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IVAN CHEREDNIK AND ROSS ELLIOT

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Refined composite invariants of torus knots via DAHA

Ivan Cherednik[(1)], Ross Elliot[(2)]

Dedicated to Vadim Schechtman on the occasion of his 60th birthday

Résumé. – Nous définissons les DAHA-superpolynômes composites associés aux nœuds toriques, en fonction des paires de diagrammes de Young qui généralisent les polynômes de HOMFLY-PT composites dans la théorie de skein de l'anneau. Nous donnons divers exemples. Nos superpolynômes étendent les polynômes (raffinés) de DAHA-Jones et satisfont toutes les symétries standards des DAHA-superpolynômes des nœuds toriques. Ces derniers sont conjecturalement liés à l'homologie de HOMFLY-PT. À la fin, nous construisons deux DAHA-hyperpolynômes en étendant les polynômes de DAHA-Jones de type E. Ils sont étroitement liés à l'approche de Deligne-Gross des systèmes de racines exceptionnels ; ce thème est de nature expérimentale.

Abstract. – We define composite DAHA-superpolynomials of torus knots, depending on pairs of Young diagrams and generalizing the composite HOMFLY-PT polynomials in the skein theory of the annulus. We provide various examples. Our superpolynomials extend the DAHA-Jones (refined) polynomials and satisfy all standard symmetries of the DAHA-superpolynomials of torus knots. The latter are conjecturally related to the HOMFLY-PT homology. At the end, we construct two DAHA-hyperpolynomials extending the DAHA-Jones polynomials of type E closely related to the Deligne-Gross approach to the exceptional root systems; this theme is of experimental nature.

[1]Department of Mathematics, UNC Chapel Hill, North Carolina 27599, USA
chered@email.unc.edu
Partially supported by NSF grant DMS–1363138
[2]California Institute of Technology, Pasadena, California 91125, USA
relliot@caltech.edu
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0. **Introduction**

We introduce and study the *composite DAHA-superpolynomials* for torus knots and arbitrary *composite weights* [17], i.e. pairs of Young diagrams. They depend on a, q, t and unify the corresponding n–series of (refined) DAHA-Jones q, t– polynomials of type A_n; all symmetries of superpolynomials from [3, 4] hold for them. When $t = q$ and $a \mapsto -a$, we establish their relation to the *composite HOMFLY-PT polynomials*, studied in [13, 20, 1, 11].

The topological composite theory is based on the *full HOMFLY-PT skein* of the annulus, which is an algebra generated by link diagrams drawn there. The adjoint representation is the simplest composite weight, which connects our results with two examples of adjoint DAHA-superpolynomials for the Deligne-Gross exceptional series of root systems considered at the end of the paper.

Topological origins. In the full HOMFLY-PT skein, the orientations of the components of the links can be simultaneously clockwise and counterclockwise around the annulus, which eventually results in pairs of Young diagrams. It is isomorphic to the tensor square of the ring of symmetric functions. The (non-full) skein has all orientations in the same direction, which is insufficient for the composite theory. The diagonalization of the *meridian maps* in the full skein of the annulus provides a natural and systematic way to define the composite HOMFLY-PT polynomials for any knots and colors.

The role of the annulus can be clearly seen in the theory of *satellite links*, which is of fundamental value in low-dimensional topology (including our paper). Given a knot $K \subset S^3$ and a Young diagram, such a link is generally constructed from both a diagram $D(K)$ of K, called a *companion*, and a link diagram Q in the annulus, called a *pattern*. The annulus inevitably emerges here due to the framing of K, an important ingredient of this construction (which influences the output).

Superpolynomials. The uncolored DAHA-superpolynomials of torus knots in S^3 are conjectured to coincide with the Poincaré polynomials for the reduced *HOMFLY-PT homology* or, equivalently, stable reduced Khovanov-Rozansky polynomials. See e.g. [8, 14, 15, 16, 24, 28] for the corresponding knot homology theories and categorification. This is expected to hold for any rectangular Young diagrams, though adding colors to HOMFLY-PT homology is a theoretical and practical challenge. Rectangular diagrams are natural here, since the DAHA-superpolynomials are conjecturally positive for such diagrams and arbitrary algebraic knots.

We note that the DAHA-superpolynomials were recently defined for iterated torus knots [6], which includes all algebraic knots (links are in progress). This is a natural setting for the composite DAHA-superpolynomials, but we focus here only on torus knots.

The theory of DAHA-Jones polynomials is uniform for any root systems and arbitrary weights; accordingly, the DAHA-superpolynomials are defined for any Young diagrams. They are studied reasonably well by now; at least, all conjectures about them from [3] are verified, but the positivity. This is generally beyond what topology provides, especially upon adding arbitrary colors to the theory.

The key open question in the composite direction we present concerns the relation of our composite DAHA-superpolynomials to HOMFLY-PT *homology* in the case of annulus. A theory in the annulus is in progress, and it seems capable of practically producing invariants for simple knots and colors; see [22]. However, we hesitate to conjecture any explicit connection because of the absence of such examples so far. Also, the composite DAHA-superpolynomials lack positivity, as do those for the non-rectangular diagrams and non-algebraic knots. It is not clear how to address this challenge, though we provide some approach of experimental nature in [6].

Exceptional series. We conclude this paper with hypothetical adjoint (quasi-minuscule) *DAHA-hyperpolynomials* for the torus knots $T^{3,2}, T^{4,3}$ for the *exceptional "magic" series*:

$$\{e \subset A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8\}$$

from [7]. This is for the maximal short root ϑ, which is the highest weight of the adjoint representation. Thus, for the root systems of type A, we make contact with the composite DAHA-superpolynomials.

The root systems G_2, F_4 are beyond our reach so far and we managed to find such hyperpolynomials only for simple torus knots (though $T(4,3)$ is not too simple). Nevertheless, we believe that even such examples demonstrate that the final theory of DAHA-hyperpolynomials will eventually incorporate all types of root systems (not only classical).

The hyperpolynomials we found based on the functoriality from [7] are non-positive but have rich symmetries. We note that there are (quite a few) other series where the existence of the superpolynomials can be expected, not only for those of Deligne-Gross type. For instance, we found (joint with Sergei Gukov) the minuscule superpolynomials for $\{E_6, A_6, D_5\}$. Here, as for the exceptional series above, only small torus knots and the simplest weights can be managed.

Generally, $\deg_a$+1 root systems are needed to determine the corresponding polynomial uniquely. They provide its evaluations at the corresponding values of a, which was used in [3] to define superpolynomials and hyperpolynomials for $ABCD$ (infinite families). This is not the case with E. However, a very small number of evaluations appeared sufficient in the examples we managed. For instance, only E_8 and E_7 are needed to determine the exceptional DAHA-hyperpolynomial of $T^{3,2}$ (assuming that it satisfies some natural properties). There is no general understanding at the moment of how to proceed for arbitrary torus knots and weights for exceptional root systems.

The structure of the paper. In Section 1.1, the composite weights $[\lambda, \mu]$ (pairs of Young diagrams) and the corresponding representations are defined, following [17]. Then we provide the definition of composite HOMFLY-PT polynomials $\mathcal{H}_{[\lambda,\mu]}(K)$ for any knot K from [13], via the full HOMFLY-PT skein algebra $\mathcal{C}$ of link diagrams in the annulus. Finally, Proposition 1.1, a generalization of the Rosso-Jones formula, gives effective means of producing $\mathcal{H}_{[\lambda,\mu]}(T^{\mathsf{r},\mathsf{s}})$ for $T^{\mathsf{r},\mathsf{s}}$ via. It essentially coincides with formula (C.6) from [11]; we give its proof.

In Section 2, we recall the main definitions and results from the DAHA theory used to introduce the DAHA-Jones (also called refined) polynomials and DAHA-superpolynomials from [3, 4]. Then, we offer the main body of results of this paper. Theorem 2.3 is the existence (stabilization) of composite DAHA-superpolynomials and their evaluations at $q = 1$. Theorem 2.4 is the composite super-duality, which is proved using a reduction to the DAHA-Jones polynomials, closely related to the color exchange from Theorem 2.5. The connection to the composite HOMFLY-PT polynomials is Theorem 2.6.

Section 3 is devoted to various examples of composite DAHA superpolynomials and discussion of their symmetries from the previous section. Our examples confirm the stabilization, connection, super-duality and evaluation theorems for a selection of seven composite partitions and simple torus knots. Section 4 is devoted to the examples of hyperpolynomials for the "magic" exceptional series from [7] (the bottom line of the triangle considered there).

The key construction. We begin with the definition of (reduced, tilde-normalized) *DAHA-Jones polynomials* $\widetilde{JD}^{R}_{\mathsf{r},\mathsf{s}}(b\,;\,q,t)$, associated to any torus knot $T^{\mathsf{r},\mathsf{s}}$, root system R, and (dominant) weight $b \in P_+$ for R. This is unchanged vs. [3, 4]. We mention that they conjecturally coincide with the corresponding Quantum Group invariants for torus knots upon $t = q$ (for both t, in the non-simply-laced case). This was checked for A_n for any Young diagrams in [3] and in various other cases, including the formulas conjectured there for E_6 (by R. E.).

When R is of type A_n, the DAHA-Jones polynomials are uniform with respect to n; see [3, 10]. Namely, the corresponding superpolynomials are defined as follows:

$$HD_{\mathsf{r},\mathsf{s}}(\lambda\,;\,q,t,a \mapsto -t^{n+1}) = \widetilde{JD}^{A_n}_{\mathsf{r},\mathsf{s}}(\lambda\,;\,q,t),$$

where the Young diagram λ is interpreted naturally as an A_n–weight for any sufficiently large n. This definition is generalized in the present paper to the case of the pairs $[\lambda, \mu]$ of Young diagrams, placed at the ends of the corresponding Dynkin graph for A_n.

The *uncolored case* corresponds to the adjoint representation:

$$HD_{\mathsf{r},\mathsf{s}}([\omega_1, \omega_1]\,;\,q,t,a \mapsto -t^{n+1}) = \widetilde{JD}^{A_n}_{\mathsf{r},\mathsf{s}}(\omega_1 + \omega_n\,;\,q,t).$$

The stabilization is a more subtle issue in the composite case. We prove that all symmetries from [3, 4] of the resulting *composite DAHA-superpolynomials* hold. The key result of this paper is the coincidence of $HD_{\mathsf{r},\mathsf{s}}([\lambda,\mu]\,;\,q,q,-a)$ with the HOMFLY-PT polynomials defined in [13] for any composite diagrams $[\lambda,\mu]$ via the skein theory of link diagrams in the annulus $S^1\times I$.

1. HOMFLY-PT polynomials

1.1. Composite representations

An irreducible (finite-dimensional) representation V of $\mathfrak{sl}_N(\mathbb{C})$ is uniquely specified by its *highest weight*:

$$b=\sum_{i=1}^{N-1} b_i\omega_i \in P_+ \stackrel{\mathbf{def}}{=\!=} \bigoplus_{i=1}^{N-1} \mathbb{Z}_+\,\omega_i,\ \ \mathbb{Z}_+=\mathbb{Z}_{\ge 0},$$

where $\{\omega_i\}$ are the fundamental dominant weights for A_{N-1}.

Equivalently, we can represent b (and V) by a *partition* or its corresponding *Young diagram* $\lambda=\lambda_1\ge\lambda_2\ldots\ \lambda_{N-1}\ge\lambda_N=0$ with at most $N-1$ nonempty rows and kth row of length $\lambda_k \stackrel{\mathbf{def}}{=\!=} b_k+\cdots+b_{N-1}$. The highest weight b is recovered from λ by taking $b_i=\lambda_i-\lambda_{i+1}$; i.e. b_i is the number of columns of λ of height i.

The dual representation V^* is specified by the highest weight $b^* \stackrel{\mathbf{def}}{=\!=} \iota(b)$, where $\iota:\omega_i\mapsto\omega_{N-i}$. Alternatively, the Young diagram λ^* has rows of length $\lambda^*_k=\lambda_1-\lambda_{N+1-k}$ (this operation depends on N).

A weight $b\in P_+$ for $\mathfrak{sl}_N(\mathbb{C})$ can be interpreted for $\mathfrak{sl}_M(\mathbb{C})$ by setting $b_i=0$ for $i\ge\min\{M,N\}$. Accordingly, we can interpret the corresponding Young diagram λ as a dominant weight for $\mathfrak{sl}_M(\mathbb{C})$ by removing any columns of height $\ge M$. It is precisely this sort of "packaging" of representations for all ranks that leads to the HOMFLY-PT polynomial and its generalizations.

One can generalize this procedure to any number of Young diagrams, "placing" them in the Dynkin diagram of type A_{N-1} with breaks in between. The *composite representations* are labeled by pairs of partitions (or Young diagrams) "placed" at the ends of the Dynkin diagram. Namely, for Young diagrams λ and μ with $\ell(\lambda)$ and $\ell(\mu)$ rows, $N\ge\ell(\lambda)+\ell(\mu)$ (always assumed), and P_+ of type A_{N-1}, let

$$[\lambda,\mu]_N=b^*+c\in P_+=P_+^{A_{N-1}} \ \text{ for } \ b,c \ \text{associated with}\ \lambda,\mu. \tag{1.1}$$

We call the pair $[\lambda,\mu]$ a *composite diagram/partition* and will constantly identify dominant weights $[\lambda,\mu]_N$ and the corresponding Young diagrams (with no greater than $N-1$ rows).

1.1.1. Schur functions. In what follows, we will require some basic facts about Schur functions and their generalization to composite representations in [17].

Let $\Lambda_n \overset{\mathbf{def}}{=\!=} \mathbb{Z}[x_1,\ldots,x_n]^{S_n}$ denote the *ring of symmetric functions in n–variables*, where the action of S_n is by permuting the variables. For any $m \geq n$, the map which sends $x_i \mapsto 0$ for $i > n$, and $x_i \mapsto x_i$ otherwise, is the restriction homomorphism $\Lambda_m \to \Lambda_n$. Then the *ring of symmetric functions* is

$$\Lambda_x \overset{\mathbf{def}}{=\!=} \varprojlim_n \Lambda_n,$$

where the projective limit is taken with respect to the restriction homomorphisms.

If λ is a partition with length at most n, one can define the corresponding *Schur function* $s_\lambda(x_1,\ldots,x_n) \in \Lambda_n$. The set of Schur functions for all such partitions is a $\mathbb{Z}$–basis for Λ_n. We may naturally interpret a given $s_\lambda(x_1,\ldots,x_n)$ as having infinitely-many variables, for which we write $s_\lambda(\vec{x}) \in \Lambda_x$. The set of all $s_\lambda(\vec{x})$ is a $\mathbb{Z}$–basis for Λ_x.

The Schur functions satisfy many interesting properties. For our purposes, we will interpret $s_\lambda(\vec{x}) \in \Lambda_x$ as a character for the irreducible polynomial representation V_λ. Consequently, the *Littlewood-Richardson rule*, that is

$$s_\lambda(\vec{x})s_\mu(\vec{x}) = \sum_\nu N^\nu_{\lambda,\mu} s_\nu(\vec{x}), \tag{1.2}$$

shows that the multiplicity of an irreducible summand V_ν in the tensor product decomposition of $V_\lambda \otimes V_\mu$ is equal to the *Littlewood-Richardson coefficient* $N^\nu_{\lambda,\mu}$.

1.1.2. The composite case. In [17], the author introduces $s_{[\lambda,\mu]}(\vec{x},\vec{y}) \in \Lambda_x \otimes \Lambda_y$, which generalize the Schur functions and provide characters for irreducible representations $V_{[\lambda,\mu]}$ corresponding to composite partitions. Their natural projection onto the character ring for $\mathfrak{sl}_N$ is the (ordinary) Schur function $s_{[\lambda,\mu]_N}(x_1,\ldots,x_{N-1}) \in \Lambda_{N-1}$. Recall that we always assume that $N \geq \ell(\lambda)+\ell(\mu)$ for the length $\ell(\lambda)$ of λ; see (1.1).

The following formulas, proved in [17], will be used as definitions in our paper:

$$s_{[\lambda,\mu]}(\vec{x},\vec{y}) \stackrel{\mathbf{def}}{=\!=} \sum_{\tau,\nu,\xi}(-1)^{|\tau|}N^{\lambda}_{\nu,\tau}N^{\mu}_{\tau,\xi}s_{\nu}(\vec{x})s_{\xi}(\vec{y}), \tag{1.3}$$

$$\text{where } s_{\eta}(\vec{x})s_{\delta}(\vec{y}) = \sum_{\alpha,\beta,\delta} N^{\eta}_{\beta,\alpha}N^{\delta}_{\gamma,\alpha}s_{[\beta,\gamma]}(\vec{x},\vec{y}); \tag{1.4}$$

the sums here are over arbitrary triples of Young diagrams.

1.2. Skein theory in the annulus

1.2.1. Composite HOMFLY-PT polynomials. The colored HOMFLY-PT polynomial for a knot K and a partition λ is the integer Laurent polynomial $\mathcal{H}_\lambda(K;q,a) \in \mathbb{Z}[q^{\pm 1}, a^{\pm 1}]$ satisfying $\mathcal{H}_\lambda(K;q;q^N) = \mathcal{J}^{\mathfrak{sl}_N}_\lambda(K;q)$ to the corresponding Jones polynomial for $\mathfrak{sl}_N$ and partition (dominant weight) λ. The latter is also called the Quantum Group knot invariant or WRT invariant.

