point $x \in |\Gamma^d|^{\mathrm{i}}$. Then $\frac{1}{h^{|v|}} \Delta_h^v f(x)$ converges to the following differential:*

$$\lim_{h \to 0} \frac{1}{h^{|v|}} \Delta_h^v f(x) = D^v f(x) := \left(\frac{\partial}{\partial x_1}\right)^{v_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{v_d} f(x).$$

Proof. — It suffices to prove the proposition for the standard graph $\Gamma = I$. We use the multidimensional Taylor series of f in x: Since f is smooth in a neighborhood of x, we have for each vector $w \in \mathbb{F}_2^d$, each $h \in \mathbb{R}$ and $l \in \mathbb{N}$:

$$\begin{aligned} f(x + h^w) &= \sum_{\substack{\lambda \in \mathbb{N}_0^d \\ 0 \leqslant |\lambda| \leqslant l}} \frac{(h^w)^\lambda}{\lambda!} D^\lambda f(x) + \mathrm{o}(\|(h^w)\|^l) \\ &= \sum_{\substack{\lambda \in \mathbb{N}_0^d \\ 0 \leqslant |\lambda| \leqslant l}} (-1)^{\langle w, \lambda \bmod 2 \rangle} \frac{h^{|\lambda|}}{\lambda!} D^\lambda f(x) + \mathrm{o}(h^l). \end{aligned}$$

The terms $\lambda!$, $(h^w)^\lambda$, $|\lambda|$ are understood in the usual multi-index notation.

For the Fourier transform $\Delta_h^v f$ this implies:

$$\begin{aligned}\Delta_h^v f(x) &= \frac{1}{2^d}\sum_{w\in\mathbb{F}_2^d}(-1)^{\langle v,w\rangle} f(x+h^w)\\ &= \frac{1}{2^d}\sum_{w\in\mathbb{F}_2^d}(-1)^{\langle v,w\rangle}\sum_{0\leqslant|\lambda|\leqslant|v|}(-1)^{\langle w,\lambda \bmod 2\rangle}\frac{h^{|\lambda|}}{\lambda!}D^\lambda f(x)+\mathrm{o}(h^{|v|})\\ &= \frac{1}{2^d}\sum_{0\leqslant|\lambda|\leqslant|v|}\left(\sum_{w\in\mathbb{F}_2^d}(-1)^{\langle w,v-(\lambda \bmod 2)\rangle}\right)\frac{h^{|\lambda|}}{\lambda!}D^\lambda f(x)+\mathrm{o}(h^{|v|})\\ &= h^{|v|}D^v f(x)+\mathrm{o}(h^{|v|}).\end{aligned}$$

We conclude

$$\lim_{h\to 0}\frac{1}{h^{|v|}}\Delta_h^v f(x) = D^v f(x) = \left(\frac{\partial}{\partial x_1}\right)^{v_1}\cdots\left(\frac{\partial}{\partial x_d}\right)^{v_d} f(x).$$

□

A similar proposition can be stated for functions, which are smooth on simplices. For these functions, however, we get a weaker convergence result on the generalized diagonals.

Proposition 3.9. — *Let $f\in\mathcal{C}^\infty_\Delta(\Gamma^d)$ be a function in Γ^d which is smooth on simplices. Let $v\in\mathbb{F}_2^d$ and $\mathcal{P}=\{A_1,\dots,A_k\}$ be a partition of $\{1,\dots,d\}$. We set*

$$\alpha=\alpha(\mathcal{P},v):=\#\{i\in\{1,\dots,k\}\mid \exists a\in A_i : v_a=1\}.$$

Then for each point $x\in\mathcal{D}_\mathcal{P}(\Gamma^d)$ the limit

$$\lim_{h\to 0}\frac{1}{h^\alpha}\Delta_h^v f(x) \tag{3.1}$$

exists and is continuous in x on $\mathcal{D}_\mathcal{P}(\Gamma^d)$.

The proof is very technical. Of course it is enough to show the proposition for the standard graph $\Gamma=I$. In the next lemma we split the sum $\Delta_n^v(f)(x)$ from Definition 3.7 into parts such that each part contains only contributions of one simplex. Then f can be replaced by a smooth function and the proposition is a consequence of Proposition 3.8.

Let $J\subseteq\{1,\dots,d\}$ be a subset and $H:=\{1,\dots,d\}\setminus J$ its complement. For the proof we denote by V_J (resp. V_H) the subspace of $V:=\mathbb{F}_2^d$ spanned by the base vectors e_i with $i\in J$ ($i\in H$). The projections in the direct sum $V=V_J\oplus V_H$ are denoted by pr_J and pr_H.

Lemma 3.10. — *Let $f \in \mathcal{C}^\infty_\triangle(I^d)$ be smooth on simplices and $x \in |I^d|$ with $d(x) = \{P_1, \dots, P_k\}$. Let $J = \{j_1, \dots, j_k\}$ with $j_1 \in P_1, \dots, j_k \in P_k$. Denote by $H := \{1, \dots d\} \setminus J$ its complement. Then for each $v \in V$ and $v' \in V_H$ there exists an $\epsilon > 0$, an open neighborhood $U \subseteq |I^d|$ of x and a function F smooth on U such that*

$$\sum_{v_H \in V_H} (-1)^{\langle v_H, v' \rangle} \Delta_h^{v+v_H}(f)(x') = \sum_{v_H \in V_H} (-1)^{\langle v_H, v' \rangle} \Delta_h^{v+v_H}(F)(x')$$

holds for all $x' \in U$ with $d(x') = d(x)$ and all $h < \epsilon$.

Proof. — We choose $\epsilon > 0$ small enough such that $\epsilon < \frac{1}{4}|x_i - x_j|$ for all i, j with $x_i \neq x_j$ and set $U' = \{x' \in |I^d| \mid |x' - x| \leqslant \epsilon\}$. For each $x' \in U', d(x') = d(x)$ we have

$$\begin{aligned}
\sum_{v_H \in V_H} (-1)^{\langle v_H, v' \rangle} \Delta_h^{v+v_H}(f)(x') &= \sum_{v_H \in V_H} (-1)^{\langle v_H, v' \rangle} \sum_{w \in \mathbb{F}_2^d} (-1)^{\langle v+v_H, w \rangle} f^h_{x'}(w) \\
&= \sum_{\substack{v_H, w_H \in V_H \\ w_J \in V_J}} (-1)^{\langle v_H, v' \rangle + \langle v+v_H, w_H \rangle + \langle v, w_J \rangle} f^h_{x'}(w_J + w_H) \\
&= \sum_{\substack{w_H \in V_H \\ w_J \in V_J}} 2^{|H|} \delta_{v', w_H} (-1)^{\langle v, w_H \rangle + \langle v, w_J \rangle} f^h_{x'}(w_J + w_H) \\
&= 2^{|H|} \sum_{w_J \in V_J} (-1)^{\langle v, w_J + v' \rangle} f(x' + h^{w_J + v'}).
\end{aligned}$$

The points occurring in the last term,

$$Q := \{x' + h^{w_J + v'} \mid w_J \in V_J, h < \epsilon, x' \in U', d(x') = d(x)\},$$

all lie in the same simplex: For all points $q \in Q$ the coordinates $q = (q_1, \dots, q_d) \in \mathbb{R}^d$ satisfy the inequalities

$$\begin{aligned}
q_{k_1} < q_{k_2} &\quad \text{if } x_{k_1} < x_{k_2}, \\
q_j \leqslant q_k &\quad \text{if } x_j = x_k, j \in H, v_j = 1, \\
q_j \geqslant q_k &\quad \text{if } x_j = x_k, j \in H, v_j = 0.
\end{aligned}$$

The coordinates of the points in P can therefore be simultaneously sorted by one permutation $\sigma \in S_d$ and thus all points of Q are in the corresponding simplex $S_\sigma \subseteq |I^d|$ (see Proposition A.6). Since f is smooth on simplices, there is an open neighborhood $U'' \subseteq S_\sigma$ and a smooth continuation F of f. By choosing $U := U' \cap U''$ we get the proposition. □

Corollary 3.11. — *Let $f \in \mathcal{C}^\infty_\vartriangle(I^d)$, $v \in \mathbb{F}_2^d$ and $x \in |I^d|$. Let $J, H \subseteq \{1, \dots, d\}$ as in Lemma 3.10. Then there exists an open neighborhood $U \subseteq |I^d|$ of x and smooth functions $(F_{v_H})_{v_H \in V_H}$ on U such that*

$$\Delta_h^v(f)(x') = \sum_{v_H \in V_H} \Delta_h^{v+v_H}(F_{v_H})(x')$$

holds for all $x' \in U$ with $d(x') = d(x)$.

Proof. — According to Lemma 3.10 there is an open neighborhood U of x, an $\epsilon > 0$ and for each $w_H \in V_H$ a smooth function G_{w_H} on U such that

$$\sum_{v_H \in V_H} (-1)^{\langle v_H, w_H \rangle} \Delta_h^{v+v_H}(f) = \sum_{v_H \in V_H} (-1)^{\langle v_H, w_H \rangle} \Delta_h^{v+v_H}(G_{w_H})$$

holds for each $x' \in U$ with $d(x') = d(x)$ and each $h < \epsilon$.

