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Energy decay for a locally undamped wave equation (*)

Matthieu Léautaud (1) and Nicolas Lerner (2)

Abstract. — We study the decay rate for the energy of solutions of a damped wave equation in a situation where the *Geometric Control Condition* is violated. We assume that the set of undamped trajectories is a flat torus of positive codimension and that the metric is locally flat around this set. We further assume that the damping function enjoys locally a prescribed homogeneity near the undamped set in transversal directions. We prove a sharp decay estimate at a polynomial rate that depends on the homogeneity of the damping function. Our method relies on a refined microlocal analysis linked to a second microlocalization procedure to cut the phase space into tiny regions respecting the uncertainty principle but way too small to enter a standard semi-classical analysis localization. Using a multiplier method, we obtain the energy estimates in each region and we then patch the microlocal estimates together.

Résumé. — Nous étudions le taux de décroissance de l'énergie des solutions de l'équation des ondes amorties dans une situation où la Condition de Contrôle Géométrique n'est pas satisfaite. Nous supposons que l'ensemble des trajectoires non amorties forme un sous-tore plat, et que la métrique est localement plate dans un voisinage. Nous supposons aussi que la fonction d'amortissement est localement homogène dans les directions transverses. Nous démontrons la décroissance à un taux polynomial optimal, qui dépend de l'homogénéité de la fonction d'amortissement. Notre méthode repose sur une procédure de deuxième microlocalisation, qui consiste à découper l'espace des phases en toutes petites régions respectant le principe d'incertitude, mais bien trop petites pour entrer dans

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le cadre de l'analyse microlocale semi-classique standard. Une méthode de multiplicateurs nous permet, dans chaque région, d'obtenir des estimées d'énergie, que nous recollons finalement.

1. Introduction and main results

1.1. Introduction

We consider a smooth connected compact Riemannian manifold (M, g) of dimension n, and denote by Δ_g the associated negative Laplace–Beltrami operator. Given $b \in L^\infty(M)$, we study the decay rates for the damped wave equation on M:

$$\begin{cases} \partial_t^2 u - \Delta_g u + b(x)\partial_t u = 0 & \text{in } \mathbb{R}^+ \times M, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) & \text{in } M. \end{cases} \tag{1.1}$$

The energy of a solution is defined by

$$E(u(t)) = \frac{1}{2}(\|\nabla_g u(t)\|^2_{L^2(M)} + \|\partial_t u(t)\|^2_{L^2(M)}), \tag{1.2}$$

(see for instance Appendix A for a definition of Δ_g and ∇_g) and evolves as

$$\frac{d}{dt}E(u(t)) = -\int_M b|\partial_t u|^2 dx.$$

The energy is thus actually damped when $b \geqslant 0$ a.e. on M, what we assume from now on. We define the subset of M on which the damping is effective as

$$\omega_b := \bigcup \left\{U \subset M, U \text{ open}, \operatorname{essinf}_U(b) > 0\right\}. \tag{1.3}$$

Notice that ω_b is an open set included in the interior of $\operatorname{supp} b$ and thus $\overline{\omega}_b \subset \operatorname{supp} b$.(1) In the usual case where b is continuous, we have $\omega_b = \{b > 0\}$ and $\overline{\omega}_b = \operatorname{supp} b$. As soon as $\omega_b \neq \emptyset$ one has $E(u(t)) \to 0$ as $t \to +\infty$ (see for instance [28, 29]). Moreover, a criterion for uniform (and hence exponential) decay is due to Rauch–Taylor [37] (see also [3] and Lemma 5.1 below): there exist $C > 0, \gamma > 0$ such that for all data,

$$E(u(t)) \leqslant Ce^{-\gamma t}E(u(0)),$$

if the Geometric Control Condition (GCC) holds: every geodesic starting from S^*M (see Appendix A for a precise statement) enters the set ω_b in finite time. Conversely, if there is a geodesic that never meets $\operatorname{supp}(b)$, then uniform decay does not hold (see for instance [36]). In the case $b \in \mathscr{C}^0(M)$, the situation is simpler since uniform decay is *equivalent* to the fact that

(1) Remark that the equality may fail, taking for instance $b = \mathbb{1}_K$ where K is a compact Cantor set with positive measure satisfying $\mathring{K} = \emptyset$, in which case $\operatorname{supp}(b) = K$ and $\omega_b = \emptyset$.

$\omega_b(= \{b > 0\})$ satisfies (GCC), as remarked by Burq and Gérard [9][2]. As a consequence, when (GCC) is not satisfied, we cannot expect a uniform decay of the energy with respect to all data in $H^1(M) \times L^2(M)$. However, Lebeau [28, 29] proved that there is always a uniform decay rate of the energy, with respect to smoother data, say in $H^2(M) \times H^1(M)$. This motivates the following definition.

Definition 1.1. — *Given $a \in \mathbb{R}$ and a decreasing function $f : [a, +\infty) \to \mathbb{R}_+^*$ such that $f(t) \to 0$ as $t \to +\infty$, we say that the solutions of* (1.1) *decay at rate $f(t)$ if there exists $C > 0$ such that for all $(u_0, u_1) \in H^2(M) \times H^1(M)$, for all $t \geqslant a$, we have*

$$E(u(t))^{\frac{1}{2}} \leqslant C f(t) \big(\|u_0\|_{H^2(M)} + \|u_1\|_{H^1(M)} \big).$$

Note that decay at a rate $f(t)$ depends only on (M, g) and on the damping function b. Note also that $f(t)^2$ characterizes the decay of the energy and $f(t)$ that of the associated norm. Lebeau [28, 29] proved that decay at rate $1/\log t$ always holds, independently of (M, g) and b as soon as $\omega_b \neq \emptyset$.

As noticed for instance in [5], decay at a rate $f(t)$ implies faster decay for "smoother" data: taking for example $b \in \mathscr{C}^\infty(M)$, decay at rate $f(t)$ implies that for all $s > 0$, there exists $C_s > 0$ such that for all $(u_0, u_1) \in H^{s+1}(M) \times H^s(M)$, we have for large t,

$$E(u(t))^{\frac{1}{2}} \leqslant C_s f(t/s)^s (\|u_0\|_{H^{s+1}(M)} + \|u_1\|_{H^s(M)}).$$

In view of the Rauch-Taylor theorem mentioned above, it is convenient to introduce the subset of phase-space consisting in points-directions that are never brought into the damping region ω_b by the geodesic flow. Namely, the *undamped set* is defined by

$$S = \{\rho \in S^*M, \text{ for all } t \in \mathbb{R}, \ \phi_t(\rho) \cap T^*\omega_b = \emptyset\},$$

where ϕ_t is the geodesic flow (see for instance Appendix A). With this definition, (GCC) is equivalent to $S = \emptyset$. In this article, we are concerned with the damped wave equation in a geometric situation where the undamped set S is the cotangent space to a *flat subtorus* of M (of dimension $1 \leqslant n'' \leqslant n-1$) under two main additional assumptions: the metric is locally flat around this subtorus; the damping function b only depends on variables transverse to this torus and enjoys locally a prescribed homogeneity. As a particular case, we can consider situations where the geodesic flow has a *single undamped trajectory* if the metric is locally flat around this trajectory; the damping function b only depends on variables transverse to the flow and enjoys a prescribed

[2] This is no longer the case in general if b is not continuous, as proved in [27].

homogeneity in a neighbourhood of the undamped trajectory. Such situations may for instance occur on the torus $M = \mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ endowed with the flat metric, as in the following examples. One of our motivations is to understand the optimal decay rate in the following model problems.

Example 1.2. — Let $M = \mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2 \equiv [-\pi,\pi]^2$, endowed with the flat metric, let $\gamma > 0$, and let $b(x_1,x_2) = x_1^{2\gamma}$ near $x_1 = 0$, positive elsewhere, depending only on x_1. The undamped set consists in two undamped trajectories:

$$S = \{0\}_{x_1} \times \mathbb{T}^1_{x_2} \times \{0\}_{\xi_1} \times \{\pm 1\}_{\xi_2} = S^*(\{0\} \times \mathbb{T}^1).$$

For the case where $b = \sin^2 x_1$, Wen Deng communicated to us a direct study in [16].

Example 1.3. — Let $M = \mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3 \equiv [-\pi,\pi]^3$, endowed with the flat metric, let $\gamma > 0$, and let $b(x_1,x_2,x_3) = (x_1^2 + x_2^2)^\gamma$. The undamped set consists in two undamped trajectories:

$$S = \{0,0\}_{x_1,x_2} \times \mathbb{T}^1_{x_3} \times \{0,0\}_{\xi_1,\xi_2} \times \{\pm 1\}_{\xi_3} = S^*(\{0,0\} \times \mathbb{T}^1).$$

Example 1.4. — Let $M = \mathbb{T}^3 \equiv [-\pi,\pi]^3$, endowed with the flat metric, let $\gamma > 0$, and let

$$b(x_1,x_2,x_3) = x_1^{2\gamma}.$$

The undamped set is a 2-dimensional subtorus given by:

$$S = \{0\}_{x_1} \times \mathbb{T}^2_{x_2,x_3} \times \{0\}_{\xi_1} \times \{(\xi_2,\xi_3) \in \mathbb{R}^2, \xi_2^2 + \xi_3^2 = 1\} = S^*(\{0\} \times \mathbb{T}^2).$$

Decay rates for the damped wave equation on a flat metric with a lack of (GCC) have already been studied in [1, 10, 31, 34]. In [1] it is proved that, on $M = \mathbb{T}^n$, decay at a rate $t^{-1/2}$ always occurs if $\omega_b \neq \emptyset$. On the other hand, the decay cannot be better than t^{-1} as soon as (GCC) is strongly violated, i.e. as soon as there exists a neighbourhood $\mathcal{N}$ of a geodesic such that $\mathcal{N} \cap \operatorname{supp}(b) = \emptyset$, see [1]. In this paper, we are studying the opposite situation, i.e. the case of a weak lack of damping on $M = \mathbb{T}^n$: only a positive codimension invariant torus is undamped. In the situation of Examples 1.2, 1.3, and 1.4, for instance, we may expect (and we shall prove) a decay at a stronger polynomial rate than t^{-1}. Functions on $\mathbb{T}^{n''}$ shall be identified in the whole paper with $2\pi\mathbb{Z}^{n''}$-periodic functions on $\mathbb{R}^{n''}$.

According to [1, 5, 8, 29], proving a decay rate for solutions of (1.1) reduces to proving a high-energy estimate for the operators

$$P_\lambda = -\Delta_g - \lambda^2 + i\lambda b, \quad \lambda \in \mathbb{R}^*, \quad D(P_\lambda) = H^2(M). \tag{1.4}$$

The latter are for instance obtained by performing a Fourier transform in the time variable of the damped wave operator $\partial_t^2 - \Delta_g + b(x)\partial_t$, λ being the frequency variable dual to the time t. More precisely, concerning polynomial decay, the optimal result was proved by [8] (see also [4] for generalizations) and can be stated as follows (see [1, Proposition 2.4]).

Proposition 1.5. — *Given $\alpha > 0$, the solutions of* (1.1) *decay at rate $t^{-\frac{1}{\alpha}}$ if and only if there exist C, λ_0 positive, such that for all $u \in H^2(M)$, for all $\lambda \geqslant \lambda_0$, we have*

$$C\|P_\lambda u\|_{L^2(M)} \geqslant \lambda^{1-\alpha}\|u\|_{L^2(M)}. \tag{1.5}$$

Recall that uniform decay is equivalent to the estimate (1.5) with $\alpha = 0$.

1.2. Main results

We first have a negative result.

Theorem 1.6. — *Assume that there exists $1 \leqslant n'' \leqslant n-1$, $\varepsilon_0 > 0$, and $C_1 > 0$ such that with $n' = n - n''$, we have*

- $\left(B_{\mathbb{R}^{n'}}(0,\varepsilon_0) \times \mathbb{T}^{n''}, |dx'_1|^2 + \cdots + |dx'_{n'}|^2 + |dx''_1|^2 + \cdots + |dx''_{n''}|^2\right) \subset (M,g), \quad (1.6)$
- $\nabla_{x''} b = 0$ *in* $\mathcal{N} = B_{\mathbb{R}^{n'}}(0,\varepsilon_0) \times \mathbb{T}^{n''}, \quad (1.7)$
- $0 \leqslant b(x') \leqslant C_1 |x'|^{2\gamma}$ *in* $\mathcal{N}. \quad (1.8)$

Then, there exist $C_0 > 0$ and $(u_k)_{k\in\mathbb{N}} \in H^2(M)^{\mathbb{N}}$ with $\|u_k\|_{L^2(M)} = 1$ such that

$$\|P_k u_k\|_{L^2(M)} \leqslant C_0 k^{\frac{1}{\gamma+1}}, \quad \text{for } k \in \mathbb{N}^*.$$

As a consequence, the best estimate we could expect is

$$C\|P_\lambda u\|_{L^2(M)} \geqslant \lambda^{\frac{1}{\gamma+1}}\|u\|_{L^2(M)}, \tag{1.9}$$

i.e. (1.5) with $\alpha = 1 - \frac{1}{\gamma+1}$. Moreover, (see also [5, Proposition 3]), our Theorem 1.6 prevents decay at a rate $o\left(t^{-(1+\frac{1}{\gamma})}\right)$: the best expected decay rate is

$$t^{-(1+\frac{1}{\gamma})}.$$

Let us now state our partial converse of Theorem 1.6: under some additional global assumptions on M and b, decay at rate $t^{-(1+\frac{1}{\gamma})}$ indeed holds. We first provide a simpler result in the case $n'' = 1$ under a global invariance assumption on b. We then give our more general result in Theorem 1.8.

Theorem 1.7. — *Let* $(M,g) = (M' \times \mathbb{T}^1, g' + |dx_n|^2)$ *where* (M',g') *is a smooth compact Riemannian manifold of dimension* $n-1$*. Assume that there exist* $y' \in M'$, $C_1 \geqslant 1$ *and a neighbourhood* $\mathcal{N}'$ *of* y' *such that*

- $g' = |dx_1|^2 + \cdots + |dx_{n-1}|^2$ *is flat in* $\mathcal{N}'$, (1.10)
- $b = b \otimes 1$ *does not depend on the variable* $x_n \in \mathbb{T}^1$, *and* $b \in L^\infty(M')$, (1.11)
- $C_1^{-1}|x'-y'|^{2\gamma} \leqslant b(x') \leqslant C_1|x'-y'|^{2\gamma}$ *for* $x' \in \mathcal{N}'$, (1.12)
- $b \geqslant C_1^{-1}$ *a.e. on* $M' \setminus \mathcal{N}'$. (1.13)

Then, Property (1.5) *holds with* $\alpha = 1 - \frac{1}{\gamma+1}$*, i.e. decay occurs at rate* $t^{-(1+\frac{1}{\gamma})}$.

This theorem tackles in particular the case of Examples 1.2 and 1.3. Note that simple examples of functions satisfying the assumptions are given by $b(x') = Q(x'-y')^\gamma$ locally around y', where Q is a positive definite quadratic form. We note that very little regularity is required for the damping coefficient b: its "vanishing rate" is prescribed here (1.12) in a relatively weak sense. One may however discuss its global invariance property in the x_n-direction. It can indeed be removed: Theorem 1.7 is a particular case of the following result, where $\mathbb{T}^1$ is replaced by $\mathbb{T}^{n''}$ (adding no significant difficulty) and b is not supposed to be globally invariant anymore, but instead satisfies (GCC) outside the undamped trajectory. We presented Theorem 1.7 separately as its proof is simpler and contains nevertheless the key ideas for the next result.

Theorem 1.8. — *Take* $1 \leqslant n'' \leqslant n-1$ *and assume that* $(M,g) = (M' \times \mathbb{T}^{n''}, g' + |dx''_1|^2 + \cdots + |dx''_{n''}|^2)$ *where* (M',g') *is a smooth compact Riemannian manifold of dimension* $n' = n - n''$ *and* $(x''_1, \ldots, x''_{n''})$ *denote variables in* $\mathbb{T}^{n''}$*. Assume that there exist* $y' \in M'$, $C_1 \geqslant 1$ *and a neighbourhood* $\mathcal{N}'$ *of* y' *such that*

- $g' = |dx'_1|^2 + \cdots + |dx'_{n'}|^2$ *is flat in* $\mathcal{N}'$, (1.14)
- $b \in L^\infty(M), \nabla_{x''} b \in L^\infty(M)$, *and* $\nabla_{x''} b = 0$ *in* $\mathcal{N}' \times \mathbb{T}^{n''}$, (1.15)
- $C_1^{-1}|x'-y'|^{2\gamma} \leqslant b(x') \leqslant C_1|x'-y'|^{2\gamma}$ *for* $x' \in \mathcal{N}'$, (1.16)
- *any geodesic starting from* $S^*M \setminus S^*(\{y'\} \times \mathbb{T}^{n''})$ *intersects* ω_b *in finite time.* (1.17)

Then, Property (1.5) *holds with* $\alpha = 1 - \frac{1}{\gamma+1}$*, i.e. decay at rate* $t^{-(1+\frac{1}{\gamma})}$.

Remark 1.9. — The proof of this theorem (as well as those of the previous ones) also holds without significant modification if the torus $\mathbb{T}^{n''} = (\mathbb{R}/2\pi\mathbb{Z})^{n''}$ is replaced by any compact connected Riemannian manifold M''.

In this situation, Fourier series on $\mathbb{T}^{n''}$ have to be replaced by the spectral decomposition for the Laplace–Beltrami operator $\Delta_{M''}$ on M'', and Fourier multipliers at the end of Section 5.2 have to be replaced by functional calculus for $\Delta_{M''}$ (for this argument, more smoothness than (1.15) for b in the x''-direction is required however). The results also remain essentially unchanged if b vanishes near *finitely many* points $y'_1, y'_2, \cdots$ (instead of a single one y') assuming Assumptions (1.14)-(1.17) around each point (with possibly different vanishing rates $\gamma_1, \gamma_2, \cdots$, in which case the decay rate is given by $t^{-(1+\frac{1}{\max \gamma_i})}$).

This result applies for instance on the torus: assume $M = \mathbb{T}^n$ and that there is a single undamped trajectory Γ. Assume that there exists a neighbourhood $\mathcal{N}$ of this trajectory such that b is invariant in $\mathcal{N}$ in the direction of Γ, and that it is positive homogeneous of degree 2γ in $\mathcal{N}$ in variables orthogonal to Γ. Then, we have the property (1.5) with $\alpha = 1 - \frac{1}{\gamma+1}$, i.e. decay at rate $t^{-(1+\frac{1}{\gamma})}$.

Since the work of Lebeau [29] (see also the introduction of [1] and the references therein), it is quite well established that the main parameters governing the decay rates when (GCC) fails are the global and local dynamics of the geodesic flow. Our results confirm the idea, raised in [10, 1], that once the geometry (and hence the dynamics) is fixed, the next relevant feature when regarding the best decay rate is the rate at which the damping coefficient b vanishes.

Observe that the bigger γ, the worse is Estimate (1.9). This is consistent with the fact that for large γ, the function b is very flat on $\{y'\} \times \mathbb{T}^{n''}$ so that much energy may keep concentrated on the set where b is small. Note that formally, when taking $\gamma \to 0^+$ in Estimate (1.9) (and forgetting that the constant C we obtain depends on γ), we recover the uniform decay estimate (i.e. (1.5) with $\alpha = 0$), equivalent to (GCC). Indeed, if b is positive homogeneous of degree zero, it does not vanish at y' so that (GCC) is satisfied. It would certainly be interesting to prove Estimate (1.9) with a constant C uniform with respect to γ to make this remark rigorous.

The plan of the article is as follows. Taking advantage of the homogeneity of b, the sought estimate near the undamped set may be reduced to an estimate on $\mathbb{R}^{n'}$ for a non-selfadjoint Schrödinger operator (with purely imaginary potential) of the form

$$-\Delta + iW(x), \quad W(x) \sim |x|^{2\gamma}.$$

This key and optimal estimate, which is of independent interest, is proved in Section 2. In turn, it provides a bound on the size of the pseudospectrum for this operator, generalizing results of E. B. Davies [14] and K. Pravda-Starov [35] in the case of the 1D complex harmonic oscillator, $-\frac{d^2}{dx^2}+e^{i\theta}x^2$.

Section 3 is devoted to the proof of two simple technical lemmata, one of them being the scaling argument. The proof of Theorem 1.7 is given in Section 4. The proof of the main result, namely Theorem 1.8, is completed in Section 5 in two steps: first, we prove a geometric control lemma in Section 5.1. Then, in Section 5.2, we patch together the estimates obtained in the different microlocal regions. Section 6 provides a proof of the lower bound of Theorem 1.6. In Section 7, we discuss the spirit of the proof, which relies on some kind of second microlocalization. In particular, our proof could not work with a standard semi-classical localization procedure: we are left with a region in the phase space, near the undamped set S, where further cutting of the phase space is necessary, with a stopping procedure linked to the Heisenberg Uncertainty Principle. To patch together the estimates, we use implicitly a metric which should satisfy some admissibility properties. Although we have avoided in the main part of the text to resort to very general tools of pseudodifferential calculus, we hope that Section 7 could bring a more conceptual vision of the technicalities included in the previous sections. The paper ends with three appendices recalling some facts of geometry and pseudodifferential calculus.

Note. — This article was written in 2014 [24] and a short version presented in [25]. Some time after this article was submitted, N. Burq and C. Zuily [11] and W. Deng [17] managed to weaken some of the assumptions of our Theorem 1.8.