The *composite HOMFLY-PT polynomial* for $[\lambda,\mu]$ is defined similarly via the specializations $\mathcal{H}_{[\lambda,\mu]}(K;q,q^N) = \mathcal{J}^{\mathfrak{sl}_N}_{[\lambda,\mu]_N}(K;q)$ for all sufficiently large N. In particular, $\mathcal{H}_{[\varnothing,\mu]}(K) = \mathcal{H}_\mu(K)$. Recall that the composite diagram $[\lambda,\mu]_N$ is from (1.1).

The HOMFLY-PT polynomial has two normalizations. For connection with DAHA, as in Theorem 2.6, we will be interested in the *normalized* polynomial $\mathcal{H}$. However, for many of our intermediate calculations, we will also need the *unnormalized* HOMFLY-PT polynomial $\bar{\mathcal{H}}$. These are generally defined and related by:

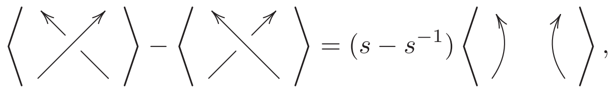
$$\bar{\mathcal{H}}(K) = \bar{\mathcal{H}}(U)\mathcal{H}(K),\ \bar{\mathcal{H}}(U) = \dim_{q,a}(V), \tag{1.5}$$

where K is any knot, U is the unknot, and $\dim_{q,a}$ is defined in Section 1.3.5 for $V = V_{[\lambda,\mu]}$. Observe that with this definition, $\mathcal{H}(U) = 1$. In the specializations described earlier in this section, the normalized (resp. unnormalized) HOMFLY-PT polynomials coincide with the reduced (resp. unreduced) Quantum Group knot invariants.

We will briefly recall the approach to composite HOMFLY-PT polynomials from [13]. The *full HOMFLY-PT skein algebra* $\mathcal{C}$ is a commutative algebra over the coefficient ring $\Upsilon = \mathbb{Z}[v^{\pm 1}, s^{\pm 1}](\{s^k - s^{-k}\}_{k\geq 1})^{-1}$. It consists in Υ−linear combinations of oriented link diagrams in $S^1 \times I$.

The *product* of two diagrams in $\mathcal{C}$ is the diagram obtained by identifying the outer circle of one annulus with the inner circle of the other; the identity with respect to this product is the empty diagram (with coefficient 1).

The relations in $\mathcal{C}$ are the (framed) HOMFLY-PT skein relation



together with the relation that accompanies a type-I Reidemeister move on a positively (resp. negatively) oriented loop with multiplication by a factor of v^{-1} (resp. v). As a consequence, observe that

$$\left\langle K \sqcup \bigcirc \right\rangle = \left(\frac{v^{-1} - v}{s - s^{-1}} \right) \langle K \rangle .$$

Furthermore, for a given diagram $D = D(K)$ of a knot K,

$$\langle D \rangle = a^{\frac{1}{2}\mathrm{wr}(D)} \bar{\mathcal{H}}(K; q, a) \text{ under } s \mapsto q^{\frac{1}{2}}, v \mapsto a^{-\frac{1}{2}},$$

tying the variables s, v used in [13] to the variables q, a used elsewhere in this paper; $\mathrm{wr}(D)$ is the *writhe* of D (see there).

1.2.2. The meridian maps. Let $\varphi : \mathcal{C} \to \mathcal{C}$ be the *meridian map* induced by adding a single oriented, unknotted meridian to any diagram in $S^1 \times I$ and extending linearly to $\mathcal{C}$. Let $\bar{\varphi}$ be the map induced by adding a meridian with an orientation opposite that of φ. Then, $\varphi, \bar{\varphi}$ are diagonal in their common eigenbasis $\{Q_{\lambda,\mu}\} \subset \mathcal{C}$ indexed by pairs λ, μ of partitions.

The subalgebras of $\mathcal{C}$ spanned by $\{Q_{\lambda,\varnothing}\}$ and $\{Q_{\varnothing,\mu}\}$ are each isomorphic to the ring of symmetric functions in infinitely many variables. Under these isomorphisms, these bases are identified with the basis of Schur polynomials. Accordingly, the full basis $\{Q_{\lambda,\mu}\}$ is the skein-theoretic analog of the characters for composite partitions in [17] that we discussed in Section 1.1.

Now to a diagram D of a knot K and a composite partition $[\lambda, \mu]$, associate the satellite link $D \star Q_{\lambda,\mu}$, whose companion is D and whose pattern is $Q_{\lambda,\mu}$. We then have that

$$\bar{\mathcal{H}}_{[\lambda,\mu]}(K) = v^{\mathrm{wr}(D)} \langle D \star Q_{\lambda,\mu} \rangle, \ \mathrm{wr}(D) = \text{writhe of } D,$$

i.e. the corresponding composite, unnormalized HOMFLY-PT polynomial for K is equal to the framed, uncolored HOMFLY-PT polynomial for $D \star Q_{\lambda,\mu}$.

The pattern $Q_{\lambda,\mu}$ can be computed explicitly as the determinant of a matrix whose entries are certain idempotents $\{h_i, h_i^*\} \subset \mathcal{C}$. For the convenience of the reader, some patterns for $[\lambda,\mu]$ considered in this paper are included in the table below.

$[\lambda,\mu]$	$Q_{\lambda,\mu}$
$[\square,\square]$	$h_1h_1^*-1$
$[\square,\ydiagram{1,1}]$	$h_1h_1^*h_1^*-h_1h_2^*-h_1^*$
$[\ydiagram{2},\square]$	$h_2h_1^*-h_1$
$[\square,\ydiagram{1,1,1}]$	$h_1h_1^*h_1^*h_1^*+h_1h_3^*+h_2^*-h_1h_1^*h_1^*-h_1h_1^*h_2^*-h_1^*h_1^*$
$[\ydiagram{2,1},\square]$	$h_1h_2h_1^*-h_1h_1-h_3h_1^*$

The idempotents h_i are closures of linear combinations of upward-oriented braids $b_i \in \Upsilon[B_i]$:

$$b_1 = 1 = \uparrow \in \Upsilon[B_1],\;\; b_2 = \frac{1}{s[2]}(1+s\sigma_1) \in \Upsilon[B_2],$$

$$b_3 = \frac{1}{s^3[2][3]}(1+s\sigma_1)(1+s\sigma_2+s^2\sigma_2\sigma_1) \in \Upsilon[B_3],$$

in the annulus by homotopically nontrivial, counterclockwise-oriented strands. Here B_i is the ordinary braid group on i strands, and the quantum integers are denoted by $[k] \stackrel{\mathbf{def}}{=\!=} \frac{s^k-s^{-k}}{s-s^{-1}}$ (only in this section). The elements h_i^* are then obtained by rotating the diagrams for h_i about their horizontal axes. That is, h_i^* are linear combinations of closures of downward-oriented braids by clockwise-oriented strands.

In fact, the pattern $Q_{\lambda,\mu}$ for a *composite* partition $[\lambda,\mu]$ is distinguished by the fact that, in general, it contains strands oriented in both directions (clockwise and counterclockwise) around $S^1 \times I$. On the other hand, the pattern $Q_\lambda = Q_{[\lambda,\varnothing]}$ for an ordinary partition will consist in strands oriented all in the same direction.

Let $K_{[\lambda,\mu]} \stackrel{\mathbf{def}}{=\!=} \frac{\langle K \star Q_{[\lambda,\mu]}\rangle}{\langle Q_{[\lambda,\mu]}\rangle}$, which is well-defined on diagrams for K up to a framing coefficient, i.e. power of v. In [13] the authors compute

$$K_{[\square,\square]}(z,v) = v^2-4v^4+4v^6+z^2(1+2v^2-7v^4+4v^6) \\ +z^4(v^2-2v^4+v^6) \;\text{ for }\; K=T^{3,2} \tag{1.6}$$

in terms of variables v and $z \stackrel{\mathbf{def}}{=\!=} s-s^{-1}$. The relation to a,q we use in this paper is $v=a^{-\frac{1}{2}}$ and $z=q^{\frac{1}{2}}-q^{-\frac{1}{2}}$; see below.

1.3. Rosso-Jones formula

1.3.1. The usual theory. The *Rosso-Jones formula* [23] and its variants, e.g. [9, 18, 27, 20], expand the HOMFLY-PT polynomial for the (r,s)–torus knot and a partition $\lambda \vdash n$ in terms of the quantum dimensions of certain irreducible representations:

$$\theta_\lambda^{\mathsf{rs}} \bar{\mathcal{H}}_\lambda(T^{\mathsf{r},\mathsf{s}}) = \sum_{\mu \vdash \mathsf{r}n} c^{\mu}_{\lambda;\mathsf{r}} \theta_\mu^{\frac{\mathsf{s}}{\mathsf{r}}} \mathrm{dim}_{q,a}(V_\mu). \tag{1.7}$$

The formulas for $\theta_\lambda, \theta_\mu$ and the coefficients $c^{\mu}_{\lambda;\mathsf{r}}$ are provided below in (1.10), (1.13); $c^{\mu}_{\lambda;\mathsf{r}}$ is nonzero only if V_μ is an irreducible summand of $V_\lambda^{\otimes \mathsf{r}}$. Here $\theta_\lambda^{\mathsf{rs}}, \theta_\mu^{\frac{\mathsf{s}}{\mathsf{r}}}$ are powers, fractional for the latter. Note that (1.7) gives the *unnormalized* polynomial as defined in (1.5).

1.3.2. The composite theory. We are going to generalize the Rosso-Jones formula to the case of composite partitions $[\lambda, \mu]$. The stabilization of the corresponding expansion is not *a priori* clear. We will use the results of [17] described in Section 1.1. The following proposition matches formula (C.6) [11] (Chern-Simons theory).

Proposition 1.1. — *For any torus knot $T^{\mathsf{r},\mathsf{s}}$ and composite partition $[\lambda, \mu]$ the corresponding (unnormalized) HOMFLY-PT polynomial admits an expansion:*

$$\theta_{[\lambda,\mu]}^{\mathsf{rs}} \bar{\mathcal{H}}_{[\lambda,\mu]}(T^{\mathsf{r},\mathsf{s}}) = \sum_{[\beta,\gamma]} c^{[\beta,\gamma]}_{[\lambda,\mu];\mathsf{r}} \theta_{[\beta,\gamma]}^{\frac{\mathsf{s}}{\mathsf{r}}} dim_{q,a}(V_{[\beta,\gamma]}), \tag{1.8}$$

into finitely many terms for which the $c^{[\beta,\gamma]}_{[\lambda,\mu];\mathsf{r}}$ are nonzero. Here $\theta_{[\lambda,\mu]}$ and $\theta_{[\beta,\gamma]}$ and the coefficients $c^{[\beta,\gamma]}_{[\lambda,\mu];\mathsf{r}}$ are provided in (1.11) and (1.15).

Proof. — First of all, it is clear from (1.15) that $c^{[\beta,\gamma]}_{[\lambda,\mu];\mathsf{r}}$ is nonzero for only finitely many $[\beta, \gamma]$. Then, by construction, the resulting expansion (1.8) will satisfy the (infinitely many) specializations

$$\mathcal{H}_{[\lambda,\mu]}(T^{\mathsf{r},\mathsf{s}}; q, q^N) = \mathcal{H}_{[\lambda,\mu]_N}(T^{\mathsf{r},\mathsf{s}}; q, q^N) = \mathcal{J}^{\mathfrak{sl}_N}_{[\lambda,\mu]_N}(T^{\mathsf{r},\mathsf{s}}; q), \tag{1.9}$$

which (uniquely) define the corresponding composite HOMFLY-PT polynomial.

We will divide the proof of (1.8) into several intermediate steps. In what follows, any occurrences of q^N will be replaced by a; all fractional exponents of N will cancel in the final formula.

1.3.3. Braiding eigenvalues. The constants $\theta_\lambda \in \mathbb{Z}[q^{\pm 1}, a^{\pm 1}]$ in (1.7) are *braiding eigenvalues* from [1] , and they are

$$\theta_\lambda = q^{-(\kappa_\lambda + nN - \frac{n^2}{N})/2} \text{ for } \kappa_\lambda \overset{\textbf{def}}{=\!=} \sum_{x\in\lambda} 2c(x), \tag{1.10}$$

where the *content* of the box $x \in \lambda$ in the ith row and jth column is $c(x) \overset{\textbf{def}}{=\!=} j - i$.

Now, for a composite partition $[\lambda, \mu]$ such that $\lambda \vdash m$ and $\mu \vdash n$, observe that $[\lambda, \mu]_N \vdash c \overset{\textbf{def}}{=\!=} (n - m + \lambda_1 N)$. We would like to construct a $\kappa_{[\lambda,\mu]}$ such that $\kappa_{[\lambda,\mu]}|_{N=k} = \kappa_{[\lambda,\mu]_k}$ for any k. To this end, we divide the Young diagram for $[\lambda, \mu]_N$ into two natural parts and count their individual contributions to $\kappa_{[\lambda,\mu]_N}$. Namely,

(1) μ contributes $\kappa_\mu + 2\lambda_1|\mu|$ to $\kappa_{[\lambda,\mu]_N}$ and
(2) λ^* contributes $\kappa_{\lambda^*} = \kappa_\lambda + N\lambda_1(\lambda_1 + 1) - \lambda_1 N(N+1) - 2|\lambda|(\lambda_1 - N)$.

Thus, we can set

$$\kappa_{[\lambda,\mu]} \overset{\textbf{def}}{=\!=} \kappa_\lambda + \kappa_\mu + N\lambda_1(\lambda_1 + 1) - \lambda_1 N(N+1) + 2\lambda_1|\mu| - 2|\lambda|(\lambda_1 - N),$$

so that $\kappa_{[\lambda,\mu]}|_{N=k} = \kappa_{[\lambda,\mu]_k}$ for any k, as desired. Furthermore we can define the composite braiding eigenvalues:

$$\theta_{[\lambda,\mu]} \overset{\textbf{def}}{=\!=} q^{-(\kappa_{[\lambda,\mu]} + cN - \frac{c^2}{N})/2}. \tag{1.11}$$

One has that $\theta_{[\lambda,\mu]} \overset{a\mapsto q^N}{=\!=\!=} \theta_{[\lambda,\mu]_N}$ by construction.

The following is the key part of the proof of Proposition 1.1.

1.3.4. Adams operation. We will use Section 1.1, where we explained that the Schur functions $s_\lambda(\vec{x}) \in \Lambda_x$ are characters for the irreducible polynomial representations V_λ and described some of their properties. For applications to the Rosso-Jones formula we need to understand the r–*Adams operation* ψ_r on s_λ; see [9, 20].

Let $p_\mathsf{r} \overset{\textbf{def}}{=\!=} \sum_{i\geq i} x_i^\mathsf{r} \in \Lambda_x$ be the *degree-*r *power sum symmetric function.* Then the r–Adams operation on s_λ may be defined formally by the plethysm $\psi_\mathsf{r}(s_\lambda) \overset{\textbf{def}}{=\!=} p_\mathsf{r} \circ s_\lambda$. This means that $\psi_\mathsf{r}(s_\lambda)$ is determined by the coefficients $c^\nu_{\lambda;r} \in \mathbb{Z}$ in the expansion

$$s_\lambda(\vec{x}^\mathsf{r}) = \sum_\nu c^\nu_{\lambda;r} s_\nu(\vec{x}), \tag{1.12}$$

where $\vec{x}^{\mathsf{r}} \stackrel{\textbf{def}}{=\!=} (x_1^{\mathsf{r}}, x_2^{\mathsf{r}}, x_3^{\mathsf{r}}, \ldots)$. The coefficients here are given an explicit description in [18]:

$$c^{\nu}_{\lambda;r} = \sum_{\mu} \frac{|C_\mu|\chi^\lambda(C_\mu)\chi^\nu(C_{r\mu})}{|\mu|}, \tag{1.13}$$

where χ^λ is the character of the symmetric group corresponding to λ, and C_μ is the conjugacy class corresponding to μ.

We need an analog of ψ_{r} for composite partitions $[\lambda, \mu]$, which must agree with the ordinary Adams operation upon specification of N. Thus, we need to switch from (1.12) to the expansion

$$s_{[\lambda,\mu]}(\vec{x}^r, \vec{y}^r) = \sum_{\nu} c^{[\beta,\gamma]}_{[\lambda,\mu];r} s_{[\beta,\gamma]}(\vec{x}, \vec{y}), \tag{1.14}$$

where $s_{[\lambda,\mu]}(\vec{x}, \vec{y}) \in \Lambda_x \otimes \Lambda_y$ is the universal character of [17], described in Section 1.1. Applying here the natural projection onto Λ_{N-1}, one recovers the following specialization of (1.12):

$$s_{[\lambda,\mu]_N}(x_1^r, \ldots, x_{N-1}^r) = \sum_{\nu} c^{[\beta,\gamma]_N}_{[\lambda,\mu]_N;r} s_{[\beta,\gamma]_N}(x_1, \ldots, x_{N-1}).$$

This demonstrates that $c^{[\beta,\gamma]}_{[\lambda,\mu];r}$ from (1.14) are exactly what we need, i.e. this formula agrees with (1.12) upon specification of N and therefore can be used for the proof of Proposition 1.1.