By an application of the Fourier transform we get

$$\begin{aligned}
\Delta_h^v(f)(x') &= 2^{|H|} \sum_{w_H \in V_H} \left(\sum_{v_H \in V_H} (-1)^{\langle v_H, w_H \rangle} \Delta_h^{v+v_H}(f)(x') \right) \\
&= 2^{|H|} \sum_{w_H \in V_H} \left(\sum_{v_H \in V_H} (-1)^{\langle v_H, w_H \rangle} \Delta_h^{v+v_H}(G_{w_H})(x') \right) \\
&= \sum_{v_H \in V_H} \Delta_h^{v+v_H} \left(\sum_{w_H \in V_H} 2^{|H|} (-1)^{\langle v_H, w_H \rangle} G_{w_H} \right)(x') \\
&= \sum_{v_H \in V_H} \Delta_h^{v+v_H}(F_{v_H})(x')
\end{aligned}$$

with

$$F_{v_H} := \sum_{w_H \in V_H} 2^{|H|} (-1)^{\langle v_H, w_H \rangle} G_{w_H}.$$

□

Proof of Proposition 3.9. — Since $x \in \mathcal{D}_\mathcal{P}(\Gamma^d)$, we have $d(x) = \{A_1, \dots, A_l\}$. From each block A_i we choose an element $j_i \in A_i$ with $v_{j_i} = 1$, in case such an element exists. Otherwise we choose an arbitrary element. The subset $J := \{j_1, \dots, j_l\} \subseteq \{1, \dots, d\}$ satisfies the conditions of Corollary 3.11. Therefore there is an open neighborhood U of x and smooth functions F_{v_H} for each $v_H \in V_H$ such that

$$\Delta_h^v(f)(x') = \sum_{v_H \in V_H} \Delta_h^{v+v_H}(F_{v_H})(x')$$

holds. The minimal length occurring in the Δ on the right hand side, i.e. the value of the set $\{|v+v_H|, v_H \in V_H\}$ is reached only with $v_H = w := \mathrm{pr}_H(v)$. Then $|v+w| = \alpha(\mathcal{P}, v) = \alpha$ and by Proposition 3.8 for each $x' \in U$ with $d(x') = d(x)$ we get

$$\begin{aligned}\lim_{h\to 0} \frac{1}{h^\alpha}\Delta_h^v(f)(x') &= \sum_{v_H \in V_H} \lim_{h\to 0}\frac{1}{h^\alpha}\Delta_h^{v+v_H}(F_{v'})(x')\\ &= D^{v+w}(F_w)(x')\end{aligned}$$

and thus the proposition. □

Motivated by Proposition 3.8 and Proposition 3.9 we define the generalized differential:

Definition 3.12. — *Let $f \in \mathcal{C}^0(\Gamma^d)$ be a continuous function on $|\Gamma^d|$. If for $x \in |\Gamma^d|$, $v \in \mathbb{F}_2^d$ and $\alpha \in \mathbb{N}$ the limit*

$$D_\alpha^v(f)(x) := \lim_{h\to 0}\frac{1}{h^\alpha}\Delta_h^v(f)(x)$$

exists, we call f differentiable *at x to v in degree α and D_α^v the* generalized differential *of f to v in degree α.*

By Proposition 3.9, for each $f \in \mathcal{C}_\triangle^\infty(I^d)$, $v \in \mathbb{F}_2^d$ and each point $x \in |I^d|^{\mathrm{i}}$ the generalized differential in degree $\alpha := \alpha(d(x), v)$ exists. For $d = 2$ we can give a connection with the term $\delta(f)$ defined by Zhang in [8, 3.4]:

Example 3.13. — Let $d = 2$ and $f \in \mathcal{C}^0(I^2)$. By the diagonal $\mathcal{D} = \mathcal{D}(\{\{1,2\}\}) = \{(x_1, x_2) \in (0,1) \mid x_1 = x_2\}$ the square (fig. 1) is split into two triangles $S^+, S^- \subseteq |I^2|$, on each of which the function f is smooth. By S^+ we denote the upper triangle, see fig. 1.

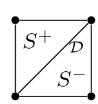

Figure 1. — Square split into two triangles S^+, S^- by diagonal $\mathcal{D}$.

As in Zhang [8, 3.4] let f^+ and f^- denote a continuation of the smooth function $f\mid_{S^+}$ resp. $f\mid_{S^-}$ above the diagonal. For each point $x \in \mathcal{D}$ and each $0 < h \in \mathbb{R}$ small enough, Lemma 3.10 implies

$$\begin{aligned}\Delta_h^{(1,1)}(f)(x) + \Delta_h^{(1,0)}(f)(x) &= \Delta_h^{(1,1)}(f^+)(x) + \Delta_h^{(1,0)}(f^+)(x),\\ \Delta_h^{(1,1)}(f)(x) - \Delta_h^{(1,0)}(f)(x) &= \Delta_h^{(1,1)}(f^-)(x) - \Delta_h^{(1,0)}(f^-)(x).\end{aligned}$$

We conclude

$$\begin{aligned}\Delta_h^{(1,1)}(f)(x) &= \frac{1}{2}\Big(\Delta_h^{(1,1)}(f^+)(x)+\Delta_h^{(1,0)}(f^+)(x)+\Delta_h^{(1,1)}(f^-)(x)-\Delta_h^{(1,0)}(f^-)(x)\Big)\\ &= \frac{1}{2}\Delta_h^{(1,1)}(f^+ + f^-)(x) + \frac{1}{2}\Delta_h^{(1,0)}(f^+ - f^-)(x)\end{aligned}$$

and therefore the following limit converges:

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h}\Delta_h^{(1,1)}(f)(x) &= \lim_{h\to 0}\frac{1}{2h}\Delta_h^{(1,0)}(f^+ - f^-)(x)\\ &= \frac{1}{2}\frac{\partial}{\partial x_1}(f^+ - f^-)(x).\end{aligned}$$

Using the notation of Zhang $\delta(f) := \frac{\partial}{\partial x_1}(f^+ - f^-)$ this means for each $x \in \mathcal{D}$:

$$D_1^{(1,1)}(f)(x) = \frac{1}{2}\delta(f)(x).$$

To conclude this section we study a discretization of the generalized differential. Let $n \in \mathbb{N}$. We subdivide the standard cube into cubes with edges of length $\frac{1}{n}$. Let $x \in [0,1]^d$ be a point in the standard cube given by its coordinates $x = (x_1, \dots x_d) \in \mathbb{R}^d$. Let $\lfloor x \rfloor$ be the vector $(\lfloor x_1 \rfloor, \dots, \lfloor x_d \rfloor)$, where $\lfloor\cdot\rfloor$ denotes the usual floor function. Then the cube of edge length $\frac{1}{n}$ surrounding the point x has the center coordinates $\tilde{x}^{(n)} := \frac{1}{n}\lfloor nx \rfloor + \frac{1}{2n}(1, \dots, 1)$.

Definition 3.14. —

1. *Let $f \in \mathcal{C}^0(I^d)$ be a continuous function on $[0,1]^d$ and $x \in (0,1)^d$. We call the term*

   $$\tilde{\Delta}_n^v f(x) := \Delta_{1/2n}^v f(\tilde{x}^{(n)}) = \Delta_{1/2n}^v f\Big(\frac{1}{n}\lfloor nx \rfloor + \frac{1}{2n}(1, \dots, 1)\Big)$$

   the n-th lattice approximation of the derivative to $v \in \mathbb{F}_2^d$ at x.

2. *Let Γ be a graph and $f \in \mathcal{C}^0(\Gamma^d)$ a continuous function on $|\Gamma^d|$ and $x \in |\Gamma^d|^{\mathrm{i}}$ an inner point. As in Definition 3.7 there are unique $\gamma \in \Gamma_1^d$, $x' \in (I^d)^{\mathrm{i}}$, such that $(i_\gamma)_*(x') = x$. The* n-th lattice approximation of the derivative to $v \in \mathbb{F}_2^d$ at x *is defined by*

   $$\tilde{\Delta}_n^v f(x) := \tilde{\Delta}_n^v (f \circ (i_\gamma)_*)(\tilde{x}').$$

Proposition 3.15. — *Let $f \in \mathcal{C}^\infty_\triangle \Gamma^d$ be a function smooth on simplices, $x \in |\Gamma^d|^{\mathrm{i}}$ an inner point and $\alpha := \alpha(\mathcal{P}(x), v)$ as in Proposition 3.9, then*

$$\lim_{n\to\infty} (2n)^\alpha \tilde{\Delta}^v_n f(x) = D^v_\alpha f(x)$$

holds.

Proof. — It suffices to study $I = \Gamma$. Let $N \in \mathbb{N}$ big enough such that the closed line $V := [\tilde{x}^{(N)} - \frac{1}{2N}(1,\ldots,1), \tilde{x}^{(N)} + \frac{1}{2N}(1,\ldots,1)]$ lies in I^d and $d(x') = d(x)$ holds for all $x \in V$. Then for each $x \in V$ and each $n \geqslant N$ we have $\tilde{x}^{(n)} \in V$. By Proposition 3.9 the limit $D^v_\alpha(f)(x) = \lim_{h\to 0} \frac{1}{h^\alpha} \Delta^v_h f(x)$ exists for each $x \in V$ and since V is compact this convergence is uniform. Again by Proposition 3.9 the function $D^v_\alpha(f)$ is continuous on V and thus

$$\begin{aligned}\lim_{n\to\infty} (2n)^\alpha \Delta^v_n f(\tilde{x}^{(n)}) &= \lim_{m\to\infty} \lim_{n\to\infty} (2n)^\alpha \Delta^v_n f(\tilde{x}^{(m)}) \\ &= \lim_{m\to\infty} D^v_\alpha(f)(\tilde{x}^{(m)}) = D^v_\alpha(f)(x).\end{aligned}$$

□

Lemma 3.16. — *Let $n \in \mathbb{N}$ and $f \in \mathcal{C}^0(\Gamma^d)$ be a function with $f(x) = f(\tilde{x}^{(n)})$ for each $x \in |\Gamma^d|$. Then for each partition $\mathcal{P}$ of $\{1,\ldots,d\}$ the equation*

$$\int_{\Gamma^d} f \mathbb{1}_{\{x \mid \mathcal{P}(\tilde{x}^{(n)}) = \mathcal{P}\}} = n^{|\mathcal{P}|-d} \int_{\mathcal{D}_\mathcal{P}} f \qquad (3.2)$$

holds.