2. A sharp estimate for a non-selfadjoint operator on $\mathbb{R}^d$

2.1. Statements

After a Fourier transformation in the periodic direction and a scaling argument (see the following sections), our main result is reduced to the following theorem. We define on $L^2(\mathbb{R}^d)$ (below, we shall take $d=n'$) the unbounded operator

$$Q_0^\lambda = -\Delta + iW_\lambda(x), \quad \lambda > 0, \tag{2.1}$$

where W_λ is a family of real-valued measurable functions and

$$D(Q_0^\lambda) = \{u \in H^2(\mathbb{R}^d), W_\lambda u \in L^2(\mathbb{R}^d)\}.$$

THEOREM 2.1. — *Suppose that W_λ is a family of real-valued measurable functions on $\mathbb{R}^d$ and that there exist $C_1 \geqslant 1$ and $\gamma > 0$ such that for all $\lambda > 0$, we have*

$$C_1^{-1}|x|^{2\gamma} \leqslant W_\lambda(x) \leqslant C_1\langle x\rangle^{2\gamma} = C_1(1+|x|^2)^\gamma. \tag{2.2}$$

Then, there exists $C_0 > 0$ such that for all $\mu \in \mathbb{R}$, all $\lambda > 0$ and $u \in \mathscr{C}_c^2(\mathbb{R}^d)$, we have

$$C_0\|(Q_0^\lambda - \mu)u\|_{L^2(\mathbb{R}^d)} \geqslant \left(\mu^{\frac{\gamma}{2\gamma+1}}\mathbf{1}(\mu \geqslant 1) + |\mu|\mathbf{1}(\mu \leqslant -1) + 1\right)\|u\|_{L^2(\mathbb{R}^d)}. \tag{2.3}$$

We stress the fact that the sole uniform Assumption (2.2) yields the uniform estimate (2.3). The power $\frac{\gamma}{2\gamma+1}$ is optimal in this estimate. Although not needed for the application to the damped wave equation, we provide for completeness a direct proof of this fact in Lemma C.1. The papers by E.B. Davies [14] and K. Pravda-Starov [35] gave a version of the above estimates in the case of the 1D complex harmonic oscillator, $-\frac{d^2}{dx^2} + e^{i\theta}x^2$.

To prove Theorem 2.1, we need the following lemma.

LEMMA 2.2. — *Suppose that W_λ satisfy the uniform Assumption* (2.2) *and let a be a smooth function on $\mathbb{R}^{2d}$, bounded as well as all its derivatives. Then, there exists $C > 0$ such that for all $\lambda > 0$ and all $u \in \mathscr{C}_c^0(\mathbb{R}^d)$, we have*

$$\|V_\lambda a^w u\|_{L^2(\mathbb{R}^d)} \leqslant C\left(\|u\|_{L^2(\mathbb{R}^d)} + \|V_\lambda u\|_{L^2(\mathbb{R}^d)}\right),$$

where $V_\lambda = W_\lambda^{1/2}$ and a^w stands for the Weyl quantization of the symbol a.

Proof of Lemma 2.2. — Using the upper bound in Assumption (2.2) yields

$$\begin{aligned}\|V_\lambda(x)a^w u\|_{L^2(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} V_\lambda^2(x)|a^w u|^2 dx \\ &\leqslant C_1\int_{\mathbb{R}^d}\langle x\rangle^{2\gamma}|a^w u|^2 dx = C_1\|\langle x\rangle^\gamma a^w u\|_{L^2(\mathbb{R}^d)}^2.\end{aligned}$$

Then, we notice that $\langle x\rangle^\gamma$ and $\langle x\rangle^{-\gamma}$ are admissible weight functions for the metric $|dx|^2 + |d\xi|^2$ in the sense of [30, Definition 2.2.15]. As a consequence of symbolic calculus, we have

$$\langle x\rangle^\gamma \sharp a \sharp \langle x\rangle^{-\gamma} \in S(1, |dx|^2 + |d\xi|^2),$$

where $S(1, |dx|^2 + |d\xi|^2)$ is the space of smooth functions on $\mathbb{R}^{2d}$ which are bounded as well as all their derivatives (see Section B.1 in the Appendix for more on this topic). Calderón–Vaillancourt Theorem (see e.g. [30, Theorem 1.1.4]) yields

$$\langle x\rangle^\gamma a(x,\xi)^w\langle x\rangle^{-\gamma} \in \mathcal{L}(L^2(\mathbb{R}^d)),$$

which implies $\|\langle x\rangle^\gamma a^w u\|_{L^2(\mathbb{R}^d)} \lesssim \|\langle x\rangle^\gamma u\|_{L^2(\mathbb{R}^d)}$. This finally gives

$$\begin{aligned}\|V_\lambda(x)a(x,\xi)^w u\|^2_{L^2(\mathbb{R}^d)} &\lesssim \|\langle x\rangle^\gamma u\|^2_{L^2(\mathbb{R}^d)}\\ &\lesssim \|u\|^2_{L^2(\mathbb{R}^d)} + \||x|^\gamma u\|^2_{L^2(\mathbb{R}^d)}\\ &\lesssim \|u\|^2_{L^2(\mathbb{R}^d)} + \|V_\lambda u\|^2_{L^2(\mathbb{R}^d)},\end{aligned}$$

according to the uniform lower bound in Assumption (2.2). This concludes the proof of the lemma. □

Now, the proof of Theorem 2.1 follows from the next two lemmata.

Lemma 2.3. — *There exists $C > 0$ and $\mu_0 \geqslant 0$ such that for all $\mu \geqslant \mu_0$, all $\lambda > 0$ and $u \in \mathscr{C}^2_c(\mathbb{R}^d)$, we have*

$$C\|(Q_0^\lambda - \mu)u\|_{L^2(\mathbb{R}^d)} \geqslant \mu^{\frac{\gamma}{2\gamma+1}}\|u\|_{L^2(\mathbb{R}^d)}. \tag{2.4}$$

Lemma 2.4. — *For any $\mu_0 \geqslant 0$, there exists $C > 0$ such that for all $\mu \leqslant \mu_0$, all $\lambda > 0$ and $u \in \mathscr{C}^2_c(\mathbb{R}^d)$, we have*

$$C\|(Q_0^\lambda - \mu)u\|_{L^2(\mathbb{R}^d)} \geqslant \big(|\mu|\mathbf{1}(\mu \leqslant -1) + 1\big)\|u\|_{L^2(\mathbb{R}^d)}. \tag{2.5}$$

Let us first prove the simpler Lemma 2.4, the proof of the more involved Lemma 2.3 being postponed to the end of the section.

Proof of Lemma 2.4. — We start with the case $\mu \leqslant -1$. We have then

$$\begin{aligned}\|(Q_0^\lambda - \mu)u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)} &\geqslant \operatorname{Re}\langle (Q_0^\lambda - \mu)u, u\rangle_{L^2(\mathbb{R}^d)}\\ &\geqslant -\mu\langle u,u\rangle_{L^2(\mathbb{R}^d)} = |\mu|\|u\|^2_{L^2(\mathbb{R}^d)},\end{aligned}$$

so that Estimate (2.5) holds for $\mu \leqslant -1$.

Next, let us prove that there exists $C > 0$ such that for all $\mu \in [-1, \mu_0]$, all $\lambda > 0$ and $u \in \mathscr{C}^2_c(\mathbb{R}^d)$, we have

$$C\|(Q_0^\lambda - \mu)u\|_{L^2(\mathbb{R}^d)} \geqslant \|u\|_{L^2(\mathbb{R}^d)}. \tag{2.6}$$

If not, there exist sequences $(\mu_k)_{k\in\mathbb{N}} \in [-1,\mu_0]^{\mathbb{N}}$, $(\lambda_k)_{k\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$, and $(u_k)_{k\in\mathbb{N}} \in \mathscr{C}^2_c(\mathbb{R}^d)^{\mathbb{N}}$ such that

$$\|(Q_0^{\lambda_k} - \mu_k)u_k\|_{L^2(\mathbb{R}^d)} < \frac{1}{k+1}, \quad \|u_k\|_{L^2(\mathbb{R}^d)} = 1. \tag{2.7}$$

This implies

$$\begin{aligned}0 \leftarrow \|(Q_0^{\lambda_k} - \mu_k)u_k\|_{L^2(\mathbb{R}^d)}\|u_k\|_{L^2(\mathbb{R}^d)} &\geqslant \mathrm{Re}\langle (Q_0^{\lambda_k} - \mu_k)u_k, u_k\rangle_{L^2(\mathbb{R}^d)}\\ &= \|\nabla u_k\|^2_{L^2(\mathbb{R}^d)} - \mu_k\|u_k\|^2_{L^2(\mathbb{R}^d)}\\ &\geqslant \|\nabla u_k\|^2_{L^2(\mathbb{R}^d)} - \mu_0\|u_k\|^2_{L^2(\mathbb{R}^d)}\\ 0 \leftarrow \|(Q_0^{\lambda_k} - \mu_k)u_k\|_{L^2(\mathbb{R}^d)}\|u_k\|_{L^2(\mathbb{R}^d)} &\geqslant \mathrm{Im}\langle Q_0 u, u\rangle_{L^2(\mathbb{R}^d)} = \|V_\lambda u_k\|^2_{L^2(\mathbb{R}^d)}\\ &\geqslant C_1^{-1}\||x|^\gamma u_k\|^2_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{2.8}$$

Since $H^1_\gamma(\mathbb{R}^d) := \{u \in H^1(\mathbb{R}^d), |x|^\gamma u \in L^2(\mathbb{R}^d)\}$ injects compactly in $L^2(\mathbb{R}^d)$ and $(u_k)_{k\in\mathbb{N}}$ is bounded in $H^1_\gamma(\mathbb{R}^d)$, we may extract a subsequence such that $(u_k)_{k\in\mathbb{N}}$ converges strongly in $L^2(\mathbb{R}^d)$, $u_k \to u_\infty \in L^2(\mathbb{R}^d)$. According to (2.8), we also obtain $u_k \rightharpoonup 0$ weakly in $L^2(\mathbb{R}^d)$: in fact we have for $\phi \in \mathscr{C}^0_c(\mathbb{R}^d\backslash\{0\})$,

$$\left|\int u_\infty(x)\phi(x)dx\right| \leftarrow \left|\int u_k(x)\phi(x)dx\right| \leqslant \||x|^\gamma u_k\|_{L^2(\mathbb{R}^d)}\||x|^{-\gamma}\phi\|_{L^2(\mathbb{R}^d)} \to 0,$$

proving that $\operatorname{supp} u_\infty \subset \{0\}$ and thus the L^2 function $u_\infty = 0$. This contradicts (2.7) which implies $\|u_\infty\|_{L^2(\mathbb{R}^d)} = 1$. This proves (2.6). As a consequence, Estimate (2.3) is now proven to hold for all $\mu \in (-\infty, \mu_0]$. □

We are now left to proving Lemma 2.3, i.e. to study the most substantial case where $\mu > \mu_0$, but we may keep in mind that we can freely choose the large fixed constant μ_0. We set

$$\nu = \mu^{1/2}, \quad Q_\nu^\lambda = Q_0^\lambda - \nu^2, \tag{2.9}$$

and study the asymptotics when $\nu \to +\infty$. From the above remarks, we have only to prove the estimate (2.3) for $\nu \geqslant \nu_0$, where ν_0 can be chosen arbitrarily large. First of all, we note that

$$\begin{aligned}\|Q_\nu^\lambda u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)} &\geqslant \mathrm{Im}\langle Q_\nu^\lambda u, u\rangle_{L^2(\mathbb{R}^d)} = \langle W_\lambda u, u\rangle_{L^2(\mathbb{R}^d)}\\ &\geqslant C_1^{-1}\langle |x|^{2\gamma}u, u\rangle_{L^2(\mathbb{R}^d)}, \end{aligned} \tag{2.10}$$

which will be used several times during the proof. In particular, this estimate provides the right scaling in the region $|x| \geqslant \nu^{1/(2\gamma+1)}$, according to the lower bound in Assumption (2.2). Next, we split the phase space in two different regions.

2.2. The propagative region

Let $\chi \in \mathscr{C}^\infty_c(\mathbb{R}^+;[0,1])$, such $\chi = 1$ on $[1/2, 3/2]$ and $\chi = 0$ on $[0, 1/4]$. Let $\varphi \in \mathscr{C}^\infty_c(\mathbb{R}^d;[0,1])$ such that $\varphi(x) = \frac{1}{2}$ if $|x| \leqslant \frac{1}{2}$ and $\varphi(x) = 0$ if $|x| \geqslant 1$.

We define

$$\psi(x,\xi)=\int_{-\infty}^{0}\varphi(x+\tau\xi)d\tau\in\mathscr{C}^\infty\big(\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})\big),$$

which is bounded on $\mathbb{R}^d\times\big(\mathbb{R}^d\setminus B(0,\frac14)\big)$ since φ is compactly supported. We set

$$m_\nu(x,\xi)=\chi(\frac{|\xi|^2}{\nu^2})\psi(\frac{x}{\nu^{\frac{1}{2\gamma+1}}},\frac{\xi}{\nu})\in S(1,\frac{|dx|^2}{\nu^{\frac{2}{2\gamma+1}}}+\frac{|d\xi|^2}{\nu^2}),$$

where each seminorm of the symbol m_ν is bounded above independently of $\nu\geqslant 1$; in particular, we get that m_ν^w is bounded on $L^2(\mathbb{R}^d)$ with $\sup_{\nu\geqslant1}\|m_\nu^w\|_{\mathcal{L}(L^2)}<+\infty$. Next, we have with $\{a,b\}$ standing for the Poisson bracket of the functions a,b,

$$\begin{aligned}
&2\operatorname{Re}\langle Q_\nu^\lambda u, im_\nu^w u\rangle_{L^2(\mathbb{R}^d)}\\
&\quad=\big\langle i\left[(|\xi|^2-\nu^2)^w,m_\nu^w\right]u,u\big\rangle_{L^2(\mathbb{R}^d)}+2\operatorname{Re}\langle V_\lambda^2u,m_\nu^wu\rangle_{L^2(\mathbb{R}^d)}\\
&\quad=\langle\big\{|\xi|^2-\nu^2,m_\nu\big\}^w u,u\rangle_{L^2(\mathbb{R}^d)}+2\operatorname{Re}\langle V_\lambda^2u,m_\nu^wu\rangle_{L^2(\mathbb{R}^d)},
\end{aligned}\tag{2.11}$$

since the symbol $|\xi|^2-\nu^2$ is quadratic. Moreover, we can compute

$$\big\{|\xi|^2-\nu^2,m_\nu\big\}=2\xi\cdot\partial_x m_\nu=2\chi(\frac{|\xi|^2}{\nu^2})\xi\cdot\frac{\partial}{\partial x}\left(\psi(\frac{x}{\nu^{\frac{1}{2\gamma+1}}},\frac{\xi}{\nu})\right)$$

with

$$\begin{aligned}
&\chi(\frac{|\xi|^2}{\nu^2})\xi\cdot\frac{\partial}{\partial x}\Big(\psi(x\nu^{-\frac{1}{2\gamma+1}},\xi\nu^{-1})\Big)\\
&\qquad=\int_{-\infty}^{0}\nu^{-\frac{1}{2\gamma+1}}\chi(\frac{|\xi|^2}{\nu^2})(\xi\cdot d\varphi)(x\nu^{-\frac{1}{2\gamma+1}}+\tau\xi\nu^{-1})d\tau\\
&\qquad=\int_{-\infty}^{0}\nu\nu^{-\frac{1}{2\gamma+1}}\chi(\frac{|\xi|^2}{\nu^2})\frac{d}{d\tau}\Big(\varphi(x\nu^{-\frac{1}{2\gamma+1}}+\tau\xi\nu^{-1})\Big)d\tau\\
&\qquad=\nu^{\frac{2\gamma}{2\gamma+1}}\chi(\frac{|\xi|^2}{\nu^2})\varphi(x\nu^{-\frac{1}{2\gamma+1}}),
\end{aligned}$$

since $|\xi\nu^{-1}|^2\geqslant\frac14$ on $\operatorname{supp}\chi(\frac{|\xi|^2}{\nu^2})$. Hence, we obtain

$$\begin{aligned}
\big\{|\xi|^2-\nu^2,m_\nu\big\}&=2\nu^{\frac{2\gamma}{2\gamma+1}}\chi(\frac{|\xi|^2}{\nu^2})\varphi(x\nu^{-\frac{1}{2\gamma+1}})\\
&\geqslant\begin{cases}\nu^{\frac{2\gamma}{2\gamma+1}} & \text{if } ||\xi|^2\nu^{-2}-1|\leqslant 1/2 \text{ and } |x|\nu^{-\frac{1}{2\gamma+1}}\leqslant 1/2,\\ 0 \text{ on } T^*\mathbb{R}^d.\end{cases}
\end{aligned}$$

Moreover we have,

$$2\nu^{\frac{2\gamma}{2\gamma+1}}\chi(\frac{|\xi|^2}{\nu^2})\varphi(x\nu^{-\frac{1}{2\gamma+1}})\in S(\nu^{\frac{2\gamma}{2\gamma+1}},\frac{|dx|^2}{\nu^{\frac{2}{2\gamma+1}}}+\frac{|d\xi|^2}{\nu^2}).$$

As a consequence, using the sharp Gårding inequality in (2.11) yields

$$\begin{aligned}
&C\|Q_\nu^\lambda u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)}\\
&\qquad \geqslant \nu^{\frac{2\gamma}{2\gamma+1}}\left\langle\Big(\chi_0(|\xi|^2\nu^{-2}-1)\chi_0(|x|^2\nu^{-\frac{2}{2\gamma+1}})\Big)^w u,u\right\rangle_{L^2(\mathbb{R}^d)}\\
&\qquad\qquad -\left|2\operatorname{Re}\langle V_\lambda^2 u, m_\nu^w u\rangle_{L^2(\mathbb{R}^d)}\right| - C\nu^{-\frac{2}{2\gamma+1}}\|u\|^2_{L^2(\mathbb{R}^d)}, \quad (2.12)
\end{aligned}$$

where, for some $\epsilon_0\in(0,1/8)$,

$$\begin{cases}\chi_0\in\mathscr{C}_c^\infty(\mathbb{R};[0,1]) \text{ is such that } \{\chi_0=1\}=[-\epsilon_0,\epsilon_0], \text{ and}\\ \{\chi_0=0\}=[-2\epsilon_0,2\epsilon_0]^c,\ \{0<\chi_0(t)<1\}=\{\epsilon_0<|t|<2\epsilon_0\}.\end{cases} \quad (2.13)$$

Next, we check the term $2\operatorname{Re}\langle V_\lambda^2 u, m_\nu^w u\rangle_{L^2(\mathbb{R}^d)}$. We have

$$\begin{aligned}
\left|2\operatorname{Re}\langle V_\lambda^2 u, m_\nu^w u\rangle_{L^2(\mathbb{R}^d)}\right| &= \left|2\operatorname{Re}\langle V_\lambda u, V_\lambda m_\nu^w u\rangle_{L^2(\mathbb{R}^d)}\right|\\
&\leqslant 2\|V_\lambda u\|_{L^2(\mathbb{R}^d)}\|V_\lambda m_\nu^w u\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Recalling that $m_\nu\in S(1,\frac{|dx|^2}{\nu^{\frac{2}{2\gamma+1}}}+\frac{|d\xi|^2}{\nu^2})\subset S(1,|dx|^2+|d\xi|^2)$ as $\nu\geqslant 1$, we may apply Lemma 2.2 to obtain

$$\left|\operatorname{Re}\langle V_\lambda^2 u, m_\nu^w u\rangle_{L^2(\mathbb{R}^d)}\right| \lesssim \|V_\lambda u\|_{L^2(\mathbb{R}^d)}\left(\|u\|_{L^2(\mathbb{R}^d)}+\|V_\lambda u\|_{L^2(\mathbb{R}^d)}\right),$$

where the constant involved is uniform w.r.t. ν and λ. Next, using (2.10), we obtain

$$\begin{aligned}
\left|2\operatorname{Re}\langle V_\lambda^2 u, m_\nu^w u\rangle_{L^2(\mathbb{R}^d)}\right| &\lesssim \|u\|^{\frac{3}{2}}_{L^2(\mathbb{R}^d)}\|Q_\nu^\lambda u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^d)}+\|Q_\nu^\lambda u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)}\\
&\lesssim \|u\|^2_{L^2(\mathbb{R}^d)}+\|Q_\nu^\lambda u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)}. \quad (2.14)
\end{aligned}$$

Combining this estimate with (2.12), we have, for $\nu\geqslant\nu_0$ and ν_0 large enough,

$$\begin{aligned}
&C\|u\|^2_{L^2(\mathbb{R}^d)}+C\|Q_\nu^\lambda u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)}\\
&\qquad\geqslant \nu^{\frac{2\gamma}{2\gamma+1}}\left\langle\Big(\chi_0(|\xi|^2\nu^{-2}-1)\chi_0(|x|^2\nu^{-\frac{2}{2\gamma+1}})\Big)^w u,u\right\rangle_{L^2(\mathbb{R}^d)}. \quad (2.15)
\end{aligned}$$

2.3. The elliptic region.

We now check the regions where $|\xi|^2\ll\nu^2$ or $|\xi|^2\gg\nu^2$. Let $\epsilon_0\in(0,1/2)$; we consider a function $\theta\in\mathscr{C}^\infty(\mathbb{R};[-1,1])$ such that

$$\theta(\sigma)=\begin{cases}1 & \text{for } \sigma\geqslant 1+2\epsilon_0,\\ \{0<\theta<1\} & \text{for } \sigma\in(1+\epsilon_0,1+2\epsilon_0),\\ 0 & \text{for } 1-\epsilon_0\leqslant\sigma\leqslant 1+\epsilon_0,\\ \{-1<\theta<0\} & \text{for } \sigma\in(1-2\epsilon_0,1-\epsilon_0),\\ -1 & \text{for } \sigma\leqslant 1-2\epsilon_0.\end{cases} \quad (2.16)$$

We claim that

$$\forall \sigma \in \mathbb{R}, \quad (\sigma-1)\theta(\sigma) \geqslant |\theta(\sigma)|(\sigma+1)\frac{\epsilon_0}{2+\epsilon_0}. \tag{2.17}$$

In fact, (2.17) is obvious whenever $\theta(\sigma)=0$ and if $\theta(\sigma)>0$, i.e. if $\sigma>1+\epsilon_0$, since it amounts to verifying

$$\sigma(1-\frac{\epsilon_0}{2+\epsilon_0}) \geqslant 1+\frac{\epsilon_0}{2+\epsilon_0} \quad \text{i.e.} \quad \sigma \geqslant 1+\epsilon_0, \quad \text{which holds true.}$$

