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# Stochastic partial differential equations: a rough paths view on weak solutions via Feynman-Kac 

Joscha Diehl ${ }^{(1)}$, Peter K. Friz ${ }^{(2)}$ and Wilhelm Stannat ${ }^{(3)}$


#### Abstract

We discuss regular and weak solutions to rough partial differential equations (RPDEs), thereby providing a (rough path-)wise view on important classes of SPDEs. In contrast to many previous works on RPDEs, our definition gives honest meaning to RPDEs as integral equations, based on which we are able to obtain existence, uniqueness and stability results. The case of weak "rough" forward equations, may be seen as robustification of the (measure-valued) Zakai equation in the rough path sense. Feynman-Kac representation for RPDEs, in formal analogy to similar classical results in SPDE theory, play an important role.

Résumé. - Nous discutons des solutions régulières et faibles d'équations aux dérivées partielles rugueuses (EDPR), fournissant ainsi un point de vue «chemins rugueux » sur des classes importantes d' EDPS. Contrairement à de nombreux travaux antérieurs sur le sujet, notre définition donne un sens honnête aux EDPR en tant qu'équations intégrales, sur la base duquel nous sommes en mesure d'obtenir l'existence, l'unicité et la stabilité des résultats. Le cas d'équations forward faibles «rugueuses» peut être vu comme une robustification de l'équation de Zakai à valeurs mesure, au sens des chemins rugueux. Des représentations de type Feynman-Kac pour EDPR, par analogie formelle avec les résultats classiques similaires dans la théorie des EDPS, jouent un rôle important.


Keywords: stochastic partial differential equations, Zakai equation, Feynman-Kac formula, rough partial differential equations, rough paths.

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${ }^{(1)}$ University of California San Diego, University of California San Diego, Dept of Mathematics - MC 0112, 9500 Gilman Drive La Jolla, CA 92093, USA joscha.diehl@gmail.com
${ }^{(2)}$ TU Berlin, Institut für Mathematik, MA 7-2, Strasse des 17. Juni 136, 10623 Berlin, Germany - ifriz@math.tu-berlin.de - Weierstrass-Institut für Angewandte Analysis und Stochastik, Mohrenstrasse 39, 10117 Berlin, Germany -friz@wias-berlin.de
(3) TU Berlin, Institut für Mathematik, MA 7-2, Strasse des 17. Juni 136, 10623 Berlin, Germany - stannat@math.tu-berlin.de

## 1. Introduction

Consider a diffusion process $X$ on $\mathbb{R}^{d}$ with generator given by a second order differential operator $L$. In its simplest form, the Feynman-Kac formula asserts that, for suitable data $g$,

$$
\begin{equation*}
u(t, x)=\mathbb{E}^{t, x}\left[g\left(X_{T}\right)\right], \quad t \leqslant T, x \in \mathbb{R}^{d} \tag{1.1}
\end{equation*}
$$

solves a parabolic partial differential equation, namely the terminal value problem

$$
\left\{\begin{aligned}
-\partial_{t} u_{t} & =L u_{t} \\
u(T, \cdot) & =g
\end{aligned}\right.
$$

(Below we will consider slightly more general operators including zero order terms, causing additional exponential factors in the Feynman-Kac formula.) On the other hand, the law of $X_{t}$ started at $X_{0}=x$, solves the forward (or Fokker-Planck) equation

$$
\left\{\begin{aligned}
\partial_{t} \rho_{t} & =L^{*} \rho_{t} \\
\rho(0, \cdot) & =\delta_{x}
\end{aligned}\right.
$$

Formally at least, an infinitesimal version of (1.1) is given by

$$
\partial_{t}\left\langle u_{t}, \rho_{t}\right\rangle=\left\langle-L u_{t}, \rho_{t}\right\rangle+\left\langle u_{t}, L^{*} \rho_{t}\right\rangle=0
$$

and indeed the resulting duality $\left\langle u_{T}, \rho_{T}\right\rangle=\left\langle u_{0}, \rho_{0}\right\rangle$ is nothing than restatement of (1.1), at $t=0$.