Proof. — By definition of the integral it suffices to show this lemma for the standard graph $\Gamma = I$. Since f is constant on the set $\{x \in |I|^d \mid \tilde{x}^{(n)} = y\}$ for each $y \in \left(\frac{1}{n}\mathbb{Z} \cap [0,1]\right)^d$, we get

$$\int_{I^d} f \mathbb{1}_{\{x \mid \mathcal{P}(\tilde{x}^{(n)}) = \mathcal{P}\}} = \frac{1}{n^d} \sum_{\substack{x \in \left(\frac{1}{n}\mathbb{Z}\cap[0,1]\right)^d \\ d(x) = \mathcal{P}}} f(x).$$

Using the chart $c_\mathcal{P} : \mathcal{D}_\mathcal{P} \to \mathbb{R}^{|\mathcal{P}|}$ of the generalized diagonal $\mathcal{D}_\mathcal{P}$ from Remark 3.5 we conclude

$$\begin{aligned}\int_{I^d} f \mathbb{1}_{\{x \mid \mathcal{P}(\tilde{x}^{(n)}) = \mathcal{P}\}} &= \frac{1}{n^d} \sum_{x \in \left(\frac{1}{n}\mathbb{Z}\cap[0,1]\right)^{|\mathcal{P}|}} f(c_\mathcal{P}^{-1}(x)) \\ &= \frac{1}{n^{d-|\mathcal{P}|}} \int_{[0,1]^{|\mathcal{P}|}} f \circ c_\mathcal{P}^{-1} \mathrm{d}\mu \\ &= \frac{1}{n^{d-|\mathcal{P}|}} \int_{\mathcal{D}(\mathcal{P})} f.\end{aligned}$$

□

Let $\mathrm{sd}_n\Gamma$ denote the n-fold subdivision of the graph according to Proposition A.7 and $\mathrm{sd}_n : |\Gamma_n| \to |\Gamma|$ the canonical homeomorphism of the geometric realizations from Proposition A.8. The d-th power of this homeomorphism yields $(\mathrm{sd}_n)^d : |(\Gamma_n)^d| \to |\Gamma^d|$ and by functoriality there is a homeomorphism

$$(\mathrm{sd}_n)^* : \mathcal{C}^0(\Gamma^d) \to \mathcal{C}^0((\Gamma_n)^d).$$

This morphism maps all subsets of $\mathcal{C}^0(\Gamma^d)$ defined so far on their counterparts:

Proposition 3.17. — *The homeomorphism* $(\mathrm{sd}_n)^* : \mathcal{C}^0(\Gamma^d) \to \mathcal{C}^0((\Gamma_n)^d)$ *satisfies*

$$\begin{aligned}(\mathrm{sd}_n)^*(\mathcal{C}^\infty_\square(\Gamma^d)) &\subseteq \mathcal{C}^\infty_\square(\Gamma^d_n),\\ (\mathrm{sd}_n)^*(\mathcal{C}^\infty_\triangle(\Gamma^d)) &\subseteq \mathcal{C}^\infty_\triangle(\Gamma^d_n),\\ (\mathrm{sd}_n)^*(\mathcal{C}^{\mathrm{lin}}_\triangle(\Gamma^d)) &\subseteq \mathcal{C}^{\mathrm{lin}}_\triangle(\Gamma^d_n).\end{aligned} \tag{3.3}$$

The diagram

$$\begin{array}{ccc} \mathcal{C}^0(\Gamma^d) & \xrightarrow{(\mathrm{sd}_n)^*} & \mathcal{C}^0((\Gamma_n)^d) \\ {\scriptstyle\tilde\Delta^v_n}\downarrow & & {\scriptstyle\tilde\Delta^v_1}\downarrow \\ \mathcal{C}^0(\Gamma^d) & \xrightarrow{(\mathrm{sd}_n)^*} & \mathcal{C}^0((\Gamma_n)^d) \end{array} \tag{3.4}$$

commutes and each function $f \in \mathcal{C}^0(\Gamma^d)$ *satisfies*

$$n^d \int_{\Gamma^d} f = \int_{\Gamma^d_n} (\mathrm{sd}_n)^* f. \tag{3.5}$$

Proof. — It suffices to proof the claim for $\Gamma = I$. The n-fold subdivision of $|I^d|$ is in this case a lattice with distance $1/n$ in $[0,1]^d = |I^d|$. Since the image of each simplex of $(I_n)^d$ by $(\mathrm{sd}_n)^*$ is contained in one simplex of I^d, the relations (3.3) result immediately.

To proof the commutativity of (3.4) it suffices to note that the edges of $(I_n)^d$ are mapped by sd_n onto the points with rational coordinates $[0, 1/n, \ldots, n/n]^d$.

Equation (3.5) is an easy implication of integration theory. $\square$

3.2. The Intersection Pairing

Let X be a proper regular strict semi-stable S-curve with total ordering $<$ on $X^{(0)}$. Let its reduction graph $\Gamma(X)$ be without multiple edges. We denote by $W = W(X, <, d)$ the model of $(X_\eta)^d$ constructed in Algorithm 2.1. The norm $|\cdot|$ is chosen such that $|\pi| = 1/b$ for a fixed $b \in \mathbb{R}_{>0}$.

Let $\mathrm{CH}^i_{W_s}(W)$ denote the Chow group of cycles of codimension i in W with support in W_s. By intersection theory there is an intersection product

$$\cdot : \mathrm{CH}^i_{W_s}(W) \times \mathrm{CH}^j_{W_s}(W) \to \mathrm{CH}^{i+j}_{W_s}(W)_{\mathbb{Q}},$$

and since the special fiber W_s is proper over a field we have also a local degree map

$$\mathrm{ldeg} : \mathrm{CH}^{d+1}_{W_s}(W) = \mathrm{CH}_0(W_s) \xrightarrow{deg} \mathbb{Z}.$$

(For details see [5, Def 4.6]). We view this local intersection product of divisors with support in W_s as pairing between affine functions on the reduction set $\mathcal{C}^{\mathrm{lin}}_{\vartriangle}(\Gamma(X)^d)$ using the following isomorphism:

Definition 3.18. — *Let X be proper regular strict semi-stable curve and $d \geqslant 2$. We denote by* $\mathrm{CaDiv}_{W_s}(W)_{\mathbb{R}}$ *the tensor product* $\mathrm{CaDiv}_{W_s}(W) \otimes_{\mathbb{Z}} \mathbb{R}$ *and by $\phi_1^{|\cdot|}$ the morphism*

$$\phi_1^{|\cdot|} : \mathrm{CaDiv}_{W_s}(W)_{\mathbb{R}} \to \mathcal{C}^{\mathrm{lin}}_{\vartriangle}(\Gamma(X)^d)$$

given by

$$C \otimes r \mapsto r f_C^{|\cdot|}.$$

By Proposition 2.17 this is an isomorphism.

Remark 3.19. — Since the intersection product

$$\begin{aligned} \mathrm{ldeg}(\cdot, \ldots, \cdot) : \left(\mathrm{CaDiv}_{W_s}(W)\right)^{d+1} &\to \mathbb{Q}, \\ (D_0, \ldots, D_d) &\mapsto \mathrm{ldeg}(D_0 \cdot \cdots \cdot D_d) \end{aligned}$$

is a multi-linear mapping, we can continue it by linearity on $\mathrm{CaDiv}_{W_s}(W)_{\mathbb{R}}$. We denote this continuation again by ldeg:

$$\mathrm{ldeg}(\cdot, \ldots, \cdot) : \left(\mathrm{CaDiv}_{W_s}(W)_{\mathbb{R}}\right)^{d+1} \to \mathbb{R}.$$

Definition 3.20. — *By*

$$\begin{aligned} \langle \cdot, \ldots, \cdot \rangle_{W,1} : \mathcal{C}^{\mathrm{lin}}_{\vartriangle}(\Gamma^d)^{d+1} &\to \mathbb{R}, \\ (f_0, \ldots, f_d) &\mapsto \mathrm{ldeg}_{W_s}((\phi_1^{|\cdot|})^{-1}(f_0), \ldots, (\phi_1^{|\cdot|})^{-1}(f_d)) \end{aligned}$$

a multi-linear pairing is defined on $\mathcal{C}^{\mathrm{lin}}_{\vartriangle}(\Gamma^d)$, the set of piecewise affine functions on the reduction set. This pairing is called the intersection pairing.

By the bijection $(\mathrm{sd}_n)^* : \mathcal{C}^0(\Gamma^d) \to \mathcal{C}^0(\Gamma^d_n)$ of (3.3) the set $\mathcal{C}^{\mathrm{lin}}_{\triangle}(\Gamma^d_n)$ can be seen as subset of $\mathcal{C}^0(\Gamma^d)$ containing $\mathcal{C}^{\mathrm{lin}}_{\triangle}(\Gamma^d)$. We continue the intersection pairing to functions of $\mathcal{C}^{\mathrm{lin}}_{\triangle}(\Gamma^d_n)$. Let K_n/K be an algebraic field extension of degree n, R_n the ring of integers in K_n and $S_n := \operatorname{Spec} R_n$. By X_n we denote the regular strict semi-stable model of $X_\eta \times_S S_n$ defined in Theorem 2.18. Similarly we denote by $W_n := W(X_n, <, d)$ the model of $(X_\eta)^d \times_S S_n$ defined in Algorithm 2.1. Since the reduction sets $\mathscr{R}(X_n)$ and $\mathscr{R}(W_n)$ are determined combinatorially, they are independent of the choice of K_n. We denote by $|\cdot|$ also the unique continuation of the valuation $|\cdot| : K \to \mathbb{R}$ to K_n.