Now using (1.3), (1.4) and (1.12) we obtain an explicit expression for these coefficients:

$$c^{[\beta,\gamma]}_{[\lambda,\mu];r} = \sum_{\tau,\nu,\xi,\eta,\delta,\alpha} (-1)^{|\tau|} N^{\lambda}_{\nu,\tau} N^{\mu}_{\tau,\xi} c^{\eta}_{\nu;r} c^{\delta}_{\xi;r} N^{\eta}_{\beta,\alpha} N^{\delta}_{\gamma,\alpha}, \tag{1.15}$$

where the sum is over arbitrary sextuples of Young diagrams. Recall that $N^{\lambda}_{\nu,\tau}$, are the Littlewood-Richardson coefficients from (1.2).

Although this formula appears rather complicated, observe that the terms are only nonzero for relatively few (and finitely many) choices of $(\tau, \nu, \xi, \eta, \delta, \alpha)$. In light of (1.13) and the combinatorial nature of the Littlewood-Richardson rule, these formula provides a completely combinatorial description of $c^{[\beta,\gamma]}_{[\lambda,\mu];r}$.

The following is the last step of the proof.

1.3.5. Quantum dimensions. We define the q, a–*integer* by

$$[uN + v]_{q,a} \stackrel{\textbf{def}}{=\!=} \frac{a^{\frac{u}{2}} q^{\frac{v}{2}} - a^{-\frac{u}{2}} q^{-\frac{v}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},$$

for $u, v \in \mathbb{Z}$, where N is "generic", i.e. it is treated here as a formal variable. Setting here $a = q^N$ for $N \in \mathbb{N}$, we obtain the ordinary quantum integer $[uN + v]_q$. We will suppress the subscript "q, a" in this and the next subsection, simply writing $[\,\cdot\,]$.

For an irreducible representation V_μ, its *stable quantum dimension* is given by the quantum Weyl dimension formula

$$\dim_{q,a}(V_\mu) = \prod_{\alpha \in A_{N-1}^+} \frac{[(\mu + \rho, \alpha)]}{[(\rho, \alpha)]}, \tag{1.16}$$

where the Young diagram μ is interpreted in the usual way as a weight for $\mathfrak{sl}_N$ for generic N and $\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$ for A_{N-1}.

Then it only depends on the diagram μ, which includes the actual number of factors due to the cancelations. We note that such a stabilization holds in the theory of Macdonald polynomials of type A_{N-1} as well; see formula (2.12) and Theorem 2.3, (i).

The stable quantum dimension for a composite partition $[\beta, \gamma]$ is defined as follows:

$$\dim_{q,a}(V_{[\beta,\gamma]}) \stackrel{\mathbf{def}}{=\!=} \prod_{\alpha \in A_{N-1}^+} \frac{[([\beta,\gamma]_N + \rho, \alpha)]}{[(\rho, \alpha)]}. \tag{1.17}$$

Similarly to (1.16), we claim that there is no actual dependence of N in this formula (including the actual number of factors). However the justification is somewhat more involved because the weight

$$[\beta, \gamma]_N = \sum_{j=1}^{\ell(\gamma)} (\gamma_i - \gamma_{i+1})\omega_i + \sum_{j=1}^{\ell(\beta)} (\beta_j - \beta_{j+1})\omega_{N-j},$$

depends on N (in contrast to the case of one diagram). We will omit a straightforward justification; see table (1.19) below and the general formula (C.3) from [11] (a calculation of normalized open-string stretched annulus amplitudes). Finally, the relation $\dim_{q,a}(V_{[\beta,\gamma]})|_{a \mapsto q^N} = \dim_q(V_{[\beta,\gamma]_N})$ concludes the proof of Proposition 1.1. $\square$

Formula (1.8) provides a purely combinatorial and computationally effective way of producing HOMFLY-PT polynomials for arbitrary torus knots and composite representations. See examples below and also Section C from [11].

1.3.6. Simplest examples. First, we evaluate the (ordinary) Rosso-Jones formula (1.7) for the trefoil $T^{3,2}$ and $\lambda = \square$. The necessary values are contained in table (1.18):

μ	θ_μ	$c^{\mu}_{\square;2}$	$\dim_{q,a}(V_\mu)$
$\square$	$a^{-\frac{1}{2}}q^{\frac{1}{2N}}$	0	$[N]$
$\square\square$	$a^{-1}q^{\frac{2}{N}-1}$	1	$\frac{[N][N+1]}{[2]}$
$\boxminus$	$a^{-1}q^{\frac{2}{N}+1}$	-1	$\frac{[N-1][N]}{[2]}$

(1.18)

Inserting the components of (1.18) into formula (1.7), we obtain the familiar expression:

$$\begin{aligned}\mathcal{H}_{\square}(T^{3,2};q,a) &= \frac{\theta_{\square}^{-6}(\theta_{\square\square}^{\frac{3}{2}}\dim_{q,a}(V_{\square\square}) - \theta_{\boxminus}^{\frac{3}{2}}\dim_{q,a}(V_{\boxminus}))}{\dim_{q,a}(V_{\square})} \\ &= aq^{-1} - a^2 + aq,\end{aligned}$$

the normalized HOMFLY-PT polynomial of $T^{3,2}$. Note that although $\square$ appears with coefficient 0 in the expansion (1.7), we include it in table (1.18) since both $\theta_{\square}$ and $\dim_{q,a}(V_{\square})$ are needed to give the final, normalized polynomial, as defined in (1.5).

Similarly, we evaluate our composite Rosso-Jones formula (1.8) for the trefoil $T^{3,2}$ and $[\square,\square]$ using table (1.19):

$[\beta,\gamma]$	$\theta_{[\beta,\gamma]}$	$c^{[\beta,\gamma]}_{[\square,\square];2}$	$\dim_{q,a}(V_{[\beta,\gamma]})$
$[\square,\square]$	a^{-1}	0	$[N-1][N+1]$
$[\square\square,\square\square]$	$q^{-2}a^{-2}$	1	$\frac{[N-1][N]^2[N+3]}{[2][2]}$
$[\square\square,\boxminus]$	a^{-2}	-1	$\frac{[N-2][N-1][N+1][N+2]}{[2][2]}$
$[\boxminus,\square\square]$	a^{-2}	-1	$\frac{[N-2][N-1][N+1][N+2]}{[2][2]}$
$[\boxminus,\boxminus]$	q^2a^{-2}	1	$\frac{[N-3][N]^2[N+1]}{[2][2]}$
$[\varnothing,\varnothing]$	1	1	1

(1.19)

Inserting the components of (1.19) into formula (1.8), we obtain

$$\begin{aligned}\mathcal{H}_{[\square,\square]}(T^{3,2};q,a) &= a^2(q^{-2}+q^2+2) + a^3(-2q^{-2}+q^{-1}+q-2q^2-2) \\ &\quad + a^4(q^{-2}-2q^{-1}-2q+q^2+3) + a^5(q^{-1}+q-2),\end{aligned}$$

where we include $[\square,\square]$ in table (1.19) for the same reason that we included $\square$ in table (1.18).

Observe that we can touch base with formula (1.6) from [13] by

$$a^5 T^{3,2}_{[\square,\square]}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}, a^{-\frac{1}{2}}) = \mathcal{H}_{[\square,\square]}(T^{3,2};q,a).$$

Our expression for $\mathcal{H}_{[\square,\square]}(T^{3,2};q,a)$ agrees with that obtained in [21]. See also examples (C.8-16) from [11], obtained there via Chern-Simons theory (open-string amplitudes); they match our ones.

2. DAHA superpolynomials

2.1. Definition of DAHA

2.1.1. Affine root systems. Let $R=\{\alpha\}\subset\mathbb{R}^n$ be a root system of type $A_n,\ldots,G_2$ with respect to a euclidean form $(\,,\,)$ on $\mathbb{R}^n$, normalized by the condition $(\alpha,\alpha)=2$ for *short* roots. Let $W=\langle s_\alpha\rangle$ be its Weyl group, and let R_+ be the set of positive roots corresponding to a fixed set $\{\alpha_1,...,\alpha_n\}$ of simple roots for R. The weight lattice is $P=\oplus_{i=1}^n\mathbb{Z}\omega_i$, where $\{\omega_i\}$ are fundamental weights: $(\omega_i,\alpha_j^\vee)=\delta_{ij}$ for the coroots $\alpha^\vee=2\alpha/(\alpha,\alpha)$; $P_\pm=\oplus_{i=1}^n\mathbb{Z}_\pm\omega_i$, for $\mathbb{Z}_\pm=\{m\in\mathbb{Z},\pm m\geq 0\}$.

Setting $\nu_\alpha\stackrel{\mathbf{def}}{=}(\alpha,\alpha)/2$, the vectors $\widetilde{\alpha}=[\alpha,\nu_\alpha j]\in\mathbb{R}^n\times\mathbb{R}\subset\mathbb{R}^{n+1}$ for $\alpha\in R, j\in\mathbb{Z}$ form the *twisted affine root system* $\widetilde{R}\supset R$ ($z\in\mathbb{R}^n$ are identified with $[z,0]$). We add $\alpha_0\stackrel{\mathbf{def}}{=}[-\vartheta,1]$ to the simple roots for the *maximal short root* $\vartheta\in R_+$. The corresponding set $\widetilde{R}_+$ of positive roots is $R_+\cup\{[\alpha,\nu_\alpha j],\ \alpha\in R,\ j>0\}$.

The set of the indices of the images of α_0 by all automorphisms of the affine Dynkin diagram will be denoted by O; let $O'\stackrel{\mathbf{def}}{=}\{r\in O, r\neq 0\}$. The elements ω_r for $r\in O'$ are *minuscule weights*. We set $\omega_0=0$.

2.1.2. Extended Weyl group. Given $\widetilde{\alpha}=[\alpha,\nu_\alpha j]\in\widetilde{R},\ b\in P$, let

$$s_{\widetilde{\alpha}}(\widetilde{z})\ =\ \widetilde{z}-(z,\alpha^\vee)\widetilde{\alpha},\ \ b'(\widetilde{z})\ =\ [z,\zeta-(z,b)] \qquad (2.1)$$

for $\widetilde{z}=[z,\zeta]\in\mathbb{R}^{n+1}$. The *affine Weyl group* $\widetilde{W}=\langle s_{\widetilde{\alpha}},\widetilde{\alpha}\in\widetilde{R}_+\rangle$ is the semidirect product $W\ltimes Q$ of its subgroups $W=\langle s_\alpha,\alpha\in R_+\rangle$ and Q, where α is identified with

$$s_\alpha s_{[\alpha,\,\nu_\alpha]}\ =\ s_{[-\alpha,\,\nu_\alpha]}s_\alpha\ \text{ for }\ \alpha\in R.$$

The *extended Weyl group* $\widehat{W}$ is $W\ltimes P$, where the corresponding action is

$$(wb)([z,\zeta])\ =\ [w(z),\zeta-(z,b)]\ \text{ for }\ w\in W, b\in P. \qquad (2.2)$$

It is isomorphic to $\widetilde{W}\ltimes\Pi$ for $\Pi\stackrel{\mathbf{def}}{=}P/Q$. The latter group consists of $\pi_0=$id and the images π_r of minuscule ω_r in P/Q.

The group Π is naturally identified with the subgroup of $\widehat{W}$ of the elements of the length zero; the *length* is defined as follows:

$$l(\widehat{w}) = |\lambda(\widehat{w})| \ \text{ for } \ \lambda(\widehat{w}) \stackrel{\mathbf{def}}{=\!=} \widetilde{R}_+ \cap \widehat{w}^{-1}(-\widetilde{R}_+).$$

One has $\omega_r = \pi_r u_r$ for $r \in O'$, where u_r is the element $u \in W$ of minimal length such that $u(\omega_r) \in P_-$.

Setting $\widehat{w} = \pi_r \widetilde{w} \in \widehat{W}$ for $\pi_r \in \Pi$, $\widetilde{w} \in \widetilde{W}$, $l(\widehat{w})$ coincides with the length of any reduced decomposition of $\widetilde{w}$ in terms of the simple reflections s_i, $0 \le i \le n$.

2.1.3. Parameters. We follow [4, 3, 5]. Let m, be the least natural number such that $(P, P) = (1/\mathsf{m})\mathbb{Z}$. Thus $\mathsf{m} = |\Pi|$ unless $\mathsf{m} = 2$ for D_{2k} and $\mathsf{m} = 1$ for B_{2k}, C_k.

The double affine Hecke algebra, *DAHA*, depends on the parameters q, t_ν $(\nu \in \{\nu_\alpha\})$ and is naturally defined over the ring $Z_{q,t} \stackrel{\mathbf{def}}{=\!=} \mathbb{Z}[q^{\pm 1/\mathsf{m}}, t_\nu^{\pm 1/2}]$ formed by polynomials in terms of $q^{\pm 1/\mathsf{m}}$ and $\{t_\nu^{1/2}\}$.

For $\widetilde{\alpha} = [\alpha, \nu_\alpha j] \in \widetilde{R}$, $0 \le i \le n$, we set

$$t_{\widetilde{\alpha}} = t_\alpha = t_{\nu_\alpha} = q_\alpha^{k_\nu}, \ \ q_{\widetilde{\alpha}} = q^{\nu_\alpha}, \ \ t_i = t_{\alpha_i}, \ \ q_i = q_{\alpha_i},$$

Also, using here (and below) *sht, lng* instead of ν, we set

$$\rho_k \stackrel{\mathbf{def}}{=\!=} \frac{1}{2} \sum_{\alpha>0} k_\alpha \alpha = k_{\mathrm{sht}} \rho_{\mathrm{sht}} + k_{\mathrm{lng}} \rho_{\mathrm{lng}}, \ \ \rho_\nu = \frac{1}{2} \sum_{\nu_\alpha = \nu} \alpha = \sum_{\nu_i = \nu,\, i>0} \omega_i.$$

For pairwise commutative $X_1, \ldots, X_n$,

$$X_{\widetilde{b}} \stackrel{\mathbf{def}}{=\!=} \prod_{i=1}^{n} X_i^{l_i} q^j \ \text{ if } \ \widetilde{b} = [b, j], \ \widehat{w}(X_{\widetilde{b}}) = X_{\widehat{w}(\widetilde{b})}, \tag{2.3}$$
$$\text{where } b = \sum_{i=1}^{n} l_i \omega_i \in P, \ j \in \frac{1}{m}\mathbb{Z}, \ \widehat{w} \in \widehat{W}.$$

For instance, $X_0 \stackrel{\mathbf{def}}{=\!=} X_{\alpha_0} = q X_\vartheta^{-1}$.

2.1.4. The main definition. Recall that $\omega_r = \pi_r u_r$ for $r \in O'$ (see above). We will use that π_r^{-1} is $\pi_{\iota(i)}$, where ι is the standard involution of the nonaffine Dynkin diagram, induced by $\alpha_i \mapsto -w_0(\alpha_i)$. Generally, $\iota(b) = -w_0(b) = b^\iota$, where w_0 is the longest element in W. Finally, we set $m_{ij} =$

$2,3,4,6$ when the number of links between α_i and α_j in the affine Dynkin diagram is $0,1,2,3$.

DEFINITION 2.1. — *The double affine Hecke algebra* $\mathcal{H}$ *is generated over* $\mathbb{Z}_{q,t}$ *by the elements* $\{T_i,\ 0 \le i \le n\}$, *pairwise commutative* $\{X_b,\ b \in P\}$ *satisfying (2.3) and the group* Π, *where the following relations are imposed:*

(o) $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0,\ 0 \le i \le n$;

(i) $T_iT_jT_i... = T_jT_iT_j...,\ m_{ij}$ *factors on each side;*

(ii) $\pi_r T_i \pi_r^{-1} = T_j$ *if* $\pi_r(\alpha_i) = \alpha_j$;

(iii) $T_iX_b = X_bX_{\alpha_i}^{-1}T_i^{-1}$ *if* $(b, \alpha_i^\vee) = 1,\ 0 \le i \le n$;

(iv) $T_iX_b = X_bT_i$ *if* $(b, \alpha_i^\vee) = 0$ *for* $0 \le i \le n$;

(v) $\pi_r X_b \pi_r^{-1} = X_{\pi_r(b)} = X_{u_r^{-1}(b)} q^{(\omega_{\iota(r)}, b)},\ r \in O'$.