If $\theta(\sigma)<0$, i.e. if $\sigma<1-\epsilon_0$, it amounts to verify

$$\sigma(1+\frac{\epsilon_0}{2+\epsilon_0}) \leqslant 1-\frac{\epsilon_0}{2+\epsilon_0} \quad \text{i.e.} \quad \sigma \leqslant \frac{1}{1+\epsilon_0},$$

$$\text{which holds true since } 1-\epsilon_0 \leqslant \frac{1}{1+\epsilon_0}.$$

A consequence of (2.17) is that, with $c_0=\frac{\epsilon_0}{2+\epsilon_0}$, we have

$$\forall \xi \in \mathbb{R}^d, \forall \nu \geqslant 1, \quad (|\xi|^2-\nu^2)\theta(|\xi|^2\nu^{-2}) \geqslant c_0|\theta(|\xi|^2\nu^{-2})|\big(|\xi|^2+\nu^2\big). \tag{2.18}$$

We compute

$$\begin{aligned}
&\operatorname{Re}\langle Q_\nu^\lambda u, \theta(|\xi|^2\nu^{-2})^w u\rangle_{L^2(\mathbb{R}^d)} \\
&= \left\langle (|\xi|^2-\nu^2)^w u, \theta(|\xi|^2\nu^{-2})^w u\right\rangle_{L^2(\mathbb{R}^d)} + \operatorname{Re}\left\langle iV_\lambda^2 u, \theta(|\xi|^2\nu^{-2})^w u\right\rangle_{L^2(\mathbb{R}^d)} \\
&\geqslant c_0 \left\langle \big((|\xi|^2+\nu^2)|\theta(|\xi|^2\nu^{-2})|\big)^w u, u\right\rangle_{L^2(\mathbb{R}^d)} \\
&\qquad\qquad - \left|\operatorname{Re}\left\langle iV_\lambda^2 u, \theta(|\xi|^2\nu^{-2})^w u\right\rangle_{L^2(\mathbb{R}^d)}\right|.
\end{aligned}$$

Following (2.14), we have

$$\left|\operatorname{Re}\langle iV_\lambda^2 u, \theta(|\xi|^2\nu^{-2})^w u\rangle_{L^2(\mathbb{R}^d)}\right| \lesssim \|u\|^2_{L^2(\mathbb{R}^d)} + \|Q_\nu^\lambda u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)},$$

so that we finally obtain, as $\theta(|\xi|^2\nu^{-2})^w$ is bounded on $L^2(\mathbb{R}^d)$,

$$\begin{aligned}
&C\|u\|^2_{L^2(\mathbb{R}^d)} + C\|Q_\nu^\lambda u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)} \\
&\qquad\qquad \geqslant \left\langle \big((|\xi|^2+\nu^2)|\theta(|\xi|^2\nu^{-2})|\big)^w u, u\right\rangle_{L^2(\mathbb{R}^d)}.
\end{aligned} \tag{2.19}$$

2.4. Patching the estimates together.

Combining (2.10), (2.15) and (2.19), we obtain the following estimate

$$\begin{aligned}
&C\|u\|^2_{L^2(\mathbb{R}^d)} + C\|Q_\nu^\lambda u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)} \\
&\quad \geqslant \|V_\lambda u\|^2_{L^2(\mathbb{R}^d)} + \nu^2 \left\langle \big(|\theta(|\xi|^2\nu^{-2})|\big)^w u, u\right\rangle_{L^2(\mathbb{R}^d)} \\
&\qquad + \nu^{\frac{2\gamma}{2\gamma+1}} \left\langle \Big(\chi_0(|\xi|^2\nu^{-2}-1)\chi_0(|x|^2\nu^{-\frac{2}{2\gamma+1}})\Big)^w u, u\right\rangle_{L^2(\mathbb{R}^d)}.
\end{aligned} \tag{2.20}$$

Since χ_0 is given and satisfies (2.13), we define now

$$\theta(\sigma) = \operatorname{sign}(\sigma - 1)\big(1 - \chi_0(\sigma - 1)\big).$$

Since χ_0 is smooth and vanishes near 0, the function θ is smooth and such that

$$\begin{cases} \text{for } \sigma \geqslant 1 + 2\epsilon_0, & \theta(\sigma) = 1, \\ \text{for } 1 + \epsilon_0 < \sigma < 1 + 2\epsilon_0, & \theta(\sigma) \in (0,1), \\ \text{for } 1 - \epsilon_0 \leqslant \sigma \leqslant 1 + \epsilon_0, & \theta(\sigma) = 0, \\ \text{for } 1 - 2\epsilon_0 < \sigma < 1 - \epsilon_0, & \theta(\sigma) \in (-1, 0), \\ \text{for } \sigma \leqslant 1 - 2\epsilon_0, & \theta(\sigma) = -1. \end{cases}$$

That function θ satisfies (2.16) so that (2.17) holds. We note that

$$|\theta(\sigma)| + \chi_0(\sigma - 1) = 1,$$

since $|1 - \chi_0(\sigma - 1)| + \chi_0(\sigma - 1) = 1 - \chi_0(\sigma - 1) + \chi_0(\sigma - 1) = 1$. As a consequence, we write

$$\begin{aligned} 1 &= |\theta(|\xi|^2\nu^{-2})| + \chi_0(|\xi|^2\nu^{-2} - 1) \\ &= |\theta(|\xi|^2\nu^{-2})| + \chi_0(|x|^2\nu^{-\frac{2}{2\gamma+1}})\chi_0(|\xi|^2\nu^{-2} - 1) \\ &\qquad + \big(1 - \chi_0(|x|^2\nu^{-\frac{2}{2\gamma+1}})\big)\chi_0(|\xi|^2\nu^{-2} - 1), \end{aligned}$$

and hence

$$\begin{aligned} \nu^{\frac{2\gamma}{2\gamma+1}} \leqslant \nu^{\frac{2\gamma}{2\gamma+1}} |\theta(|\xi|^2\nu^{-2})| + \nu^{\frac{2\gamma}{2\gamma+1}} \chi_0(|x|^2\nu^{-\frac{2}{2\gamma+1}})\chi_0(|\xi|^2\nu^{-2} - 1) \\ + \nu^{\frac{2\gamma}{2\gamma+1}} (1 - \chi_0(|x|^2\nu^{-\frac{2}{2\gamma+1}})). \end{aligned}$$

Since the symbols on both sides of the inequality belong to the class

$$S\Big(\nu^{\frac{2\gamma}{2\gamma+1}}, \frac{|dx|^2}{\nu^{\frac{2}{2\gamma+1}}} + \frac{|d\xi|^2}{\nu^2}\Big),$$

we can apply Gårding's inequality. Note that the gain in the pseudodifferential calculus for symbols in this class is given by $\nu^{-\frac{1}{2\gamma+1}}\nu^{-1} = \nu^{-\frac{2\gamma+2}{2\gamma+1}}$. This gives, for $\nu \geqslant \nu_0$ and ν_0 large enough,

$$\begin{aligned} \nu^{\frac{2\gamma}{2\gamma+1}} \Big\langle \big(\chi_0(|\xi|^2\nu^{-2} - 1)\chi_0(|x|^2\nu^{-\frac{2}{2\gamma+1}})\big)^w u, u \Big\rangle_{L^2(\mathbb{R}^d)} \\ + \nu^{\frac{2\gamma}{2\gamma+1}} \langle \big(|\theta(|\xi|^2\nu^{-2})|\big)^w u, u\rangle_{L^2(\mathbb{R}^d)} \\ + \Big\langle \nu^{\frac{2\gamma}{2\gamma+1}} (1 - \chi_0(|x|^2\nu^{-\frac{2}{2\gamma+1}}))u, u \Big\rangle_{L^2(\mathbb{R}^d)} \\ \geqslant \frac{1}{2}\nu^{\frac{2\gamma}{2\gamma+1}} \|u\|^2_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Next, we note that, using the properties of χ_0, given in (2.13), we find

$$\nu^{\frac{2\gamma}{2\gamma+1}} (1 - \chi_0(|x|^2\nu^{-\frac{2}{2\gamma+1}})) \leqslant CV_\lambda(x)^2,$$

according to the lower bound in Assumption (2.2). Using the last two inequalities together with (2.20) gives

$$C\|u\|^2_{L^2(\mathbb{R}^d)} + C\|Q^\lambda_\nu u\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)} \geqslant \nu^{\frac{2\gamma}{2\gamma+1}}\|u\|^2_{L^2(\mathbb{R}^d)}, \tag{2.21}$$

which concludes the proof of the theorem, dividing by $\|u\|_{L^2(\mathbb{R}^d)}$ and taking $\nu \geqslant \nu_0$ with ν_0 large enough.

3. Two lemmata

In this section, we state and prove two simple technical lemmata that will be used in the proofs of both Theorems 1.7 and 1.8.

3.1. Scaling argument

First, we prove the following lemma, which is a consequence of Theorem 2.1 together with a scaling argument. We define on $L^2(\mathbb{R}^d)$ (below, we shall take $d = n'$) the operator

$$\widetilde{P}_{\lambda,\omega} = -\Delta - \omega + i\lambda W(x). \tag{3.1}$$

Lemma 3.1. — *Let $\gamma > 0$ be given. Assume that there exist $C_1 \geqslant 1$ and $\gamma > 0$ such that*

$$C_1^{-1}|x|^{2\gamma} \leqslant W(x) \leqslant C_1|x|^{2\gamma}, \quad x \in \mathbb{R}^d. \tag{3.2}$$

Then, there exists $C > 0$ such that for all $u \in \mathscr{C}^2_c(\mathbb{R}^d)$, for all $\lambda > 0$ and for all $\omega \in \mathbb{R}$, we have

$$C\|\widetilde{P}_{\lambda,\omega}u\|_{L^2(\mathbb{R}^d)} \geqslant \lambda^{\frac{1}{\gamma+1}}\left(1 + H(\omega)\Big(\frac{\omega}{\lambda^{\frac{1}{\gamma+1}}}\Big)^{\frac{\gamma}{2\gamma+1}}\right)\|u\|_{L^2(\mathbb{R}^d)},$$

where $H = \mathbf{1}_{\mathbb{R}_+}$ is the Heaviside function.

Remark 3.2. — Note that this lemma does not use either λ large, or $0 \leqslant \omega \leqslant \lambda^2$.

Proof of Lemma 3.1. — First, we remark that for all $\alpha > 0$, the operator

$$\begin{array}{rcl} T_\alpha : L^2(\mathbb{R}^d) & \to & L^2(\mathbb{R}^d) \\ u(x) & \mapsto & \alpha^{\frac{d}{2}}u(\alpha x) \end{array}$$

is an isometry, with inverse $(T_\alpha)^{-1} = T_{\alpha^{-1}}$. As a consequence, we have

$$T_\alpha \tilde{P}_{\lambda,\omega}(T_\alpha)^{-1} = -\alpha^{-2}\Delta - \omega + i\lambda W(\alpha x),$$

where, according to Assumption (3.2), we have

$$C_1^{-1}\lambda\alpha^{2\gamma}|x|^{2\gamma} \leqslant \lambda W(\alpha x) \leqslant C_1\lambda\alpha^{2\gamma}|x|^{2\gamma}, \quad x \in \mathbb{R}^d.$$

Now, we choose $\alpha = \lambda^{-\frac{1}{2(\gamma+1)}}$, so that we have $\alpha^{-2} = \alpha^{2\gamma}\lambda = \lambda^{\frac{1}{\gamma+1}}$. Setting

$$W_\lambda(x) := \lambda^{-\frac{1}{\gamma+1}}\lambda W(\alpha x),$$

we obtain the uniform estimates

$$C_1^{-1}|x|^{2\gamma} \leqslant W_\lambda(x) \leqslant C_1|x|^{2\gamma}, \quad x \in \mathbb{R}^d, \lambda > 0. \tag{3.3}$$

With Q_0^λ defined in (2.1), this now yields

$$T_\alpha \tilde{P}_{\lambda,\omega}(T_\alpha)^{-1} = \lambda^{\frac{1}{\gamma+1}}\left(Q_0^\lambda - \omega\lambda^{-\frac{1}{\gamma+1}}\right), \qquad \alpha = \lambda^{-\frac{1}{2(\gamma+1)}}.$$

Since (3.3) implies Assumption (2.2), we can apply Theorem 2.1 (where $\mu = \omega\lambda^{\frac{1}{\gamma+1}}$). This yields (still with $\alpha = \lambda^{-\frac{1}{2(\gamma+1)}}$),

$$\begin{aligned}
\|\widetilde{P}_{\lambda,\omega}u\|_{L^2(\mathbb{R}^d)} &= \|T_\alpha \widetilde{P}_{\lambda,\omega}(T_\alpha)^{-1}T_\alpha u\|_{L^2(\mathbb{R}^d)} \\
&= \lambda^{\frac{1}{\gamma+1}}\left\|\left(Q_0^\lambda - \omega\lambda^{-\frac{1}{\gamma+1}}\right)T_\alpha u\right\|_{L^2(\mathbb{R}^d)} \\
&\geqslant C_0^{-1}\lambda^{\frac{1}{\gamma+1}}\left(1 + H(\omega)\left(\frac{\omega}{\lambda^{\frac{1}{\gamma+1}}}\right)^{\frac{\gamma}{2\gamma+1}}\right)\|T_\alpha u\|_{L^2(\mathbb{R}^d)},
\end{aligned}$$

concluding the proof of the lemma since T_α is an isometry. □

3.2. An elementary lemma

Let $\gamma > 0, c_0 > 0$ be given. We define for $\lambda > 0, \omega \geqslant 0$,

$$f(\lambda,\omega) = 1 + \left(\frac{\omega}{\lambda^{\frac{1}{\gamma+1}}}\right)^{\frac{2\gamma}{2\gamma+1}} - c_0\ \omega\lambda^{-\frac{2}{\gamma+1}}. \tag{3.4}$$

Lemma 3.3. — *$\forall c_0 > 0, \forall\lambda > 0, \forall\omega \in [0, c_0^{-(2\gamma+1)}\lambda^2]$, we have $f(\lambda,\omega) \geqslant 1$.*

Proof. — For $\omega \geqslant 0, \lambda > 0$, the inequality

$$c_0\ \omega\lambda^{-\frac{2}{\gamma+1}} \leqslant \left(\frac{\omega}{\lambda^{\frac{1}{\gamma+1}}}\right)^{\frac{2\gamma}{2\gamma+1}}$$

is equivalent to

$$c_0\omega^{\frac{1}{2\gamma+1}} \leqslant \lambda^{\frac{2}{\gamma+1} - \frac{2\gamma}{(2\gamma+1)(\gamma+1)}} = \lambda^{\frac{2}{2\gamma+1}}$$

i.e. to

$$c_0^{2\gamma+1}\omega \leqslant \lambda^2.$$ □

4. Proof of Theorem 1.7: the invariant case

4.1. Reduction of Theorem 1.7 to a $(n-1)$ dimensional problem

After a Fourier transform in the x_n variable, Theorem 1.7 reduces to the following result.

Theorem 4.1. — *Assume* (1.11), (1.13) *and define the operator acting on* $L^2(M')$

$$P_{\lambda,\omega} = -\Delta_{M'} - \omega + i\lambda b, \quad D(P_{\lambda,\omega}) = H^2(M'). \tag{4.1}$$

Then, there exist $C > 0$ and $\lambda_0 > 0$ such that for all $u \in H^2(M')$, for all $\lambda \geqslant \lambda_0$ and for all $\omega \leqslant \lambda^2$, we have

$$\|P_{\lambda,\omega} u\|_{L^2(M')} \geqslant C\lambda^{\frac{1}{\gamma+1}} \|u\|_{L^2(M')}. \tag{4.2}$$

In this section, we only prove that Theorem 4.1 implies Theorem 1.7. The proof of Theorem 4.1 needs more work and is completed in Section 4.2.

Proof that Theorem 4.1 ⇒ Theorem 1.7. — We perform a Fourier transform in the variable $x_n \in \mathbb{T}^1$:

$$u(x', x_n) = \sum_{k\in\mathbb{Z}} \hat{u}_k(x') e^{ikx_n}, \quad \text{with} \quad \hat{u}_k(x') = \frac{1}{2\pi}\int_0^{2\pi} u(x', x_n) e^{-ikx_n} dx_n.$$

Then, for $u \in H^2(M)$, we have with P_λ defined in (1.4) and $P_{\lambda,\omega}$ in (4.1),

$$\begin{aligned}(P_\lambda u)(x', x_n) &= \sum_{k\in\mathbb{Z}} \left((-\Delta_{M'} + k^2 - \lambda^2 + i\lambda b)\hat{u}_k\right)(x') e^{ikx_n} \\ &= \sum_{k\in\mathbb{Z}} \left(P_{\lambda,\lambda^2-k^2}\hat{u}_k\right)(x') e^{ikx_n},\end{aligned}$$

as $b = b(x')$ does not depend on the x_n-variable. We hence obtain

$$\|P_\lambda u\|^2_{L^2(M)} = 2\pi \sum_{k\in\mathbb{Z}} \|P_{\lambda,\lambda^2-k^2}\hat{u}_k\|^2_{L^2(M')}.$$

Finally, as a consequence of Theorem 4.1, we have $\|P_{\lambda,\lambda^2-k^2} w\|_{L^2(M')} \geqslant C\lambda^{\frac{1}{\gamma+1}}\|w\|_{L^2(M')}$ where $C > 0$ does not depend on k. This yields

$$\|P_\lambda u\|^2_{L^2(M)} \geqslant 2\pi C^2 \sum_{k\in\mathbb{Z}} \lambda^{\frac{2}{\gamma+1}} \|\hat{u}_k\|^2_{L^2(M')} = C^2\lambda^{\frac{2}{\gamma+1}}\|u\|^2_{L^2(M)},$$

which proves Theorem 1.7. □

4.2. Proof of Theorem 4.1

We now want to use Lemma 3.1 in a neighbourhood of $\{y'\} \times \mathbb{T}^1$ and to patch estimates together to complete the proof of Theorem 4.1.

Proof of Theorem 4.1. — Let $\chi_0 \in \mathscr{C}_c^\infty(B(y',\varepsilon_0);[0,1])$ such that $\chi_0 = 1$ in a neighbourhood of y' and set $\chi_1 = 1-\chi_0 \in \mathscr{C}^\infty(M')$. On the one hand, we have, with $P_{\lambda,\omega}$ given by (4.1),

$$C\|P_{\lambda,\omega}\chi_1 u\|_{L^2(M')} \geqslant \lambda\|\chi_1 u\|_{L^2(M')}, \tag{4.3}$$

since, according to Assumption (1.13), b is bounded from below on $\mathrm{supp}(\chi_1)$ and hence

$$\begin{aligned}\lambda\|\chi_1 u\|^2_{L^2(M')} &\leqslant C\lambda\langle b\chi_1 u,\chi_1 u\rangle_{L^2(M')} = C\,\mathrm{Im}\langle P_{\lambda,\omega}\chi_1 u,\chi_1 u\rangle_{L^2(M')}\\ &\leqslant C\|P_{\lambda,\omega}\chi_1 u\|_{L^2(M')}\|\chi_1 u\|_{L^2(M')}.\end{aligned}$$

On the other hand, we have

$$\|P_{\lambda,\omega}\chi_0 u\|^2_{L^2(M')} = \|P_{\lambda,\omega}\chi_0 u\|^2_{L^2(B(y',\varepsilon_0))}.$$

We write $x' = (x_1,\cdots x_{n-1})$. According to Assumption (1.12), we can extend (for instance by homogeneity in the variable $x'-y'$ outside $B_{\mathbb{R}^{n-1}}(y',\varepsilon_0)$) the function b from $B_{\mathbb{R}^{n-1}}(y',\varepsilon_0)$ to the whole $\mathbb{R}^{n-1}$ as a measurable function W satisfying

$$C_1^{-1}|x'-y'|^{2\gamma} \leqslant W(x') \leqslant C_1|x'-y'|^{2\gamma},\ \ x'\in\mathbb{R}^{n-1},\ \text{and } b = W \text{ on } B(y',\varepsilon_0).$$

Hence, we have

$$P_{\lambda,\omega}\chi_0 = \tilde{P}_{\lambda,\omega}\chi_0,$$

where $\tilde{P}_{\lambda,\omega}$ is given by (3.1). We may then apply Lemma 3.1 to the operator $\tilde{P}_{\lambda,\omega}$ in $\mathbb{R}^{n-1}$. This yields, for some $C>0$,

$$\begin{aligned}C\|P_{\lambda,\omega}\chi_0 u\|^2_{L^2(M')} &= C\|\tilde{P}_{\lambda,\omega}\chi_0 u\|^2_{L^2(\mathbb{R}^{n-1})}\\ &\geqslant \lambda^{\frac{2}{\gamma+1}}\left(1+H(\omega)\left(\frac{\omega}{\lambda^{\frac{1}{\gamma+1}}}\right)^{\frac{2\gamma}{2\gamma+1}}\right)\|\chi_0 u\|^2_{L^2(\mathbb{R}^{n-1})}\\ &= \lambda^{\frac{2}{\gamma+1}}\left(1+H(\omega)\left(\frac{\omega}{\lambda^{\frac{1}{\gamma+1}}}\right)^{\frac{2\gamma}{2\gamma+1}}\right)\|\chi_0 u\|^2_{L^2(M')}.\end{aligned} \tag{4.4}$$

We now want to estimate the remainder term

$$\begin{aligned}\|[P_{\lambda,\omega},\chi_1]u\|_{L^2(M')} &= \|[P_{\lambda,\omega},\chi_0]u\|_{L^2(M')} = \|[-\Delta_{M'},\chi_0]u\|_{L^2(M')}\\ &\leqslant \|(\Delta_{M'}\chi_0)u\|_{L^2(M')} + 2\|\nabla_{x'}\chi_0\cdot\nabla_{x'}u\|_{L^2(M')}.\end{aligned} \tag{4.5}$$