In this situation Proposition 2.17 yields again an isomorphism of $\mathbb{R}$-algebras by Proposition 2.17,

$$\phi^{|\cdot|}_n : \mathrm{CaDiv}_{(W_n)_s}(W_n)_{\mathbb{R}} \to \mathcal{C}^{\mathrm{lin}}_{\triangle}((\Gamma(X)_n)^d),$$

defined by

$$C \otimes r \mapsto r f^{|\cdot|}_C.$$

Definition 3.21. — *By*

$$\begin{aligned} \langle \cdot, \dots, \cdot \rangle_{W,n} : \mathcal{C}^{\mathrm{lin}}_{\triangle}(\Gamma^d_n)^{d+1} &\to \mathbb{R}, \\ (f_0, \dots, f_d) &\mapsto 1/n \, \mathrm{ldeg}_{W_s}\left((\phi^{|\cdot|}_n)^{-1}(f_0), \dots, (\phi^{|\cdot|}_n)^{-1}(f_d)\right) \end{aligned}$$

a multi-linear pairing is defined on $\mathcal{C}^{\mathrm{lin}}_{\triangle}(\Gamma^d_n)$. Since local intersection numbers depend only on the structure of $\mathscr{R}(W_n)$, $\langle \cdot, \dots, \cdot \rangle_{W,n}$ is independent of the choice of K_n.

Proposition 3.22. — *The pairing from Definition 3.21 is a continuation of the pairing from Definition 3.20: Let $f_0, \dots, f_d \in \mathcal{C}^{\mathrm{lin}}_{\triangle}(\Gamma^d)$ be functions affine on the simplices of Γ^d. Then*

$$\langle f_0, \dots, f_d \rangle_{W,n} = \langle f_0, \dots, f_d \rangle_{W,1}$$

holds.

Proof. — By linearity we may assume that there are Cartier divisors $D'_0, \dots D'_d \in \mathrm{CaDiv}_{W_s}(W)$, such that $f_i = f^{|\cdot|}_{D'_i}$ for each i. By Theorem 2.19 there is a morphism $g : W_n \to W$ between the models. The pull-back divisors $g^* D_i$ satisfy $\phi^{|\cdot|}_n(g^* D_i) = f_i$ by Proposition 2.8, thus $\langle f_0, \dots, f_d \rangle_{W,n} = \mathrm{ldeg}_{W_n}(g^*(D_0) \cdot \dots \cdot g^*(D_d))$. As $g : W_n \to W$ is generic flat and $(W_n)_\eta = W_\eta \times_{S_\eta} (S_n)_\eta$ we may apply [5, Lemma 4.7]:

$$\mathrm{ldeg}_W(D_0 \cdot \dots \cdot D_d) = n \, \mathrm{ldeg}_{W_n}(g^*(D_0) \cdot \dots \cdot g^*(D_d)).$$

This proofs the proposition. ☐

We may also apply Definition 3.20 directly to a model X_n on S_n. The pairing defined in this way is denoted by $\langle \cdot, \ldots, \cdot \rangle_{W_n,1}$ and coincides with Definition 3.21 up to a constant factor:

PROPOSITION 3.23. — *For $f_0, \ldots, f_d \in \mathcal{C}^{\mathrm{lin}}_{\triangle}(\Gamma^d_n)$ the equation*

$$\langle f_0, \ldots, f_d \rangle_{W,n} = n^d \langle f_0, \ldots, f_d \rangle_{W_n,1}$$

holds.

Proof. — Let π_n denote a uniformizer of the discrete valuation ring R_n. To define $\langle \cdot, \ldots, \cdot \rangle_{W_n,1}$ one has to use the valuation $|\cdot|_n$ on R_n which is normalized by $|\pi_n| = 1/e$. Therefore we have $|\cdot|_n = (|\cdot|)^n$ and thus for each Cartier divisor $C \in \mathrm{CaDiv}_{(W_n)_s}(W_n)_{\mathbb{R}}$:

$$f_C^{|\cdot|_n} = n f_C^{|\cdot|}.$$

We have

$$(\phi_n^{|\cdot|})^{-1}(f_i) = D_i = (\phi_1^{|\cdot|_n})^{-1}(n f_i).$$

This implies

$$\begin{aligned}\langle f_0, \ldots, f_d \rangle_{W,n} &= \frac{1}{n} \operatorname{ldeg}_{W_n}(D_0, \ldots, D_d) = \frac{1}{n} \langle n f_0, \ldots, n f_d \rangle_{W_{n,1}} \\ &= n^d \langle f_0, \ldots, f_d \rangle_{W_{n,1}}.\end{aligned}$$

□

For our further calculation we need a more explicit description of the Chow ring $\mathrm{CH}^1_{W_s}(W)$. We use the combinatorial Chow ring of [5, Def 4.12] defined as:

DEFINITION 3.24. — *Let Γ be a finite graph without multiple edges and Γ^d be the d-fold product. We denote with $Z(\Gamma^d)$ the polynomial ring $Z(\Gamma^d) := \mathbb{Z}[C \mid C \in (\Gamma^d)_0]$ generated by the 0-simplices. It is supplied with the usual grading, which gives all generators $C \in (\Gamma^d)_0$ degree one.*

We define a graded ideal $\mathrm{Rat}(\Gamma^d)$ on $Z(\Gamma^d)$ generated by the polynomials

$$C_1 \cdot \cdots \cdot C_k \text{ for } \{C_1, \ldots, C_k\} \notin (\Gamma^d)_S, \tag{3.6}$$

$$\Big(\sum_{C' \in (\Gamma^d)_0} C' \Big) C_1, \tag{3.7}$$

$$\sum_{\substack{C' \in (\Gamma^d)_0 \\ \mathrm{pr}_i(C') = \mathrm{pr}_i(C_2)}} C_1 C_2 C' \text{ for } i \in \{1, \ldots, d\} \text{ with } \mathrm{pr}_i(C_1) \neq pr_i(C_2). \tag{3.8}$$

We call $\mathrm{Rat}(\Gamma^d)$ *the ideal of cycles rationally equivalent to zero.*

The graded ring

$$\mathcal{C}(\Gamma^d) := Z(\Gamma^d)/\mathrm{Rat}(\Gamma^d)$$

is called the combinatorial Chow ring.

The combinatorial Chow ring has the following properties:

THEOREM 3.25. —

1. *There exists a morphism of* $\mathbb{Z}$*-modules*

$$\mathrm{ldeg}_{\Gamma^d} : \mathcal{C}(\Gamma^d) \to \mathbb{Z}$$

such that for each set $\{C_0, \ldots, C_d\}$ *of* $d+1$ *distinct vertices, which form a simplex of* Γ^d*, the degree* $\mathrm{ldeg}_{\Gamma^d}(C_0 \cdot \ldots \cdot C_d) = 1$.

2. *The local degree map can be calculated locally: By Remark A.2 we associate to each* $\gamma = (\gamma_1, \ldots, \gamma_d)$ *with* $\gamma_1, \ldots, \gamma_d \in \Gamma(X)^1$ *an embedding* $i_\gamma : I^d \to \Gamma^d$*. This gives a covering of* Γ^d *and the local degree satisfies*

$$\mathrm{ldeg}_{\Gamma^d}(\alpha) = \sum_{\gamma \in (\Gamma_1)^d} \mathrm{ldeg}_{I^d}(i_\gamma^* \alpha),$$

where $i_\gamma^* : \mathcal{C}(\Gamma^d) \to \mathcal{C}(I^d)$ *denotes the morphism given by functoriality of* $\mathcal{C}$.

Proof. — [5, 4.4] ☐

In the local situation $\Gamma = I$ we can describe the situation more precisely:

Let the vertices of $I = \Delta[1]$ be denoted by C_0 and C_1 with the ordering $C_0 < C_1$. Then the vertices of I^d can be described using vectors $v \in \mathbb{F}_2^d$: For each vector $v = (v_1, \ldots, v_d)$ let $C_v \in (I^d)_0$ denote the vertex with $\mathrm{pr}_i(C_v) = C_{v_i}$. The set $\{C_v \mid v \in \mathbb{F}_2^d\}$ is a generating set for $\mathcal{C}(I^d)_{\mathbb{Q}}$. Using Fourier transforms we get yet another generating set:

$$F_v := \sum_{w \in \mathbb{F}_2^d} (-1)^{\langle v,w \rangle} C_w.$$

It turns out that the generating set $\{F_v \mid v \in \mathbb{F}_2^d\}$ of $\mathcal{C}(I^d)$ is appropriate for the further calculations. We first note the following isomorphism:

THEOREM 3.26. — *Consider $\mathcal{C}(I^d)$ with the generating sets $\{C_v \mid v \in \mathbb{F}_2^d\}$ and $\{F_v \mid v \in \mathbb{F}_2^d\}$ as defined above. There is an isomorphism of graded rings*

$$\psi : \mathcal{C}(I^d) \xrightarrow{\sim} \mathcal{C}(I^d),$$

which is uniquely determined by $\psi(C_v) = C_{v+(1,\dots,1)}$. The equation

$$\psi(F_v) = (-1)^{\langle v,(1,\dots,1)\rangle} F_v.$$

holds and ψ is compatible with the local degree $\operatorname{ldeg}_{I^d}$.