In both cases, forward and backward, there may not exist a classical $C^{1,2}$ solution. Indeed, it suffices to consider the case of degenerate $X$ so that $\rho_{t}$ remains a measure; in the backward case consider $g \notin C^{2}$. In both cases one then needs a concept of weak solutions. A natural way to do this, consists in testing the equation in space; that is, to consider the evolution for $\left\langle u_{t}, \phi\right\rangle$ and $\left\langle\rho_{t}, f\right\rangle$ where $\phi$ and $f$ are suitable test functions defined on $\mathbb{R}^{d}$.

Applications from filtering theory lead to (backward) SPDEs of the form

$$
\left\{\begin{aligned}
-\mathrm{d} u_{t} & =L\left[u_{t}\right] \mathrm{d} t+\Gamma\left[u_{t}\right] \circ \mathrm{d} W_{t} \\
u(T, \cdot) & =g
\end{aligned}\right.
$$

where $W=\left(W^{1}, \ldots, W^{e}\right)$ and $\Gamma=\left(\Gamma_{1}, \ldots, \Gamma_{e}\right)$ are first order differential operators, ${ }^{(1)}$ in duality with the forward (or Zakai) equation

$$
\left\{\begin{aligned}
\mathrm{d} \rho_{t} & =L^{*}\left[\rho_{t}\right] \mathrm{d} t+\Gamma^{*}\left[\rho_{t}\right] \circ \mathrm{d} W_{t} \\
\rho(0, \cdot) & =\delta_{x} .
\end{aligned}\right.
$$

[^0]Such SPDEs were studied extensively in classical works [22, 24, 25]. It is a natural question, studied for instance in a series of papers by Gyöngy [17, 18], to what extent such SPDEs are approximated by (random) PDEs, upon replacing the (Stratonovich) differential $\mathrm{d} W=\mathrm{d} W(\omega)$ by $\dot{W}^{\varepsilon}(\omega) \mathrm{d} t$, given a suitable family of smooth approximation $\left(W^{\varepsilon}\right)$ to Brownian motion. In recent works [10, 14], also [12, Ch. 12], it was shown that the backward solutions $u^{\varepsilon}$, interpreted as viscosity solution (assuming $g \in C_{b}$ ) actually converge locally uniformly, with limit $u$ only depending on the rough path limit of $\left(W^{\varepsilon}\right)$. Writing $\mathbf{W}=(W, \mathbb{W})$ for such a (deterministic!) rough path (see the Appendix and [12] for notation) say, $\alpha$-Hölder for $1 / 3<\alpha<1 / 2$, the question arises if one can give an honest meaning to the equations

$$
\begin{align*}
-\mathrm{d} u_{t} & =L\left[u_{t}\right] \mathrm{d} t+\Gamma\left[u_{t}\right] \mathrm{d} \mathbf{W}_{t}, \\
\mathrm{~d} \rho_{t} & =L^{*}\left[\rho_{t}\right] \mathrm{d} t+\Gamma^{*}\left[\rho_{t}\right] \mathrm{d} \mathbf{W}_{t} . \tag{1.2}
\end{align*}
$$

In the aforementioned works, these "rough partial differential equations" (RPDEs) had only formal meaning. The actual definition was then given either in terms of a (flow)transformed equation in the spirit of Kunita (e.g. [14], also [12, p. 177]) or in terms of a unique continuous extension of the PDE solution as function of driving noise, $[5,14]$.

There are two difficulties with such rough partial differential equations. The first one is the temporal roughness of $\mathbf{W}$, a problem that has been wellunderstood from the rough path analysis of SDEs. Indeed, following Davie's approach to RDEs [8], the (rough) pathwise meaning of

$$
\mathrm{d} X=\beta(X) \mathrm{d} \mathbf{W}
$$

is, by definition, and writing $X_{s, t}=X_{t}-X_{s}$ for path increments,

$$
X_{s, t}=\beta\left(X_{s}\right) W_{s, t}+\beta^{\prime} \beta\left(X_{s}\right) \mathbb{W}_{s, t}+o(|t-s|)
$$