Given $\widetilde{w} \in \widetilde{W}, r \in O$, the product

$$T_{\pi_r\widetilde{w}} \stackrel{\textbf{def}}{=\!=} \pi_r T_{i_l} \cdots T_{i_1}, \text{ where } \widetilde{w} = s_{i_l} \cdots s_{i_1} \text{ for } l = l(\widetilde{w}), \tag{2.4}$$

does not depend on the choice of the reduced decomposition. Moreover,

$$T_{\widehat{v}}T_{\widehat{w}} = T_{\widehat{v}\widehat{w}} \text{ whenever } l(\widehat{v}\widehat{w}) = l(\widehat{v}) + l(\widehat{w}) \text{ for } \widehat{v}, \widehat{w} \in \widehat{W}. \tag{2.5}$$

In particular, we arrive at the pairwise commutative elements

$$Y_b \stackrel{\textbf{def}}{=\!=} \prod_{i=1}^n Y_i^{l_i} \text{ if } b = \sum_{i=1}^n l_i\omega_i \in P,\ Y_i \stackrel{\textbf{def}}{=\!=} T_{\omega_i}, b \in P. \tag{2.6}$$

When acting in the polynomial representation, they are called *difference Dunkl operators.*

2.1.5. Automorphisms. The following maps can be (uniquely) extended to an automorphism of $\mathcal{H}$, fixing t_ν, q and their fractional powers; see [5], (3.2.10)-(3.2.15). Adding $q^{1/(2\mathsf{m})}$ to $\mathbb{Z}_{q,t}$,

$$\tau_+ : X_b \mapsto X_b,\ T_i \mapsto T_i\,(i > 0),\ \ Y_r \mapsto X_rY_rq^{-\frac{(\omega_r,\omega_r)}{2}}, \tag{2.7}$$

$$\tau_+ : T_0 \mapsto q^{-1}X_\vartheta T_0^{-1},\ \pi_r \mapsto q^{-\frac{(\omega_r,\omega_r)}{2}}X_r\pi_r\ (r \in O'),$$

$$\tau_- : Y_b \mapsto Y_b,\ T_i \mapsto T_i\,(i \ge 0),\ \ X_r \mapsto Y_rX_rq^{\frac{(\omega_r,\omega_r)}{2}}, \tag{2.8}$$

$$\tau_-(X_\vartheta) = qT_0X_\vartheta^{-1}T_{s_\vartheta}^{-1};\ \ \sigma \stackrel{\textbf{def}}{=\!=} \tau_+\tau_-^{-1}\tau_+ = \tau_-^{-1}\tau_+\tau_-^{-1},$$

$$\sigma(X_b) = Y_b^{-1},\ \sigma(Y_b) = T_{w_0}^{-1}X_{b^\iota}^{-1}T_{w_0},\ \sigma(T_i) = T_i(i > 0). \tag{2.9}$$

The group $PSL_2^{\wedge}(\mathbb{Z})$ generated by $\tau_\pm$, the *projective* $PSL_2(\mathbb{Z})$ due to Steinberg, has a natural projection onto $PSL_2(\mathbb{Z})$, corresponding to taking $t_\nu^{1/(2\mathsf{m})} = 1 = q^{1/(2\mathsf{m})}$: $\tau_+ \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\tau_- \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\sigma \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

2.2. DAHA-Jones polynomials

2.2.1. Coinvariant. Following [5], we use the PBW Theorem to express any $H \in \mathcal{H}$ in the form $\sum_{b,w,c} d_{b,w,c}\, X_b T_w Y_c$ for $w \in W$, $b, c \in P$ (this presentation is unique). Then we substitute:

$$\{\,\}_{ev} : \ X_b \ \mapsto \ q^{-(\rho_k,b)}, \ Y_c \ \mapsto \ q^{(\rho_k,c)}, \ T_i \ \mapsto \ t_i^{1/2}. \tag{2.10}$$

The functional $\mathcal{H} \ni H \mapsto \{H\}_{ev}$, called *coinvariant*, acts via the projection $H \mapsto H(1)$ of $\mathcal{H}$ onto the *polynomial representation* $\mathcal{V}$, which is the $\mathcal{H}$–module induced from the one-dimensional character $T_i(1) = t_i^{-1/2} = Y_i(1)$ for $1 \le i \le n$ and $T_0(1) = t_0^{-1/2}$. Recall that $t_0 = t_{\rm sht}$; see [5, 3].

2.2.2. Macdonald polynomials. The polynomial representation is isomorphic to $\mathbb{Z}_{q,t}[X_b]$ as a vector space, and the action of $T_i (0 \le i \le n)$ there is given by the *Demazure-Lusztig operators*:

$$T_i \ = \ t_i^{1/2} s_i \ + \ (t_i^{1/2} - t_i^{-1/2})(X_{\alpha_i} - 1)^{-1}(s_i - 1), \ 0 \le i \le n. \tag{2.11}$$

The elements X_b become the multiplication operators and $\pi_r (r \in O')$ act via the general formula $\widehat{w}(X_b) = X_{\widehat{w}(b)}$ for $\widehat{w} \in \widehat{W}$.

The Macdonald polynomials $P_b(X)$ are uniquely defined as follows. Let c_+ be the unique element such that $c_+ \in W(c) \cap P_+$. For $b \in P_+$,

$$P_b - \sum_{b' \in W(b)} X_{b'} \in \oplus_{b_+ \neq c_+ \in b+Q_+} \mathbb{Q}(q, t_\nu) X_c \ \text{ and } \ CT\big(P_b X_{c^\iota}\, \mu(X; q, t)\big) = 0$$

for such c, where $\mu(X; q, t) \stackrel{\mathbf{def}}{=\!=} \prod_{\alpha \in R_+} \prod_{j=0}^{\infty} \dfrac{(1 - X_\alpha q_\alpha^j)(1 - X_\alpha^{-1} q_\alpha^{j+1})}{(1 - X_\alpha t_\alpha q_\alpha^j)(1 - X_\alpha^{-1} t_\alpha q_\alpha^{j+1})}$.

Here CT is the constant term; μ is considered a Laurent series in X_b with the coefficients expanded in terms of positive powers of q. The coefficients of P_b belong to the field $\mathbb{Q}(q, t_\nu)$. One has:

$$P_b(X^{-1}) \ = \ P_{b^\iota}(X) \ = \ P_b(q^{-\rho_k}) = P_b(q^{\rho_k}) \tag{2.12}$$

$$= \ q^{-(\rho_k,b)} \prod_{\alpha > 0} \prod_{j=0}^{(\alpha^\vee,b)-1} \Big(\frac{1 - q_\alpha^j t_\alpha X_\alpha(q^{\rho_k})}{1 - q_\alpha^j X_\alpha(q^{\rho_k})} \Big). \tag{2.13}$$

See [5], formula (3.3.23); recall that $\iota(b) = b^\iota = -w_0(b)$ for $b \in P$.

2.2.3. DAHA-Jones polynomials. We begin with the following theorem, which is from [3, 4].

Torus knots $T^{\mathsf{r},\mathsf{s}}$ are naturally represented by $\gamma_{\mathsf{r},\mathsf{s}} \in PSL_2(\mathbb{Z})$ with the first column $(\mathsf{r},\mathsf{s})^{tr}$ (tr is the transposition) for $\mathsf{r},\mathsf{s} \in \mathbb{N}$, assuming that $\gcd(\mathsf{r},\mathsf{s}) = 1$. Let $\widehat{\gamma}_{\mathsf{r},\mathsf{s}} \in PSL_2^{\wedge}(\mathbb{Z})$ be any pullback of $\gamma_{\mathsf{r},\mathsf{s}}$.

For a polynomial F in terms of fractional powers of q and t_ν, the *tilde-normalization* $\widetilde{F}$ will be the result of the division of F by the lowest $q, t_\nu-$ monomial, assuming that it is well defined. We put $q^\bullet t^\bullet$ for a monomial factor (possibly fractional) in terms of q, t_ν.

Theorem 2.2. — *Given a torus knot $T^{\mathsf{r},\mathsf{s}}$, we lift $(\mathsf{r},\mathsf{s})^{tr}$ to γ and then to $\widehat{\gamma} \in PSL_2^{\wedge}(\mathbb{Z})$ as above.*

(i) The DAHA-Jones (or refined) polynomial *for a reduced irreducible root system R and $b \in P_+$ is defined as follows:*

$$JD^R_{\mathsf{r},\mathsf{s}}(b\,;\,q,t) \;=\; JD_{\mathsf{r},\mathsf{s}}(b\,;\,q,t) \stackrel{\mathbf{def}}{=\!=} \big\{\widehat{\gamma}(P_b)\big\}_{ev}. \tag{2.14}$$

(ii) It does not depend on the ordering of r,s *or on the particular choice of $\gamma \in PSL_2(\mathbb{Z})$, $\widehat{\gamma} \in PSL_2^{\wedge}(\mathbb{Z})$. The tilde-normalization $\widetilde{JD}_{\mathsf{r},\mathsf{s}}(b\,;\,q,t)$ is well defined and is a polynomial in terms of q, t_ν with constant term* 1.

(iii) Specialization at the trivial center charge. *For $b = \sum_{i=1}^n b_i\omega_i$,*

$$JD_{\mathsf{r},\mathsf{s}}\big(b\,;\,q\!=\!1,t\big) \!=\! \prod_{i=1}^n JD_{\mathsf{r},\mathsf{s}}(\omega_i\,;\,q\!=\!1,t)^{b_i} \;\; \textit{for any } \mathsf{r},\mathsf{s}. \tag{2.15}$$

□

It was conjectured in [3] in general (and checked there for A_n) that $JD_{\mathsf{r},\mathsf{s}}(b\,;\,q,t_\nu \mapsto q_\nu)$ coincide up to $q^\bullet$ with the reduced Quantum Group (WRT) invariants for the corresponding $T^{\mathsf{r},\mathsf{s}}$ and any colors $b \in P_+$. The Quantum Group is associated with the *twisted* root system $\widetilde{R}$. The *shift operator* was used there to deduce this coincidence from [18, 27] in the case of A_n and torus knots. The papers [27, 2] provide the necessary tools to establish this coincidence for D_n. Quite a few further confirmations for other root systems are known by now; the second author (R. E.) checked such a coincidence with the DAHA formulas provided in [3] for the minuscule and quasi-minuscule weights for E_6 (unpublished).

2.3. DAHA superpolynomials

Theorem 2.2 leads to the theory of *DAHA-superpolynomials*, which are the result of the *stabilization* of $\widetilde{JD}^{A_n}(b;q,t)$ with respect to n. This stabilization was announced in [3]; its proof was published in [10]. Both approaches use [26]; we note that the stabilization holds for arbitrary torus iterated knots.

Following [26] (see also [10, 4]), we can generalize the stabilization construction to the torus knots in the annulus.

The pairs $\{\mathsf{r},\mathsf{s}\}$ remains the same, but now colored torus knots $T^{\mathsf{r},\mathsf{s}}$ will be treated as link diagrams in the annulus; see Section 1.

Theorem 2.3. — *We switch to* A_n*, setting* $t = t_{\rm sht} = q^k$*. Let* $b, c \in P_+^n \stackrel{\mathbf{def}}{=\!=} P_+^{A_n}$ *and* λ, μ *be the corresponding Young diagrams (with no greater than n rows). Recall that* $[\lambda,\mu]_N \in P_+^{N-1}$ *is* b^*+c*, where* $N \geq \ell(\lambda)+\ell(\mu)$ *and* $(\omega_i)^* = \omega_{N-i}$*; see (1.1).*

(i) Stabilization. *Given a pair* $\{\mathsf{r},\mathsf{s}\}$ *as above, there exists a polynomial* $HD_{\mathsf{r},\mathsf{s}}([\lambda,\mu]\,;q,t,a)$ *from* $\mathbb{Z}[q,t^{\pm 1},a]$ *such that its coefficient of* a^0 *is tilde-normalized (i.e. in the form* $\sum_{u,v\geq 0} C_{u,v} q^u t^v$ *with* $C_{0,0}=1$*) and*

$$HD_{\mathsf{r},\mathsf{s}}([\lambda,\mu]\,;q,t,a\!=\!-t^N) = \widetilde{JD}_{\mathsf{r},\mathsf{s}}^{A_{N-1}}(b^*+c\,;q,t) \ \ \textit{for any}\ \ N>n. \quad (2.16)$$

This polynomial does not depend on the ordering of r,s *or that of* λ,μ*.*

(ii) Specialization at $q=1$. *Setting* $HD_{\mathsf{r},\mathsf{s}}(\lambda) = HD_{\mathsf{r},\mathsf{s}}([\varnothing,\lambda])$,

$$\begin{aligned} &HD_{\mathsf{r},\mathsf{s}}([\lambda,\mu]\,;q\!=\!1,t,a) \\ &\quad = HD_{\mathsf{r},\mathsf{s}}(\lambda\,;q\!=\!1,t,a)\,HD_{\mathsf{r},\mathsf{s}}(\mu\,;q\!=\!1,t,a), \quad \textit{where} \end{aligned} \quad (2.17)$$

$$HD_{\mathsf{r},\mathsf{s}}(\lambda\,;\,q\!=\!1,t,a) \,=\, \prod_{i=1}^{n} HD_{\mathsf{r},\mathsf{s}}(\omega_i\,;\,q\!=\!1,t,a)^{b_i} \ \ \textit{for}\ \ b\!=\!\sum_{i=1}^{n} b_i\omega_i,$$

b corresponds to λ and ω_i means the column with i boxes. □

2.3.1. Degree of $\boldsymbol{a}$ *and duality.* Assuming that $\mathsf{r}>\mathsf{s}$, we conjecture that

$$\deg_a HD_{\mathsf{r},\mathsf{s}}([\lambda,\mu]\,;\,q,t,a) \,=\, \mathsf{s}(|\lambda|+|\mu|) - |\lambda\vee\mu|, \quad (2.18)$$

where $\lambda\vee\mu$ (the join operation) is the smallest Young diagram containing them, $|\lambda|$ is the number of boxes in λ. This is based on the numerical evidence and on a generalization of the construction from [10] to the composite case (though we did not check all details).

Let us generalize the DAHA-duality from [3] (justified in [10]) to the composite case; see also [12, 4].

Theorem 2.4. — Composite super-duality. *Up to a power of q and t,*

$$HD_{\mathsf{r,s}}([\lambda,\mu]\,;q,t,a) = q^{\bullet}t^{\bullet}HD_{\mathsf{r,s}}([\lambda^{tr},\mu^{tr}]\,;t^{-1},q^{-1},a), \tag{2.19}$$

where λ^{tr} is the transposition of λ.

Proof. — According to the remark after the super-duality formula (1.44) from Section 1.6 of [4], the standard type A (one-diagram) duality is equivalent to $q^{\bullet}$-proportionality between $\mathcal{J}^{A_n}_{\mathsf{r,s}}(\lambda\,;q,t)$ and $\mathcal{J}^{A_m}_{\mathsf{r,s}}(\lambda^{tr}\,;t^{-1},q^{-1})$ for $t = q^{-(m+1)/(n+1)}$ (i.e. for $k = -\frac{m+1}{n+1}$) and all possible relatively prime $m+1, n+1 \in \mathbb{N}$. This is directly connected with the generalized *level-rank duality*. Using that q,n,m are essentially arbitrary, we conclude that these proportionality conditions (all of them) are equivalent to the duality. The latter was proved in [10]; the above argument (and the theory of perfect DAHA modules at roots of unity from [5]) can be used for the justification of the standard super-duality as well (unpublished).

This reformulation of the super-duality in terms of the DAHA-Jones polynomials (i.e. without a) gives the composite super-duality upon considering the diagrams in the form $[\lambda,\mu]_N$. □

Combining the evaluation formula (2.17) with the duality:

$$\begin{aligned}&HD_{\mathsf{r,s}}([\lambda,\mu]\,;\,q,t{=}1,a) \\ &\qquad = HD_{\mathsf{r,s}}(\lambda\,;\,q,t{=}1,a)\,HD_{\mathsf{r,s}}(\mu\,;\,q,t{=}1,a).\end{aligned} \tag{2.20}$$

2.3.2. Color exchange. The following theorem can be proved following Sections 1.6, 1.7 from [4].

Theorem 2.5. — Color Exchange. *Let $t = q^k$ for $k \in -\mathbb{Q}_+$. For λ,μ as above, we assume the existence of permutations $v,w \in \mathbf{S}_n$ satisfy the following conditions. Setting $\lambda = \{l_1 \geq l_2 \ldots \geq l_n \geq 0\}$,*

$$\lambda' = \{l'_1,\ldots,l'_n\} \stackrel{\mathbf{def}}{=\!=} \{l_{u(i)} + k(i-u(i)),\ i=1,2,\ldots,n\} \tag{2.21}$$

must be a diagram, i.e we require $l'_i \geq l'_{i+1}$ and $l'_i \in \mathbb{Z}_+$. Similarly, μ' defined by μ, w (for the same k) is assumed a Young diagram. Then $HD_{\mathsf{r,s}}([\lambda,\mu]\,;\,q,t,a) = HD_{\mathsf{r,s}}([\lambda',\mu']\,;\,q,t,a)$ for such q,t and any $\mathsf{r,s}$.

□

Let us provide an example for $t = q^{-\kappa}$, $\kappa \in \mathbb{N}$ (see [4], formula (1.47) for details). For any $p > 0$ and $i \in \{1, 2\} \ni j$, one has:

$$HD_{\mathsf{r},\mathsf{s}}(\,[\kappa b^{(i)}, \kappa b^{(j)}]\,;\, q, q^{-\kappa}, a) = q^{\bullet}\, HD_{\mathsf{r},\mathsf{s}}(\,[\kappa c^{(i)}, \kappa c^{(j)}]\,;\, q, q^{-\kappa}, a) \;\text{ for}$$
$$b^{(1)} = \omega_{p+1}, c^{(1)} = (p+1)\omega_1 \;\text{ and }\; b^{(2)} = p\omega_{p+1}, c^{(2)} = (p+1)\omega_p\,,$$

where the weights are identified with the corresponding diagrams. If $\kappa = 1$, then $t = q^{-1}$ and these relations follow from the duality.