For this, we take $\psi = \psi(x') \in \mathscr{C}_c^\infty(B(0,\varepsilon_0);[0,1])$ such that $\psi = 1$ on $\operatorname{supp}(\nabla_{x'}\chi_0)$ and $\psi = 0$ in a neighbourhood of 0. We compute

$$\begin{aligned}
&\operatorname{Re}\langle P_{\lambda,\omega}u, \psi^2 u\rangle_{L^2(M')} \\
&\qquad = \operatorname{Re}\langle(-\Delta_{M'} - \omega)u, \psi^2 u\rangle_{L^2(M')} + \overbrace{\operatorname{Re}\langle i\lambda\psi^2 bu, u\rangle_{L^2(M')}}^{=0} \\
&\qquad = \operatorname{Re}\langle -\Delta_{M'}u, \psi^2 u\rangle_{L^2(M')} - \omega\|\psi u\|^2_{L^2(M')}. \qquad (4.6)
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\operatorname{Re}\langle -\Delta_{\mathbb{R}^{n-1}}u, \psi^2 u\rangle_{L^2(M')} \\
&\qquad = \langle \nabla_{x'}u, \psi^2\nabla_{x'}u\rangle_{L^2(M')} + \operatorname{Re}\langle \nabla_{x'}u, u\nabla_{x'}\psi^2\rangle_{L^2(M')} \\
&\qquad = \langle \nabla_{x'}u, \psi^2\nabla_{x'}u\rangle_{L^2(M')} + \sum_{j=1}^{n-1}\operatorname{Re}\langle iD_{x_j}u, u\partial_{x_j}\psi^2\rangle_{L^2(M')} \\
&\qquad = \langle \nabla_{x'}u, \psi^2\nabla_{x'}u\rangle_{L^2(M')} - \sum_{j=1}^{n-1}\frac{i}{2}\langle [D_{x_j}, \partial_{x_j}\psi^2]u, u\rangle_{L^2(M')} \\
&\qquad = \langle \nabla_{x'}u, \psi^2\nabla_{x'}u\rangle_{L^2(M')} - \frac{1}{2}\langle (\Delta_{\mathbb{R}^{n-1}}\psi^2)u, u\rangle_{L^2(M')},
\end{aligned}$$

since the two operators D_{x_j} and $\partial_{x_j}\psi^2$ are selfadjoint. We have thus

$$\begin{aligned}
\|\psi\nabla_{x'}u\|^2_{L^2(M')} &= \operatorname{Re}\big\langle\big(-\Delta_{\mathbb{R}^{n-1}} - \omega + i\lambda b\big)u, \psi^2 u\big\rangle_{L^2(M')} \\
&\quad + \overbrace{\operatorname{Re}\big\langle\big(\omega - i\lambda b\big)u, \psi^2 u\big\rangle_{L^2(M')}}^{=\omega\|\psi u\|^2_{L^2(M')}} \\
&\quad + \frac{1}{2}\langle(\Delta_{\mathbb{R}^{n-1}}\psi^2)u, u\rangle_{L^2(M')},
\end{aligned}$$

and consequently we obtain

$$\|\psi\nabla_{x'}u\|^2_{L^2(M')} \leqslant \|P_{\lambda,\omega}u\|_{L^2(M')}\|u\|_{L^2(M')} + C_1\|u\|^2_{L^2(M')} + \omega\|\psi u\|^2_{L^2(M')}.$$

As a result, we can estimate the commutator of (4.5) by

$$\begin{aligned}
&\|[P_{\lambda,\omega}, \chi_0]u\|^2_{L^2(M')} \\
&\qquad \leqslant C_2\Big(\|P_{\lambda,\omega}u\|_{L^2(M')}\|u\|_{L^2(M')} + \|u\|^2_{L^2(M')} + \omega\|\psi u\|^2_{L^2(M')}\Big). \qquad (4.7)
\end{aligned}$$

Now, we have

$$\|P_{\lambda,\omega}(\chi_j u)\|^2_{L^2(M')} \leqslant 2\|[P_{\lambda,\omega}, \chi_j]u\|^2_{L^2(M')} + 2\|\chi_j P_{\lambda,\omega}u\|^2_{L^2(M')},$$

so that

$$\begin{aligned}
&2\|P_{\lambda,\omega}u\|^2_{L^2(M')} \\
&\qquad \geqslant \|P_{\lambda,\omega}\chi_0 u\|^2_{L^2(M')} + \|P_\lambda\chi_1 u\|^2_{L^2(M')} - 4\|[P_{\lambda,\omega}, \chi_0]u\|^2_{L^2(M')}, \qquad (4.8)
\end{aligned}$$

which, combined with the estimates (4.3), (4.4) and (4.7), yields

$$
\begin{aligned}
C_3\Big(\|P_{\lambda,\omega}u\|^2_{L^2(M')} + \|u\|^2_{L^2(M')}\Big) \\
\geqslant \lambda^{\frac{2}{\gamma+1}} \left(1 + H(\omega)\left(\frac{\omega}{\lambda^{\frac{1}{\gamma+1}}}\right)^{\frac{2\gamma}{2\gamma+1}}\right) \|\chi_0 u\|^2_{L^2(M')} \\
+ \lambda^2 \|\chi_1 u\|^2_{L^2(M')} - c_1\omega\|\psi u\|^2_{L^2(M')}, \quad (4.9)
\end{aligned}
$$

where c_1 is a fixed positive constant. In the régime $\omega \leqslant 0$ (or, more generally, $\omega \leqslant \omega_0$ for any given ω_0), this suffices to prove (4.2).

Let us now study the régime $\omega \geqslant 0$. We notice that, for $\omega \leqslant \lambda^2$, $\lambda \geqslant 1$,

$$
\begin{aligned}
\lambda^{\frac{2}{\gamma+1}}\left(1 + \left(\frac{\omega}{\lambda^{\frac{1}{\gamma+1}}}\right)^{\frac{2\gamma}{2\gamma+1}}\right) &= \lambda^{\frac{2}{\gamma+1}} + \lambda^{\frac{2}{\gamma+1} - \frac{2\gamma}{(2\gamma+1)(\gamma+1)}} \omega^{\frac{2\gamma}{2\gamma+1}} \\
&\leqslant \lambda^{\frac{2}{\gamma+1}} + \lambda^{\frac{2}{\gamma+1} - \frac{2\gamma}{(2\gamma+1)(\gamma+1)} + \frac{4\gamma}{2\gamma+1}} \\
&= \lambda^{\frac{2}{\gamma+1}} + \lambda^2 \leqslant 2\lambda^2,
\end{aligned}
$$

and that, for all $v \in L^2(M')$ we have

$$
\begin{aligned}
\|\psi v\|^2_{L^2(M')} \leqslant C_4\|v\|^2_{L^2(M')} &= C_4\|(\chi_0+\chi_1)v\|^2_{L^2(M')} \\
&\leqslant 2C_4\|\chi_0 v\|^2_{L^2(M')} + 2C_4\|\chi_1 v\|^2_{L^2(M')}.
\end{aligned}
$$

This, together with (4.9) then yields

$$
\begin{aligned}
C_5\Big(\|P_{\lambda,\omega}u\|^2_{L^2(M')} + \|u\|^2_{L^2(M')}\Big) &\geqslant \lambda^{\frac{2}{\gamma+1}} f(\lambda,\omega)\|(\chi_0+\chi_1)u\|^2_{L^2(M')} \\
&= \lambda^{\frac{2}{\gamma+1}} f(\lambda,\omega)\|u\|^2_{L^2(M')},
\end{aligned}
$$

where $f(\lambda,\omega)$ is defined in (3.4) with a fixed positive constant c_0. According to Lemma 3.3, there exists $\lambda_0 > 0$ and

$$
\delta = c_0^{-(2\gamma+1)} > 0, \tag{4.10}
$$

such that for all $\lambda \geqslant \lambda_0$ and $\omega \in [0, \delta\lambda^2]$, we have $f(\lambda,\omega) \geqslant 1$. As a consequence, (4.2) is satisfied in this régime. Finally, suppose that $\delta\lambda^2 \leqslant \omega \leqslant \lambda^2$, where δ is given by (4.10). In this régime, the estimate (4.2) is a direct consequence of the usual (stronger) 1-microlocal estimate (see Lemma 4.2 below). □

Lemma 4.2. — *Let $\delta > 0$, $y' \in M'$ and suppose that $b(y') = 0$ and $b > 0$ on $M' \setminus \{y'\}$. Then, there exists $\lambda_0 > 0$ and $C > 0$ such that for all $\lambda \geqslant \lambda_0$, for all $\omega \in [\delta\lambda^2, \lambda^2]$, we have*

$$
\|P_{\lambda,\omega}u\|_{L^2(M')} \gtrsim \lambda\|u\|_{L^2(M')}.
$$

This lemma states the classical estimate associated to a "one-microlocal" propagation result in the presence of geometric control. We do not provide a proof here since it is simpler than the proof of Lemma 5.1 below and would follow exactly the same lines. The only additional difficulty with respect to the proof of Lemma 5.1 is that the constants are uniform with respect to the parameter $\omega \in [\delta\lambda^2, \lambda^2]$ (whereas Lemma 5.1 only tackles the case $\omega = \lambda^2$). It only requires a simple change of definition of the compact K in the geometric definitions in the first part of the proof of Lemma 5.1.

5. Proof of Theorem 1.8: the non-invariant case

5.1. Proof of a geometric control lemma

In this section, we prove the following lemma. All definitions and tools of geometry and pseudodifferential calculus used in the proof are introduced in Appendices A and B respectively.

Lemma 5.1. — *Assume that $b \in L^\infty(M;\mathbb{R}^+)$ and recall that ω_b is defined in* (1.3)*. Take a non-negative function $\alpha \in S^0_{0,0}(T^*M)$. Assume that for all $\rho \in \operatorname{supp}(\alpha) \cap S^*M$ there exists $t \in \mathbb{R}$ such that $\phi_t(\rho) \in T^*\omega_b$. Then, there exist $C, \lambda_0 > 0$ such that for all $\lambda \geqslant \lambda_0$, we have*

$$\lambda \| \operatorname{Op}\big(\alpha(x, \frac{\xi}{\lambda})\big) u\|_{L^2(M)} \leqslant C \|P_\lambda \operatorname{Op}\big(\alpha(x, \frac{\xi}{\lambda})\big) u\|_{L^2(M)} + C\|u\|_{L^2(M)}.$$

This Lemma states a "one-microlocal" estimate in the presence of a partial geometric control, which is adapted to our needs. We give a proof here to check that no smoothness is required on b. Moreover, the proof below uses multiplier estimates and is hence of constructive type. Note that if ω_b satisfies (GCC), then the assumption of the lemma is satisfied by $\alpha = 1$ and the lemma yields the optimal estimate

$$\lambda \|u\|_{L^2(M)} \leqslant C \|P_\lambda u\|_{L^2(M)}.$$

This estimate is equivalent to the uniform (and hence exponential) decay of the associated problem (1.1) (see for instance [1] and the references therein).

Proof. — The proof is divided in several steps.

Some geometric facts. — Set $K := \operatorname{supp}(\alpha) \cap S^*M \subset T^*M$ and, for $\rho \in K$ denote $t_\rho \in \mathbb{R}$ a time such that $\phi_{t_\rho}(\rho) \in T^*\omega_b$. The compact set K is hence such that for all $\rho \in K$, $\rho \in \phi_{-t_\rho}(T^*\omega_b)$, i.e.

$$K \subset \bigcup_{t\in\mathbb{R}} \phi_{-t}(T^*\omega_b).$$

Moreover, for $\ell \in \mathbb{N}^*$, we let ω^ℓ be a family of open sets such that $\overline{\omega}^\ell \subset \omega_b$, $\overline{\omega}^\ell \subset \omega^{\ell+1}$, and $\bigcup_{\ell\in\mathbb{N}^*} \omega^\ell = \omega_b$. For any $t \in \mathbb{R}$, we have $\phi_{-t}(T^*\omega_b) = \bigcup_{\ell\in\mathbb{N}^*} \phi_{-t}(T^*\omega^\ell)$, where $\phi_{-t}(T^*\omega^\ell)$ are open subsets of T^*M. This reads

$$K \subset \bigcup_{t\in\mathbb{R}} \bigcup_{\ell\in\mathbb{N}^*} \phi_{-t}(T^*\omega^\ell).$$

Since K is compact, we may hence extract a finite open cover of K, that is a finite number of times t_j, $j \in \{1, \cdots, J\}$ and of ω^ℓ, $\ell \in \{1, \cdots, L\}$ such that

$$K \subset \bigcup_{j=1}^{J} \bigcup_{\ell=1}^{L} \phi_{-t_j}(T^*\omega^\ell).$$

We then remark that $\omega^\ell \subset \omega^{\ell+1}$, so that, for all $t \in \mathbb{R}$, $\bigcup_{\ell=1}^{L} \phi_{-t}(T^*\omega^\ell) = \phi_{-t}(T^*\omega^L)$. This finally yields

$$K \subset \bigcup_{j=1}^{J} \phi_{-t_j}(T^*\omega^L), \qquad \text{with } \omega^L \text{ open such that } \overline{\omega}^L \subset \omega_b.$$

As $\bigcup_{j=1}^{J} \phi_{-t_j}(T^*\omega^L)$ is open, we also have

$$K_\gamma := \operatorname{supp}(\alpha) \cap \{(x,\xi) \in T^*M, 1-\gamma \leqslant |\xi|_x \leqslant 1+\gamma\} \subset \bigcup_{j=1}^{J} \phi_{-t_j}(T^*\omega^L)$$

for $\gamma \in (0,1)$ sufficiently small (fixed from now on).

Note that, at this point, we have in particular proved that the assumption of the Lemma, satisfied by the set ω_b in arbitrarily large time (depending on the point ρ), is actually satisfied by the smaller set ω^L in a uniform time (namely $\max_{1\leqslant i,j\leqslant J} |t_j - t_i|$).

Since $\overline{\omega}^L$ is a compact subset of M such that

$$\overline{\omega}^L \subset \omega_b = \bigcup \{U \subset M, U \text{ open}, \operatorname{essinf}_U(b) > 0\},$$

we can extract a finite cover of $\overline{\omega}^L$:

$$\overline{\omega}^L \subset \bigcup_{k=1}^{\kappa} \{U_k \subset M, U_k \text{ open}, \operatorname{essinf}_{U_k}(b) > 0\}.$$

Hence, we have

$$b \geqslant \min\{\operatorname{essinf}_{U_k}(b), 1 \leqslant k \leqslant \kappa\} > 0, \quad \text{a.e. on } \overline{\omega}^L.$$

Now we define $\tilde{b} = \varepsilon\chi$ with $\chi \in \mathscr{C}_c^\infty(\omega_b; [0,1])$ such that $\chi = 1$ on ω^L and $\varepsilon = \min\{\operatorname{essinf}_{U_k}(b), 1 \leqslant k \leqslant \kappa\}$ so that we have

$$\tilde{b} \in \mathscr{C}^\infty(M), \quad \omega^L \subset \{\tilde{b} > 0\}, \quad 0 \leqslant \tilde{b} \leqslant b.$$

This yields in particular

$$\lambda\langle\tilde{b}u,u\rangle_{L^2(M)} \lesssim \lambda\langle bu,u\rangle_{L^2(M)} = \operatorname{Im}\langle P_\lambda u,u\rangle_{L^2(M)} \\ \leqslant \|P_\lambda u\|_{L^2(M)}\|u\|_{L^2(M)}. \quad (5.1)$$

Definition of the multipliers. — Denoting by $O_j = \phi_{-t_j}(T^*\omega^L)$, the sets $(O_j)_{j\in\{1,\cdots,J\}}$ form a finite open cover of K_γ such that $\phi_{t_j}(O_j) \subset T^*\omega^L$. We denote $(\chi_j)_{j\in\{1,\cdots,J\}}$ a partition of unity of K_γ subordinated to $(O_j)_{j\in\{1,\cdots,J\}}$. We set

$$\psi_j(x,\xi) = -\int_0^{t_j} \chi_j \circ \phi_\tau(x,\xi)d\tau.$$

Taking $\chi \in \mathscr{C}_c^\infty(\mathbb{R};[0,1])$ such that $\chi = 1$ a neighbourhood of 1 and $\operatorname{supp}\chi \subset (1-\gamma,1+\gamma)$, we now define the multipliers

$$m_j(x,\frac{\xi}{\lambda}) = \chi(\frac{|\xi|_x^2}{\lambda^2})\psi_j(x,\frac{\xi}{\lambda}), \quad j\in\{1,\cdots,J\}.$$

To m_j, we associate an operator $\operatorname{Op}(m_j)$, bounded on $L^2(M)$ (see Appendix B).

Estimate in the propagative region. — We have, with P_λ defined in (1.4),

$$\begin{aligned} &2\operatorname{Re}\langle P_\lambda u, i\operatorname{Op}(m_j)u\rangle_{L^2(M)} \\ &\quad= \left\langle (\left[(-\Delta-\lambda^2), i\operatorname{Op}(m_j)\right] u,u\right\rangle_{L^2(M)} + 2\operatorname{Re}\left\langle \lambda bu, \operatorname{Op}(m_j)u\right\rangle_{L^2(M)} \\ &\quad= \operatorname{Re}\left\langle \operatorname{Op}\left(\left\{|\xi|_x^2-\lambda^2, m_j\right\}\right)u,u\right\rangle_{L^2(M)} + O(1)\|u\|_{L^2(M)}^2 \\ &\qquad + 2\lambda\operatorname{Re}\left\langle bu, \operatorname{Op}(m_j)u\right\rangle_{L^2(M)}, \end{aligned} \quad (5.2)$$

according to symbolic calculus. Moreover, we have

$$\begin{aligned} \left\{|\xi|_x^2-\lambda^2, m_j\right\} = H_p m_j &= \chi(\frac{|\xi|_x^2}{\lambda^2}) H_p\left(\psi_j(x,\frac{\xi}{\lambda})\right) \\ &= -\chi(\frac{|\xi|_x^2}{\lambda^2})\int_0^{t_j} H_p\left(\chi_j\circ\phi_\tau(x,\frac{\xi}{\lambda})\right)d\tau \\ &= -\chi(\frac{|\xi|_x^2}{\lambda^2})\int_0^{t_j} \lambda\frac{d}{d\tau}\left(\chi_j\circ\phi_\tau(x,\frac{\xi}{\lambda})\right)d\tau \\ &= \lambda\chi(\frac{|\xi|_x^2}{\lambda^2})\left(\chi_j(x,\frac{\xi}{\lambda}) - \chi_j\circ\phi_{t_j}(x,\frac{\xi}{\lambda})\right), \end{aligned}$$

as $\dot{\phi}_\tau = H_p(\phi_\tau)$. Coming back to (5.2), and using the boundedness of $\mathrm{Op}(m_j)$, we obtain

$$\begin{aligned}
&\|P_\lambda u\|_{L^2(M)}\|u\|_{L^2(M)} + \|u\|^2_{L^2(M)} + \lambda\|bu\|_{L^2(M)}\|u\|_{L^2(M)} \\
&\qquad\gtrsim \lambda \left\langle \mathrm{Op}\big(\chi(\frac{|\xi|_x^2}{\lambda^2})\chi_j(x,\frac{\xi}{\lambda})\big)u, u\right\rangle_{L^2(M)} \\
&\qquad\quad - \lambda \left\langle \mathrm{Op}\left(\chi(\frac{|\xi|_x^2}{\lambda^2})\chi_j\circ\phi_{t_j}(x,\frac{\xi}{\lambda})\right) u, u\right\rangle_{L^2(M)}. \qquad (5.3)
\end{aligned}$$

Next, by construction, the function $(x,\xi)\mapsto \chi(|\xi|_x^2)\chi_j\circ\phi_{t_j}(x,\xi)$ is supported in $T^*\omega^L$ where $\tilde{b}>0$, so that we have

$$\chi(\frac{|\xi|_x^2}{\lambda^2})\chi_j\circ\phi_{t_j}(x,\frac{\xi}{\lambda}) \lesssim \tilde{b}(x)$$

uniformly on T^*M. According to the sharp Gårding inequality, this yields

$$\begin{aligned}
&\lambda \left\langle \mathrm{Op}\left(\chi(\frac{|\xi|_x^2}{\lambda^2})\chi_j\circ\phi_{t_j}(x,\frac{\xi}{\lambda})\right) u, u\right\rangle_{L^2(M)} \\
&\qquad\qquad \lesssim \lambda\left\langle \tilde{b}u, u\right\rangle_{L^2(M)} + \|u\|^2_{L^2(M)} \\
&\qquad\qquad \lesssim \|P_\lambda u\|_{L^2(M)}\|u\|_{L^2(M)} + \|u\|^2_{L^2(M)},
\end{aligned}$$

when using (5.1). Combined with (5.3) and

$$\lambda\|bu\|^2_{L^2(M)} \lesssim \lambda\left\langle bu,u\right\rangle_{L^2(M)} \leqslant \|P_\lambda u\|_{L^2(M)}\|u\|_{L^2(M)},$$

this implies, for all $j\in\{1,\cdots,J\}$,

$$\begin{aligned}
&\|P_\lambda u\|_{L^2(M)}\|u\|_{L^2(M)} + \|u\|^2_{L^2(M)} + \lambda^{\frac12}\|P_\lambda u\|^{\frac12}_{L^2(M)}\|u\|^{\frac32}_{L^2(M)} \\
&\qquad\qquad \gtrsim \lambda \left\langle \mathrm{Op}\left(\chi(\frac{|\xi|_x^2}{\lambda^2})\chi_j(x,\frac{\xi}{\lambda})\right) u, u\right\rangle_{L^2(M)}. \qquad (5.4)
\end{aligned}$$

Estimate in the elliptic region. — Next, we estimate

$$\mathrm{Op}(1-\chi(\frac{|\xi|_x^2}{\lambda^2}))u.$$

Take $\theta\in\mathscr{C}^\infty(\mathbb{R}^+;[-1,1])$ such that $\theta=0$ in a neighbourhood of 1, $\theta=-1$ in a neighbourhood of 0 and $\theta=1$ ouside of a neighbourhood of