Proof. — [5, Prop 4.30] □

The combinatorial Chow ring can be compared with the Chow ring of W:

THEOREM 3.27. *([5, Prop 4.14, Prop 4.23])* . — *Let $d \in \mathbb{N}$, X be a proper regular strict semi-stable curve over S with a fixed ordering of $X_s^{(0)}$. Let X be the Gross-Schoen desingularization of X^d by Algorithm 2.1. Then there exists a morphism of graded rings*

$$\varphi : \mathcal{C}(\mathscr{R}(W)) \to \mathrm{CH}_{W_s}(W)_{\mathbb{Q}}$$

such that φ is an isomorphism in degree 1 and the equation

$$\operatorname{ldeg}_W(\varphi(\alpha)) = \operatorname{ldeg}_{\mathscr{R}(W)}(\alpha)$$

holds for each $\alpha \in \mathcal{C}(\mathscr{R}(W))^{d+1}$.

We may now formulate the intersection pairing in analytical terms using the n-th lattice approximation of the functions $f_0, \dots, f_d$.

PROPOSITION 3.28. — *Let $n \in \mathbb{N}$ and $f_0, \dots, f_d \in \mathcal{C}^{\mathrm{lin}}_{\vartriangle}(\Gamma_n)^d$. Then*

$$\langle f_0, \dots, f_d\rangle_{W,n} = n^{2d} \sum_{v_0,\dots,v_d \in \mathbb{F}_2^d} \operatorname{ldeg}_{I^d}\Big(\prod_{i=0}^d F_{v_i}\Big) \int_{\Gamma^d} \prod_{i=0}^d \tilde{\Delta}_n^{v_i}(f_i) \tag{3.9}$$

holds.

Proof. — **Case 1: $n = 1$.**

Denote by $\varphi_W : \mathcal{C}(\Gamma(X)^d) \to \mathrm{CH}_{W_s}(W)$ the canonical morphism of the combinatorial Chow ring to $\mathrm{CH}_{W_s}(W)$ of Theorem 3.27. By linearity we may assume $f_i = f_{D_i}^{|\cdot|}$ for Cartier divisors $D'_0, \dots D'_d \in \mathrm{CaDiv}_{W_s}(W)$. These divisors are in the image of φ_W and we choose pre-images $D_0, \dots D_d \in$

$\mathcal{C}(\Gamma(X)^d)$ such that $D_i' = \varphi_W(D_i)$. By Theorem 3.27 we may calculate the local degree in $\mathcal{C}(\Gamma(X)^d)$ and get together with Theorem 3.25(2)

$$\operatorname{ldeg}_W(\varphi_W(D_0 \cdot \cdots \cdot D_d)) = \sum_{\gamma \in \Gamma_1(X)^d} \operatorname{ldeg}_{I^d} \circ i_\gamma^*(D_0 \cdot \cdots \cdot D_d).$$

Therefore both sides of (3.9) are additive on cubes and it suffices to deal with the case $\mathscr{R}(X) = I$.

Let $\Gamma(X) = I$. The center of the standard cube $|\Gamma(X)^d| = |I^d| = [0,1]^d$ is denoted by $x_M := 1/2(1, \ldots, 1)$. We describe the vertices of I^d as usual by $\{C_v \mid v \in \mathbb{F}_2^d\}$ and study for each function $f \in \mathcal{C}^0(I^d)$ the associated divisor by Proposition 2.17:

$$D_f := \phi_1^{-1}(f) = \sum_{v \in \mathbb{F}_2^d} f_i(C_v) C_v.$$

The coordinates of C_v as point in I^d are given by $x_M - 1/2(-1)^v$, where $(-1)^v = ((-1)^{v_1}, \ldots, (-1)^{v_d})$ for $v = (v_1, \ldots, v_d)$. We have $x_M - 1/2(-1)^v = x_M + (1/2)(-1)^{v+(1,\ldots,1)}$ and thus $f(C_v) = f_{x_M}^{1/2}(v + (1, \ldots, 1))$. Using the morphism $\psi : \mathcal{C}(I^d) \to \mathcal{C}(I^d), C_v \mapsto C_{v+(1,\ldots,1)}$ from Theorem 3.26 we calculate:

$$\begin{aligned}
D_f &= \sum_{v \in \mathbb{F}_2^d} f_{x_M}^{1/2}(v + (1, \ldots, 1)) C_v = \sum_{v \in \mathbb{F}_2^d} f_{x_M}^{1/2}(v) \psi(C_v) \\
&= \psi\left(\sum_{v \in \mathbb{F}_2^d} \frac{1}{2^d} f_{x_M}^{1/2}(v) C_v \right) \\
&= \psi\left(\sum_{v \in \mathbb{F}_2^d} \Big(\frac{1}{2^d} \sum_{w \in \mathbb{F}_2^d} (-1)^{\langle v, w\rangle} f_{x_M}^{1/2}(w) \Big) \Big(\sum_{w \in \mathbb{F}_2^d} (-1)^{\langle v, w\rangle} C_w \Big) \right) \\
&= \sum_{v \in \mathbb{F}_2^d} \Delta_{1/2}^v(f)(x_M) \psi(F_v).
\end{aligned}$$

Putting this into the definition of the intersection paring we get

$$\begin{aligned}
\langle f_0, \ldots, f_d \rangle &= \operatorname{ldeg}_{I^d} \left(\sum_{v_0, \ldots, v_d \in \mathbb{F}_2^d} \left(\prod_{i=0}^d \Delta_{1/2}^{v_i}(f)(x_M) \psi(F_{v_i}) \right) \right) \\
&= \sum_{v_0, \ldots, v_d \in \mathbb{F}_2^d} \operatorname{ldeg}_{I^d}\Big(\prod_{i=0}^d F_{v_i} \Big) \prod_{i=0}^d \Delta_{1/2}^{v_i}(f_i)(x_M).
\end{aligned}$$

We used hereby that the degree is invariant under ψ (Theorem 3.26). By the definition of the lattice approximation $\tilde{\Delta}$ we finally get

$$\prod_{i=0}^{d} \Delta_{\frac{1}{2}}^{v_i}(f_i)(x_M) = \int_{\Gamma_1^d} \prod_{i=0}^{d} \tilde{\Delta}_1^{v_i}(f_i),$$

the claim in the case $n = 1$.

Case 2: $n > 1$

By Proposition 3.23 we have

$$\langle f_0, \ldots, f_d \rangle_{W,n} = n^d \langle f_0, \ldots, f_d \rangle_{W_n,1}$$

and by (3.5) for each function $f \in \mathcal{C}^0(\Gamma^d)$

$$n^d \int_{|\Gamma^d|} f = \int_{|\Gamma_n^d|} f$$

holds. Using Proposition 3.17 and the claim in the case $n = 1$ we conclude

$$\begin{aligned}
\langle f_0, \ldots, f_d \rangle_{W,n} &= n^d \langle f_0, \ldots, f_d \rangle_{W_n,1} \\
&= n^d \sum_{v_0,\ldots,v_d \in \mathbb{F}_2^d} \operatorname{ldeg}_{I^d}\Big(\prod_{i=0}^{d} F_{v_i}\Big) \int_{\Gamma_n^d} \prod_{i=0}^{d} \tilde{\Delta}_1^{v_i}(f_i) \\
&= n^{2d} \sum_{v_0,\ldots,v_d \in \mathbb{F}_2^d} \operatorname{ldeg}_{I^d}\Big(\prod_{i=0}^{d} F_{v_i}\Big) \int_{\Gamma^d} \prod_{i=0}^{d} \tilde{\Delta}_n^{v_i}(f_i).
\end{aligned}$$

□

This description of the intersection pairing is used to generalize onto a bigger set of functions. For this purpose we use the following approximation of continuous functions by piecewise affine functions.

Definition 3.29. — *Let $f \in \mathcal{C}_{\vartriangle}^{\infty}(\Gamma^d)$ and $n \in \mathbb{N}$. The function $f^{(n)} \in \mathcal{C}_{\vartriangle}^{\mathrm{lin}}(\Gamma_n^d)$ which is uniquely defined by $f^{(n)}(p) = f(p)$ for each $p \in (\gamma_n^d)_0$ is called the* n-th standard approximation *of f.*

Remark 3.30. — To calculate $\tilde{\Delta}_n^v(f)$ we need only the values of f on the vertices of the n-th subdivision. Thus we have for each $v \in \mathbb{F}_2^d$

$$\tilde{\Delta}_n^v(f) = \tilde{\Delta}_n^v(f^{(n)}).$$

We can now formulate a general convergence result for the standard approximation of functions in $\mathcal{C}^0(\Gamma^d)$. As guarantee for all results we need the vanishing condition from [5, 4.7]:

Definition 3.31. — *Let $d \in \mathbb{N}$. We say d satisfies the* vanishing condition, *if for each partition $\mathcal{P}$ of the set $\{1, \dots, d\}$ and each $v_0, \dots, v_d \in \mathbb{F}_2^d$ the relation $\sum_{i=0}^{d} \alpha(\mathcal{P}, v_i) < d + |\mathcal{P}|$ implies the equation*

$$\mathrm{ldeg}_{I^d}(\prod_{i=0}^{d} F_{v_i}) = 0 \tag{3.10}$$

in the combinatorial Chow ring $\mathcal{C}(I^d)$.