2.3.3. Obtaining HOMFLY-PT polynomials.

THEOREM 2.6. HOMFLY-PT via DAHA. *For* r, s *and* λ, μ *as above,*

$$HD_{\mathsf{r},\mathsf{s}}([\lambda, \mu]; q, t \mapsto q, a \mapsto -a) = \mathcal{H}_{[\lambda,\mu]}(T^{\mathsf{r},\mathsf{s}}; q, a), \tag{2.22}$$

where $\mathcal{H}_{[\lambda,\mu]}(T^{\mathsf{r},\mathsf{s}}; q, a)$ *is the composite HOMFLY-PT polynomial for* $[\lambda, \mu]$ *normalized by the condition* $\mathcal{H}(U) = 1$ *for the unknot* U*.*

Proof. — This theorem formally results from the coincidence of the $\widetilde{JD}$–polynomials in type A with the corresponding (reduced) Jones polynomials for torus knots under the tilde-normalization. Generally, this claim is from Conjecture 2.1 in [3]; it was verified there for A_{N-1} using the DAHA *shift operator* (Proposition 2.3) and papers [18, 27]. The weights were arbitrary there; we need them here for $[\lambda, \mu]_N$. □

3. Examples and confirmations

We provide here examples of the composite DAHA-superpolynomials and discuss their symmetries. The first 5 particular composite representations considered below are contained in the following table.

$[b,c]$	$[\omega_1,\omega_1]$	$[\omega_1,\omega_2]$	$[2\omega_1,\omega_1]$	$[\omega_{\mathbf{1}},\omega_{\mathbf{3}}]$	$[\omega_1+\omega_2,\omega_1]$
$[\lambda,\mu]$	$[\square,\square]$	$[\square,(1,1)]$	$[(2),\square]$	$[\square,(1,1,1)]$	$[(2,1),\square]$
l	2	3	2	4	3
A_1	(2)	—	(3)	—	—
A_2	$(2,1)$	$(2,2)$	$(3,2)$	—	$(3,1)$
A_3	$(2,1,1)$	$(2,2,1)$	$(3,2,2)$	$(2,2,2)$	$(3,2,1)$
A_4	$(2,1,1,1)$	$(2,2,1,1)$	$(3,2,2,2)$	$(2,2,2,1)$	$(3,3,2,1)$
A_5	$(2,1,1,1,1)$	$(2,2,1,1,1)$	$(3,2,2,2,2)$	$(2,2,2,1,1)$	$(3,3,3,2,1)$
A_6	$(2,1,1,1,1,1)$	$(2,2,1,1,1,1)$	$(3,2,2,2,2,2)$	$(2,2,2,1,1,1)$	$(3,3,3,3,2,1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

3.1. The adjoint representation

The adjoint representation has the weight $\omega_1+\omega_n$ and is represented in our notation by the pair $[\omega_1,\omega_1]=[\square,\square]$. We consider this representation for two knots.

3.1.1. Trefoil. The adjoint DAHA superpolynomial for the trefoil is given by the formula

$$HD_{3,2}([\omega_1,\omega_1];q,t,a) \;=\; 1+2qt+q^2t^2+a(3q^2-q^3+2qt^{-1}-q^2t^{-1}-q^3t^{-1} + 2q^3t) + a^2(q^4+q^2t^{-2}-2q^3t^{-2}+q^4t^{-2}+2q^3t^{-1}-2q^4t^{-1}) + a^3(-q^4t^{-3} + q^5t^{-3}+q^4t^{-2}-q^5t^{-2}).$$

Recall that it is defined by the relations

$$HD_{\mathsf{r,s}}([\lambda,\mu];q,t,a\mapsto -t^{n+1}) = \widetilde{JD}^{A_n}_{\mathsf{r,s}}(\lambda^*+\mu;q,t) \tag{3.1}$$

for $\lambda=\omega_1,\mu=\omega_1$ and all $n\geq 1$.

The corresponding normalized adjoint HOMFLY-PT polynomial for the unframed trefoil is given by formula (2.17) from [21]; see also Section 1.3.6.

One has:

$$\mathcal{H}_{[\square,\square]}(T^{3,2}) \;=\; a^2(q^{-2}+q^2+2)+a^3(-2q^{-2}+q^{-1}+q-2q^2-2)+a^4(q^{-2} - 2q^{-1}-2q+q^2+3)+a^5(q^{-1}+q-2),$$

and we have the following confirmation of Theorem 2.6:

$$a^2q^{-2}HD_{3,2}([\omega_1,\omega_1]\,;\,q,t\mapsto q,a\mapsto -a)=\mathcal{H}_{[\square,\square]}(T^{3,2}).$$

The super-duality from (2.19) in this case is as follows:

$$t^{-2}HD_{3,2}([\omega_1,\omega_1];q,t,a)=q^2HD_{3,2}([\omega_1,\omega_1];t^{-1},q^{-1},a).$$

The evaluation formula (2.20) reads

$$HD_{3,2}([\omega_1,\omega_1];q,1,a) \;=\; \big(1+q+aq\big)^2 \;=\; HD_{3,2}(\omega_1;q,1,a)^2.$$

3.1.2. The case of $T^{4,3}$. The adjoint DAHA superpolynomial for the $(4,3)$–torus knot $T(4,3)$ is given by the formula

$$\begin{aligned}
&HD_{4,3}([\omega_1,\omega_1]\,;\,q,t,a) \;=\\
&1+2qt+2q^2t+3q^2t^2+2q^3t^2+q^4t^2+4q^3t^3+2q^4t^3+3q^4t^4+2q^5t^4+2q^5t^5+q^6t^6\\
&+a(5q^2+5q^3-q^4-3q^5-2q^6+2qt^{-1}+q^2t^{-1}-q^3t^{-1}-q^4t^{-1}-q^5t^{-1}+8q^3t\\
&+7q^4t+q^5t-3q^6t-q^7t+9q^4t^2+7q^5t^2-q^6t^2-q^7t^2+8q^5t^3+5q^6t^3-q^7t^3\\
&+5q^6t^4+q^7t^4+2q^7t^5)+a^2(7q^4+9q^5-2q^6-8q^7+q^2t^{-2}+2q^3t^{-2}-2q^4t^{-2}\\
&-3q^5t^{-2}+q^7t^{-2}+q^8t^{-2}+4q^3t^{-1}+5q^4t^{-1}-2q^5t^{-1}-8q^6t^{-1}+q^8t^{-1}+8q^5t\\
&+9q^6t-2q^7t-3q^8t+7q^6t^2+5q^7t^2-2q^8t^2+4q^7t^3+2q^8t^3+q^8t^4)+a^3(3q^6+5q^7\\
&-q^8-3q^9+q^4t^{-3}-q^5t^{-3}-3q^6t^{-3}+3q^7t^{-3}+q^8t^{-3}-q^9t^{-3}+q^4t^{-2}+3q^5t^{-2}\\
&-q^6t^{-2}-8q^7t^{-2}+4q^8t^{-2}+q^9t^{-2}+2q^5t^{-1}+5q^6t^{-1}-2q^7t^{-1}-8q^8t^{-1}+3q^9t^{-1}\\
&+2q^7t+3q^8t-q^9t+q^8t^2+q^9t^2)+a^4(q^9-q^7t^{-4}+2q^8t^{-4}-q^9t^{-4}+q^6t^{-3}\\
&-4q^8t^{-3}+4q^9t^{-3}-q^{10}t^{-3}+q^7t^{-2}+q^8t^{-2}-4q^9t^{-2}+2q^{10}t^{-2}+q^8t^{-1}-q^{10}t^{-1})\\
&+a^5(-q^{10}t^{-5}+q^{11}t^{-5}-q^9t^{-4}+2q^{10}t^{-4}-q^{11}t^{-4}+q^9t^{-3}-q^{10}t^{-3}),
\end{aligned}$$

defined as for the trefoil. Computed using (1.8), the corresponding normalized HOMFLY-PT polynomial is

$$\begin{aligned}
&\mathcal{H}_{[\square,\square]}(T^{4,3}) \;=\; q^{-6}\big(a^6(q^{12}+2q^{10}+2q^9+3q^8+2q^7+5q^6+2q^5+3q^4+2q^3\\
&+2q^2+1)+a^7(-2q^{12}-q^{11}-4q^{10}-4q^9-6q^8-4q^7-8q^6-4q^5-6q^4-4q^3-4q^2\\
&-q-2)+a^8(q^{12}+2q^{11}+2q^{10}+2q^9+5q^8+2q^7+7q^6+2q^5+5q^4+2q^3+2q^2+2q\\
&+1)+a^9(-q^{11}-4q^8+2q^7-4q^6+2q^5-4q^4-q)+a^{10}(2q^8-4q^7+5q^6-4q^5\\
&+2q^4)+a^{11}(2q^7-4q^6+2q^5)\big).
\end{aligned}$$

We have the *connection formula*

$$a^6 q^{-6} HD_{4,3}([\omega_1,\omega_1]\,;\,q, t\mapsto q, a\mapsto -a) = \mathcal{H}_{[\square,\square]}(T^{4,3}).$$

The *super-duality* reads

$$t^{-6} HD_{4,3}([\omega_1,\omega_1];q,t,a) = q^6 HD_{4,3}([\omega_1,\omega_1];t^{-1},q^{-1},a),$$

and the evaluation at $t=1$ is as follows:

$$\begin{aligned} HD_{4,3}([\omega_1,\omega_1];q,1,a) &= \big(1+q+2q^2+q^3+a(q+2q^2+2q^3)+a^2q^3\big)^2\\ &= HD_{4,3}(\omega_1;q,1,a)^2. \end{aligned}$$

3.2. Column/row and a box

Such diagrams correspond to the symmetric and wedge powers of the fundamental representation.

3.2.1. Two-row and a box: $[2\omega_1,\omega_1] = [\square\square,\square]$.

Then the composite DAHA superpolynomial for the trefoil is

$$\begin{aligned} HD_{3,2}([2\omega_1,\omega_1];q,t,a) \;=\; & 1+qt+q^2t+q^3t+q^3t^2+2q^4t^2+q^5t^3+a(3q^3\\ &+3q^4-2q^6-q^7+qt^{-1}+q^2t^{-1}-q^4t^{-1}-q^5t^{-1}+q^4t+4q^5t+2q^6t-q^7t+q^6t^2\\ &+2q^7t^2)+a^2(2q^6+4q^7-q^8-2q^9+q^3t^{-2}-q^5t^{-2}-q^6t^{-2}+q^8t^{-2}+q^4t^{-1}\\ &+3q^5t^{-1}+q^6t^{-1}-4q^7t^{-1}-2q^8t^{-1}+q^9t^{-1}+2q^8t+q^9t)+a^3(q^{10}-q^7t^{-3}+q^9t^{-3}\\ &+q^6t^{-2}+q^7t^{-2}-2q^8t^{-2}-2q^9t^{-2}+q^{10}t^{-2}+q^{11}t^{-2}+2q^8t^{-1}+q^9t^{-1}-2q^{10}t^{-1}\\ &-q^{11}t^{-1})+a^4(-q^{10}t^{-3}+q^{12}t^{-3}+q^{10}t^{-2}-q^{12}t^{-2}), \end{aligned}$$

defined by (3.1) for $\lambda = 2\omega_1, \mu=\omega_1$ and all $n\geq 1$.

The corresponding normalized HOMFLY-PT polynomial is given by formula (A.1) from [21], as well as computed using (1.8). It is

$$\begin{aligned} \mathcal{H}_{[\square\square,\square]}(T^{3,2}) \;=\; & q^{-3}\big(a^3(q^8+2q^6+q^5+q^4+q^3+q^2+1)+a^4(-1-q-2q^3\\ &-2q^4-q^5-2q^6-q^7-2q^9)+a^5(q+2q^4+q^5-q^6+2q^7+q^{10})+a^6(-q^5+q^6\\ &-2q^8+q^9)+a^7(-q^7+q^8+q^9-q^{10})\big), \end{aligned}$$

and we have the relationship

$$a^3q^{-3}HD_{3,2}([2\omega_1,\omega_1]\,;\,q,t\mapsto q, a\mapsto -a) = \mathcal{H}_{[\square\square,\square]}(T^{3,2}),$$

confirming Theorem 2.6. The super-duality here requires $[\omega_1, \omega_2]$, which will be considered next. The evaluation at $t = 1$ reads

$$HD_{3,2}([2\omega_1, \omega_1]\,;\, q, 1, a) = \big(1+ q+ aq\big) \times \big(1+ q^2+ q^3+ q^4+ a(q^2 + q^3+ q^4+ q^5) + a^2 q^5\big) = HD_{3,2}(\omega_1; q, 1, a) \times HD_{3,2}(2\omega_1; q, 1, a).$$

3.2.2. Two-column and a box: $[\omega_1, \omega_2] = [\square, \begin{smallmatrix}\square\\ \square\end{smallmatrix}]$.

The DAHA superpolynomial for the trefoil reads

$$\begin{aligned}HD_{3,2}([\omega_1, \omega_2]\,;\, q, t, a) \;=\; & 1 + 2qt + qt^2 + q^2t^2 + q^2t^3 + q^2t^4 + q^3t^5 + a(4q^2 \\ & -q^4 + 2qt^{-2} - q^2t^{-2} - q^3t^{-2} + qt^{-1} + 2q^2t^{-1} - 2q^3t^{-1} + q^2t + 3q^3t - q^4t + 3q^3t^2 \\ & + q^4t^3 + q^4t^4) + a^2(3q^4 - q^5 + q^2t^{-4} - 2q^3t^{-4} + q^4t^{-4} + 2q^2t^{-3} - q^3t^{-3} - 2q^4t^{-3} \\ & + q^5t^{-3} + 4q^3t^{-2} - 4q^4t^{-2} + 2q^3t^{-1} + q^4t^{-1} - q^5t^{-1} + q^4t + q^5t^2) + a^3(-q^4t^{-6} \\ & + q^5t^{-6} + q^3t^{-5} - 2q^4t^{-5} + q^5t^{-5} + q^4t^{-4} - 2q^5t^{-4} + q^6t^{-4} + 2q^4t^{-3} - 2q^5t^{-3} \\ & + q^5t^{-2} - q^6t^{-2} + q^5t^{-1}) + a^4(-q^5t^{-7} + q^6t^{-7} + q^5t^{-5} - q^6t^{-5}),\end{aligned}$$

where the specialization relations for all $n \geq 2$ are

$$HD_{3,2}([\omega_1, \omega_2]; q, t, a \mapsto -t^{n+1}) = \widetilde{JD}^{A_n}_{3,2}(\omega_2 + \omega_n; q, t).$$

The corresponding normalized HOMFLY-PT polynomial is given by formula (A.4) from [21], as well as computed using (1.8):

$$\begin{aligned}\mathcal{H}_{[\square,\begin{smallmatrix}\square\\ \square\end{smallmatrix}]}(T^{3,2}) \;=\; & q^{-7}\big(a^3(q^2+2q^4+q^5+q^6+q^7+q^8+q^{10}) + a^4(-2q-q^3-2q^4 \\ & -q^5-2q^6-2q^7-q^9-q^{10}) + a^5(1+2q^3-q^4+q^5+2q^6+q^9) + a^6(q-2q^2+q^4-q^5) \\ & + a^7(-1+q+q^2-q^3)\big),\end{aligned}$$

and we have the *connection formula*

$$a^3 q^{-5} HD_{3,2}([\omega_1, \omega_2]; q, t \mapsto q, a \mapsto -a) = \mathcal{H}_{[\square,\begin{smallmatrix}\square\\ \square\end{smallmatrix}]}(T^{3,2}).$$

The *super-duality* and *evaluation* are as follows:

$$t^{-3} HD_{4,3}([2\omega_1, \omega_1]; q, t, a) = q^5 HD_{4,3}([\omega_1, \omega_2]; t^{-1}, q^{-1}, a),$$

$$\begin{aligned}HD_{3,2}([\omega_1, \omega_2]; q, 1, a) &= \big(1 + q + aq\big)\big(1 + q + aq\big)^2 \\ &= HD_{3,2}(\omega_1; q, 1, a) \times HD_{3,2}(\omega_2; q, 1, a).\end{aligned}$$

The corresponding standard superpolynomials are

$$HD_{3,2}(\omega_1; q, t, a) = 1 + qt + aq,$$

$$HD_{3,2}(\omega_2; q, t, a) = 1+ \frac{a^2q^2}{t} + qt+ qt^2+ q^2t^4 + a\left(q+ \frac{q}{t} + q^2t+ q^2t^2\right).$$

See e.g. [3] and references therein.