$[0,1]$, so that $|\theta| = 1-\chi$. We compute

$$\begin{aligned}\operatorname{Re}\left\langle P_\lambda u, \operatorname{Op}\left(\theta(\frac{|\xi|_x^2}{\lambda^2})\right) u\right\rangle_{L^2(M)} \\ = \operatorname{Re}\left\langle (-\Delta_g-\lambda^2)u, \operatorname{Op}\left(\theta(\frac{|\xi|_x^2}{\lambda^2})\right) u\right\rangle_{L^2(M)} \\ + \operatorname{Re}\left\langle i\lambda b u, \operatorname{Op}\left(\theta(\frac{|\xi|_x^2}{\lambda^2})\right) u\right\rangle_{L^2(M)}. \end{aligned} \tag{5.5}$$

Then, using symbolic calculus, we have

$$\begin{aligned}&\left(\operatorname{Op}\left(\theta(\frac{|\xi|_x^2}{\lambda^2})\right)\right)^*(-\Delta_g-\lambda^2) \\ &= \operatorname{Op}\big(\theta(\frac{|\xi|_x^2}{\lambda^2})(|\xi|_x^2-\lambda^2)\big) + O_{\mathcal{L}(L^2(M))}(\lambda) + O_{\mathcal{L}(H^1(M);L^2(M))}(1) \\ &= \operatorname{Op}\big(|\theta(\frac{|\xi|_x^2}{\lambda^2})|(|\xi|_x^2+\lambda^2)\big) + O_{\mathcal{L}(L^2(M))}(\lambda) + O_{\mathcal{L}(H^1(M);L^2(M))}(1),\end{aligned}$$

so that sharp Gårding inequality yields

$$\begin{aligned}&\operatorname{Re}\left\langle (-\Delta_g-\lambda^2)u, \operatorname{Op}\left(\theta(\frac{|\xi|_x^2}{\lambda^2})\right) u\right\rangle_{L^2(M)} \\ &\gtrsim \left\langle \operatorname{Op}\big(|\theta(\frac{|\xi|_x^2}{\lambda^2})|(|\xi|_x^2+\lambda^2)\big)u, u\right\rangle_{L^2(M)} - O(\lambda)\|u\|_{L^2(M)}^2 \\ &\quad - O(1)\|u\|^2_{H^{\frac12}(M)} - O(1)\|u\|_{H^1(M)}\|u\|_{L^2(M)} \\ &\gtrsim \lambda^2\left\langle \operatorname{Op}\big(1-\chi(\frac{|\xi|_x^2}{\lambda^2})\big)u, u\right\rangle_{L^2(M)} - O(\lambda)\|u\|_{L^2(M)}^2 \\ &\quad - O(1)\|u\|_{H^1(M)}\|u\|_{L^2(M)},\end{aligned} \tag{5.6}$$

since $\|u\|^2_{H^{\frac12}(M)} \leqslant \|u\|_{H^1(M)}\|u\|_{L^2(M)}$. To estimate $\|u\|_{H^1(M)}\|u\|_{L^2(M)}$, we simply write

$$\begin{aligned}\|P_\lambda u\|_{L^2(M)}\|u\|_{L^2(M)} \geqslant \operatorname{Re}\langle P_\lambda u, u\rangle_{L^2(M)} &= \big\langle (-\Delta_g-\lambda^2)u, u\big\rangle_{L^2(M)} \\ &= \|\nabla_g u\|^2_{L^2(M)} - \lambda^2\|u\|^2_{L^2(M)},\end{aligned}$$

so that

$$\|u\|_{H^1(M)}\|u\|_{L^2(M)} \lesssim \lambda\|u\|^2_{L^2(M)} + \|P_\lambda u\|^{\frac12}_{L^2(M)}\|u\|^{\frac32}_{L^2(M)}.$$

Coming back to (5.6), this yields

$$\begin{aligned}\left\langle(-\Delta_g-\lambda^2)u,\operatorname{Op}\left(\theta(\frac{|\xi|_x^2}{\lambda^2})\right)u\right\rangle_{L^2(M)}&\\ +\lambda\|u\|_{L^2(M)}^2+\|P_\lambda u\|_{L^2(M)}\|u\|_{L^2(M)}&\\ \gtrsim\lambda^2\left\langle\operatorname{Op}\left(1-\chi(\frac{|\xi|_x^2}{\lambda^2})\right)u,u\right\rangle_{L^2(M)}&.\end{aligned}$$

Moreover, as above, we have

$$\begin{aligned}\left|\operatorname{Re}\left\langle i\lambda bu,\operatorname{Op}\left(\theta(\frac{|\xi|_x^2}{\lambda^2})\right)u\right\rangle\right|&\lesssim\lambda\|bu\|_{L^2(M)}\|u\|_{L^2(M)}\\ &\lesssim\lambda^{\frac12}\|P_\lambda u\|_{L^2(M)}^{\frac12}\|u\|_{L^2(M)}^{\frac32}\\ &\lesssim\|P_\lambda u\|_{L^2(M)}\|u\|_{L^2(M)}+\lambda\|u\|_{L^2(M)}^2.\end{aligned}$$

Coming back to (5.5), this implies

$$\|P_\lambda u\|_{L^2(M)}\|u\|_{L^2(M)}+\lambda\|u\|_{L^2(M)}^2\gtrsim\lambda^2\left\langle\operatorname{Op}\left(1-\chi(\frac{|\xi|_x^2}{\lambda^2})\right)u,u\right\rangle_{L^2(M)}.$$

Dividing this estimate by λ, we finally obtain

$$\begin{aligned}\lambda^{-1}\|P_\lambda u\|_{L^2(M)}\|u\|_{L^2(M)}+\|u\|_{L^2(M)}^2&\\ \gtrsim\lambda\left\langle\operatorname{Op}\left(1-\chi(\frac{|\xi|_x^2}{\lambda^2})\right)u,u\right\rangle_{L^2(M)}&.\end{aligned}\tag{5.7}$$

Patching estimates together. — Finally, according to the construction of χ_j,χ and denoting

$$\Omega:=\left\{(x,\xi)\in T^*M,\quad\left(1-\chi(|\xi|_x^2)\right)+\chi(|\xi|_x^2)\sum_{j=1}^J\chi_j(x,\xi)>0\right\},$$

we have $\operatorname{supp}(\alpha)\subset\Omega$. Let $\beta\in\mathscr{C}^\infty(T^*M)$ be a function which is homogeneous of degree zero in each fiber for $|\xi|_x$ large, such that $\operatorname{supp}(\beta)\subset\Omega$ and $\beta=1$ on a neighbourhood of $\operatorname{supp}(\alpha)$. We have

$$\left(1-\chi(|\xi|_x^2)\right)+\chi(|\xi|_x^2)\sum_{j=1}^J\chi_j(x,\xi)\gtrsim\beta^2(x,\xi),$$

uniformly on T^*M. As a consequence, according to sharp Gårding inequality, we have

$$\lambda\Big\langle \operatorname{Op}\Big(\chi(\frac{|\xi|_x^2}{\lambda^2})\sum_{j=1}^{J}\chi_j(x,\frac{\xi}{\lambda}) + \big(1-\chi(\frac{|\xi|_x^2}{\lambda^2})\big)\Big)u, u\Big\rangle_{L^2(M)} \gtrsim \lambda\Big\|\operatorname{Op}\big(\beta(x,\frac{\xi}{\lambda})\big)u\Big\|^2_{L^2(M)} - C\|u\|^2_{L^2(M)}.$$

Using inequalities (5.4) and (5.7) yields

$$\lambda\Big\|\operatorname{Op}\big(\beta(x,\frac{\xi}{\lambda})\big)u\Big\|^2_{L^2(M)} \lesssim \|P_\lambda u\|_{L^2(M)}\|u\|_{L^2(M)} + \|u\|^2_{L^2(M)} + \lambda^{\frac12}\|P_\lambda u\|^{\frac12}_{L^2(M)}\|u\|^{\frac32}_{L^2(M)}.$$

Applying this estimate to u replaced by

$$Au := \operatorname{Op}\big(\alpha(x,\frac{\xi}{\lambda})\big)u$$

and using that

$$\operatorname{Op}\big(\beta(x,\frac{\xi}{\lambda})\big)\operatorname{Op}\big(\alpha(x,\frac{\xi}{\lambda})\big) = \operatorname{Op}\big(\alpha(x,\frac{\xi}{\lambda})\big) + O_{\mathcal{L}(L^2(M))}(\lambda^{-1}),$$

we obtain

$$\begin{aligned}\lambda\|Au\|^2_{L^2(M)} &\lesssim \|P_\lambda Au\|_{L^2(M)}\|Au\|_{L^2(M)} + \|Au\|^2_{L^2(M)}\\ &\quad + \lambda^{\frac12}\|P_\lambda Au\|^{\frac12}_{L^2(M)}\|Au\|^{\frac32}_{L^2(M)} + O(\lambda^{-1})\|u\|^2_{L^2(M)}\\ &\lesssim (1+\varepsilon^{-1})\|P_\lambda Au\|_{L^2(M)}\|Au\|_{L^2(M)}\\ &\quad + (1+\varepsilon\lambda)\|Au\|^2_{L^2(M)} + O(\lambda^{-1})\|u\|^2_{L^2(M)}\end{aligned}$$

for all $\varepsilon > 0$. Choosing ε sufficiently small, yields

$$\begin{aligned}\lambda\|Au\|^2_{L^2(M)} &\lesssim \|P_\lambda Au\|_{L^2(M)}\|Au\|_{L^2(M)} + \|Au\|^2_{L^2(M)} + O(\lambda^{-1})\|u\|^2_{L^2(M)}\\ &\lesssim (\varepsilon\lambda)^{-1}\|P_\lambda Au\|^2_{L^2(M)} + (1+\varepsilon\lambda)\|Au\|^2_{L^2(M)} + O(\lambda^{-1})\|u\|^2_{L^2(M)}\end{aligned}$$

for all $\varepsilon > 0$. Taking again ε sufficiently small, and then $\lambda \geqslant \lambda_0$ for λ_0 large enough provides the proof of the lemma. □

5.2. End of the proof of Theorem 1.8

Next, we want to patch estimates together to complete the proof of Theorem 1.8.

Proof of Theorem 1.8. — Let $U_{\varepsilon_0} = B(y',\varepsilon_0) \times \mathbb{T}^{n''} \subset M$ and $\chi_0 \in \mathscr{C}_c^\infty(B(y',\varepsilon_0);[0,1])$ such that $\chi_0 = 1$ in the neighbourhood of y'. We shall also write χ_0 instead of $\chi_0 \otimes 1 \in \mathscr{C}_c^\infty(U_{\varepsilon_0})$, and instead of $\chi_0 \otimes 1 \in \mathscr{C}^\infty(M)$ where this function has been extended by 0 in $M \setminus U_{\varepsilon_0}$. We denote by $\chi_1 = 1 - \chi_0 \in \mathscr{C}^\infty(M)$. On the one hand, as a consequence of Lemma 5.1 and Assumption (1.17), we have for $\lambda \geqslant \lambda_0$,

$$O(\lambda^{-1})\|u\|_{L^2(M)} + \|P_\lambda \chi_1 u\|_{L^2(M)} \gtrsim \lambda \|\chi_1 u\|_{L^2(M)}. \tag{5.8}$$

On the other hand, we have

$$\|P_\lambda \chi_0 u\|^2_{L^2(M)} = \|P_\lambda \chi_0 u\|^2_{L^2(U_{\varepsilon_0})}.$$

Notation. — In U_{ε_0}, we note the coordinates (x',x'') with $x' = (x'_1,\cdots, x'_{n'}) \in \mathbb{R}^{n'}$ and $x'' = (x''_1,\cdots,x''_{n''}) \in \mathbb{R}^{n''}$.

According to Assumption (1.16), we extend (for instance by homogeneity in the variable $x'-y'$ outside $B_{\mathbb{R}^{n'}}(y',\varepsilon_0)$) the function $b = b(x')$ from $B_{\mathbb{R}^{n'}}(y',\varepsilon_0)$ to the whole $\mathbb{R}^{n'}$ as a measurable function W satisfying

$$C_1^{-1}|x'-y'|^{2\gamma} \leqslant W(x') \leqslant C_1|x'-y'|^{2\gamma}, \quad x' \in \mathbb{R}^{n'}, \text{ and } b = W \text{ on } B(y',\varepsilon_0).$$

Hence, we have

$$P_\lambda \chi_0 = (-\Delta_{\mathbb{R}^{n'}} - \Delta_{\mathbb{T}^{n''}} - \lambda^2 + i\lambda W(x')) \circ \chi_0.$$

As a consequence, we have

$$\begin{aligned}\|P_\lambda \chi_0 u\|^2_{L^2(M)} &= \|(-\Delta_{\mathbb{R}^{n'}} - \Delta_{\mathbb{T}^{n''}} - \lambda^2 + i\lambda W(x'))(\chi_0 u)\|^2_{L^2(\mathbb{R}^{n'}\times\mathbb{T}^{n''})}\\ &= \sum_{k\in\mathbb{Z}^{n''}} \|(-\Delta_{\mathbb{R}^{n'}} + |k|^2 - \lambda^2 + i\lambda W(x'))(\chi_0(x')\hat{u}_k)\|^2_{L^2(\mathbb{R}^{n'})},\end{aligned}$$

where we have denoted by $\hat{u}_k(x') = \frac{1}{(2\pi)^{n''}}\int_{\mathbb{T}^{n''}} u(x',x'')e^{-ik\cdot x''}dx''$, $k \in \mathbb{Z}^{n''}$ the partial Fourier transform in the periodic directions.

Next, we apply Lemma 3.1 to the operator $-\Delta_{\mathbb{R}^{n'}} - (\lambda^2 - |k|^2) + i\lambda W(x')$ in $\mathbb{R}^{n'}$ for any $k \in \mathbb{Z}^{n''}$. This yields, for some $C > 0$,

$$\begin{aligned}C\|P_\lambda\chi_0 u\|^2_{L^2(M)} &\geqslant \lambda^{\frac{2}{\gamma+1}} \sum_{\lambda^2-|k|^2\leqslant 0} \|\chi_0(x')\hat{u}_k\|^2_{L^2(\mathbb{R}^{n'})}\\ &+ \lambda^{\frac{2}{\gamma+1}} \sum_{\lambda^2-|k|^2>0} \left(1 + \left(\frac{\lambda^2-|k|^2}{\lambda^{\frac{1}{\gamma+1}}}\right)^{\frac{2\gamma}{2\gamma+1}}\right) \|\chi_0(x')\hat{u}_k\|^2_{L^2(\mathbb{R}^{n'})}.\end{aligned} \tag{5.9}$$

We now want to estimate the remainder term

$$\begin{aligned}\|[P_\lambda,\chi_1]u\|_{L^2(M)} &= \|[P_\lambda,\chi_0]u\|_{L^2(M)} = \|[-\Delta_{\mathbb{R}^{n'}},\chi_0(x')]u\|_{L^2(\mathbb{R}^{n'}\times\mathbb{T}^{n''})}\\ &\leqslant \|(\Delta_{\mathbb{R}^{n'}}\chi_0)u\|_{L^2(\mathbb{R}^{n'}\times\mathbb{T}^{n''})} + 2\|\nabla_{x'}\chi_0\cdot\nabla_{x'}u\|_{L^2(\mathbb{R}^{n'}\times\mathbb{T}^{n''})}.\end{aligned} \tag{5.10}$$

To estimate this remainder, we take now $\psi = \psi(x') = \psi \otimes 1 \in \mathscr{C}_c^\infty(U_{\varepsilon_0})$ such that $\psi = 1$ on $\operatorname{supp}(\nabla_{x'}\chi_0)$. We compute

$$\begin{aligned}
&\operatorname{Re}\langle P_\lambda u, \psi^2 u\rangle_{L^2(M)} \\
&\qquad = \operatorname{Re}\langle(-\Delta - \lambda^2)u, \psi^2 u\rangle_{L^2(M)} + \operatorname{Re}\langle i\lambda\psi^2 bu, u\rangle_{L^2(M)} \\
&\qquad = \operatorname{Re}\langle -\Delta_{\mathbb{R}^{n'}} u, \psi^2 u\rangle_{L^2(\mathbb{R}^{n'}\times\mathbb{T}^{n''})} + \langle \psi^2 \nabla_{x''} u, \nabla_{x''} u\rangle_{L^2(\mathbb{R}^{n'}\times\mathbb{T}^{n''})} \\
&\qquad\qquad - \lambda^2\langle \psi^2 u, u\rangle_{L^2(\mathbb{R}^{n'}\times\mathbb{T}^{n''})}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\operatorname{Re}\langle -\Delta_{\mathbb{R}^{n'}} u, \psi^2 u\rangle_{L^2(\mathbb{R}^{n'}\times\mathbb{T}^{n''})} \\
&\quad = \langle \nabla_{x'} u, \psi^2 \nabla_{x'} u\rangle_{L^2(\mathbb{R}^{n'}\times\mathbb{T}^{n''})} + \operatorname{Re}\langle \nabla_{x'} u, u\nabla_{x'}\psi^2\rangle_{L^2(\mathbb{R}^{n'}\times\mathbb{T}^{n''})} \\
&\quad = \langle \nabla_{x'} u, \psi^2 \nabla_{x'} u\rangle_{L^2(\mathbb{R}^{n'}\times\mathbb{T}^{n''})} + \sum_{j=1}^{n'} \operatorname{Re}\langle iD_{x_j} u, u\partial_{x_j}\psi^2\rangle_{L^2(\mathbb{R}^{n'}\times\mathbb{T}^{n''})} \\
&\quad = \langle \nabla_{x'} u, \psi^2 \nabla_{x'} u\rangle_{L^2(\mathbb{R}^{n'}\times\mathbb{T}^{n''})} + \sum_{j=1}^{n'} \frac{i}{2}\langle [D_{x_j}, \partial_{x_j}\psi^2]u, u\rangle_{L^2(\mathbb{R}^{n'}\times\mathbb{T}^{n''})} \\
&\quad = \langle \nabla_{x'} u, \psi^2 \nabla_{x'} u\rangle_{L^2(\mathbb{R}^{n'}\times\mathbb{T}^{n''})} + \frac{1}{2}\langle (\Delta_{\mathbb{R}^{n'}}\psi^2)u, u\rangle_{L^2(\mathbb{R}^{n'}\times\mathbb{T}^{n''})},
\end{aligned}$$

when using that the two operators D_{x_j} and $\partial_{x_j}\psi^2$ are selfadjoint. Now, we write

$$\begin{aligned}
&\langle \psi^2 \nabla_{x''} u, \nabla_{x''} u\rangle_{L^2(\mathbb{R}^{n'}\times\mathbb{T}^{n''})} - \lambda^2\langle \psi^2 u, u\rangle_{L^2(\mathbb{R}^{n'}\times\mathbb{T}^{n''})} \\
&\qquad\qquad = \sum_{k\in\mathbb{Z}^{n''}} (|k|^2 - \lambda^2)\|\psi(x')\hat{u}_k\|^2_{L^2(\mathbb{R}^{n'})},
\end{aligned}$$

and coming back to (4.6), we obtain

$$\begin{aligned}
\|\psi\nabla_{x'} u\|^2_{L^2(\mathbb{R}^{n'}\times\mathbb{T}^{n''})} &\leqslant \|P_\lambda u\|_{L^2(M)}\|u\|_{L^2(M)} + C\|u\|^2_{L^2(M)} \\
&\qquad + \sum_{\lambda^2-|k|^2>0} (\lambda^2 - |k|^2)\|\psi(x')\hat{u}_k\|^2_{L^2(\mathbb{R}^{n'})}.
\end{aligned}$$

As a consequence, we estimate the commutator of (4.5) by

$$\begin{aligned}
\|[P_\lambda, \chi_0]u\|^2_{L^2(M)} \leqslant c_0\Big(&\|P_\lambda u\|_{L^2(M)}\|u\|_{L^2(M)} + \|u\|^2_{L^2(M)} \\
&+ \sum_{\lambda^2-|k|^2>0} (\lambda^2 - |k|^2)\|\psi(x')\hat{u}_k\|^2_{L^2(\mathbb{R}^{n'})}\Big).
\end{aligned} \tag{5.11}$$

Now, we have

$$\|P_\lambda u\|^2_{L^2(M)} \gtrsim \|P_\lambda\chi_0 u\|^2_{L^2(M)} + \|P_\lambda\chi_1 u\|^2_{L^2(M)} - C\|[P_\lambda, \chi_0]u\|^2_{L^2(M)}, \tag{5.12}$$

which, combined with the estimates, (5.8) (5.9) and (4.7), yields

$$\begin{aligned}
\|P_\lambda u\|^2_{L^2(M)} &+ \|u\|^2_{L^2(M)} \\
&\gtrsim \lambda^2 \|\chi_1 u\|^2_{L^2(M)} + \lambda^{\frac{2}{\gamma+1}} \sum_{\lambda^2-|k|^2\leqslant 0} \|\chi_0(x')\hat{u}_k\|^2_{L^2(\mathbb{R}^{n'})} \\
&\quad + \lambda^{\frac{2}{\gamma+1}} \sum_{\lambda^2-|k|^2>0} \left(1+\left(\frac{\lambda^2-|k|^2}{\lambda^{\frac{1}{\gamma+1}}}\right)^{\frac{2\gamma}{2\gamma+1}}\right) \|\chi_0(x')\hat{u}_k\|^2_{L^2(\mathbb{R}^{n'})} \\
&\quad - c_0 \sum_{\lambda^2-|k|^2>0} (\lambda^2-|k|^2)\|\psi(x')\hat{u}_k\|^2_{L^2(\mathbb{R}^{n'})}.
\end{aligned}$$

Next, we remark that the cutoff functions are chosen so that, for all $v \in L^2(M')$ we have

$$\|\psi v\|^2_{L^2(M')} \lesssim \|(\chi_0+\chi_1)v\|^2_{L^2(M')} \lesssim \|\chi_0 v\|^2_{L^2(M')} + \|\chi_1 v\|^2_{L^2(M')}.$$