Under suitable assumptions on $\beta$, uniqueness and local/global existence results are well-known. This quantifies the statement that $X$ is controlled by $W$, with "Gubinelli derivative" $\beta(X)$, and in turn implies the integral representation in terms of a bona-fide rough integral (cf. [12, Ch. 4])

$$
X_{t}-X_{s}=\int_{s}^{t} \beta(X) \mathrm{d} \mathbf{W}=\lim \sum_{[u, v] \in P} \beta\left(X_{u}\right) W_{u, v}+\beta^{\prime} \beta\left(X_{u}\right) \mathbb{W}_{u, v}
$$

This suggests that the meaning of the backward equation (1.2) is

$$
u(s, x)-u(t, x)=\int_{s}^{t} L\left[u_{r}\right] \mathrm{d} r+\int_{s}^{t} \Gamma\left[u_{r}\right] \mathrm{d} \mathbf{W}_{r}
$$

provided $u$ is sufficiently regular (in space) such as to make $L[u], \Gamma[u]$ meaningful, and provided the last term makes sense as rough integral. The other difficulty is exactly that $u$ may not be regular in space so that $L[u], \Gamma[u]$
require a weak meaning. More precisely, we propose the following spatially weak ${ }^{(2)}$ formulation, of the form

$$
\left\langle u_{s}, \phi\right\rangle-\left\langle u_{t}, \phi\right\rangle=\int_{s}^{t}\left\langle u_{r}, L^{*} \phi\right\rangle \mathrm{d} r+\int_{s}^{t}\left\langle u_{r}, \Gamma^{*} \phi\right\rangle \mathrm{d} \mathbf{W}_{r},
$$

where, again, we can hope to understand the last term as rough integral. (Everything said for backward equations translates, mutatis mutandis, to the forward setting.)

The main result of this paper is that - in all cases - one has existence and uniqueness results. Loosely speaking (and subject to suitable regularity assumptions on the coefficients of $L, \Gamma$; but no ellipticity assumptions) we have

Theorem 1.1. - For nice terminal data $g$ there exists a unique (spatially) regular solution to the backward RPDE. Similarly, for nice initial data $\rho_{0}$ (with nice density $p_{0}$, say) the forward RPDE has a unique (spatially) regular solution.

If the terminal data $g$ is only bounded and continuous, we have existence and uniqueness of a weak solution to the backward RPDE. Similarly, if the initial data $\rho_{0}$ of the forward $R P D E$ is only a finite measure, we have existence and uniqueness of a weak (here: measure-valued) solution to the forward RPDE.

In all cases, the (unique) solution depends continuously on the driving rough path and we have Feynman-Kac type representation formulae.

Let us briefly discuss the strategy of proof. In all cases (regular/weak, forward/backward) existence of a solution is verified via an explicit FeynmanKac type formula, based on a notion of "hybrid" Itô/rough differential equation, which already appeared in previous works [7, 10], see also [12]. We then use regular forward existence to show weak backward uniqueness (Theorem 2.8 ), which actually requires us to work with exponentially decaying test functions. Next, regular backward existence leads to weak (actually, measurevalued) forward uniqueness (Theorem 3.5), here we just need boundedness and some control in the sense of Gubinelli. Then weak (measure-valued) forward existence gives regular backward uniqueness. At last, we note that, subject to suitable smoothness assumptions on the coefficients, regular forward equations can be reformulated as regular backward equations, from which we deduce regular forward uniqueness.

[^1]It is a natural question what the above RPDE solutions have to do with classical SPDE solutions. To this end, recall [12, Ch. 9] consistency of RDEs with SDEs in the following sense: RDE solutions driven by $\mathbf{W}=\mathbf{W}^{\text {Strato }}(\omega)$, the usual (random) geometric rough path associated to Brownian motion $W$ via iterated Stratonovich integration, are solutions to the corresponding (Stratonovich) SDEs. Consider now - for the sake of argument - a regular backward RPDE solution; that is, the unique solution $u=u(t, x ; \mathbf{W})$ to

$$
-\mathrm{d} u_{t}=L\left[u_{t}\right] \mathrm{d} t+\Gamma\left[u_{t}\right] \mathrm{d} \mathbf{W}_{t}
$$