3.2.3. Three-column and a box: $[\omega_1,\omega_3]=[\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}]$. This example is of $\deg_a=5$, which matches our conjecture. The corresponding DAHA-superpolynomial for the trefoil is as follows:

$$
\begin{aligned}
&HD_{3,2}([\omega_1,\omega_3];q,t,a)\ =\\
&1+2qt+qt^2+q^2t^2+qt^3+q^2t^3+2q^2t^4+q^2t^5+q^3t^5+q^2t^6+q^3t^6+q^3t^7+q^3t^9\\
&+q^4t^{10}+a(5q^2+q^3-2q^4+2qt^{-3}-q^2t^{-3}-q^3t^{-3}+qt^{-2}+2q^2t^{-2}-2q^3t^{-2}\\
&+qt^{-1}+3q^2t^{-1}-q^3t^{-1}-q^4t^{-1}+2q^2t+5q^3t-2q^4t+q^2t^2+4q^3t^2+4q^3t^3+q^4t^3\\
&-q^5t^3+q^3t^4+3q^4t^4-q^5t^4+q^3t^5+2q^4t^5+3q^4t^6+q^5t^7+q^5t^9)+a^2(q^3+6q^4\\
&-3q^5+q^2t^{-6}-2q^3t^{-6}+q^4t^{-6}+2q^2t^{-5}-q^3t^{-5}-2q^4t^{-5}+q^5t^{-5}+2q^2t^{-4}\\
&+q^3t^{-4}-4q^4t^{-4}+q^5t^{-4}+q^2t^{-3}+5q^3t^{-3}-5q^4t^{-3}+5q^3t^{-2}-3q^5t^{-2}+q^6t^{-2}\\
&+3q^3t^{-1}+4q^4t^{-1}-4q^5t^{-1}+4q^4t+q^5t-q^6t+2q^4t^2+2q^5t^2+3q^5t^3-q^6t^3\\
&+q^5t^4+q^5t^5+q^6t^6)+a^3(q^5+q^6-q^7-q^4t^{-9}+q^5t^{-9}+q^3t^{-8}-2q^4t^{-8}+q^5t^{-8}\\
&+q^3t^{-7}-2q^4t^{-7}+q^6t^{-7}+2q^3t^{-6}-4q^5t^{-6}+2q^6t^{-6}+4q^4t^{-5}-5q^5t^{-5}+q^6t^{-5}\\
&+3q^4t^{-4}-2q^5t^{-4}-q^6t^{-4}+2q^4t^{-3}+2q^5t^{-3}-3q^6t^{-3}+q^7t^{-3}+4q^5t^{-2}-2q^6t^{-2}\\
&+2q^5t^{-1}-q^6t^{-1}+q^6t+q^6t^2)+a^4(-q^5t^{-11}+q^6t^{-11}-q^5t^{-10}+q^6t^{-10}+q^4t^{-9}\\
&-2q^5t^{-9}+q^6t^{-9}+q^5t^{-8}-2q^6t^{-8}+q^7t^{-8}+q^5t^{-7}-2q^6t^{-7}+q^7t^{-7}+2q^5t^{-6}\\
&-2q^6t^{-6}+q^6t^{-5}-q^7t^{-5}+q^6t^{-4}-q^7t^{-4}+q^6t^{-3})+a^5(-q^6t^{-12}+q^7t^{-12}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+q^6t^{-9}-q^7t^{-9}),
\end{aligned}
$$

which is defined by (3.1) for all $n\ge 3$ and $\lambda=2\omega_1,\mu=\omega_3$:

$$HD_{3,2}([\omega_1,\omega_3];q,t,a\mapsto -t^{n+1})=\widetilde{JD}^{A_n}_{3,2}(\omega_3+\omega_n;q,t).$$

The corresponding normalized HOMFLY-PT polynomial is

$$
\begin{aligned}
&\mathcal{H}_{[\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}]}(T^{3,2})\ =\ q^{-16}\big(a^4(q^{20}+q^{18}+q^{16}+q^{15}+2q^{14}+q^{13}+2q^{12}+q^{11}+2q^{10}\\
&+q^9+2q^8+q^6)+a^5(-q^{20}-q^{18}-3q^{16}-q^{15}-3q^{14}-2q^{13}-4q^{12}-2q^{11}-4q^{10}\\
&-2q^9-4q^8-q^7-2q^6-2q^4)+a^6(q^{18}+q^{16}+3q^{14}+q^{13}+3q^{12}+q^{11}+3q^{10}+2q^9\\
&+3q^8+q^7+2q^6+2q^4+q^2)+a^7(-q^3-q^5-q^7-q^8-q^9-q^{10}-q^{12}-q^{14})+a^8(q^7\\
&\qquad\qquad-q^6+q^5-q^4+2q^3-q^2+q-1)+a^9(q^4-q^3-q+1)\big).
\end{aligned}
$$

One has: $a^4q^{-10}HD_{3,2}([\omega_1,\omega_3];q,t\mapsto q,a\mapsto -a)=\mathcal{H}_{[\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}]}(T^{3,2})$ and

$$
\begin{aligned}
HD_{3,2}([\omega_1,\omega_3]\,;\,q,1,a)&=\big(1+q+aq\big)\times\big(1+q+aq\big)^3\\
&=\ HD_{3,2}(\omega_1;q,1,a)\times HD_{3,2}(\omega_3;q,1,a).
\end{aligned}
$$

3.3. Three-hook and a box

The last case is $[\omega_1+\omega_2,\omega_1] = [\,\square\!\square/\square\,,\square]$.

The corresponding DAHA-superpolynomial for the trefoil is

$$\begin{aligned}
&HD_{3,2}([\omega_1+\omega_2,\omega_1]\,;\,q,t,a) \;=\\
&1+3qt-qt^2+4q^2t^2+q^3t^2-2q^2t^3+4q^3t^3+q^4t^3-2q^3t^4+4q^4t^4-q^4t^5+3q^5t^5\\
&+q^6t^6+a(-2q^2+12q^3-4q^5-q^6+2qt^{-2}-q^2t^{-2}-q^3t^{-2}+6q^2t^{-1}-2q^3t^{-1}\\
&-2q^4t^{-1}-q^5t^{-1}-4q^3t+16q^4t+q^5t-4q^6t-q^7t-6q^4t^2+16q^5t^2-2q^7t^2\\
&-4q^5t^3+12q^6t^3-2q^7t^3-q^8t^3-2q^6t^4+6q^7t^4-q^8t^4+2q^8t^5)+a^2(-6q^5+26q^6\\
&-8q^7-6q^8+q^9+q^2t^{-4}-2q^3t^{-4}+q^4t^{-4}+5q^3t^{-3}-5q^4t^{-3}-2q^5t^{-3}+q^6t^{-3}\\
&+q^7t^{-3}-q^3t^{-2}+13q^4t^{-2}-8q^5t^{-2}-6q^6t^{-2}+q^7t^{-2}+q^8t^{-2}-4q^4t^{-1}+22q^5t^{-1}\\
&-8q^6t^{-1}-9q^7t^{-1}+q^8t^{-1}+q^9t^{-1}-6q^6t+22q^7t-8q^8t-2q^9t-4q^7t^2+13q^8t^2\\
&-5q^9t^2+q^{10}t^2-q^8t^3+5q^9t^3-2q^{10}t^3+q^{10}t^4)+a^3(-3q^8+14q^9-9q^{10}+q^{11}\\
&-q^4t^{-6}+q^5t^{-6}+2q^4t^{-5}-4q^5t^{-5}+q^6t^{-5}+q^7t^{-5}+6q^5t^{-4}-9q^6t^{-4}+3q^8t^{-4}\\
&-2q^5t^{-3}+14q^6t^{-3}-14q^7t^{-3}-3q^8t^{-3}+5q^9t^{-3}-3q^6t^{-2}+21q^7t^{-2}-18q^8t^{-2}\\
&-3q^9t^{-2}+3q^{10}t^{-2}-5q^7t^{-1}+21q^8t^{-1}-14q^9t^{-1}+q^{11}t^{-1}-2q^9t+6q^{10}t\\
&-4q^{11}t+q^{12}t+2q^{11}t^2-q^{12}t^2)+a^4(q^{12}-q^{13}-q^6t^{-7}+q^7t^{-7}+q^6t^{-6}-3q^7t^{-6}\\
&+q^8t^{-6}+q^9t^{-6}+4q^7t^{-5}-7q^8t^{-5}+2q^9t^{-5}+2q^{10}t^{-5}-q^{11}t^{-5}-q^7t^{-4}+8q^8t^{-4}\\
&-11q^9t^{-4}+2q^{10}t^{-4}+2q^{11}t^{-4}-2q^8t^{-3}+10q^9t^{-3}-11q^{10}t^{-3}+2q^{11}t^{-3}\\
&+q^{12}t^{-3}-2q^9t^{-2}+8q^{10}t^{-2}-7q^{11}t^{-2}+q^{12}t^{-2}-q^{10}t^{-1}+4q^{11}t^{-1}-3q^{12}t^{-1}\\
&+q^{13}t^{-1})+a^5(-q^9t^{-7}+q^{10}t^{-7}+q^9t^{-6}-2q^{10}t^{-6}+q^{11}t^{-6}+2q^{10}t^{-5}-3q^{11}t^{-5}\\
&\quad+q^{12}t^{-5}-q^{10}t^{-4}+2q^{11}t^{-4}-2q^{12}t^{-4}+q^{13}t^{-4}+q^{12}t^{-3}-q^{13}t^{-3}),
\end{aligned}$$

defined by (3.1) for $\lambda=\omega_1+\omega_2, \mu=\omega_1$ and all $n\geq 2$. The corresponding normalized HOMFLY-PT polynomial is

$$\begin{aligned}
&\mathcal{H}_{[\square\!\square/\square,\square]}(T^{3,2}) \;=\; q^{-8}\big(a^4(q^{14}+3q^{12}-q^{11}+4q^{10}-q^9+4q^8-q^7+4q^6-q^5\\
&+3q^4+q^2)+a^5(-2q^{15}+q^{14}-5q^{13}+4q^{12}-10q^{11}+5q^{10}-12q^9+6q^8-12q^7\\
&+5q^6-10q^5+4q^4-5q^3+q^2-2q)+a^6(q^{16}-2q^{15}+6q^{14}-6q^{13}+11q^{12}-11q^{11}\\
&+17q^{10}-13q^9+18q^8-13q^7+17q^6-11q^5+11q^4-6q^3+6q^2-2q+1)+a^7(q^{16}\\
&-3q^{15}+4q^{14}-7q^{13}+10q^{12}-14q^{11}+14q^{10}-18q^9+18q^8-18q^7+14q^6-14q^5\\
&+10q^4-7q^3+4q^2-3q+1)+a^8(-q^{15}+2q^{14}-3q^{13}+5q^{12}-7q^{11}+10q^{10}-11q^9
\end{aligned}$$

$$+11q^8-11q^7+10q^6-7q^5+5q^4-3q^3+2q^2-q)+a^9(q^{12}-2q^{11}+2q^{10}-3q^9 \\ +4q^8-3q^7+2q^6-2q^5+q^4)\big),$$

which reduces to the HOMFLY-PT polynomial as follows:

$$a^4q^{-6}HD_{3,2}([\omega_1+\omega_2,\omega_1];q,t\mapsto q,a\mapsto -a)=\mathcal{H}_{[\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square]}(T^{3,2}).$$

The exact super-duality identity from (2.19) is

$$t^{-6}HD_{3,2}([\omega_1+\omega_2,\omega_1];q,t,a)=q^6HD_{3,2}([\omega_1+\omega_2,\omega_1];t^{-1},q^{-1},a).$$

The evaluation at $t=1$ from (2.20) reads

$$\begin{aligned} HD_{3,2}([\omega_1+\omega_2,\omega_1]\,;\,q,1,a) \;&=\; \big(1+q+aq\big) \\ \times\,(1+q+aq)\,\big(1+q^2+q^3+q^4+a^2q^5+a\,&(q^2+q^3+q^4+q^5)\big) \\ =HD_{3,2}(\omega_1;q,1,a)\times HD_{3,2}&(\omega_1{+}\omega_2;q,1,a). \end{aligned}$$

3.4. Two-rows and two-columns

One of the two diagrams in the previous examples was always a box. Let us discuss the cases when two-row and two-column diagrams are combined. They match well our conjectural formula (2.18) for $\deg_a$; we also checked directly the super-duality and other properties provided by the theorems above.

3.4.1. Two two-columns: $[\omega_2,\omega_2]=[\begin{smallmatrix}\square\\\square\end{smallmatrix},\begin{smallmatrix}\square\\\square\end{smallmatrix}]$.

$$HD_{3,2}([\omega_2,\omega_2]\,;\,q,t,a)=$$

$$1+a^6\big(\tfrac{q^8}{t^{14}}-\tfrac{q^9}{t^{14}}-\tfrac{q^8}{t^{13}}+\tfrac{q^{10}}{t^{13}}-\tfrac{q^8}{t^{12}}+\tfrac{2q^9}{t^{12}}-\tfrac{q^{10}}{t^{12}}+\tfrac{q^8}{t^{11}}-\tfrac{q^{10}}{t^{11}}-\tfrac{q^9}{t^{10}}+\tfrac{q^{10}}{t^{10}}\big)+a^5\big(-\tfrac{q^6}{t^{13}}+ \\ \tfrac{q^7}{t^{13}}-\tfrac{q^6}{t^{12}}+\tfrac{3q^7}{t^{12}}-\tfrac{2q^8}{t^{12}}+\tfrac{q^6}{t^{11}}-\tfrac{q^7}{t^{11}}-\tfrac{q^8}{t^{11}}+\tfrac{q^9}{t^{11}}+\tfrac{q^6}{t^{10}}-\tfrac{5q^7}{t^{10}}+\tfrac{5q^8}{t^{10}}-\tfrac{q^9}{t^{10}}+\tfrac{2q^8}{t^9}-\tfrac{2q^9}{t^9}+ \\ \tfrac{2q^7}{t^8}-\tfrac{4q^8}{t^8}+\tfrac{2q^9}{t^8}-\tfrac{q^8}{t^7}+\tfrac{q^9}{t^7}+\tfrac{q^8}{t^6}-\tfrac{q^9}{t^6}\big)+a^4\big(-\tfrac{2q^5}{t^{11}}+\tfrac{2q^6}{t^{11}}+\tfrac{q^4}{t^{10}}-\tfrac{4q^5}{t^{10}}+\tfrac{4q^6}{t^{10}}-\tfrac{q^7}{t^{10}}+ \\ \tfrac{q^6}{t^9}-\tfrac{q^8}{t^9}+\tfrac{4q^5}{t^8}-\tfrac{10q^6}{t^8}+\tfrac{7q^7}{t^8}-\tfrac{q^8}{t^8}+\tfrac{2q^5}{t^7}-\tfrac{6q^6}{t^7}+\tfrac{4q^7}{t^7}+\tfrac{6q^6}{t^6}-\tfrac{9q^7}{t^6}+\tfrac{3q^8}{t^6}+\tfrac{3q^6}{t^5}-\tfrac{6q^7}{t^5}+ \\ \tfrac{3q^8}{t^5}+\tfrac{3q^7}{t^4}-\tfrac{3q^8}{t^4}+\tfrac{2q^7}{t^3}-\tfrac{2q^8}{t^3}+\tfrac{q^8}{t^2}\big)+2qt+2qt^2+q^2t^2+2q^2t^3+3q^2t^4+2q^3t^5+ \\ 2q^3t^6+q^4t^8+a^3\big(q^6+q^7-\tfrac{q^4}{t^9}+\tfrac{q^5}{t^9}+\tfrac{2q^3}{t^8}-\tfrac{5q^4}{t^8}+\tfrac{3q^5}{t^8}+\tfrac{2q^3}{t^7}-\tfrac{4q^4}{t^7}+\tfrac{3q^6}{t^7}-\tfrac{q^7}{t^7}+\tfrac{6q^4}{t^6}- \\ \tfrac{12q^5}{t^6}+\tfrac{7q^6}{t^6}-\tfrac{q^7}{t^6}+\tfrac{7q^4}{t^5}-\tfrac{11q^5}{t^5}+\tfrac{2q^6}{t^5}+\tfrac{2q^7}{t^5}+\tfrac{q^4}{t^4}+\tfrac{7q^5}{t^4}-\tfrac{12q^6}{t^4}+\tfrac{4q^7}{t^4}+\tfrac{10q^5}{t^3}-\tfrac{12q^6}{t^3}+ \\ \tfrac{2q^7}{t^3}+\tfrac{2q^5}{t^2}+\tfrac{4q^6}{t^2}-\tfrac{4q^7}{t^2}+\tfrac{7q^6}{t}-\tfrac{5q^7}{t}+2q^7t\big)+a^2\big(5q^4+3q^5-3q^6+\tfrac{q^2}{t^6}-\tfrac{2q^3}{t^6}+\tfrac{q^4}{t^6}+ \\ \tfrac{4q^2}{t^5}-\tfrac{4q^3}{t^5}-\tfrac{2q^4}{t^5}+\tfrac{2q^5}{t^5}+\tfrac{q^2}{t^4}+\tfrac{4q^3}{t^4}-\tfrac{9q^4}{t^4}+\tfrac{3q^5}{t^4}+\tfrac{q^6}{t^4}+\tfrac{10q^3}{t^3}-\tfrac{9q^4}{t^3}-\tfrac{2q^5}{t^3}+\tfrac{q^6}{t^3}+\tfrac{4q^3}{t^2}+ \\ \tfrac{5q^4}{t^2}-\tfrac{10q^5}{t^2}+\tfrac{2q^6}{t^2}+\tfrac{15q^4}{t}-\tfrac{10q^5}{t}-\tfrac{q^6}{t}+10q^5t-4q^6t+4q^5t^2-q^6t^2+4q^6t^3+q^6t^4\big)+ \\ a\big(5q^2+3q^3-4q^4+\tfrac{2q}{t^3}-\tfrac{q^2}{t^3}-\tfrac{q^3}{t^3}+\tfrac{2q}{t^2}+\tfrac{q^2}{t^2}-\tfrac{3q^3}{t^2}+\tfrac{7q^2}{t}-\tfrac{3q^3}{t}-\tfrac{2q^4}{t}+10q^3t- \\ 3q^4t-q^5t+6q^3t^2+q^4t^2-q^5t^2+7q^4t^3-q^5t^3+5q^4t^4-q^5t^4+2q^5t^5+2q^5t^6\big).$$