We hence obtain

$$\begin{aligned}
\|P_\lambda u\|^2_{L^2(M)} &+ \|u\|^2_{L^2(M)} \\
&\gtrsim \lambda^2 \|\chi_1 u\|^2_{L^2(M)} \\
&\quad + \lambda^{\frac{2}{\gamma+1}} \sum_{\lambda^2-|k|^2\leqslant 0} \|\chi_0 \hat{u}_k\|^2_{L^2(M')} + \|\chi_1 \hat{u}_k\|^2_{L^2(M')} \\
&\quad + \lambda^{\frac{2}{\gamma+1}} \sum_{\lambda^2-|k|^2>0} f(\lambda,\lambda^2-|k|^2)\|(\chi_0+\chi_1)\hat{u}_k\|^2_{L^2(M')},
\end{aligned}$$

where $f(\lambda,\omega)$ is defined in (3.4). According to Lemma 3.3, there exists $\delta>0$ such that for all $\lambda>0$ and $\omega \in [0,\delta\lambda^2]$, we have $f(\lambda,\omega)\geqslant 1$. Moreover, we always have $\lambda^{\frac{2}{\gamma+1}} f(\lambda,\omega) \geqslant -c_0\omega \geqslant -c_0\lambda^2$. As a consequence, we have

$$\begin{aligned}
\|P_\lambda u\|^2_{L^2(M)} &+ \|u\|^2_{L^2(M)} \\
&\gtrsim \lambda^{\frac{2}{\gamma+1}} \sum_{\lambda^2-|k|^2\leqslant \delta\lambda^2} \|\hat{u}_k\|^2_{L^2(M')} - c_0\lambda^2 \sum_{\delta\lambda^2<\lambda^2-|k|^2\leqslant\lambda^2} \|\hat{u}_k\|^2_{L^2(M')}, \qquad (5.13)
\end{aligned}$$

Next, we study the last term in this estimate. The range $\delta\lambda^2 < \lambda^2-|k|^2 \leqslant \lambda^2$ may be rewritten as $\left|\frac{k}{\lambda}\right|^2 < (1-\delta)$. Take $\chi \in \mathscr{C}^\infty_c(\mathbb{R})$ such that $\chi = 1$ in a neighbourhood of $[-(1-\delta),(1-\delta)]$ and $\chi = 0$ in a neighbourhood of $(-\infty,-1]\cup[1,+\infty)$. We then have

$$\begin{aligned}
\sum_{\delta\lambda^2<\lambda^2-|k|^2\leqslant\lambda^2} \|\hat{u}_k\|^2_{L^2(M')} &\leqslant \sum_{k\in\mathbb{Z}^{n''}} \chi^2(|k|^2/\lambda^2)\|\hat{u}_k\|^2_{L^2(M')} \\
&= \|\chi(-\Delta_{\mathbb{T}^{n''}}/\lambda^2)u\|^2_{L^2(M)}, \qquad (5.14)
\end{aligned}$$

where the last identity is a consequence of Lemma B.8. Using (B.7), we moreover have

$$\begin{aligned}\|\chi(-\Delta_{\mathbb{T}^{n''}}/\lambda^2)u\|^2_{L^2(M)}\\ = \|\operatorname{Op}\left(\chi(|\xi''|^2/\lambda^2)\right)u\|^2_{L^2(M)} + O(\lambda^{-2})\|u\|^2_{L^2(M)}. \quad (5.15)\end{aligned}$$

As $\operatorname{supp}(\chi) \subset (-1, 1)$, Assumption (1.17) implies that for all $\rho \in \operatorname{supp}(1 \otimes \chi) \cap S^*M$ there exists $t > 0$ such that $\phi_t(\rho) \in T^*\omega_b$. We may hence apply Lemma 5.1 with $\alpha = 1 \otimes \chi$ to obtain

$$\begin{aligned}\lambda^2\|\operatorname{Op}\left(\chi(|\xi''|^2/\lambda^2)\right)u\|^2_{L^2(M)}\\ \leqslant C\|P_\lambda \operatorname{Op}\left(\chi(|\xi''|^2/\lambda^2)\right)u\|^2_{L^2(M)} + O(\lambda^{-1})\|u\|^2_{L^2(M)}. \quad (5.16)\end{aligned}$$

We now have

$$\begin{aligned}P_\lambda \operatorname{Op}\left(\chi(|\xi''|^2/\lambda^2)\right)\\ &= (-\Delta_{M'} - \Delta_{\mathbb{T}^{n''}} - \lambda^2 + i\lambda b)\operatorname{Op}\left(\chi(|\xi''|^2/\lambda^2)\right)\\ &= \operatorname{Op}\left(\chi(|\xi''|^2/\lambda^2)\right)P_\lambda + [-\Delta_{\mathbb{T}^{n''}}, \operatorname{Op}\left(\chi(|\xi''|^2/\lambda^2)\right)]\\ &\qquad + i\lambda[b, \operatorname{Op}\left(\chi(|\xi''|^2/\lambda^2)\right)], \quad (5.17)\end{aligned}$$

as $[\Delta_{M'}, \operatorname{Op}\left(\chi(|\xi''|^2/\lambda^2)\right)] = 0$. Next, we have $[-\Delta_{\mathbb{T}^{n''}}, \operatorname{Op}\left(\chi(|\xi''|^2/\lambda^2)\right)] \in \Psi^0_{sc}(M)$ as its principal symbol is $\{|\xi''|^2, \chi(|\xi''|^2/\lambda^2)\} = 0$. Finally, we have

$$\left\|i\lambda[b, \operatorname{Op}\left(\chi(|\xi''|^2/\lambda^2)\right)]\right\|_{\mathcal{L}(L^2(M))} \leqslant C\left(\|b\|_{L^\infty(M)} + \|\nabla_{x''}b\|_{L^\infty(M)}\right), \quad (5.18)$$

uniformly with respect to λ according to [13, Chapter IV, Proposition 7]. Combining all estimates (5.14)-(5.18), we now obtain

$$\lambda^2 \sum_{\delta\lambda^2 < \lambda^2 - |k|^2 \leqslant \lambda^2} \|\hat{u}_k\|^2_{L^2(M')} \lesssim \|P_\lambda u\|^2_{L^2(M)} + \|u\|^2_{L^2(M)}.$$

Together with (5.13), this implies

$$\begin{aligned}\|P_\lambda u\|^2_{L^2(M)} + \|u\|^2_{L^2(M)} &\gtrsim \lambda^{\frac{2}{\gamma+1}} \sum_{\lambda^2 - |k|^2 \leqslant \delta\lambda^2} \|\hat{u}_k\|^2_{L^2(M')}\\ &\qquad + \lambda^2 \sum_{\delta\lambda^2 < \lambda^2 - |k|^2 \leqslant \lambda^2} \|\hat{u}_k\|^2_{L^2(M')}\\ &\gtrsim \lambda^{\frac{2}{\gamma+1}}\|u\|^2_{L^2(M)},\end{aligned}$$

which concludes the proof of the theorem when taking $\lambda \geqslant \lambda_0$ for λ_0 large enough. ☐

6. Proof of Theorem 1.6: the lower bound

In this section, we prove Theorem 1.6, i.e. we construct a sequence of quasimode that saturates Inequality (1.9). Let $\chi \in \mathscr{C}_c^\infty(B(0,\varepsilon_0))$ be a real-valued function such that $\chi = 1$ in a neighbourhood of 0. Take $v = (1,0,\cdots,0) \in \mathbb{Z}^{n''}$, $\lambda = k$ for $k \in \mathbb{N}$ and set

$$u_k(x',x'') = c_1 k^{\frac{\alpha(n-1)}{2}} \chi(k^\alpha x') \frac{e^{ikv\cdot x''}}{(2\pi)^{\frac{n''}{2}}} = c_1 k^{\frac{\alpha(n-1)}{2}} \chi(k^\alpha x') \frac{e^{ikx_1''}}{(2\pi)^{\frac{n''}{2}}},$$

where $c_0 = \|\chi\|^{-1}_{L^2(B(0,\varepsilon_0))}$ is chosen so that for any $k \in \mathbb{N}$, we have $\|u_k\|_{L^2(M)} = 1$, and $\alpha > 0$ is to be fixed later on (in terms of γ). Now, we have

$$(P_k u_k)(x) = c_0 k^{\frac{\alpha n'}{2}} \frac{e^{ikx_1''}}{(2\pi)^{\frac{n''}{2}}} \left(-k^{2\alpha}(\Delta_{M'}\chi)(k^\alpha x') + ikb(x')\chi(k^\alpha x')\right),$$

and

$$\begin{aligned}
&\|P_k u_k\|^2_{L^2(M)} \\
&= c_0^2 \int_{M'} k^{\alpha n'} \left|-k^{2\alpha}(\Delta_{M'}\chi)(k^\alpha x') + ikb(x')\chi(k^\alpha x')\right|^2 dx' \\
&= c_0^2 \int_{\mathbb{R}^{n'}} k^{\alpha n'} \left|k^{2\alpha}(\Delta_{M'}\chi)(k^\alpha x')\right|^2 dx' + c_0^2 \int_{\mathbb{R}^{n'}} k^{\alpha n'} \left|kb(x')\chi(k^\alpha x')\right|^2 dx'
\end{aligned}$$

since χ is real-valued and compactly supported. After a change of variables, this yields

$$\|P_k u_k\|^2_{L^2(M)} = c_0^2 \int_{\mathbb{R}^{n'}} \left|k^{2\alpha}(\Delta_{M'}\chi)(x')\right|^2 dx' + c_0^2 \int_{\mathbb{R}^{n'}} \left|kb(\frac{x'}{k^\alpha})\chi(x')\right|^2 dx'.$$

Using then Assumption (1.8) on the vanishing rate of b on $\operatorname{supp}(\chi)$, we obtain

$$\begin{aligned}
&\|P_k u_k\|^2_{L^2(M)} \\
&\qquad \leqslant c_0^2 k^{4\alpha} \int_{\mathbb{R}^{n'}} \left|(\Delta_{M'}\chi)(x')\right|^2 dx' + c_0^2 \int_{\mathbb{R}^{n'}} \left|kC_1 \left|\frac{x'}{k^\alpha}\right|^{2\gamma} \chi(x')\right|^2 dx' \\
&\qquad \leqslant c_0^2 k^{4\alpha} \int_{\mathbb{R}^{n'}} \left|(\Delta_{M'}\chi)(x')\right|^2 dx' + c_0^2 C_1^2 k^{2-4\alpha\gamma} \int_{\mathbb{R}^{n'}} \left||x'|^{2\gamma}\chi(x')\right|^2 dx'.
\end{aligned}$$

Minimizing the exponent w.r.t. α gives $\alpha = \frac{1}{2(1+\gamma)}$, and hence

$$\|P_k u_k\|^2_{L^2(M)} \leqslant k^{\frac{2}{1+\gamma}} c_0^2 \left(\|(\Delta_{M'}\chi)\|^2_{L^2(M')} + C_1^2 \||x'|^{2\gamma}\chi\|^2_{L^2(M')}\right).$$

Finally, we have $\|P_k u_k\|_{L^2(M)} \leqslant C_0 k^{\frac{1}{1+\gamma}}$ for some $C_0 > 0$, which concludes the proof of Theorem 1.6.

7. Second microlocalization, a key tool for the proof

The proofs above contain several steps of microlocalizations, i.e. of cutting the phase space into pieces, proving the key estimates for symbols supported in these pieces and finally patching together the whole set of inequalities. We are willing in this section to (hopefully) illuminate these technicalities by resorting to various concepts related to the so-called second microlocalization. These notions were first developed in the analytic category in M. Kashiwara & T. Kawai's article [22], followed by G. Lebeau's paper [26]. J.-M. Bony's article [6] and J.-M. Delort's book [15] displayed striking applications to propagation of weak singularities for non-linear equations, the J.-M. Bony & N. Lerner's paper [7] provided a metrics point of view. More recently, N. Anantharaman & M. Léautaud's work on the damped wave equation [1] showed, using techniques of N. Anantharaman & F. Macià [2, 32], that the second micolocalization could be useful to tackle estimates related to some non-selfadjoint operators. The key tools in the last three papers are the 2-microlocal measures, introduced by L. Miller [33], C. Fermanian-Kammerer and P. Gérard [18, 19, 20], which allow to perform (at the level of defect measures) a second microlocalization for bounded sequences in L^2.

In the present article, we have used a multiplier method, instead of resorting to 2-microlocal measures: it means that have computed

$$\langle P_\lambda u, \mathcal{M} u\rangle,$$

with a carefully chosen *multiplier* $\mathcal{M}$. That multiplier operator $\mathcal{M}$ is in fact a second microlocalization operator and cannot be chosen as a standard semiclassical operator. It is constructed rather explicitly in the various regions of the phase space.

7.1. First microlocalization arguments and their limitations

Taking advantage of the fact that the damping term b in (1.1) does not depend on time[(3)] , a Fourier–Laplace transform yields the operator P_λ

$$P_\lambda = -\Delta_g - \lambda^2 + ib(x)\lambda, \tag{7.1}$$

where λ can be considered as a large positive parameter. This is thus a semi-classical problem where the Planck constant is $h = 1/\lambda$. P_λ is a real

(3) It would be interesting to explore the case where b depends on time and to show that a direct energy method should provide essentially the same results, at least when the time-dependence is smooth.

principal type operator whose principal symbol is $p = |\xi|_x^2 - \lambda^2$ so that the characteristic manifold is

$$\operatorname{Char} P_\lambda = \{(x,\xi) \in T^*M,\ |\xi|_x^2 = \lambda^2\}. \tag{7.2}$$

We ask the following regularity question: assuming that $P_\lambda u$ belongs to the (semi-classical) Sobolev space H^s_{sc}, could we have $u \in H^{s+1}_{sc}$ at a point γ_0 of the cotangent bundle? Of course when γ_0 is non-characteristic, we have the better result $u \in H^{s+2}_{sc}$. If γ_0 belongs to the characteristic set, we may combine three pieces of information:

(1) The point γ_0 belongs to a bicharacteristic curve of p whose endpoint γ_1 belongs to the set where the imaginary part of the subprincipal symbol is elliptic.
(2) At γ_1, we have a regularity result, due to the ellipticity of the imaginary part of the subprincipal symbol.
(3) The propagation-of-singularities theorem for real-principal type operators shows that the regularity at γ_1 propagates down to γ_0.

For condition (1) to be fulfilled, we need the hypothesis that the point γ_0 is connected by the bicharacteristic flow to the set where b is positive. The so-called *Geometric Control Condition* requires that this should be true for any point γ_0 in the characteristic set, providing then a (semi-classical) regularity result. We may thus introduce the *singular set*

$$S_\lambda = \{(x,\xi) \in \operatorname{Char} P_\lambda,\ \forall t \in \mathbb{R},\ b(\phi_t(x,\xi)) = 0\}, \tag{7.3}$$

(here ϕ_t is the bicharacteristic flow) and try to understand what will happen if S_λ is not empty. Let us describe the model situation in which we are interested: the manifold and the operator are simply $M = \mathbb{R}^n = \mathbb{R}^{n'} \times \mathbb{R}^{n''}$ and

$$P_\lambda = \operatorname{Op}(|\xi'|^2 + |\xi''|^2 - \lambda^2 + i\lambda|x'|^{2\gamma}). \tag{7.4}$$

The characteristic manifold has the equation

$$|\xi'|^2 + |\xi''|^2 = \lambda^2$$

and on the characteristic curves $\dot{x}' = 2\xi',\ \dot{x}'' = 2\xi'',\ \xi' = \text{constant},\ \xi'' = \text{constant}$. We get that $(x', x'', \xi', \xi'') \in S_\lambda$ iff $\xi' = 0, |\xi''|^2 = \lambda^2, x' = 0$. We see that S_λ is a $(2n''-1)$-dimensional submanifold of the symplectic $\mathbb{R}^{n'}_{x'} \times \mathbb{R}^{n''}_{x''} \times \mathbb{R}^{n'}_{\xi'} \times \mathbb{R}^{n''}_{\xi''}$ given by

$$S_\lambda = \{0_{\mathbb{R}^{n'}}\} \times \mathbb{R}^{n''}_{x''} \times \{0_{\mathbb{R}^{n'}}\} \times \{\xi'' \in \mathbb{R}^{n''}, |\xi''| = \lambda\}.$$

To be 1-microlocally away from S_λ would mean that for some $\epsilon_0 \in (0,1)$

$$|\xi'|^2 + |\xi''|^2 = \lambda^2, \quad |\xi''|^2 \leqslant (1-\epsilon_0)\lambda^2,\ |\xi'|^2 \geqslant \epsilon_0\lambda^2,$$

so that the (GCC) condition holds even if $x' = 0$. Of course if $|x'| > \epsilon_0$, the (GCC) condition holds.

We are left with a neighbourhood of the set S_λ and we shall not be able to take advantage of the particular behavior of b if we do not make further localization in the phase space.

7.2. Reviewing our estimates

Let us quickly review our arguments for the various estimates proven for P_λ given by (7.4). We set $|\xi'|^2 + |\xi''|^2 = \lambda^2$.

(1) $|\xi'|^2 \gtrsim |\xi''|^2$ or $|x'| \gtrsim 1$: this is the (GCC) region since the first condition implies that the characteristic curve starting at $x' = 0$ enters at once the damping set, and the second condition requires to start within the damping set.

(2) $|\xi'|^2 \ll |\xi''|^2$ and $|x'| \ll 1$: this is where we need further localization.

(2.1) $|\xi'|^2 \ll \lambda^{\frac{1}{\gamma+1}}$ and $|x'|^2 \ll \lambda^{-\frac{1}{\gamma+1}}$: a tiny piece of the phase space, with volume 1 though, thus compatible with the uncertainty principle. We use a pseudo-spectral estimate for the operator $|D'|^2 + i\lambda|x'|^{2\gamma}$, that is an estimate of type

$$\||D'|^2 u + i\lambda|x'|^{2\gamma} u\| \gtrsim \|u\|\lambda^{\frac{1}{\gamma+1}}.$$

(2.2) $|\xi'|^2 \ll |\xi''|^2$ and $\lambda^{-\frac{1}{\gamma+1}} \lesssim |x'|^2 \ll 1$: there we have $\lambda|x'|^{2\gamma} \gtrsim \lambda^{\frac{1}{\gamma+1}}$, we calculate

$$\operatorname{Re}\langle P_\lambda u, iu\rangle = \langle \lambda|x'|^{2\gamma} u, u\rangle \gtrsim \lambda^{\frac{1}{\gamma+1}}\|u\|^2.$$

(2.3) $\lambda^{\frac{1}{\gamma+1}} \lesssim |\xi'|^2 \ll |\xi''|^2$ and $|x'|^2 \ll \lambda^{-\frac{1}{\gamma+1}}$: this is the most difficult region, in which we use a propagation estimate. We study the model

$$|D'|^2 - \omega^2 + i\lambda|x'|^{2\gamma}, \quad \lambda^{\frac{1}{2\gamma+2}} \leqslant \omega \leqslant \epsilon_0\lambda.$$

Figure 7.1 illustrates the transition from region (1) to region (2) in the frequency variables for small x' and shows that localization in region (1) is no longer conical.

7.3. Second microlocalization with respect to the singular set

The first microlocalization metric on $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$ is

$$G_{x,\xi} = \frac{|dx|^2}{1} + \frac{|d\xi|^2}{\Lambda(\xi)^2}, \text{ with } \Lambda(\xi)^2 = \langle\xi\rangle^2 = 1 + |\xi|^2. \tag{7.5}$$

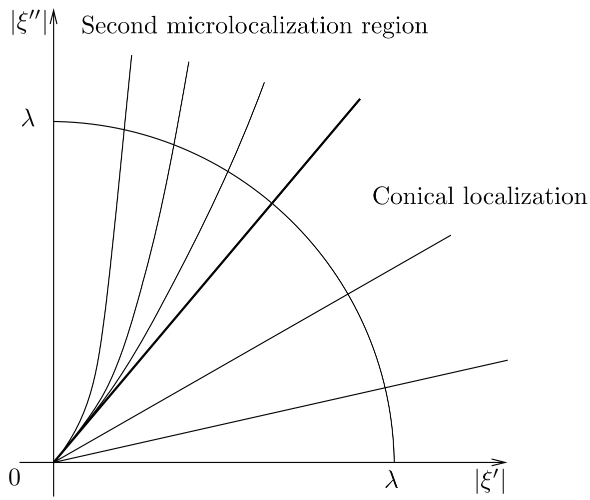

Figure 7.1. Classical conical localization and second microlocalization on the singular set.

It means that the standard symbols of order 0 used for this first microlocalization are functions $a \in C^\infty(\mathbb{R}^{2n})$ such that

$$|(\partial_x^\alpha \partial_\xi^\beta a)(x,\xi)| \leqslant C_{\alpha\beta}\Lambda(\xi)^{-|\beta|} = C_{\alpha\beta}\langle\xi\rangle^{-|\beta|}.$$

The "large parameter" of this calculus is the product of the conjugate axes, $\Lambda = \Lambda(\xi)$.