(with fixed $C_{b}^{2}$ time- $T$ terminal data). We expect that

$$
\begin{equation*}
\tilde{u}(t, x)=\tilde{u}(t, x ; \omega)=u\left(t, x ; \mathbf{W}^{\text {Strato }}(\omega)\right) \tag{1.3}
\end{equation*}
$$

is also a (and hopefully: the unique) solution to the (backward) SPDE, again with fixed terminal data,

$$
-\mathrm{d} \tilde{u}_{t}=L\left[\tilde{u}_{t}\right] \mathrm{d} t+\Gamma\left[\tilde{u}_{t}\right] \circ \mathrm{d} W_{t} .
$$

(Similar for weak backward and weak/regular forward equations.) Unfortunately, we cannot hope for a general RPDE/SPDE consistency statement for the simple reason that the choice of spaces in which SPDE existence and uniqueness statements are proven are model-dependent and therefore vary from paper to paper. In other words, checking that $\tilde{u}(t, x ; \omega)$ is a - and then the (unique) - SPDE solution within a given SPDE setting will necessarily require to check details specific to this setting. Luckily, there are arguments which do not force us into such a particular setting.

- Consider a notion of (Stratonovich) SPDE solution for which there are existence, uniqueness results and Wong-Zakai stability, by which we mean that the (unique bounded, or finite-measure valued) solutions to the random PDEs obtained by replacing $\mathrm{d} W(\omega)$ by the mollified $\dot{W}^{\varepsilon}(\omega) \mathrm{d} t$ converge to the unique SPDE solution. (Such Wong-Zakai results are found e.g. in the works of Gyöngy.) Assume also that our regularity assumptions fall within the scope of these existence and uniqueness results. Then, for fixed terminal (resp. initial) data, our unique RPDE solution, with driving rough path $\mathbf{W}=\mathbf{W}^{\text {Strato }}(\omega)$, coincides with (and in fact, may be a very pleasant version of) the unique SPDE solution. (This follows immediately from continuous dependence of our RPDE solutions on the driving rough paths, together with well-known rough path convergence of mollifier approximations [16].) In a context of viscosity solutions, this argument was spelled out in [14].
- Consider a notion of (Stratonovich) SPDE solution for which there are existence, uniqueness results and a Feynman-Kac representation formula. (This is the case in essentially every classical work on linear

SPDEs, especially in the filtering context.) Recall that such SPDE Feynman-Kac formulas are conditional expectations, given $W(\omega)$ (the observation, in the filtering context). In contrast, the FeynmanKac formula alluded to in Theorem 1.1, is of unconditional form $\mathbb{E}^{t, x}(\ldots)$, the expectation taken over some hybrid Itô-rough process (with rough driver $\mathrm{d} \mathbf{W}$ ). By a stochastic Fubini argument (similar to the one in [10]) one can show that the Feynman-Kac formula, evaluated at $\mathbf{W}=\mathbf{W}^{\text {Strato }}(\omega)$, indeed yields the SPDE FeynmanKac formula. In particular, our unique RPDE solution, with driving rough path $\mathbf{W}=\mathbf{W}^{\text {Strato }}(\omega)$, then coincides with the unique SPDE solution.

- At last, we consider an immediate consequence of our (rough path-) wise definition in case of $\mathbf{W}=\mathbf{W}^{\text {Strato }}(\omega)$. For the sake of argument, let us now focus on the weak backward equation,

$$
\left\langle u_{s}, \phi\right\rangle-\left\langle u_{t}, \phi\right\rangle=\int_{s}^{t}\left\langle u_{r}, L^{*} \phi\right\rangle \mathrm{d} r+\int_{s}^{t}\left\langle u_{r}, \Gamma^{*} \phi\right\rangle \mathrm{d} \mathbf{W}_{r}
$$