By explicit calculations of the intersection numbers we have already shown in [5, Cor 4.36]:

Lemma 3.32. — *For $d = 2$ and $d = 3$ the vanishing condition in Definition 3.31 is satisfied.*

Theorem 3.33. — *If $d \in \mathbb{N}$ satisfies the vanishing condition in Definition 3.31, then for any choice of the functions $f_0, \dots, f_d \in \mathcal{C}^\infty_\triangle(\Gamma^d)$ the limit*

$$\langle f_0, \dots, f_d \rangle := \lim_{n \to \infty} \langle f_0^{(n)}, \dots, f_n^{(n)} \rangle_{W,n}$$

exists. It can be calculated by

$$\langle f_0, \dots, f_d \rangle = \sum_{\mathcal{P} \text{ Partition}} \frac{1}{2^{d+|\mathcal{P}|}} \sum_{\substack{v_0, \dots, v_d \in \mathbb{F}_2^d, \\ \sum \alpha(v_i, \mathcal{P}) = d + |\mathcal{P}|}} \mathrm{ldeg}_{I^d}(\prod_{i=0}^{d} F_{v_i}) \int_{\mathcal{D}_\mathcal{P}} \prod_{i=0}^{d} D^{v_i}_{\alpha(v_i, \mathcal{P})}(f_i).$$

Proof. — We use the description of the intersection pairing from Proposition 3.28,

$$\langle f_0^{(n)}, \dots, f_d^{(n)} \rangle = n^{2d} \sum_{v_0, \dots, v_d} \mathrm{ldeg}_{I^d}(\prod_{i=0}^{d} F_{v_i}) \int_{\Gamma^d} \prod_{i=0}^{d} \tilde{\Delta}_n^{v_i}(f_i). \tag{3.11}$$

Since the sum of the characteristic functions $\sum_{\mathcal{P} \text{ Partition}} \mathbb{1}_{\{x \in |\Gamma|^d | d(\tilde{x}^n) = \mathcal{P}\}}$ is the constant function $\mathbb{1}_{\Gamma^d}$, we may split the integral of (3.11) into components along the different "pixelated diagonals" (fig. 2) and apply Lemma 3.16:

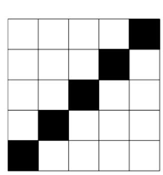

Figure 2. — The "pixelated diagonal" $\{x \mid d(\tilde{x}^{(n)}) = \mathcal{P}\}$ with $n = 5$ and $\mathcal{P} = \{\{1, 2\}\}$.

$$n^{2d}\int_{\Gamma^d}\prod_{i=0}^{d}\tilde{\Delta}_n^{v_i}(f_i) = \sum_{\mathcal{P}\text{ Partition}} n^{2d}\int_{\Gamma^d}\mathbb{1}_{\{d(\tilde{x}^{(n)})=\mathcal{P}\}}\prod_{i=0}^{d}\tilde{\Delta}_n^{v_i}(f_i)$$

$$= \sum_{\mathcal{P}\text{ Partition}} n^{d+|\mathcal{P}|}\int_{\mathcal{D}_\mathcal{P}}\mathbb{1}_{\{d(\tilde{x}^{(n)})=\mathcal{P}\}}\prod_{i=0}^{d}\tilde{\Delta}_n^{v_i}(f_i).$$

Together with (3.11) this implies

$$\langle f_0^{(n)},\ldots,f_d^{(n)}\rangle = \sum_{\mathcal{P}\text{ Partition}}\frac{1}{2^{d+|\mathcal{P}|}}\sum_{v_0,\ldots,v_d\in\mathbb{F}_2^d}\operatorname{ldeg}_{I^d}(\prod_{i=0}^{d}F_{v_i})\int_{\mathcal{D}_\mathcal{P}}T_n(\mathcal{P},v_0,\ldots,v_d)$$

with

$$T_n(\mathcal{P},v_0,\ldots,v_d) := (2n)^{d+|\mathcal{P}|}\mathbb{1}_{\{d(\tilde{x}^{(n)})=\mathcal{P}\}}\prod_{i=0}^{d}\tilde{\Delta}_n^{v_i}(f_i).$$

We study the convergence of the terms $T_n(\ldots)$: By the vanishing conjecture (3.10) we only have to deal with terms where $\sum\alpha(\mathcal{P},v_i)\geqslant d+|\mathcal{P}|$. By dominated convergence it suffices to show that all $T_n(\mathcal{P},v_0,\ldots,v_d)$ are globally bounded and converge to

$$T(\mathcal{P},v_0,\ldots,v_d) := \begin{cases}0 & \text{if } \sum\alpha(v_i,\mathcal{P})>d+|\mathcal{P}|,\\ \prod_{i=0}^{d}D^{v_i}_{\alpha(v_i,\mathcal{P})}(f_i) & \text{if } \sum\alpha(v_i,\mathcal{P})=d+|\mathcal{P}|.\end{cases}$$

For this purpose we rewrite T_n as

$$T_n(\mathcal{P},v_0,\ldots,v_d)=\mathbb{1}_{\{d(\tilde{x}^{(n)})=\mathcal{P}\}}\cdot\Big((2n)^{d+|\mathcal{P}|-\sum\alpha(v_i,\mathcal{P})}\Big)\cdot\Big(\prod_{i=0}^{d}(2n)^{\alpha(v_i,\mathcal{P})}\tilde{\Delta}_n^{v_i}(f_i)\Big)$$

and discuss each part individually. By Proposition 3.15 the function $(2n)^{\alpha(v_i,\mathcal{P})}\tilde{\Delta}_n^{v_i}(f_i)$ is bounded and converges to $D^{v_i}_{\alpha(\mathcal{P},v_i)}$. The characteristic function $\mathbb{1}_{\{\mathcal{P}(\tilde{x}^{(n)})=\mathcal{P}\}}$ is obviously bounded and converges to $\mathbb{1}_{\{x|\mathcal{D}(x)=\mathcal{P}\}}=\mathbb{1}_{\mathcal{D}(\mathcal{P})}$. Finally the behavior of the term $n^{d+|\mathcal{P}|-\sum\alpha(v_i,\mathcal{P})}$ yields the convergence

$$\lim_{n\to\infty}T_n(\mathcal{P},v_0,\ldots,v_d)=T(\mathcal{P},v_0,\ldots,v_d).$$

□

Since the vanishing condition is true for $d=2$ and $d=3$ (Lemma 3.32), we are able to give a definitive formulation of the intersection pairing for $d=2$ and $d=3$. The case $d=2$ yields the result of Zhang [8, Prop 3.3.1, Prop 3.4.1]:

COROLLARY 3.34. — *Let $d = 2$ and $f_0, f_1, f_2 \in \mathcal{C}^\infty_\Delta(\Gamma^2)$ be continuous functions which are smooth on simplices. Let $f_i^{(n)}$ be the standard approximation of f_i. Then the limit of the triple pairing $\lim_{n\to\infty}\langle f_0^{(n)}, f_1^{(n)}, f_2^{(n)}\rangle$ exists and can be calculated by*

$$\lim_{n\to\infty}\langle f_0^{(n)}, f_1^{(n)}, f_2^{(n)}\rangle = \langle f_0, f_1, f_2\rangle_{\mathrm{sm}} + \langle f_0, f_1, f_2\rangle_{\mathrm{sing}}$$

where

$$\begin{aligned}\langle f_0, f_1, f_2\rangle_{\mathrm{sm}} &= \sum_{\substack{v_0,v_1,v_2\in\mathbb{F}_2^2\\ \{v_0,v_1,v_2\}=\{(1,0),(0,1),(1,1)\}}} \int_{|\Gamma^2|} D^{v_0}_{|v_0|}(f_0) D^{v_1}_{|v_1|}(f_1) D^{v_2}_{|v_2|}(f_2),\\ \langle f_0, f_1, f_2\rangle_{\mathrm{sing}} &= \sum_{\substack{v_0,v_1,v_2\in\mathbb{F}_2^2\\ \{v_0,v_1,v_2\}=\{(1,0),(0,1),(1,1)\}}} 2\int_{\mathcal{D}} D^{v_0}_1(f_0) D^{v_1}_1(f_1) D^{v_2}_1(f_2)\\ &\quad - 4\int_{\mathcal{D}} D^{(1,1)}_1(f_0) D^{(1,1)}_1(f_1) D^{(1,1)}_1(f_2).\end{aligned}$$

Proof. — Recall the intersection numbers calculated in [5, Thm 4.32],

$$\mathrm{ldeg}(F_{v_1}F_{v_2}F_{v_3}) = \begin{cases} -32 & \text{if } v_1 = v_2 = v_3 = (1,1),\\ 16 & \text{if } \{v_1, v_2, v_3\} = \{(1,0),(0,1),(1,1)\},\\ 0 & \text{otherwise.}\end{cases}$$

The result is a direct consequence of Theorem 3.33: The summand with $\mathcal{P} = \{\{1\},\{2\}\}$ yields the non-singular part $\langle\cdot,\dots,\cdot\rangle_{\mathrm{sm}}$. Since for $v_0 = v_1 = v_2 = (1,1)$ the equation

$$\sum_{i=0} 2\alpha(v_i, \mathcal{P}) = \sum_{i=0}^{2} |v_i| = 6 > 4 = d + |\mathcal{P}|$$

holds, we only have to deal with elements $v_0, v_1, v_2 \in \mathbb{F}_2^d$ where $\{v_0, v_1, v_2\} = \{(1,0),(0,1),(1,1)\}$.