We want to provide a finer localization when we are getting close to the singular set S_λ. For this purpose, we define the metric

$$g_{x,\xi} = \frac{|dx'|^2}{\Lambda(\xi)^{-\frac{1}{\gamma+1}}\mu(\xi)^{\frac{1}{\gamma+1}}} + \frac{|d\xi'|^2}{\Lambda(\xi)^{\frac{1}{\gamma+1}}\mu(\xi)^{\frac{2\gamma+1}{\gamma+1}}} + \frac{|dx''|^2}{1} + \frac{|d\xi''|^2}{\Lambda(\xi)^2}, \tag{7.6}$$

$$\text{where} \quad \mu(\xi) = 1 + \left(|\xi'|^2\Lambda(\xi)^{-\frac{1}{\gamma+1}}\right)^{\frac{\gamma+1}{2\gamma+1}}. \tag{7.7}$$

The notation for g above means that for each $(x,\xi) \in \mathbb{R}^{2n}$, $g_{x,\xi}$ is a positive definite quadratic form on $\mathbb{R}^{2n}$ so that for $(z', z'', \zeta', \zeta'') \in \mathbb{R}^{n'} \times \mathbb{R}^{n''} \times \mathbb{R}^{n'} \times \mathbb{R}^{n''}$, we have

$$g_{x,\xi}(z,\zeta) = \frac{|z'|^2}{\Lambda(\xi)^{-\frac{1}{\gamma+1}}\mu(\xi)^{\frac{1}{\gamma+1}}} + \frac{|\zeta'|^2}{\Lambda(\xi)^{\frac{1}{\gamma+1}}\mu(\xi)^{\frac{2\gamma+1}{\gamma+1}}} + \frac{|z''|^2}{1} + \frac{|\zeta''|^2}{\Lambda(\xi)^2}.$$

We note first that

$$1 \leqslant \mu(\xi) \leqslant 1 + \Lambda(\xi) \leqslant 2\Lambda(\xi). \tag{7.8}$$

This inequality implies that

$$\Lambda^{-1}\mu \leqslant 2, \quad \Lambda^{\frac{1}{\gamma+1}}\mu^{\frac{2\gamma+1}{\gamma+1}} \leqslant \Lambda^2\, 2^{\frac{2\gamma+1}{\gamma+1}}, \text{ and hence } g \geqslant c(\gamma)G,$$

where $c(\gamma)$ is a positive constant depending only on γ. This inequality induces of course that the localization given by the metric g is finer than the one provided by G. Calculating the square of the product of conjugate axes of g, we get respectively

$$\Big(\Lambda(\xi)^{-\frac{1}{\gamma+1}}\mu(\xi)^{\frac{1}{\gamma+1}}\Big) \times \Big(\Lambda(\xi)^{\frac{1}{\gamma+1}}\mu(\xi)^{\frac{2\gamma+1}{\gamma+1}}\Big) = \mu(\xi)^2, \quad 1 \times \Lambda(\xi)^2$$

so that the "large parameter" of the metric g is (equivalent to) μ, and g satisfies the Uncertainty Principle.

Lemma 7.1. — *The metric g given by* (7.6) *is slowly varying, i.e. such that there exist r, C positive so that*

$$g_{y,\eta}\big((y,\eta)-(x,\xi)\big) \leqslant r^2 \Longrightarrow \forall T \in \mathbb{R}^{2n}, \quad C^{-1} \leqslant \frac{g_{x,\xi}(T)}{g_{y,\eta}(T)} \leqslant C. \tag{7.9}$$

Moreover, the metric g is also uniformly temperate on the balls of the metric G, i.e. there exists C, N, r positive such that $\forall X = (x,\xi) \in \mathbb{R}^{2n}, \forall Y = (y,\eta) \in \mathbb{R}^{2n}$,

$$G_X(Y-X) \leqslant r^2 \Longrightarrow \forall T \in \mathbb{R}^{2n}, \quad \frac{g_{x,\xi}(T)}{g_{y,\eta}(T)} \leqslant C\big(1+g^{\sigma}_{x,\xi}(Y-X)\big)^N, \tag{7.10}$$

where the quadratic form $g^{\sigma}_{x,\xi}$ (which is the "symplectic inverse" of $g_{x,\xi}$) is given by

$$g^{\sigma}_{x,\xi}(z,\zeta) = \Lambda(\xi)^{\frac{1}{\gamma+1}}\mu(\xi)^{\frac{2\gamma+1}{\gamma+1}}|z'|^2 + \Lambda(\xi)^{-\frac{1}{\gamma+1}}\mu(\xi)^{\frac{1}{\gamma+1}}|\zeta'|^2 + \Lambda(\xi)^2|z''|^2 + |\zeta''|^2.$$

Proof. — We have

$$\left(\frac{g_{x,\xi}}{g_{y,\eta}}\right)^{\gamma+1} \lesssim \frac{\Lambda(\eta)}{\Lambda(\xi)}\frac{\mu(\eta)^{2\gamma+1}}{\mu(\xi)^{2\gamma+1}} + \frac{\Lambda(\xi)}{\Lambda(\eta)}\frac{\mu(\eta)}{\mu(\xi)}, \tag{7.11}$$

and if $g_{y,\eta}\big((y,\eta)-(x,\xi)\big) \leqslant r^2$, this implies $G_{y,\eta}\big((y,\eta)-(x,\xi)\big) \leqslant r^2/c(\gamma)$. Since G is slowly varying, we may choose r small enough to get $\Lambda(\xi)/\Lambda(\eta) \sim 1$. Then, with fixed constants C_j, we find

$$\frac{\mu(\eta)^{2\gamma+1}}{\mu(\xi)^{2\gamma+1}} \leqslant C_1 \frac{1+|\eta'|^{2\gamma+2}\Lambda^{-1}}{1+|\xi'|^{2\gamma+2}\Lambda^{-1}} \leqslant C_2 + C_2\left(\frac{|\eta'-\xi'|^2\Lambda^{-\frac{1}{\gamma+1}}}{1+|\xi'|^2\Lambda^{-\frac{1}{\gamma+1}}}\right)^{\gamma+1},$$

and since from the assumption (7.9), we know that

$$|\xi'-\eta'|^2 \leqslant r^2\mu(\eta)^{\frac{2\gamma+1}{\gamma+1}}\Lambda^{\frac{1}{\gamma+1}},$$

we obtain

$$\frac{\mu(\eta)^{2\gamma+1}}{\mu(\xi)^{2\gamma+1}} \leqslant C_2 + C_3 r^{2\gamma+2}\frac{\mu(\eta)^{2\gamma+1}}{\mu(\xi)^{2\gamma+1}}$$

which implies

$$\frac{\mu(\eta)^{2\gamma+1}}{\mu(\xi)^{2\gamma+1}} \leqslant 2C_2,$$

if $C_3 r^{2\gamma+2} \leqslant 1/2$, providing the inequality $g_X \leqslant C_4 g_Y$ in (7.9) for $r \leqslant r_0$. Now if $g_Y(X-Y) \leqslant r_0^2/C_4$, we get $g_X(X-Y) \leqslant r_0^2$ and thus $g_Y \leqslant C_4 g_X$, completing the proof of (7.9).

To prove (7.10), we may choose r such that $G_X(Y-X) \leqslant r^2$ implies $\Lambda(\xi) \sim \Lambda(\eta)$ and from (7.11), it suffices to prove

$$|\xi' - \eta'|^2 \Lambda^{-\frac{1}{\gamma+1}} \leqslant |\xi' - \eta'|^2 \Lambda^{-\frac{1}{\gamma+1}} \mu(\xi)^{\frac{1}{\gamma+1}},$$

which is true since $\mu \geqslant 1$. □

Remark 7.2. — Lemma 7.1 may look outrageously complicated and unintuitive, but these properties, essentially introduced by L. Hörmander (see [21, Chapter 18]), are linked to some "admissibility" of the cutting of the phase space provided by this metric. The uncertainty principle (here $\mu \geqslant 1$) is the most natural condition, but Conditions (7.9) and (7.10) are important for a pseudodifferential calculus to make sense. In other words, we need some conditions to patch together the estimates that we are able to prove in each specific region described in Section 7.2. A cutting procedure will generate commutators and we have to make sure that these commutators do not destroy or spoil the basic local estimates that we are able to prove.

Appendix A. Some geometric facts

In this section, we recall some elementary geometric definitions and facts. The metric g provides in each tangent space T_xM (as well as in each cotangent space T_x^*M) an inner product denoted $(\cdot,\cdot)_{g(x)}$ (with the same notation on T_x^*M). In local coordinates, we write g_{ij} for the metric g on the tangent bundle TM. As a metric on the cotangent bundle T^*M, g is given by g^{ij} in local coordinates, i.e.

$$(\eta,\xi)_{g(x)} = \sum_{i,j} g^{ij}(x)\eta_i\xi_j, \qquad \text{where } g^{ij}(x) = (g(x)^{-1})_{ij}.$$

For $x \in M$ and $\xi \in T_x^*M$, we denote by $|\xi|_x = (\xi,\xi)_{g(x)}^{\frac{1}{2}}$ the associated norm. We also use the notation $p(x,\xi) = |\xi|_x^2$. For all $v \in T_xM$, we can define $v^* \in T_x^*M$ uniquely determined by the identity

$$(v,w)_{g(x)} = \langle v^*, w\rangle_{T_x^*(M),T_x(M)}, \qquad \text{for all } w \in T_xM,$$

which reads in local coordinates $v_i^* = \sum_j g_{ij}(x)v_j$. Note that $|v|_x = |v^*|_x$.

We now give a definition of geodesics on M associated with the metric g, as used in Theorem 1.8. We denote by $s \mapsto \phi_s(x,\xi) \in T^*M \setminus 0$ the Hamiltonian flow associated to p, that is, the (maximal) solutions of

$$\frac{d}{ds}\phi_s(x,\xi) = H_p\big(\phi_s(x,\xi)\big), \quad \phi_0(x,\xi) = (x,\xi) \in T^*M \setminus 0, \tag{A.1}$$

where the Hamilton vector field H_p is given by $H_p = (\nabla_\xi p, -\nabla_x p)$ in local coordinates. In particular we have $\frac{d}{ds}x_i(s) = 2\sum_j g^{ij}(x(s))\xi_j(s)$, that is $\xi(s) = \frac{1}{2}\big(\frac{d}{ds}x(s)\big)^*$. Note that the value of p is preserved along this integral curve as

$$\frac{d}{ds}p \circ \phi_s|_{s=s_0} = H_p(p)(\phi_{s_0}) = \{p,p\}(\phi_{s_0}) = 0.$$

As a consequence, ϕ_s is a global flow preserving the norm.

Let now $S^2M = \{(x,v) \in TM, |v|_x = (v,v)^{\frac{1}{2}}_{g(x)} = 2\}$. For $(x,v) \in S^2M$, we consider the curve $(x(s),v(s))$ given by

$$(x(s), v(s)^*) = \phi_s(x,v^*).$$

Note that we have $\frac{d}{ds}x(s) = v(s)$. In particular, $\frac{d}{ds}x(0) = v$ and moreover

$$|v(s)|_{x(s)} = |v(s)^*|_{x(s)} = 2|\xi(s)|_{x(s)} = p(x(s),v(s)^*)^{\frac{1}{2}} = p(x,v^*)^{\frac{1}{2}} = |v|_x = 2.$$

We call the curve $s \mapsto x(s)$ on M the geodesic originating from $(x,\xi) \in S^*M$ at time $s=0$. The above remarks show that $(x(t), \frac{dx}{dt}(t)) \in S^2M$: the traveling speed of the geodesic is constant (and equal to 2).

Finally, the covariant gradient and the divergence operators are given in local coordinates by

$$\nabla_g = \sum_i g^{ij}\partial_{x_i}, \qquad \operatorname{div}_g v = \frac{1}{\sqrt{\det(g)}}\sum_i \partial_{x_i}(\sqrt{\det(g)}v_i),$$

The usual (negative) Laplace–Beltrami operator on M is defined by $\Delta_g = \operatorname{div}_g \nabla_g$, that is,

$$\Delta_g = \frac{1}{\sqrt{\det(g)}}\sum_{i,j}\partial_{x_i}(g^{ij}\sqrt{\det(g)}\partial_{x_j}),$$

in local coordinates. It is selfadjoint on $L^2(M)$ endowed with the Riemannian dot-product $\langle f,g\rangle_{L^2(M)}$ given by $\int f\overline{g}\sqrt{\det(g)}dx$ in local charts.

Finally, under the additional structure assumption $(M,g) = (M' \times \mathbb{T}^{n''}, g' + |dx''_1|^2 + \cdots + |dx''_{n''}|^2)$ (where g' is a metric on M'), we obtain

$$\phi_s(x',\xi',x'',\xi'') = (\phi'_s(x',\xi'), x'' + s\xi'', \xi''),$$

(where we changed the order of the variables for readability) with ϕ'_s the flow on T^*M' associated with the Hamilton vector field $H_{p'}$ with $p'(x',\xi') = |\xi'|^2_{x'}$.

Similarly, we have

$$\Delta_g = \Delta_{M'} + \Delta_{\mathbb{T}^{n''}} = \frac{1}{\sqrt{\det(g')}} \sum_{i,j\leqslant n'} \partial_{x'_i}(g'^{ij}\sqrt{\det(g')}\partial_{x'_j}) + \sum_{j\leqslant n''} \partial^2_{x''_j}.$$

Appendix B. Toolbox of pseudodifferential calculus

B.1. Pseudodifferential operators on $\mathbb{R}^d$

Notation. — We recall that the Weyl quantization of a symbol $a(x,\xi)$ on $\mathbb{R}^{2d}$, the operator denoted by a^w, is given by

$$(a^w u)(x) = \iint e^{i(x-y)\cdot\xi} a(\frac{x+y}{2},\xi)u(y)dyd\xi(2\pi)^{-d}, \tag{B.1}$$

which is a small variation with respect to the more standard quantization

$$(a(x,D)u)(x) = \int e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)d\xi(2\pi)^{-d}.$$

One of the (many) assets of Weyl quantization is the formula for taking adjoints,

$$(a^w)^* = (\bar{a})^w,$$

a convenient feature for our computations with non-selfadjoint operators. The symplectic invariance of the Weyl quantization is an important property, useful for the proof that our estimate (2.3) is optimal, can be expressed as follows (see e.g. [30, Theorem 2.1.2]): let a be a tempered distribution on $\mathbb{R}^{2d}$ and let χ be an affine symplectic mapping of $\mathbb{R}^{2d}$. Then there exists a unitary transformation U of $L^2(\mathbb{R}^d)$ such that

$$(a\circ\chi)^w = U^* a^w U. \tag{B.2}$$

This implies for instance that, for $\alpha_0 > 0, (x_0,\xi_0) \in \mathbb{R}^{2d}$, the operator with Weyl symbol b given by

$$b(x,\xi) = a(\alpha_0 x + x_0, \alpha_0^{-1}\xi + \xi_0)$$

is unitarily equivalent to a^w. In the main part of the article, we also use the notation

$$S(\mathcal{M}, \frac{|dx|^2}{\varphi(x,\xi)^2} + \frac{|d\xi|^2}{\Phi(x,\xi)^2}) \tag{B.3}$$

for the space of smooth functions a on $\mathbb{R}^{2d}$ such that for each multi-indices α,β, there exists $C_{\alpha\beta} > 0$ such that

$$\forall (x,\xi) \in \mathbb{R}^{2d}, \quad |(\partial_x^\alpha \partial_\xi^\beta a)(x,\xi)| \leqslant C_{\alpha\beta}\mathcal{M}(x,\xi)\varphi(x,\xi)^{-|\alpha|}\Phi(x,\xi)^{-|\beta|},$$

where the positive functions $\varphi, \Phi, \mathcal{M}$ are such that the metric $\frac{|dx|^2}{\varphi(x,\xi)^2}+\frac{|d\xi|^2}{\Phi(x,\xi)^2}$ and the weight $\mathcal{M}$ are admissible (see [21, Section 18.5] or [30, Section 2.2] for precise definitions). In Section 5.1 (where we only use "one-microlocal" calculus), the following semiclassical class is used (see below for its definition on a manifold):

$$S^m_{sc}(\mathbb{R}^{2n}) := S((\lambda^2+\langle\xi\rangle^2)^{\frac{m}{2}}, |dx|^2+\frac{|d\xi|^2}{\lambda^2+\langle\xi\rangle^2}), \quad \lambda \geqslant 1.$$

Note that $|\xi|^2-\lambda^2 \in S^2_{sc}(\mathbb{R}^{2d})$, and for any $a \in \mathscr{C}^\infty(\mathbb{R}^{2d})$ which is homogeneous of degree m, we have $\lambda^{\tilde{m}} a(x,\frac{\xi}{\lambda}) \in S^{m+\tilde{m}}_{sc}(\mathbb{R}^{2d})$. We also denote by

$$S^0_{0,0}(\mathbb{R}^{2d}) := S(1, |dx|^2+|d\xi|^2), \tag{B.4}$$

the space of smooth functions on $\mathbb{R}^{2d}$ which are bounded as well as all their derivatives, and

$$S^m_\lambda(\mathbb{R}^{2d}) := S(\lambda^m, |dx|^2+\frac{|d\xi|^2}{\lambda^2}), \quad \lambda \geqslant 1.$$

We notice that if $a(x,\xi) \in S^0_{0,0}(\mathbb{R}^{2d})$, then $\lambda^m a(x,\frac{\xi}{\lambda}) \in S^m_\lambda(\mathbb{R}^{2d})$. Moreover, we have $S^m_{sc}(\mathbb{R}^{2d}) \subset S^m_\lambda(\mathbb{R}^{2d})$.

We shall use also the following identities, for a, b real valued symbols, say smooth functions on $\mathbb{R}^{2d}$ bounded with all derivatives bounded:

$$2\operatorname{Re}\langle a^w u, ib^w u\rangle = \langle [a^w, ib^w]u, u\rangle,$$

which follows from

$$2\operatorname{Re}\langle a^w u, ib^w u\rangle = \langle a^w u, ib^w u\rangle + \langle ib^w u, a^w u\rangle = \langle Cu, u\rangle,$$

with $C = -i(b^w)^* a^w + (a^w)^* ib^w = [a^w, ib^w]$. Moreover the "principal" symbol of $[a^w, ib^w]$ is the Poisson bracket

$$\{a,b\} = \sum_{1\leqslant j\leqslant d} \frac{\partial a}{\partial \xi_j}\frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial b}{\partial \xi_j}. \tag{B.5}$$

B.2. Pseudodifferential operators on a manifold

In this section, we briefly explain the semiclassical calculus used in Section 5.1 on any n-dimensional compact manifold M. More details can be found in [21, p. 81-87] for classical operators or [23, Appendix B and Appendices C9-C13] in the semiclassical setting.

Let us first recall that, given a diffeomorphism ϕ between two open sets $\phi : U_1 \to U_2$, the associated pullback (here stated for continuous functions) is

$$\phi^* : \mathscr{C}(U_2) \to \mathscr{C}(U_1),$$
$$u \mapsto u \circ \phi.$$

For a function a defined on phase-space, e.g. a symbol, the pullback is given by

$$\phi^* a(x, \xi) = a(\phi(x), {}^t\phi'(x)^{-1}\xi), \quad x \in U_1, \xi \in T_x^* U_1,\ a \in \mathscr{C}(T^* U_2). \quad \text{(B.6)}$$

Note that this transformation is symplectic.

The compact manifold M is of dimension n and is furnished with a finite atlas (U_j, ϕ_j), $j \in J$. The maps $\phi_j : U_j \to \tilde{U}_j \subset \mathbb{R}^n$ are smooth diffeomorphisms.

Definition B.1. — *We say that $a \in S^m_{sc}(T^*M)$, resp. $S^m_\lambda(T^*M)$, resp. $S^0_{0,0}(T^*M)$, if $a \in \mathscr{C}^\infty(T^*M)$ and for any $j \in J$, for any $\chi \in \mathscr{C}^\infty_c(U_j)$, we have $(\phi_j^{-1})^*(\chi a) \in S^m_{sc}(\mathbb{R}^{2n})$, resp. $S^m_\lambda(\mathbb{R}^{2n})$, resp. $S^0_{0,0}(\mathbb{R}^{2n})$.*

Note that with this definition, we have $|\xi|_x^2 - \lambda^2 \in S^2_{sc}(T^*M)$, and for any $a \in \mathscr{C}^\infty(T^*M)$ which is homogeneous of degree m in the fibers, we have $\lambda^{\tilde{m}} a(x, \frac{\xi}{\lambda}) \in S^{m+\tilde{m}}_{sc}(T^*M)$.

Next, we explain how to quantize such symbols and recall some of the properties of the quantization. Let us first denote by $(\psi_j)_j$ a partition of unity subordinated to the covering $M = \bigcup_{j \in J} U_j$:

$$\psi_j \in \mathscr{C}^\infty(M), \quad \operatorname{supp}(\psi_j) \subset U_j, \quad 0 \leqslant \psi_j \leqslant 1, \quad \sum_j \psi_j = 1.$$

We also need functions $\tilde{\psi}_j \in \mathscr{C}^\infty_c(\mathbb{R}^{2n})$ such that $\operatorname{supp}(\tilde{\psi}_j) \subset \tilde{U}_j$ and $\tilde{\psi}_j = 1$ in a neighbourhood of $\operatorname{supp}((\phi_j^{-1})^*\psi_j)$.

Definition B.2. — *Given $a = a(\lambda, x, \xi) \in \mathscr{C}^\infty(T^*M)$, we define the following operator*

$$\operatorname{Op}(a) = \operatorname{Op}\left(a(\lambda, x, \xi)\right) = \sum_{j \in J} A_j,$$
$$A_j u = \phi_j^*\big(\tilde{\psi}_j a_j^w (\phi_j^{-1})^*(\psi_j u)\big), \quad u \in \mathscr{C}^\infty(M),$$

where $a_j = (\phi_j^{-1})^* a$.

Basically, this amounts to apply the operator associated to a in local charts. This definition applies in particular if $a \in S^m_{sc}(T^*M)$ or $a \in S^m_\lambda(T^*M)$ Note that if $a = a(\lambda, x)$ does not depend on the cotangent variable ξ, then $(\operatorname{Op}(a)u)(x) = a(\lambda, x)u(x)$. An operator $\operatorname{Op}(a)$ for $a \in S^m_{sc}(T^*M)$ is a semiclassical pseudodifferential operator in the following usual sense:

Definition B.3. — *We say that the operator* $A : \mathscr{C}_c^\infty(M) \to \mathscr{C}^\infty(M)$ *belongs to the class* $\Psi_{sc}^m(M)$ *if:*

(1) *Its distribution kernel* $K(x,y)$ *is smooth outside* $\mathrm{diag}(M \times M) = \{(x,x),\ x \in M\}$ *in the semi-classical sense:* $K \in \mathscr{C}^\infty(M \times M \setminus \mathrm{diag}(M \times M))$ *and for any semi-norm* q *on* $\mathscr{C}^\infty(M \times M)$*, for any* $\chi, \hat{\chi} \in \mathscr{C}_c^\infty(M)$ *such that* $\operatorname{supp}\chi \cap \operatorname{supp}\hat{\chi} = \emptyset$*, we have* $q\big(\chi(x)\hat{\chi}(y)K(x,y)\big) = O(\lambda^{-\infty})$.

(2) *For all* $j \in J$ *and all* $\chi \in \mathscr{C}_c^\infty(U_j)$, $\tilde{\chi} \in \mathscr{C}_c^\infty(\tilde{U}_j)$*, the application*

$$u \mapsto \big(\phi_j^{-1}\big)^* \big(\chi A \phi_j^*(\tilde{\chi} u)\big)$$

belongs to $\Psi_{sc}^m(\mathbb{R}^n) = \{a^w, a \in S_{sc}^m(\mathbb{R}^{2n})\}$.