With $\tilde{u}(t, x ; \omega)=u\left(t, x ; \mathbf{W}^{\text {Strato }}(\omega)\right)$, as before it follows from consistency of rough with classical (backward) Stratonovich integration [12, Ch. 5] that

$$
\left\langle\tilde{u}_{s}, \phi\right\rangle-\left\langle\tilde{u}_{t}, \phi\right\rangle=\int_{s}^{t}\left\langle\tilde{u}_{r}, L^{*} \phi\right\rangle \mathrm{d} r+\int_{s}^{t}\left\langle\tilde{u}_{r}, \Gamma^{*} \phi\right\rangle \circ \mathrm{d} W,
$$

for the same class of spatial test functions. Such notion of weak (or distributional) SPDE solutions appear for instance in the works of Krylov, e.g. [21, Def. 4.6]. Hence, whenever such a notion of SPDE solution comes with uniqueness results, it is straight-forward to see that $\tilde{u}$, i.e. our solution constructed via rough paths, must coincide with the unique SPDE solution.

### 1.1. Notation

The second resp. first oder operators we shall consider are of the following form,

$$
\begin{aligned}
L u & :=\frac{1}{2} \operatorname{Tr}\left(\sigma(x) \sigma^{T}(x) D^{2} u\right)+\langle b(x), D u\rangle+c(x) u \\
\Gamma_{k} u & :=\left\langle\beta_{k}(x), D u\right\rangle+\gamma_{k}(x) u
\end{aligned}
$$

with $\sigma=\left(\sigma_{1}, \ldots, \sigma_{d_{B}}\right), \beta=\left(\beta_{1}, \ldots, \beta_{e}\right)$ and $b$ vector fields on $\mathbb{R}^{d}$ and scalar functions $c, \gamma_{1}, \ldots, \gamma_{e}$. We note that the formal adjoints are given as,

$$
\begin{aligned}
L^{*} \varphi & =\frac{1}{2} \operatorname{Tr}\left[\tilde{a}(x) D^{2} \varphi\right]+\langle\tilde{b}(x), D \varphi\rangle+\tilde{c}(x) \varphi \\
\Gamma_{k}^{*} \varphi & =\left\langle\tilde{\beta}_{k}(x), D \varphi\right\rangle+\tilde{\gamma}_{k}(x) \varphi
\end{aligned}
$$

where

$$
\begin{align*}
\tilde{a}(x) & :=a(x):=\sigma \sigma^{T}(x) \\
\tilde{b}_{i}(x) & :=\partial_{j} a_{j i}(x)-b_{i}(x) \\
\tilde{c}(x) & :=\frac{1}{2} \partial_{i j} a_{i j}(x)-\operatorname{div}(b)(x)+c(x)  \tag{1.4}\\
\tilde{\beta}_{k}(x) & :=-\beta_{k}(x) \\
\tilde{\gamma}_{k}(x) & :=-\operatorname{div}\left(\beta_{k}\right)(x)+\gamma_{k}(x) .
\end{align*}
$$

Precise assumptions on the coefficients will appear in the theorems below. Let us remark, however, that we did not push for optimal assumptions. As is typical in rough path theory, $C_{b}^{n}$-regularity (bounded, with bounded derivatives up to order $n$ ) can often be improved to Lip ${ }^{\gamma}$-regularity (in the sense of Stein) with $\gamma \in(n-1, n)$, depending on the Hölder exponent of the driving rough path.

## 2. The backward equation

Replacing the rough path by a smooth path, say $W \in C^{1}\left([0, T], \mathbb{R}^{e}\right)$ we certainly want to recover a solution to the PDE

$$
\left\{\begin{align*}
-\partial_{t} u_{t} & =L u_{t}+\sum_{k=1}^{e} \Gamma_{k} u_{t} \dot{W}_{t}^{k}\left(\equiv L u_{t}+\Gamma u_{t} \dot{W}_{t}\right)  \tag{2.1}\\
u(T, \cdot) & =g
\end{align*}\right.
$$

For the precise statement of the following lemma, let us now introduce a suitable class of test functions with exponential decay, that will become important in the concept of weak solutions.