For the summand with $\mathcal{P} = \{\{1,2\}\}$ we must take all non-trivial intersections $F_{v_0}F_{v_1}F_{v_2}$ into account. The resulting term gives exactly the singular part $\langle\cdot,\dots,\cdot\rangle_{\mathrm{sing}}$ in above formula. $\square$

Remark 3.35. — The exact formulation of [8] is obtained using Example 3.13, i.e., by identifying

$$\begin{aligned}
D_1^{(1,0)}(f)(x) &= \frac{\partial f}{\partial x_1}(x),\\
D_1^{(0,1)}(f)(x) &= \frac{\partial f}{\partial x_2}(x),\\
D_2^{(1,1)}(f)(x) &= \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) \quad \text{if } x \notin \mathcal{D},\\
D_1^{(1,1)}(f)(x) &= \frac{1}{2}\frac{\partial}{\partial x_1}(f^+ - f^-)(x) = \frac{1}{2}\delta(f)(x) \quad \text{if } x \in \mathcal{D}.
\end{aligned}$$

Then the formula for the smooth resp. singular part becomes

$$\begin{aligned}
\langle f_0, f_1, f_2\rangle_{\mathrm{sm}} &= \int_{\Gamma^2\setminus\mathcal{D}} \frac{\partial f_0}{\partial x_1}(x)\frac{\partial f_1}{\partial x_1}(x)\frac{\partial^2 f_2}{\partial x_1\partial x_2}(x) + \text{ permutations},\\
\langle f_0, f_1, f_2\rangle_{\mathrm{sing}} &= \int_{\mathcal{D}} \left(\frac{\partial f_0}{\partial x_1}(x)\frac{\partial f_1}{\partial x_2}(x)\delta(f_2)(x) + \text{ permutations}\right)\\
&\quad - \int_{\mathcal{D}} \left(\frac{1}{2}\delta(f_0)(x)\delta(f_1)(x)\delta(f_2)(x)\right).
\end{aligned}$$

A similar formula is deducible from Theorem 3.33 in the case $d = 3$. For clarity reasons we only calculate the non-singular part of the pairing. This is achieved by calculating the intersection pairing only of functions $f \in \mathcal{C}^\infty_\square(\Gamma^3)$, i.e., functions smooth on cubes. For these functions the singular part vanishes.

Theorem 3.36. — *Let $f_0, \dots, f_3 \in \mathcal{C}^\infty_\square(\Gamma^3)$ be functions smooth on cubes. Then the limit of the quadruple pairing $\langle f_0, f_1, f_2, f_3\rangle$ exists and can be calculated as*

$$\lim_{n\to\infty} \langle f_0^{(n)}, \dots, f_3^{(n)}\rangle = \int_{\Gamma^3} \sum_{\substack{v_0,v_1,v_2,v_3\in\mathbb{F}_2\\ \{v_0,v_1,v_2,v_3\}\in B}} \prod_{i=0}^{d} D^{v_i}_{|v_i|}(f_i),$$

where the set $B \subset \mathcal{P}(\mathbb{F}_2^3)$ is defined as follows

$$\begin{aligned}
B := \Big\{&\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\},\\
&\{(1,0,0),(0,1,0),(1,0,1),(0,1,1)\},\\
&\{(1,0,0),(0,0,1),(1,1,0),(0,1,1)\},\\
&\{(0,1,0),(0,0,1),(1,1,0),(1,0,1)\}\Big\}.
\end{aligned}$$

Proof. — Let $v_0, \ldots, v_3 \in \mathbb{F}_2^3$. By [5, Thm 4.33] $\{v_0, v_1, v_2, v_3\} \in B$ holds iff $\mathrm{ldeg}_{\mathcal{C}(I^3)}\left(\prod_{i=0}^3 F_{v_i}\right) \neq 0$ and $\sum_{i=0}^3 |v_i| = 6$ hold. For these elements we have $\mathrm{ldeg}_{\mathcal{C}(I^3)}(\left(\prod_{i=0}^3 F_{v_i}\right) = 2^6$ and thus the term in Theorem 3.33 belonging to $\mathcal{P} = \{\{1\}, \{2\}, \{3\}\}$ is given by

$$\sum_{\substack{v_0, v_1, v_2, v_3 \in \mathbb{F}_2 \\ \{v_0, v_1, v_2, v_3\} \in B}} \prod_{i=0}^{d} D^{v_i}_{|v_i|}(f_i). \tag{3.12}$$

If $\mathcal{P}$ is another partition of $\{1, 2, 3\}$, then there exists for each $v_0, \ldots, v_3 \in \mathbb{F}_2^3$ at least one $i \in \{0, 1, 2, 3\}$ such that $\alpha(v_i, \mathcal{P}) < |v_i|$. This implies $D^{v_i}_{\alpha(v_i, \mathcal{P})}(f_i) = 0$ since the functions are smooth.

Furthermore all functions are defined on $|\Gamma(X)^3|$ and $|\Gamma(X)^3| \setminus \mathcal{D}_{\{\{1\},\{2\},\{3\}\}}$ is a zero-set, thus the claim is proven. □

By Theorem 3.33 one could think that the intersection pairing converges also for a broader set of functions. Then there is a meaningful definition of positivity needed (cp. [7]). Convergence without conditions can not be expected, as the following example shows.

Example 3.37. — Let $X = \mathrm{Proj}\, R[x_0, x_1, z]/(x_0 x_1 - z^2 \pi)$ be the projective completion of the standard scheme L with the usual ordering $<$ of the components of L_s and $W = W(X, <, 2)$ the product model according to Algorithm 2.1. We identify as always $|\mathscr{R}(W)| = |\Gamma(X)|^2 = [0,1]^2$ and set

$$\begin{aligned} &\varphi_n : [0,1] \to \mathbb{R}, \\ &\qquad x \mapsto \frac{1}{2} \sum_{i=0}^{n} (-1)^i \max(0, \frac{1}{n} - |x - \frac{i}{n}|). \end{aligned}$$

The function φ_n describes a triangle wave with amplitude 1 and length $\frac{2}{n}$. We define the following sequence of functions

$$\begin{aligned} f_{0,n} &:= \varphi_n(x), \\ f_{1,n} &:= \varphi_n(y), \\ f_{2,n} &:= \varphi_n(x - y). \end{aligned}$$

They are bounded and lie in $\mathcal{C}^{\mathrm{lin}}_{\vartriangle}(\mathrm{sd}_n(\Gamma))$. We have however

$$\langle f_{0,n}, f_{1,n}, f_{2,n} \rangle = n.$$

By introducing a factor $f'_{i,n} := n^{1/3} f_{i,n}$ we get functions which converge uniformly to 0, but whose triple pairing is constant:

$$\langle f'_{0,n}, f'_{1,n}, f'_{2,n} \rangle = 1.$$

Proof. — Obviously the differential of the function φ_n satisfies

$$\varphi'_n(x) = \begin{cases} 1 & \text{if } x \in (\frac{2i}{n}, \frac{2i+1}{n}), \\ -1 & \text{if } x \in (\frac{2i+1}{n}, \frac{2i+2}{n}). \end{cases}$$

This gives for the generalized differentials

$$\begin{aligned} D_1^{(1,0)}(f_{0,n})(x,y) &= \varphi'_n(x), \\ D_1^{(0,1)}(f_{1,n})(x,y) &= \varphi'_n(y), \\ D_1^{(1,0)}(f_{2,n})(x,y) &= \varphi'_n(x-y), \\ D_1^{(0,1)}(f_{2,n})(x,y) &= -\varphi'_n(x-y). \end{aligned}$$

Furthermore

$$D_1^{(0,1)}(f_{0,n}) = D_1^{(1,0)}(f_{1,n}) = D_2^{(1,1)}(f_{i,n}) = 0.$$

The diagonals of the n-fold subdivision are given by the points $(x,y) \in [0,1]^2$ with $|x-y| = \frac{i}{n}$. At this point only $f_{2,n}$ has a singularity and we get

$$D_1^{(1,1)}(f_{2,n})(x,y) = (-1)^i \quad \text{for } |x-y| = \frac{i}{n}.$$

We may now apply the formula from Corollary 3.34: Since $D_2^{(1,1)}(f_{i,n}) = 0$ outside of the diagonal for each $i = 0,1,2$, it suffices to calculate the singular part. This is given by

$$\begin{aligned} \langle f_{0,n}, f_{1,n}, f_{2,n} \rangle &= \int_{\mathcal{D}} D_1^{(1,0)}(f_{0,n}) D_1^{(0,1)}(f_{1,n}) D_1^{(1,1)}(f_{2,n}) \\ &= \int_{\mathcal{D}} 1 = n. \end{aligned}$$

□

Appendix

A. The geometric realization of simplicial sets and their subdivision

As in [5] we need some basic facts about simplicial sets. To recall the definition and notation of *simplicial sets*, the *standard-n-simplex*, *degenerate simplices* please see the appendix of [5, Appendix A].