Note that the specializations $a = -t^{n+1}$ to A_n begins here with $A_{n=3}$. We omit the formula for $HD_{3,2}([2\omega_1, 2\omega_1]\,;\, q, t, a)$, since it can be readily obtained via the super-duality (checked numerically). Also,

$$HD_{3,2}([\omega_2, \omega_2]\,;\, q, t{=}1, a) = (1 + q + qa)^4.$$

3.4.2. Two-column and two-row: $[\omega_2, 2\omega_1] = [\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square]$. Note that the a–degree is 5 in this example vs. 6 in the previous one; $\lambda\vee\mu$ contains now 3 boxes (it is a 3–hook) in (conjectural) formula (2.18). This formula is self-dual with respect to $q \mapsto t^{-1}, t \mapsto q^{-1}, a \mapsto a$ (up to $q^\bullet t^\bullet$).

$$HD_{3,2}([\omega_2, 2\omega_1]\,;\, q, t, a) =$$

$$1+a^5\Big(-\frac{q^{11}}{t^7}+\frac{q^{13}}{t^7}+\frac{q^{11}}{t^5}-\frac{q^{13}}{t^5}\Big)+a^4\Big(-\frac{q^8}{t^7}+\frac{q^{10}}{t^7}-\frac{q^9}{t^6}-\frac{q^{10}}{t^6}+\frac{q^{11}}{t^6}+\frac{q^{12}}{t^6}+\frac{q^7}{t^5}+\frac{q^8}{t^5}-\frac{q^9}{t^5}-\frac{2q^{10}}{t^5}+\frac{q^{12}}{t^5}+\frac{q^9}{t^4}+\frac{q^{10}}{t^4}-\frac{2q^{11}}{t^4}-\frac{q^{12}}{t^4}+\frac{q^{13}}{t^4}+\frac{q^9}{t^3}+\frac{q^{10}}{t^3}-\frac{q^{11}}{t^3}-\frac{q^{12}}{t^3}+\frac{q^{11}}{t^2}-\frac{q^{13}}{t^2}+\frac{q^{11}}{t}\Big)+qt+q^2t+q^3t+qt^2+q^3t^2+2q^4t^2+q^3t^3+q^4t^3+q^5t^3+q^2t^4+q^5t^4+q^4t^5+q^5t^5+q^6t^6+a^3\Big(2q^9+2q^{10}-q^{11}-\frac{q^7}{t^6}+\frac{q^9}{t^6}+\frac{q^4}{t^5}-\frac{q^6}{t^5}-\frac{q^7}{t^5}-\frac{q^8}{t^5}+\frac{q^9}{t^5}+\frac{q^{10}}{t^5}+\frac{2q^6}{t^4}+\frac{q^7}{t^4}-\frac{4q^8}{t^4}-\frac{2q^9}{t^4}+\frac{2q^{10}}{t^4}+\frac{q^{11}}{t^4}+\frac{q^5}{t^3}+\frac{2q^6}{t^3}+\frac{q^7}{t^3}-\frac{q^8}{t^3}-\frac{3q^9}{t^3}-\frac{2q^{10}}{t^3}+\frac{q^{11}}{t^3}+\frac{q^{12}}{t^3}+\frac{2q^7}{t^2}+\frac{4q^8}{t^2}-\frac{q^9}{t^2}-\frac{4q^{10}}{t^2}-\frac{q^{11}}{t^2}+\frac{q^7}{t}+\frac{2q^8}{t}+\frac{q^9}{t}+\frac{q^{10}}{t}-\frac{q^{11}}{t}-\frac{q^{12}}{t}+q^9t+q^{11}t^2\Big)+a^2\Big(q^5+4q^6+4q^7-q^9-q^{10}+\frac{q^3}{t^4}-\frac{q^5}{t^4}-\frac{q^6}{t^4}+\frac{q^8}{t^4}+\frac{q^2}{t^3}+\frac{q^3}{t^3}-\frac{q^6}{t^3}-\frac{3q^7}{t^3}+\frac{2q^9}{t^3}+\frac{3q^4}{t^2}+\frac{4q^5}{t^2}-\frac{4q^7}{t^2}-\frac{4q^8}{t^2}+\frac{q^{10}}{t^2}+\frac{2q^4}{t}+\frac{q^5}{t}+\frac{4q^6}{t}+\frac{3q^7}{t}-\frac{4q^8}{t}-\frac{3q^9}{t}+q^5t+q^6t+q^7t+4q^8t-q^{10}t+2q^7t^2+3q^8t^2+q^9t^3+q^{10}t^3+q^9t^4\Big)+a\Big(q^2+3q^3+2q^4+2q^5-q^6-2q^7+\frac{q}{t^2}+\frac{q^2}{t^2}-\frac{q^4}{t^2}-\frac{q^5}{t^2}+\frac{q}{t}+\frac{2q^3}{t}+\frac{2q^4}{t}-\frac{q^5}{t}-\frac{2q^6}{t}-\frac{q^7}{t}+q^2t+2q^4t+5q^5t+2q^6t-q^7t-q^8t+3q^4t^2+2q^5t^2+2q^6t^2+2q^7t^2-q^8t^2+3q^6t^3+2q^7t^3+q^5t^4+q^6t^4+q^8t^4+q^7t^5+q^8t^5\Big).$$

The evaluation at $t = 1$ from formula (2.17) now reads as follows:

$$\begin{aligned}&HD_{3,2}([\omega_2, 2\omega_1]\,;\, q, t{=}1, a)\\ =&(1 + q + aq)^2(1 + q^2 + q^3 + q^4 + a(q^2 + q^3 + q^4 + q^5) + a^2q^5,\end{aligned}$$

where the standard superpolynomial for $2\omega_1$ is

$$HD_{3,2}(2\omega_1\,;\, q, t, a) = 1+ a^2q^5+ q^2t+ q^3t+ q^4t^2 + a\left(q^2+ q^3+ q^4t+ q^5t\right).$$

Here and above we omit the formulas for the composite HOMFLY-PT polynomials; they do satisfy the Connection Theorem 2.6.

4. Deligne-Gross series

4.1. General procedure

Here we consider the “exceptional series”:

$$e \subset A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8,$$

discussed in [7]. This is actually the bottom row of the triangle considered in that paper; we are going to discuss it in full elsewhere.

Recall that the algebraic groups G in this series are given a parameter ν in this paper as follows:

$$\nu(G) = \frac{h^\vee}{6},$$

where $h^\vee$ is the dual Coxeter number of G. This very quantity provides the specializations of our hyperpolynomials.

The *E–hyperpolynomials* we will construct below unify the DAHA-Jones polynomials (also called refined polynomials) for $T^{3,2}, T^{4,3}$ “colored” by the adjoint representation for the groups of type ADE in this series. The root systems G_2 and F_4 play an important role in the exceptional series, but we cannot incorporate them so far (see also the end of this section).

As with the (colored) superpolynomial and hyperpolynomials of [3, 4] and the present paper, this unification works by packaging the corresponding DAHA-Jones polynomials into a single polynomial, denoted by $H^{\mathfrak{ad}}_{\mathsf{r,s}}(q,t,a)$, with an additional parameter a, where the individual polynomials are recovered via the following specializations:

$$H^{\mathfrak{ad}}_{\mathsf{r,s}}(q,t,a=-t^{\nu(G)}) = \widetilde{JD}^{\,G}_{\mathsf{r,s}}(\mathfrak{ad};q,t), \text{ excluding } G_2, F_4. \tag{4.1}$$

Thus a is associated with the (dual) Coxeter number, rather than with the rank. Relations (4.1) appeared sufficient to determine $H^{\mathfrak{ad}}$ for $T^{3,2}$ and $T^{4,3}$, but this cannot be expected for arbitrary torus knots.

In general, such polynomials cannot be uniquely determined via these specializations for sufficiently complicated torus knots; one needs an infinite family of root systems in (4.1) to restore a for any knots. Practically speaking, however, only two specializations to E_8 and E_7 are enough for the trefoil. We will demonstrate this in detail below. Even more convincingly, the three specializations to E_8, E_7, and E_6 were enough for $T^{4,3}$; the resulting polynomial has hundreds of terms.

Here we construct $H^{\mathfrak{ad}}_{\mathsf{r,s}}$ for two knots, the trefoil $T^{3,2}$ and $T^{4,3}$. We will call this polynomial the *adjoint exceptional hyperpolynomial*, since we consider only the adjoint representations. As in [4], we use the name "hyperpolynomial", since "superpolynomial" is commonly reserved for the root systems of type A.

For the trefoil we will show explicitly how $H^{\mathfrak{ad}}_{3,2}$ is obtained from the relevant DAHA-Jones polynomials for E_8, E_7 and the adjoint representation $\mathfrak{ad}$ whose highest weight is the highest short root ϑ.

For $T^{4,3}$, we obtain $H^{\mathfrak{ad}}_{4,3}$ using the same procedure, though E_6 is also required to find some coefficients. Since the DAHA-Jones polynomials in these cases are rather long, we do not include them and instead refer the reader to [4] where they are posted.

Both $H^{\mathfrak{ad}}_{3,2}$ and $H^{\mathfrak{ad}}_{4,3}$ will satisfy all six of the defining specializations from (4.1), even though they are only constructed using two and three of these specializations, respectively. This is a convincing confirmation that the formulas we found are meaningful. See Section 4.3, where we discuss this relations and some further interesting symmetries.

4.2. E-type hyperpolynomials

4.2.1. Trefoil. Here we will demonstrate how $H^{\mathfrak{ad}}_{3,2}(q,t,a)$ is obtained from only the specializations (4.1) for G of types E_8, E_7. The relevant DAHA-Jones polynomial for E_8 from [3] is

$$\widetilde{JD}^{E_8}_{3,2}(\omega_8;q,t) = 1+q(t+t^6+t^{10}-t^{20}-t^{24}-t^{29})+q^2(t^{12}+t^{16}+t^{20}-t^{26} +t^{29}-3t^{30}-t^{34}-t^{35}-t^{39}+t^{44}+t^{49}+t^{53})+q^3(t^{29}+t^{35}-t^{36}+t^{39}-t^{40}-t^{41} -t^{45}-2t^{49}+t^{50}-t^{53}+t^{54}+t^{55}-t^{58}+2t^{59}+t^{63}-t^{73})+q^4(t^{58}-t^{59}-t^{64}+t^{65} - t^{68}+t^{69}+t^{78}-t^{79}+t^{82}-t^{83})+q^5(-t^{87}+t^{88}),$$

and the relevant DAHA-Jones polynomial for E_7 is

$$\widetilde{JD}^{E_7}_{3,2}(\omega_1;q,t) = 1+q(t+t^4+t^6-t^{12}-t^{14}-t^{17})+q^2(t^8+t^{10}+t^{12}-t^{16} +t^{17}-3t^{18}-t^{20}-t^{21}-t^{23}+t^{26}+t^{29}+t^{31})+q^3(t^{17}+t^{21}-t^{22}+t^{23}-t^{24}-t^{25} -t^{27}-2t^{29}+t^{30}-t^{31}+t^{32}+t^{33}-t^{34}+2t^{35}+t^{37}-t^{43})+q^4(t^{34}-t^{35}-t^{38}+t^{39} - t^{40}+t^{41}+t^{46}-t^{47}+t^{48}-t^{49})+q^5(-t^{51}+t^{52}).$$

The (lexicographic) order in which these two polynomials are printed gives a perfect, one-to-one correspondence between their terms. Furthermore, this correspondence respects the signs $\pm$ of these terms.

For example, in this correspondence $-q^2t^{39}$ in the E_8 polynomial is paired with $-q^2t^{23}$ in the E_7 polynomial. Determining the common exponent x of a that satisfies the right specializations from (4.1) readily reduces to finding a solution to $39-5x=23-3x$, since $\nu(E_8)=5$ and $\nu(E_7)=3$. Evidently, this solution is $x=8$, and the corresponding term in $H^{\mathfrak{ad}}_{3,2}$ will then be $-q^2t^{-1}a^8$.

Applying this procedure to every pair of terms in these two polynomials, the *adjoint exceptional hyperpolynomial* for the trefoil is

$$H^{\mathfrak{ad}}_{3,2}(q,t,a) \;=\; 1+q(t-ta+a^2-a^4+t^{-1}a^5-t^{-1}a^6)+q^2(t^2a^2-ta^3+a^4 +ta^5+t^{-1}a^6-3a^6+t^{-1}a^7+a^7-t^{-1}a^8-t^{-1}a^9+t^{-1}a^{10}-t^{-2}a^{11})+q^3(t^{-1}a^6 -a^7+ta^7+t^{-1}a^8-a^8-ta^8+a^9-2t^{-1}a^{10}+a^{10}+t^{-2}a^{11}-t^{-1}a^{11}-a^{11}-t^{-2}a^{12} +2t^{-1}a^{12}-t^{-2}a^{13}+t^{-2}a^{15})+q^4(t^{-2}a^{12}-t^{-1}a^{12}+t^{-1}a^{13}-a^{13}-t^{-2}a^{14} +t^{-1}a^{14}+t^{-2}a^{16}-t^{-1}a^{16}-t^{-3}a^{17}+t^{-2}a^{17})+q^5(-t^{-3}a^{18}+t^{-2}a^{18}).$$

4.2.2. The case of $T^{4,3}$. As it was mentioned above, we will not provide the corresponding formulas for DAHA-Jones polynomials for $E_{6,7,8}$ from [3] here, since they are long. The adjoint exceptional hyperpolynomial for the torus knot $T^{4,3}$ can be constructed using essentially the same method as that for the trefoil. However, since the DAHA-Jones polynomials $\widetilde{JD}^{E_8}_{4,3}$ and $\widetilde{JD}^{E_7}_{4,3}$ have now different numbers of terms, their lexicographic orderings are (for some powers of q) insufficient to determine a correspondence between their respective monomials. These few ambiguities are resolved by also considering $\widetilde{JD}^{E_6}_{4,3}$.

Once such a correspondence between triples of monomials is established, the a–degrees are uniquely restored using the relevant specializations from (4.1), as for the trefoil. The resulting hyperpolynomial is long, but we think that the formula must be provided, since it has various symmetries beyond those discussed in the paper and we expect that further relations will be found. For instance, its connection to the root systems F_4, G_2 is an open problem. One has: $H^{\mathfrak{ad}}_{4,3}(q,t,a)=$

$$1+q\big(-t^{-1}a^6+t^{-1}a^5-a^4+a^2-ta+t\big)+q^2\big(-t^{-2}a^{11}+t^{-1}a^{10}-t^{-1}a^9-t^{-1}a^8+ t^{-1}a^7+a^7-4a^6+ta^5+t^{-1}a^5+a^5-ta^4-ta^3+t^2a^2+ta^2+a^2-t^2a-ta+t^2+t\big)+ q^3\big(t^{-2}a^{15}-t^{-2}a^{13}+3t^{-1}a^{12}-3t^{-1}a^{11}-t^{-2}a^{11}-a^{11}+t^{-2}a^{10}+3a^{10}-t^{-1}a^9-ta^8- t^{-1}a^8-3a^8+4ta^7+2t^{-1}a^7+2a^7-t^2a^6-4ta^6+t^{-1}a^6-4a^6+t^2a^5+2ta^5+a^5+ a^4-t^3a^3-t^2a^3-2ta^3+t^3a^2+2t^2a^2+ta^2-t^3a-t^2a+t^3\big)+q^4\big(2t^{-2}a^{17}-2t^{-1}a^{16}- t^{-3}a^{16}+2t^{-1}a^{15}+t^{-2}a^{15}+2t^{-1}a^{14}-t^{-2}a^{14}-2t^{-1}a^{13}-2t^{-2}a^{13}-3a^{13}+ta^{12}+$$