Definition 2.1. - For $n \geqslant 0$ denote with $C_{\exp }^{n}\left(\mathbb{R}^{d}\right)$ the class of functions $\phi \in C^{n}\left(\mathbb{R}^{d}\right)$ such that there exists $c>0$ such that

$$
\left|D^{k} \phi(x)\right| \leqslant c e^{-\frac{1}{c}|x|}, k=0,1, \ldots, n .
$$

Define the quasinorm ${ }^{(3)}\|\cdot\|_{C_{\exp }^{n}\left(\mathbb{R}^{d}\right)}$ as the infimum over the values of $c$ satisfying the bound. Define moreover the space $C_{\mathrm{exp}}^{m, n}\left([0, T] \times \mathbb{R}^{d}\right)$ to be the

[^2]class of functions $\phi \in C^{m, n}\left([0, T] \times \mathbb{R}^{d}\right)$ such that there exists $c>0$ such that
$$
\left|D^{j, k} \phi(t, x)\right| \leqslant c e^{-\frac{1}{c}|x|}, j=0, \ldots, m, k=0,1, \ldots, n
$$

We then recall the following Feynman-Kac representation for solutions to the classical equation (2.1).

Lemma 2.2. - Assume $c, b, \sigma_{i}, \gamma_{j}, \beta_{k} \in C_{b}^{2}, i=1, \ldots, d_{B}, j, k=1, \ldots, e$. Let $u$ be given as

$$
\begin{equation*}
u(t, x)=\mathbb{E}^{t, x}\left[g\left(X_{T}\right) \exp \left(\int_{t}^{T} c\left(X_{r}\right) \mathrm{d} r+\int_{t}^{T} \gamma\left(X_{r}\right) \dot{W}_{r} \mathrm{~d} r\right)\right] \tag{2.2}
\end{equation*}
$$

with

$$
\mathrm{d} X_{t}=\sigma\left(X_{t}\right) \mathrm{d} B(\omega)+b\left(X_{t}\right) \mathrm{d} t+\beta\left(X_{t}\right) \dot{W}_{t} \mathrm{~d} t
$$

where $B$ is a $d_{B}$-dimensional Brownian motion and $W \in C^{1}\left([0, T], \mathbb{R}^{e}\right)$.
(1) If $g \in C_{b}^{2}\left(\mathbb{R}^{d}\right)$ then $u$ is the unique $C_{b}^{1,2}\left([0, T] \times \mathbb{R}^{d}\right)$ solution to (2.1). If moreover $g \in C_{\exp }^{2}\left(\mathbb{R}^{d}\right)$ then $u \in C_{\exp }^{1,2}\left([0, T] \times \mathbb{R}^{d}\right)$.
(2) If $g \in C_{b}\left(\mathbb{R}^{d}\right)$ then $u \in C_{b}\left([0, T] \times \mathbb{R}^{d}\right)$ and it is the unique bounded analytically weak solution to (2.1), that is, for $\varphi \in \mathcal{D}\left(\mathbb{R}^{d}\right)$

$$
\begin{equation*}
\left\langle u_{t}, \varphi\right\rangle=\langle g, \varphi\rangle+\int_{t}^{T}\left\langle u_{r}, L^{*} \varphi\right\rangle \mathrm{d} r+\int_{t}^{T}\left\langle u_{r}, \Gamma^{*} \varphi\right\rangle \mathrm{d} W_{r} \tag{2.3}
\end{equation*}
$$

Proof. - Let us first note that the expectation actually exists, since $g$, $c, \gamma$ and $|\dot{W}|$ are bounded.
(1): The proof amounts to taking derivatives under the expectation, see for example Theorem V.7.4 in [19], which shows that $u$ is a $C_{b}^{1,2}$ solution. ${ }^{(4)}$ Uniqueness follows from the maximum principle, see for example Theorem 8.1.4 in [20].

If $g \in C_{\exp }^{2}\left(\mathbb{R}^{d}\right)$ then one can show that actually $u \in C_{\exp }^{1,2}\left([0, T] \times \mathbb{R}^{d}\right)$. This is similar to the rough case in Theorem 2.8, so we omit the proof here.
(2): Take some $g^{n} \in C_{b}^{2}\left(\mathbb{R}^{d}\right)$ converging to $g$ locally uniformly, uniformly bounded by $2\|g\|_{\infty}$. Let $u^{n}$ be the corresponding classical solution from part (1). Then $u^{n}$ satisfies (2.3) with $g$ replaced by $g^{n}$. Now by the FeynmanKac representation, we get for every $N>0$,

$$
\begin{aligned}
\left|u^{n}(t, x)-u(t, x)\right| & \lesssim \mathbb{E}\left[\left|g^{n}\left(X_{T}^{t, x}\right)-g\left(X_{T}^{t, x}\right)\right|^{2}\right]^{1 / 2} \\
& \leqslant \sup _{|y| \leqslant N}\left|g^{n}(y)-g(y)\right|+2\|g\|_{\infty} \mathbb{E}\left[1_{[-N, N]^{C}}\left(\mid X_{T}^{t, x}\right)\right]
\end{aligned}
$$