Definition A.1. — *Let $k \in \mathbb{N}$. With s_i, for each $i \in \{0, \dots, k\}$, the morphism*

$$s_i : [0] \to [k], 0 \mapsto i$$

is denoted. A simplicial set $\mathscr{R}$ is called simplicial set without multiple simplices, *if the map*

$$\varphi : \coprod_{k=0}^{\infty} K_k^{\mathrm{nd}} \to \mathcal{P}(K_0), t \in K_k^{\mathrm{nd}} \mapsto \{K(s_0)(t), \dots, K(s_k)(t)\}$$

is a monomorphism. If this is true, we denote the image of φ by

$$\mathscr{R}_S := \mathrm{Im}(\varphi) \subseteq \mathcal{P}(\mathscr{R}_0).$$

Remark A.2. — Let Γ be a graph without multiple simplices. By functoriality we identify the 1-simplices $\gamma_1 \in \Gamma_1$ with morphisms $i_{\gamma_1} : \Delta[1] \to \Gamma$. Since Γ is without multiple simplices, the i_{γ_1} are injective for each non-degenerate 1-simplex γ_1. Let now $\gamma := (\gamma_1, \dots \gamma_d) \in (\Gamma_1^{\mathrm{nd}})^d$ be a d-tuple of 1-simplices. The product

$$i_\gamma := (i_{\gamma_1} \times \cdots \times \cdots \times i_{\gamma_d}) : I^d \to \Gamma^d$$

is injective as well and denoted by i_γ. By [5, Prop 4.18] the set of all i_γ gives a covering of Γ^d.

Definition A.3. — *Let $n \in \mathbb{N}_0$. Then $|\Delta[n]|$ denotes the topological standard-n-simplex, i.e., the space*

$$\{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1, t_i \geqslant 0\} \subseteq \mathbb{R}^{n+1}.$$

Let $n, m \in \mathbb{N}_0$ and $\varphi : [n] \to [m]$ be a morphism in the category Δ. Then φ induces a continuous morphism

$$\begin{aligned} |\Delta[\varphi]| : |\Delta[n]| &\to |\Delta[m]|, \\ (t_0, \dots, t_n) &\mapsto (t'_0, \dots t'_m) \end{aligned}$$

where $t'_j := \sum_{\varphi(i)=j} t_i$. *This makes* $|\Delta[\cdot]| : \Delta \to \mathrm{Top}$ *a covariant functor. For each simplicial set* $\mathscr{R}.$ *we call the topological space*

$$|\,\mathscr{R}\,| := \operatorname*{colim}_{\Delta\,\mathscr{R}.} |\Delta[n]|$$

geometric realization *of* $\mathscr{R}.$. *As colimit this construction is functorial in* $\mathscr{R}.$.

For simplicial sets without multiple simplices we have the following more explicit description of the geometric realization:

Proposition A.4. — *Let* $\mathscr{R}.$ *be a simplicial set without multiple simplices. The geometric realization is the subspace* $|K| \subseteq \hom_{\mathrm{set}}(\mathscr{R}_0, \mathbb{R})$ *consisting of the probability distributions on* $\mathscr{R}_0$ *(with respect to the counting measure) with support in a simplex of* $\mathscr{R}.$. *An element of* $|K|$ *is therefore a function* $f : \mathscr{R}_0 \to [0,1]$ *with*

$$\sum_{v \in \mathscr{R}_0} f(v) = 1 \text{ and } \operatorname{supp}(f) \in \mathscr{R}_S \tag{A.1}$$

with $\mathscr{R}_S$ *defined as in Definition A.1.*

If $\varphi : \mathscr{R}. \to \mathscr{R}'.$ *is a morphism of simplicial sets without multiple simplices, the induced morphism* $\varphi_* : |K| \to |K'|$ *is given on probability distributions as follows: Let* $f \in |K|$ *be as in (A.1). Then* $f' = \varphi_*(f)$ *is given by the map* $f' : \mathscr{R}'_0 \to \mathbb{R}$ *with*

$$f'(s') = \sum_{s \in \varphi_0^{-1}(s')} f(s).$$

Proof. — Denote the set of probability distributions by $\mathrm{PD}(K)$. For the standard simplices $\Delta[n]$ and morphisms of standard simplices the isomorphism is obvious. We therefore get a continuous map $|K| \to \mathrm{PD}(K)$. It is easy to see that this map is open. Therefore it is enough to show that the map is also bijective. We calculate the inverse: Let $f : \mathscr{R}_0 \to \mathbb{R}$ be a probability distribution as in (A.1). Since the simplicial set $\mathscr{R}.$ has no multiple simplices, there is an unique non-degenerate simplex $s : \Delta[j] \to \mathscr{R}.$ with $\operatorname{supp}(f) = \operatorname{Im}(s)$. Then there is a morphism $f' : \Delta[j]_0 \to \mathbb{R}$ with $f = s_*(f')$ and we map f onto the point $|s|(f')$. □

Example A.5. — For the standard-1-simplex $\Delta[1]$ we have $\Delta[1]_0 \simeq \{0,1\}$ and $\Delta[1]_S \simeq \mathcal{P}(\{0,1\})$. Thus there is a canonical isomorphism between the geometric realization $|\Delta[1]|$ and the interval $[0,1]$.

Proposition A.6. — *Let I denote the standard-1-simplex $I := \Delta[1]$ and S_d the symmetric group of degree d. Then there is a canonical bijection*

$$\psi : S_d \to (I^d)_d^{\mathrm{nd}}$$

between non-degenerate d-simplices in the product I^d and S_d. The geometric realization of the simplex $\psi(\sigma)$ is given in $I^d = [0,1]^d$ by

$$\{x = (x_1, \dots, x_d) \in [0,1]^d \mid x_{\sigma(1)} \leqslant x_{\sigma(2)} \leqslant \cdots \leqslant x_{\sigma(d)}\}.$$

Proof. — Since the product I^d is defined component by component, we have

$$I^d([n]) \simeq \prod_d \hom_{\mathrm{Poset}}([n],[1]) \simeq \hom_{\mathrm{Poset}}([n],[1]^d),$$

where the product $[1]^d$ is calculated in the category of sets with partial order (see [5, Cor A.7]). To identify the d-simplices of I^d we use

$$(I^d)_d \simeq \hom([d],[1]^d).$$

An element $\varphi \in \hom([d],[1]^d)$ is non-degenerate iff $\varphi(0) < \varphi(1) < \cdots < \varphi(d)$ holds with $<$ being the product ordering on $[1]^d$. Then there exists a unique permutation $\sigma \in S_d$ such that

$$\varphi(i) = (0, \dots, 0, \underbrace{1, \dots, 1}_{i})^{\sigma}$$

holds for each $i \in \{0, \dots, d\}$.

In the geometric realization $[0,1]^d$ the vertices of the simplex φ are given by the values of $\varphi(i)$. Each point of this simplex is a convex combination of these points, thus $x = (x_1, \dots, x_d) \in |I^d|$ is in the simplex φ iff

$$x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(d)}$$

holds. □

For the description of ramified base-change we need a subdivision of simplicial sets. This can also described completely categorical:

Definition A.7. —

1. *Let $k \in \mathbb{N}$. We denote by $\tilde{\mathrm{sd}}_k$ the functor*

$$\tilde{\mathrm{sd}}_k : \Delta \to \Delta$$

given on objects by

$$[n] \mapsto [(n+1) \cdot k - 1]$$

and on morphisms by

$$\mathrm{Hom}_{\mathrm{sSet}}([n],[m]) \ni \varphi \mapsto (ak+b \mapsto ak+\varphi(b) \text{ for } 0 \leqslant b < k)\,.$$

2. *The functor induces by* $\tilde{\mathrm{sd}}_k$

$$\mathrm{sd}_k : \mathrm{sSet} \to \mathrm{sSet},\, \mathscr{R}. \mapsto \tilde{\mathrm{sd}}_k \circ \mathscr{R}.$$

is called the k-fold subdivision functor*.*

Proposition A.8. — *For each simplicial set* $X.$ *there is a canonical isomorphism*

$$\mathrm{Sd}_n : |\mathrm{sd}_n(X)| \simeq |X|.$$

For $X. = I. = \Delta[1]$ *and with the description of the geometric realization of Proposition A.4 this isomorphism is given by the mapping*

$$[f : (\mathrm{sd}_n(I))_0 \to \mathbb{R}] \mapsto [f' : I_0 \to \mathbb{R}]$$

where

$$f'(0) = \sum_{i=0}^{n} \frac{1}{n} f(\varphi_i^k).$$

Proof. — The construction of this mapping is given in [1, Lemma 1.1]. ☐

Proposition A.9. — *Let* $\mathscr{R}.$ *be a simplicial set and* $t : \Delta \to \mathrm{sd}_n(K)$ *a simplex of* $\mathrm{sd}_n(K)$*. Then there is a simplex* $s : \Delta \to \mathscr{R}.$ *in* $\mathscr{R}.$ *such that the image of* $|t|$ *under the canonical morphism*

$$|\mathrm{sd}_n(K)| \simeq |K|$$

lies completely in $\mathrm{Im}(|s|)$*.*

Proof. — Let $i \in \mathbb{N}$ such, that $t \in (\mathrm{sd}_k(K))_i$ holds. By the definition of the subdivision we have $(\mathrm{sd}_k(K))_i \simeq \mathscr{R}_{(i+1)k-1}$ and we choose $s \in \mathscr{R}_{(i+1)k-1}$ as image of t under this isomorphism. Then t allows a factorization of the form

$$t : \Delta[i] \xrightarrow{\tilde{t}} \mathrm{sd}_k(\Delta[i]) \xrightarrow{\mathrm{sd}_k(s)} \mathrm{sd}_k(K)$$

and therefore the following diagram commutes:

$$\begin{array}{ccc} |\Delta[i]| & & \\ {\scriptstyle |\tilde{t}|}\big\downarrow & & \\ |\mathrm{sd}_k(\Delta[i])| & \xrightarrow{\simeq} & |\Delta[(i+1)k-1]| \\ {\scriptstyle |\mathrm{sd}_k(s)|}\big\downarrow & & {\scriptstyle |s|}\big\downarrow \\ |\mathrm{sd}_k(K)| & \xrightarrow{\simeq} & |K|. \end{array}$$

This finishes the proof. □

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