$3t^{-1}a^{12}+6a^{12}-3ta^{11}-5t^{-1}a^{11}+t^{-2}a^{11}-4a^{11}+2ta^{10}-2t^{-1}a^{10}+2a^{10}+t^2a^9+$
$2ta^9+t^{-1}a^9+2a^9-4t^2a^8-5ta^8+t^{-1}a^8-6a^8+t^3a^7+4t^2a^7+7ta^7+t^{-1}a^7-t^3a^6-$
$4t^2a^6-2ta^6+t^{-1}a^6-ta^5+t^4a^4+t^3a^4+3t^2a^4+ta^4+a^4-t^4a^3-2t^3a^3-t^2a^3-ta^3+$
$t^4a^2+t^3a^2+t^2a^2\big)+q^5\big(-t^{-2}a^{21}+t^{-3}a^{20}+t^{-2}a^{19}-3t^{-1}a^{18}-t^{-3}a^{18}+3t^{-1}a^{17}+$
$4t^{-2}a^{17}-t^{-3}a^{17}+2a^{17}-2t^{-1}a^{16}-t^{-2}a^{16}-3a^{16}+2t^{-1}a^{15}-2t^{-2}a^{15}+3ta^{14}+$
$5t^{-1}a^{14}-t^{-2}a^{14}+4a^{14}-t^2a^{13}-6ta^{13}-4t^{-1}a^{13}-8a^{13}+3t^2a^{12}+6ta^{12}-3t^{-1}a^{12}+$
$9a^{12}-2t^2a^{11}-4ta^{11}-t^{-1}a^{11}+2t^{-2}a^{11}+a^{11}-t^3a^{10}-2t^2a^{10}-4ta^{10}-2t^{-1}a^{10}-$
$3a^{10}+4t^3a^9+5t^2a^9+7ta^9+t^{-1}a^9+a^9-t^4a^8-4t^3a^8-7t^2a^8-2ta^8+2t^{-1}a^8-2a^8+$
$t^4a^7+2t^3a^7+2t^2a^7+ta^7-2a^7+t^4a^6+t^2a^6+ta^6+t^{-1}a^6+a^6-t^5a^5-t^4a^5-2t^3a^5-$
$t^2a^5-ta^5+t^5a^4+t^4a^4+t^3a^4+t^2a^4-t^4a^3\big)+q^6\big(-t^{-2}a^{23}+t^{-1}a^{22}+2t^{-3}a^{22}-$
$t^{-1}a^{21}-2t^{-2}a^{21}+t^{-3}a^{21}-t^{-1}a^{20}+t^{-2}a^{20}+2t^{-1}a^{19}+4t^{-2}a^{19}-t^{-3}a^{19}+3a^{19}-$
$2ta^{18}-7t^{-1}a^{18}+2t^{-2}a^{18}-t^{-3}a^{18}-4a^{18}+3ta^{17}+4t^{-1}a^{17}-t^{-2}a^{17}-2t^{-3}a^{17}+$
$5a^{17}+4t^{-1}a^{16}+t^{-3}a^{16}-2a^{16}-3t^2a^{15}-4ta^{15}-t^{-1}a^{15}-3t^{-2}a^{15}-6a^{15}+t^3a^{14}+$
$6t^2a^{14}+8ta^{14}-2t^{-2}a^{14}+6a^{14}-3t^3a^{13}-4t^2a^{13}-9ta^{13}+3t^{-2}a^{13}+a^{13}+2t^2a^{12}-$
$ta^{12}-7t^{-1}a^{12}+t^{-2}a^{12}-a^{12}+t^4a^{11}+3t^3a^{11}+2t^2a^{11}+3ta^{11}+t^{-1}a^{11}+t^{-2}a^{11}+$
$3a^{11}-4t^4a^{10}-2t^3a^{10}-6t^2a^{10}-ta^{10}+t^{-1}a^{10}-2a^{10}+t^5a^9+t^4a^9+4t^3a^9+2ta^9-$
$3a^9-t^3a^8+2ta^8+t^{-1}a^8+a^8-t^5a^7-t^3a^7-ta^7-a^7+t^6a^6+t^4a^6+t^2a^6+a^6\big)+$
$q^7\big(-t^{-3}a^{26}+t^{-1}a^{24}+t^{-3}a^{24}-t^{-1}a^{23}-4t^{-2}a^{23}+2t^{-3}a^{23}-a^{23}+3t^{-1}a^{22}-t^{-2}a^{22}+$
$t^{-3}a^{22}-t^{-4}a^{22}+a^{22}-t^{-1}a^{21}+2t^{-2}a^{21}+2t^{-3}a^{21}+a^{21}-3ta^{20}-5t^{-1}a^{20}+2t^{-2}a^{20}-$
$t^{-3}a^{20}-2a^{20}+2t^2a^{19}+3ta^{19}+t^{-2}a^{19}-3t^{-3}a^{19}+7a^{19}-2t^2a^{18}-2ta^{18}+3t^{-1}a^{18}+$
$5t^{-2}a^{18}-t^{-3}a^{18}-4a^{18}-2t^2a^{17}+2ta^{17}-t^{-1}a^{17}-6t^{-2}a^{17}+t^{-3}a^{17}-4a^{17}+3t^3a^{16}+$
$2t^2a^{16}+5ta^{16}+5t^{-1}a^{16}-3t^{-2}a^{16}+a^{16}-t^4a^{15}-3t^3a^{15}-3t^2a^{15}-4ta^{15}+2t^{-1}a^{15}+$
$t^{-2}a^{15}+t^4a^{14}+5t^2a^{14}-3ta^{14}-7t^{-1}a^{14}+t^{-2}a^{14}+t^4a^{13}-t^3a^{13}+2t^2a^{13}-ta^{13}-$
$t^{-1}a^{13}+t^{-2}a^{13}+7a^{13}-t^5a^{12}-t^4a^{12}-t^3a^{12}-t^2a^{12}-2ta^{12}-t^{-1}a^{12}+t^{-2}a^{12}-$
$a^{12}+t^5a^{11}+2t^3a^{11}-t^2a^{11}+ta^{11}-a^{11}-t^4a^{10}+t^3a^{10}-t^2a^{10}+2ta^{10}+t^{-1}a^{10}-$
$t^2a^9-a^9+ta^8\big)+q^8\big(-t^{-3}a^{28}+t^{-2}a^{27}-t^{-3}a^{27}+t^{-4}a^{26}-2t^{-2}a^{25}+t^{-3}a^{25}-a^{25}+$
$ta^{24}+4t^{-1}a^{24}-3t^{-2}a^{24}+t^{-3}a^{24}-t^{-4}a^{24}+t^{-1}a^{23}+5t^{-3}a^{23}-2t^{-4}a^{23}-2a^{23}-$
$ta^{22}-2t^{-1}a^{22}-2t^{-2}a^{22}+t^{-3}a^{22}+a^{22}+2t^2a^{21}-2t^{-1}a^{21}+3t^{-2}a^{21}-3t^{-3}a^{21}+$
$4a^{21}-t^3a^{20}-4ta^{20}-t^{-1}a^{20}+6t^{-2}a^{20}-2t^{-3}a^{20}+2a^{20}+t^2a^{19}+ta^{19}-5t^{-1}a^{19}-$
$t^{-2}a^{19}+t^{-3}a^{19}-a^{19}+t^3a^{18}-t^2a^{18}+2ta^{18}+6t^{-1}a^{18}-5t^{-2}a^{18}-t^4a^{17}-2t^2a^{17}+$
$ta^{17}+3t^{-1}a^{17}-t^{-2}a^{17}+t^{-3}a^{17}-3a^{17}+t^3a^{16}-t^{-1}a^{16}-2a^{16}-t^3a^{15}+t^2a^{15}-$
$t^{-1}a^{15}+3a^{15}-2ta^{14}-t^{-1}a^{14}+t^{-2}a^{14}+a^{14}-t^{-1}a^{13}+a^{13}+t^{-2}a^{12}\big)+q^9\big(t^{-2}a^{29}-$
$t^{-3}a^{29}-t^{-1}a^{28}+t^{-2}a^{28}-t^{-3}a^{28}+t^{-4}a^{28}-2t^{-3}a^{27}+2t^{-4}a^{27}+t^{-1}a^{26}-t^{-2}a^{26}+$
$2t^{-1}a^{25}-t^{-2}a^{25}+3t^{-3}a^{25}-2t^{-4}a^{25}-2a^{25}+ta^{24}+t^{-1}a^{24}-5t^{-2}a^{24}+5t^{-3}a^{24}-$
$t^{-4}a^{24}-a^{24}+2t^{-1}a^{23}-t^{-2}a^{23}-t^{-3}a^{23}-ta^{22}-t^{-1}a^{22}+3t^{-2}a^{22}-2t^{-3}a^{22}+a^{22}+$
$t^2a^{21}-ta^{21}-3t^{-1}a^{21}+2t^{-2}a^{21}+a^{21}-t^{-1}a^{20}-t^{-2}a^{20}+2a^{20}+t^{-1}a^{19}-a^{19}-$
$t^{-2}a^{18}+t^{-3}a^{18}\big)+q^{10}\big(-t^{-3}a^{29}+2t^{-4}a^{29}-t^{-5}a^{29}+t^{-2}a^{28}-2t^{-3}a^{28}+t^{-4}a^{28}-$
$t^{-2}a^{26}+2t^{-3}a^{26}-t^{-4}a^{26}+t^{-1}a^{25}-2t^{-2}a^{25}+t^{-3}a^{25}\big)+q^{11}\big(t^{-4}a^{30}-t^{-5}a^{30}\big).$

4.3. Specializations

For $\{\mathsf{r},\mathsf{s}\} \in \{\{3,2\},\{4,3\}\}$, the following specializations, which are special cases of (4.1), are easily verified:

$$H^{\mathfrak{ad}}_{\mathsf{r,s}}(q,t,a=-t^5) = \widetilde{JD}^{E_8}_{\mathsf{r,s}}(\omega_8;q,t),$$

$$H^{\mathfrak{ad}}_{\mathsf{r,s}}(q,t,a=-t^3) = \widetilde{JD}^{E_7}_{\mathsf{r,s}}(\omega_1;q,t),$$

$$H^{\mathfrak{ad}}_{\mathsf{r,s}}(q,t,a=-t^2) = \widetilde{JD}^{E_6}_{\mathsf{r,s}}(\omega_2;q,t),$$

$$H^{\mathfrak{ad}}_{\mathsf{r,s}}(q,t,a=-t^1) = \widetilde{JD}^{D_4}_{\mathsf{r,s}}(\omega_2;q,t),$$

$$H^{\mathfrak{ad}}_{\mathsf{r,s}}(q,t,a=-t^{\frac{1}{2}}) = \widetilde{JD}^{A_2}_{\mathsf{r,s}}(\omega_1+\omega_2;q,t),$$

$$H^{\mathfrak{ad}}_{\mathsf{r,s}}(q,t,a=-t^{\frac{1}{3}}) = \widetilde{JD}^{A_1}_{\mathsf{r,s}}(2\omega_1;q,t).$$

The DAHA-Jones polynomials for the first four specializations may be found in [4]. The last two DAHA-Jones polynomials are specializations of the DAHA superpolynomials from Section 3.1.

In addition to these defining specializations, the expressions for $H^{\mathfrak{ad}}_{\mathsf{r,s}}$ possess two structures that resemble the "canceling differentials" from [8] and other papers. On the level of polynomials, these canceling differentials correspond to specializations of the parameters with respect to which $H^{\mathfrak{ad}}_{\mathsf{r,s}}$ becomes a single monomial.

The simplest such specialization corresponds to the evaluation at $t=1$ of DAHA-Jones polynomials. On the level of hyperpolynomials, we set $a \mapsto -t^{\nu} = -1$, which readily results in the relation

$$H^{\mathfrak{ad}}_{\mathsf{r,s}}(q,t=1,a=-1) = 1.$$

The following example of a "canceling differential" is more interesting. We set $t = qa^6$. Then

$$H^{\mathfrak{ad}}_{3,2}(q,t,a) = q^3t^{-1}a^6 + (1-qt^{-1}a^6)\mathcal{Q}_{3,2}(q,t,a),$$

$$H^{\mathfrak{ad}}_{4,3}(q,t,a) = q^7t^{-1}a^6 + (1-qt^{-1}a^6)\mathcal{Q}_{4,3}(q,t,a)$$

for some polynomials $\mathcal{Q}_{\mathsf{r,s}}(q,t,a)$. Observe that $qt^{-1}a^6 \mapsto -qt^{h^\vee-1}$ in the specialization $a \mapsto -t^{\nu}$. Upon this specialization, the above relations reflect the $PSL^{\wedge}_2(\mathbb{Z})$–invariance of the image of nonsymmetric Macdonald polynomials E_{ϑ} in the quotient of the polynomial representation of the corresponding DAHA under the relation $qt^{h^\vee-1} = -1$ by its radical. However we did not check all details.

Let us also mention potential links of our hyperpolynomials evaluated at $a = -t^{-1}$ and $a = -1$ to the root systems D_6 and respectively A_3, which we are going to investigate elsewhere.

Finally, let us touch upon the root systems G_2, F_4 in the Deligne-Gross series. For $\nu(G_2) = \frac{2}{3}$ and for $\nu(F_4) = \frac{3}{2}$, the corresponding specializations of $H^{\mathfrak{ad}}_{\mathsf{r,s}}$ resemble the polynomials $\widetilde{JD}^{G_2}_{3,2}(\omega_1; q, r, t)$ and $\widetilde{JD}^{F_4}_{3,2}(\omega_1; q, r, t)$ from [3] at $r = t$, but do not coincide with them. Hopefully, these specializations are connected with the *untwisted* variants of these two DAHA-Jones polynomials, but they are known so far only in the twisted setting.

Conclusion. Let us mention that we do not touch in this paper the physics aspects of the composite superpolynomials (and those for other root systems). See [11] concerning the corresponding theory of *resolved conifold*; we thank Masoud Soroush for a discussion. In the *refined case* (related to open Gromov-Witten invariants), this approach reached so far only the simplest examples (our composite DAHA-superpolynomials are well ahead), but this is an important motivation of what we did in this paper. In contrast to conventional Gromov-Witten invariants, a systematic theory of open Gromov-Witten invariants is not yet developed. See e.g. [19] for a comprehensive account of this field.

Finally, we note that the counterparts of the HOMFLY-PT polynomials for the classical series of root systems, for instance Kauffman polynomials, can be generally addressed via Chern-Simons theory. Recall that DAHA provide a uniform theory of (refined) DAHA-Jones polynomials for any root systems and arbitrary weights (for algebraic knots/links), including the hyperpolynomials for the classical series (conjecturally for B, C). The exceptional DAHA-hyperpolynomials are quite a challenge for us (see the last section).

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Bibliography

[1] Aiston (A.K.), and Morton (H.). — Idempotents of Hecke algebras of type A, J. Knot Theory Ramif. 7, p. 463-487; arXiv:math.QA/9702017 (1998).

[2] Chen (L.), and Chen (Q.). — Orthogonal Quantum Group Invariants Of Links, arXiv: 1007.1656v1 (2010).

[3] Cherednik (I.). — DAHA-Jones polynomials of torus knots, arXiv: 1406.3959 [math.QA] (2014).

[4] Cherednik (I.) Jones polynomials of torus knots via DAHA, arXiv: 1111.6195v6 [math.QA] (2012).

[5] Cherednik (I.) Double affine Hecke algebras, London Mathematical Society Lecture Note Series, 319, Cambridge University Press, Cambridge, (2006).

[6] Cherednik (I.), and Danilenko (I.). — DAHA and iterated torus knots, arXiv: 1408.4348 (2014).

[7] Deligne (P.), and Gross (B.). — On the exceptional series, and its descendants, Comptes Rendus Acad. Sci. Paris, Ser I, 335 (2002), 877-881.

[8] Dunfield (N.), and Gukov (S.). — and J. Rasmussen,The superpolynomial for knot homologies, Experimental Mathematics, 15:2 (2006), 129-159.

[9] Garoufalidis (S.), and Morton (H.), and Vuong (T.). — The SL_3 colored Jones polynomial of the trefoil, arXiv:1010.3147v4 [math.GT] (2010).

[10] Gorsky (E.), and Negut (A.). — Refined knot invariants and Hilbert schemes, arXiv: 1304.3328v2 (2013).

[11] Gu (J.), and Jockers (H.), and Klemm (A.), and Soroush (M.). — Knot invariants from topological recursion on augmentation varieties, arXiv: 1401.5095v1 [hep-th] (2014).

[12] Gukov (S.), and Stosic (M.). — Homological algebra of knots and BPS states, arXiv: 1112.0030v1 [hep-th] (2011).

[13] Hadji (R.), and Morton (H.). — A basis for the full HOMFLY-PT skein of the annulus, Math. Proc. Camb. Philos. Soc. 141, p. 81-100 (2006).

[14] Khovanov (M.). — Triply-graded link homology and Hochschild homology of Soergel bimodules, International J. of Math. 18, p. 869–885 (2007).

[15] Khovanov (M.), and Rozansky (L.). — Matrix factorizations and link homology, Fundamenta Mathematicae, 199, p. 1-91 (2008).

[16] Khovanov (M.), and Rozansky (L.). — Matrix factorizations and link homology II, Geometry and Topology, 12, p. 1387-1425 (2008).

[17] Koike (K.). — On the decomposition of tensor products of the representations of the classical groups: by means of the universal character, Adv. Math. 74:1 p. 57–86 (1989).

[18] Lin (X. S.), and Zheng (H.). — On the Hecke algebra and the colored HOMFLY-PT polynomial, Trans. Amer. Math. Soc. 362, p. 1-18 (2010), arXiv: math.QA/0601267 (2006).

[19] Mariño (M.). — Chern-Simons theory and topological strings, Rev.Mod.Phys. 77 p. 675-720 (2005).

[20] Manchon (P.M.G.), and Morton (H.). — Geometrical relations and plethysms in the HOMFLY-PT skein of the annulus, J. London Math. Soc. 78, p. 305-328 (2008); arXiv: math.GT/0707.2851 (2007).

[21] Paul (C.), Borhade (P.), and Ramadevi (P.). — Composite invariants and unoriented topological string amplitudes, arXiv: 1003.5282v2 [hep-th] (2010).

[22] Queffelec (H.), and Rose (D.). — The $\mathfrak{sl}_n$ foam 2-category: a combinatorial formulation of Khovanov-Rozansky homology via categorical skew Howe duality, arXiv: 1405.5920 (2014).

[23] Rosso (M.), and Jones (V. F. R.). — On the invariants of torus knots derived from quantum groups, Journal of Knot Theory and its Ramifications, 2, p. 97-112 (1993).

[24] Rouquier (R.). — Khovanov-Rozansky homology and 2-braid groups, arXiv: 1203.5065 (2012).

[25] The Sage – Combinat community. — Sage – Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, Site: `http://combinat.sagemath.org`, (2008).

[26] Schiffmann (O.), and Vasserot (E.). — The elliptic Hall algebra, Cherednik Hecke algebras and Macdonald polynomials, Compos. Math. 147, p. 188-234 (2011).

[27] Stevan (S.). — Chern-Simons invariants of torus links, arXiv: 1003.2861v2 [hep-th] (2010).

[28] Webster (B.). — Knot invariants and higher representation theory, arXiv: 1309.3796 (2013).