We could have defined as well the classes $\Psi_\lambda^m(M)$ to which would belong $\mathrm{Op}(a)$ for $a \in S_\lambda^m(T^*M)$. However, in the main part of the text, we only use for such operators the Calderón–Vaillancourt Theorem [12] entailing that for $a \in S_\lambda^0(T^*M)$, $\mathrm{Op}(a)$ defines an operator bounded on $L^2(M)$ uniformly with respect to λ (in fact, this theorem is stated in that reference on $\mathbb{R}^n$; its counterpart on M follows easily when using local charts).

Next, we describe the pseudodifferential calculus for the class $\Psi_{sc}^m(M)$. We define semiclassical norms on M by

$$\|u\|_{H^s_{sc}(M)} = \| \operatorname{Op}\big((|\xi|_x^2 + \lambda^2)^{\frac{s}{2}}\big) u\|_{L^2(M)}.$$

For $s \geqslant 0$, this norm is (uniformly with respect to $\lambda \geqslant 1$) equivalent to the norm $\|u\|_{H^s(M)} + \lambda^s \|u\|_{L^2(M)}$ (where $\|u\|_{H^s(M)}$ may be defined in local charts). We have the following important property for semiclassical pseudo-differential operators in the class $\Psi_{sc}^m(M)$.

Proposition B.4. — *For any* $s, m \in \mathbb{R}$ *and any* $A \in \Psi_{sc}^m(M)$*, we have*

$$A \in \mathcal{L}(H_{sc}^s(M); H_{sc}^{s-m}(M)), \quad \textit{uniformly with respect to } \lambda \geqslant 1.$$

The quantization formula defining $\mathrm{Op}(a)$ enjoys the following properties, used in the main part of the article. We refer to [21] or [23, Appendix B and Appendices C9-C13] for detailed proofs.

Proposition B.5. — *For any* $m, \tilde{m} \in \mathbb{R}$ *and any* $a \in S_{sc}^m(T^*M)$, $\tilde{a} \in S_{sc}^{\tilde{m}}(T^*M)$*, we have*

- $\mathrm{Op}(a) \in \Psi_{sc}^m(M)$;
- $\mathrm{Op}(a)^* - \mathrm{Op}(\bar{a}) \in \Psi_{sc}^{m-1}(M)$ *(where the adjoint is taken in* $L^2(M)$*)*;
- $\mathrm{Op}(a)\,\mathrm{Op}(\tilde{a}) - \mathrm{Op}(a\tilde{a}) \in \Psi_{sc}^{m+\tilde{m}-1}(M)$;
- $[\mathrm{Op}(a), \mathrm{Op}(\tilde{a})] - \frac{1}{i}\mathrm{Op}(\{a, \tilde{a}\}) \in \Psi_{sc}^{m+\tilde{m}-2}(M)$;

We have the following version of the sharp Gårding inequality for the above quantization of symbols in $S^m_{sc}(T^*M)$, which can be easily deduced from that adapted to symbols in $S^m_{sc}(\mathbb{R}^{2n})$ (see [21, Theorem 18.1.14] or [30, Theorem 2.5.4]) by using local charts.

Theorem B.6. — *Let $m \in \mathbb{R}$ and assume that $a \in S^{2m+1}_{sc}(T^*M)$ is real valued and satisfies $a \geqslant 0$ on T^*M. Then, there exist $C, \lambda_0 > 0$ such that for all $u \in \mathscr{C}^\infty(M)$ and all $\lambda \geqslant \lambda_0$ we have*

$$\mathrm{Re}(\mathrm{Op}(a)u, u)_{L^2(M)} \geqslant -C\|u\|^2_{H^m_{sc}(M)}.$$

This last inequality can be equivalently rewritten as

$$\mathrm{Re}(\mathrm{Op}(a)u, u)_{L^2(M)} \geqslant -C\|u\|^2_{H^m(M)} - C\lambda^{2m}\|u\|^2_{L^2(M)}.$$

Note finally that the quantization formula defining $\mathrm{Op}(a)$ depends on the set of charts and of the partition of unity chosen. However, for $a \in S^m_{sc}(T^*M)$ one can show that its definition is intrinsic modulo $\Psi^{m-1}_{sc}(T^*M)$.

B.3. Pseudodifferential operators on $M = M' \times \mathbb{T}^{n''}$

The construction of pseudodifferential operators in the previous section is very general and does not take into account the particular product structure of the manifold $M = M' \times \mathbb{T}^{n''}$. When taking into account the structure of $M = M' \times \mathbb{T}^{n''}$ in the definition of the quantization $\mathrm{Op}(a)$, it is natural to choose product charts (and associated product partitions of unity): take $U_{kj} = W_k \times V_j$ where $\{W_k\}$ is an atlas of $\mathbb{T}^{n''}$ and $\{V_j\}$ an atlas of M'. With such a choice of charts and partition of unity, we note that if $a_T \in \mathscr{C}^\infty(T^*\mathbb{T}^{n''})$ and $a_{M'} \in \mathscr{C}^\infty(T^*M')$, then $a = a_T \otimes a_{M'}$ is quantified as

$$\mathrm{Op}(a) = \mathrm{Op}(a_T)\,\mathrm{Op}(a_{M'}) = \mathrm{Op}(a_{M'})\,\mathrm{Op}(a_T),$$

where $\mathrm{Op}(a_T), \mathrm{Op}(a_{M'})$ denote the quantizations on $\mathbb{T}^{n''}$ and M' defined in local charts (relative to the charts W_k and V_j respectively, and associated partitions of unity). Similarly, it is convenient to take product charts and partitions of unity on $\mathbb{T}^{n''}$, i.e. take $\{W^\pm\}$ an atlas of $\mathbb{T}^1$ and choose each W_k of the form $W^\pm \times \cdots \times W^\pm$.

B.4. Fourier multipliers on the torus $\mathbb{T}^{n''}$

Given $u \in \mathscr{C}^\infty(\mathbb{T}^{n''})$, we define by $\underline{u}$ is the unique $2\pi\mathbb{Z}^{n''}$-periodic function on $\mathbb{R}^{n''}$ coinciding with u on $(0, 2\pi]^{n''}$. We have in particular $\underline{u} \in \mathscr{S}'(\mathbb{R}^{n''})$.

We denote by $\mathscr{F}(\underline{u})$ its Fourier Transform, where, for $v \in \mathscr{S}(\mathbb{R}^{n''})$ we use the following normalization of the Fourier transform:

$$\mathscr{F}(v)(x) = \int v(x) e^{-ix\cdot\xi} dx, \quad \mathscr{F}^{-1}(v)(\xi) = \frac{1}{(2\pi)^{n''}} \int v(\xi) e^{ix\cdot\xi} d\xi.$$

Taking $\chi \in \mathscr{C}^\infty(\mathbb{R}^{n''})$ bounded, we may define the Fourier multiplier $\chi(D)$ on $\mathscr{C}^\infty(\mathbb{T}^{n''})$ by

$$(\chi(D)u)(x) = \mathscr{F}^{-1}\big(\chi(\xi)\mathscr{F}(\underline{u})(\xi)\big)(x), \quad x \in \mathbb{T}^{n''}.$$

Such Fourier multiplier are linked to the quantization $\mathrm{Op}(\chi)$ defined above on $\mathbb{T}^{n''}$ by the following proposition (for charts and partition of unity as in the previous section).

Proposition B.7. — *Take $\chi \in \mathscr{C}^\infty(\mathbb{R}^{n''})$ with bounded derivatives. Then, we have*

$$\chi(D/\lambda) - \mathrm{Op}(\chi(\xi/\lambda)) = O_{\mathcal{L}(L^2(\mathbb{T}^{n''}))}(\lambda^{-1}).$$

This implies that

$$1 \otimes \chi(D/\lambda) - \mathrm{Op}\left(1 \otimes \chi(\xi/\lambda)\right) = O_{\mathcal{L}(L^2(M))}(\lambda^{-1}) \tag{B.7}$$

Finally, this definition of Fourier multipliers is linked to Fourier series as follows.

Lemma B.8. — *Take $\chi \in \mathscr{C}^\infty(\mathbb{R}^{n''})$ with bounded derivatives. Then, for any $u \in \mathscr{C}^\infty(\mathbb{T}^{n''})$, we have*

$$(\chi(D)u)(x) = \sum_{k\in\mathbb{Z}^{n''}} \chi(k)\hat{u}_k e^{ik\cdot x}, \quad x \in \mathbb{T}^{n''},\ \hat{u}_k = \frac{1}{(2\pi)^{n''}} \int_{\mathbb{T}^{n''}} u(y) e^{-ik\cdot y} dy.$$

Proof. — We first write $u(x) = \sum_{k\in\mathbb{Z}^{n''}} \hat{u}_k e^{ik\cdot x}$ on $\mathbb{T}^{n''}$, so that $\underline{u}(x) = \sum_{k\in\mathbb{Z}^{n''}} \hat{u}_k e^{ik\cdot x}$ on $\mathbb{R}$. Hence, we obtain

$$\mathscr{F}(\underline{u})(\xi) = \mathscr{F}\Big(\sum_{k\in\mathbb{Z}^{n''}} \hat{u}_k e^{ik\cdot x}\Big)(\xi) = \sum_{k\in\mathbb{Z}^{n''}} \hat{u}_k((2\pi)^{n''}\delta_{\xi=k}).$$

This implies

$$\begin{aligned}\mathscr{F}^{-1}\big(\chi(\xi)\mathscr{F}(\underline{u})(\xi)\big)(x) &= \mathscr{F}^{-1}\bigg(\chi(\xi)\sum_{k\in\mathbb{Z}^{n''}} \hat{u}_k(2\pi)^{n''}\delta_{\xi=k}\bigg)(x) \\ &= \mathscr{F}^{-1}\bigg(\sum_{k\in\mathbb{Z}^{n''}} \hat{u}_k\chi(k)(2\pi)^{n''}\delta_{\xi=k}\bigg)(x) \\ &= \sum_{k\in\mathbb{Z}^{n''}} \chi(k)\hat{u}_k e^{ik\cdot x},\end{aligned}$$

which concludes the proof of the lemma. □

Appendix C. Sharpness of Estimate (2.3): converse of Theorem 2.1

In this appendix, we prove that our Estimate (2.3) on $\mathbb{R}^d$ is optimal.

Lemma C.1. — *Let $Q_0 := -\Delta + iW(x)$ with W satisfying (2.2). Assume that there exists a positive constant μ_0 such that*

$$\forall \mu \geqslant \mu_0, \exists \beta(\mu) > 0, \forall u \in \mathscr{C}_c^2(\mathbb{R}^d),$$
$$\|(Q_0 - \mu)u\|_{L^2(\mathbb{R}^d)} \geqslant \beta(\mu)\mu^{\frac{\gamma}{2\gamma+1}}\|u\|_{L^2(\mathbb{R}^d)}.$$

Then $\limsup_{\mu\to+\infty} \beta(\mu) < +\infty$.

Note that the optimality is proved here for any fixed potential W. The proof is the same for the operator Q_0^λ defined in (2.1), where the potential W_λ may depend on a parameter (however satisfying the uniform estimate (2.2)).

Proof. — We consider the following affine symplectic mapping $(x,\xi) \mapsto (y,\eta)$ of $\mathbb{R}^{2d}$:

$$\begin{cases} x = \mu^\kappa y, \\ \xi_1 = \mu^{-\kappa}\eta_1 + \mu^{1/2}, \quad \xi' = \mu^{-\kappa}\eta', \end{cases}$$

where κ is a positive constant to be chosen later. The operator $Q_0 - \mu$ has the Weyl symbol $|\xi|^2 + iW(x)$ and is thus unitarily equivalent to the operator b^w with

$$\begin{aligned} b(y,\eta) &= (\mu^{-\kappa}\eta_1 + \mu^{1/2})^2 + \mu^{-2\kappa}|\eta'|^2 + W(\mu^\kappa y) - \mu \\ &= 2\mu^{\frac{1}{2}-\kappa}\eta_1 + \mu^{-2\kappa}|\eta|^2 + i\mu^{2\kappa\gamma}W(\mu^\kappa y)\mu^{-2\kappa\gamma}. \end{aligned}$$

We choose now κ so that $\frac{1}{2} - \kappa = 2\kappa\gamma$, i.e. $\kappa = 1/(4\gamma+2)$ (entailing $2\kappa\gamma = \gamma/(2\gamma+1)$) and we obtain

$$b(y,\eta) = \mu^{\frac{\gamma}{2\gamma+1}}c_\mu(y,\eta), \quad c_\mu(y,\eta) = \eta_1 + i\underbrace{W(\mu^\kappa y)\mu^{-2\kappa\gamma}}_{\substack{\text{bounded from above}\\ \text{and below by positive}\\ \text{constants}}} + |\eta|^2\mu^{-\kappa-\frac{1}{2}}.$$

Let $u \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ with L^2 norm 1. From our assumption in the lemma, we find with some unitary U_μ

$$\|U_\mu^* c_\mu^w U_\mu u\|_{L^2(\mathbb{R}^d)} \geqslant \beta(\mu)\|u\|_{L^2(\mathbb{R}^d)} \Longrightarrow \|c_\mu^w U_\mu u\|_{L^2(\mathbb{R}^d)} \geqslant \beta(\mu).$$

Since the mapping U_μ is also an isomorphism of $\mathscr{C}_c^\infty(\mathbb{R}^d)$ (from the particular form of the symplectic mapping, the mapping U_μ is the composition of a multiplication by a factor $e^{i\alpha x_1}$ with a rescaling $v \mapsto v(\lambda x)\lambda^{d/2}$), we may choose

$$u = U_\mu^* w_0, \quad w_0 \in \mathscr{C}_c^\infty(\mathbb{R}^d) \text{ with } L^2 \text{ norm } 1,$$

and we obtain $\|c_\mu^w w_0\|_{L^2(\mathbb{R}^d)} \geqslant \beta(\mu)$, which implies

$$\begin{aligned}\limsup_{\mu\to+\infty} \beta(\mu) &\leqslant \limsup_{\mu\to+\infty} \|c_\mu^w w_0\|_{L^2(\mathbb{R}^d)} \\ &\leqslant \|D_1 w_0\|_{L^2(\mathbb{R}^d)} + C_1\|w_0\|_{L^2(\mathbb{R}^d)} + \limsup_{\mu\to+\infty} \mu^{-\kappa-\frac{1}{2}} \|\Delta w_0\|_{L^2(\mathbb{R}^d)} \\ &= \|D_1 w_0\|_{L^2(\mathbb{R}^d)} + C_1\|w_0\|_{L^2(\mathbb{R}^d)} \\ &< +\infty,\end{aligned}$$

which gives the result. □

Bibliography

[1] N. Anantharaman & M. Léautaud, "Sharp polynomial decay rates for the damped wave equation on the torus", *Anal. PDE* **7** (2014), no. 1, p. 159-214.

[2] N. Anantharaman & F. Macià, "Semiclassical measures for the Schrödinger equation on the torus", *J. Eur. Math. Soc. (JEMS)* **16** (2014), no. 6, p. 1253-1288.

[3] C. Bardos, G. Lebeau & J. Rauch, "Un exemple d'utilisation des notions de propagation pour le contrôle et la stabilisation de problèmes hyperboliques", *Rend. Sem. Mat. Univ. Politec. Torino* (1988), no. Special Issue, p. 11-31 (1989), Nonlinear hyperbolic equations in applied sciences.

[4] C. J. K. Batty, R. Chill & Y. Tomilov, "Fine scales of decay of operator semigroups", *J. Eur. Math. Soc.* **18** (2016), no. 4, p. 853-929.

[5] C. J. K. Batty & T. Duyckaerts, "Non-uniform stability for bounded semi-groups on Banach spaces", *J. Evol. Equ.* **8** (2008), no. 4, p. 765-780.

[6] J.-M. Bony, "Second microlocalization and propagation of singularities for semilinear hyperbolic equations", in *Hyperbolic equations and related topics (Katata/Kyoto, 1984)*, Academic Press, Boston, MA, 1986, p. 11-49.

[7] J.-M. Bony & N. Lerner, "Quantification asymptotique et microlocalisations d'ordre supérieur. I", *Ann. Sci. École Norm. Sup. (4)* **22** (1989), no. 3, p. 377-433.

[8] A. Borichev & Y. Tomilov, "Optimal polynomial decay of functions and operator semigroups", *Math. Ann.* **347** (2010), no. 2, p. 455-478.

[9] N. Burq & P. Gérard, "Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes", *C. R. Acad. Sci. Paris Sér. I Math.* **325** (1997), no. 7, p. 749-752.

[10] N. Burq & M. Hitrik, "Energy decay for damped wave equations on partially rectangular domains", *Math. Res. Lett.* **14** (2007), no. 1, p. 35-47.

[11] N. Burq & C. Zuily, "Laplace eigenfunctions and damped wave equation ii : product manifolds", preprint, `http://arxiv.org/abs/1503.05513`, 2015.

[12] A.-P. Calderón & R. Vaillancourt, "On the boundedness of pseudo-differential operators", *J. Math. Soc. Japan* **23** (1971), p. 374-378.

[13] R. Coifman & Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque, vol. 57, Société Mathématique de France, Paris, 1978, With an English summary, i+185 pages.

[14] E. B. Davies, "Pseudo-spectra, the harmonic oscillator and complex resonances", *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **455** (1999), no. 1982, p. 585-599.

[15] J.-M. Delort, *F.B.I. transformation*, Lecture Notes in Mathematics, vol. 1522, Springer-Verlag, Berlin, 1992, Second microlocalization and semilinear caustics, vi+101 pages.

[16] W. Deng, Personal communication, 2012.

[17] ———, "Resolvent Estimates and Energy Decay for the damped wave equation on the torus", preprint, 2015.

[18] C. Fermanian-Kammerer, “Mesures semi-classiques 2-microlocales”, *C. R. Acad. Sci. Paris Sér. I Math.* **331** (2000), no. 7, p. 515-518.

[19] ———, “Analyse à deux échelles d’une suite bornée de L^2 sur une sous-variété du cotangent”, *C. R. Math. Acad. Sci. Paris* **340** (2005), no. 4, p. 269-274.

[20] C. Fermanian-Kammerer & P. Gérard, “Mesures semi-classiques et croisement de modes”, *Bull. Soc. Math. France* **130** (2002), no. 1, p. 123-168.

[21] L. Hörmander, *The Analysis of Linear Partial Differential Operators III: Pseudo-differiential operators*, Grundlehren der Mathematischen Wissenschaften, vol. 274, Springer-Verlag, 1985, viii+525 pages.

[22] M. Kashiwara & T. Kawai, “Second-microlocalization and asymptotic expansions”, in *Complex analysis, microlocal calculus and relativistic quantum theory (Proc. Internat. Colloq., Centre Phys., Les Houches, 1979)*, Lecture Notes in Phys., vol. 126, Springer, Berlin-New York, 1980, p. 21-76.

[23] J. Le Rousseau, M. Léautaud & L. Robbiano, “Controllability of a parabolic system with a diffusive interface”, *J. Eur. Math. Soc. (JEMS)* **15** (2013), no. 4, p. 1485-1574.

[24] M. Léautaud & N. Lerner, “Energy decay for a locally undamped wave equation”, submitted, `http://arxiv.org/abs/1411.7271`, 2014.

[25] ———, “Sharp polynomial energy decay for locally undamped waves”, in *Séminaire Laurent Schwartz—Équations aux dérivées partielles et applications. Année 2014–2015*, Sémin. Équ. Dériv. Partielles, doi: 10.5802/slsedp.79, École Polytech, 2015, p. Exp. No. XXI,13.

[26] G. Lebeau, “Deuxième microlocalisation sur les sous-variétés isotropes”, *Ann. Inst. Fourier (Grenoble)* **35** (1985), no. 2, p. 145-216.

[27] ———, “Control for hyperbolic equations”, in *Journées “Équations aux Dérivées Partielles” (Saint-Jean-de-Monts, 1992)*, École Polytech., 1992, p. 24.

[28] ———, “Équations des ondes amorties”, in *Séminaire sur les Équations aux Dérivées Partielles, 1993–1994*, École Polytech., 1994, Exp. No. XV, 16.

[29] ———, “Équation des ondes amorties”, in *Algebraic and geometric methods in mathematical physics (Kaciveli, 1993)*, Math. Phys. Stud., vol. 19, Kluwer Acad. Publ., Dordrecht, 1996, p. 73-109.

[30] N. Lerner, *Metrics on the phase space and non-selfadjoint pseudo-differential operators*, Pseudo-Differential Operators. Theory and Applications, vol. 3, Birkhäuser Verlag, Basel, 2010, xii+397 pages.

[31] Z. Liu & B. Rao, “Characterization of polynomial decay rate for the solution of linear evolution equation”, *Z. Angew. Math. Phys.* **56** (2005), no. 4, p. 630-644.

[32] F. Macià, “High-frequency propagation for the Schrödinger equation on the torus”, *J. Funct. Anal.* **258** (2010), no. 3, p. 933-955.

[33] L. Miller, “Short waves through thin interfaces and 2-microlocal measures”, in *Journées “Équations aux Dérivées Partielles” (Saint-Jean-de-Monts, 1997)*, École Polytech., Palaiseau, 1997, p. Exp. No. XII, 12.

[34] K.-D. Phung, “Polynomial decay rate for the dissipative wave equation”, *J. Differential Equations* **240** (2007), no. 1, p. 92-124.

[35] K. Pravda-Starov, “A complete study of the pseudo-spectrum for the rotated harmonic oscillator”, *J. London Math. Soc. (2)* **73** (2006), no. 3, p. 745-761.

[36] J. Ralston, “Solutions of the wave equation with localized energy”, *Comm. Pure Appl. Math.* **22** (1969), p. 807-823.

[37] J. Rauch & M. Taylor, “Exponential decay of solutions to hyperbolic equations in bounded domains”, *Indiana Univ. Math. J.* **24** (1974), p. 79-86.