[^3]Hence for every $R>0$

$$
\sup _{|x| \leqslant R}\left|u^{n}(t, x)-u(t, x)\right| \lesssim \sup _{|y| \leqslant N}\left|g^{n}(y)-g(y)\right|+\frac{1}{N} \sup _{|x| \leqslant R} \mathbb{E}\left[\left|X_{T}^{t, x}\right|\right]
$$

from which the locally uniform convergence of $u_{t}^{n}$ to $u_{t}$ follows, uniformly in $t \leqslant T$. Taking the limit in the integral equation, we then see that $u$ satisfies (2.3).

To show uniqueness, let $u \in C_{b}\left([0, T] \times \mathbb{R}^{d}\right)$ be any solution to (2.3). It is immediate that the equation then also holds for test functions $\varphi \in C_{c}^{2}\left(\mathbb{R}^{d}\right)$. It is straightforward to show that for $\varphi \in C_{c}^{1,2}\left([0, T] \times \mathbb{R}^{d}\right)$ we have

$$
\begin{equation*}
\left\langle u_{t}, \varphi_{t}\right\rangle=\left\langle g, \varphi_{T}\right\rangle+\int_{t}^{T}\left\langle u_{r},-\partial_{t} \varphi_{r}+L^{*} \varphi_{r}\right\rangle \mathrm{d} r+\int_{t}^{T}\left\langle u_{r}, \Gamma^{*} \varphi_{r}\right\rangle \mathrm{d} W_{r} \tag{2.4}
\end{equation*}
$$

Finally, via dominated convergence, (2.4) also holds for $\varphi \in C_{\exp }^{1,2}\left([0, T] \times \mathbb{R}^{d}\right)$.
Now, an application of Lemma $3.1(2)$ yields, for every $t \in[0, T)$ and every $\phi \in C_{\exp }^{4}\left(\mathbb{R}^{d}\right)$, a $\varphi \in C_{\exp }^{1,2}\left([t, T] \times \mathbb{R}^{d}\right)$ that satisfies

$$
\begin{aligned}
\partial_{s} \varphi_{s} & =L^{*} \varphi_{s}+\Gamma^{*} \varphi_{s} \dot{W}_{s} \\
\varphi_{t} & =\phi
\end{aligned}
$$

Then, by (2.4),

$$
\left\langle u_{t}, \phi\right\rangle=\left\langle u_{t}, \varphi_{t}\right\rangle=\left\langle g, \varphi_{T}\right\rangle
$$

So, tested against $\phi \in C_{c}^{4}\left(\mathbb{R}^{d}\right)$, all solutions coincide at every $t \in[0, T]$, which gives uniqueness in $C_{b}\left([0, T] \times \mathbb{R}^{d}\right)$.

In (2.1), replacing $W$ by a rough path $\mathbf{W}$, we are interested in the following formal equation

$$
\left\{\begin{align*}
-\mathrm{d} u & =L u \mathrm{~d} t+\Gamma u \mathrm{~d} \mathbf{W}  \tag{2.5}\\
u(T, \cdot) & =g
\end{align*}\right.
$$

We will next introduce two solution concepts, weak and regular in nature (see Definitions 2.3 and 2.6 below). Both rely on the (standard) notion of a controlled rough path space $\mathscr{D}_{W}^{2 \alpha}$ - see Appendix, Definition 4.1, for a recall.

Definition 2.3 (Analytically weak backward RPDE solution). - Given an $\alpha$-Hölder rough path $\mathbf{W}=(W, \mathbb{W}), \alpha \in(1 / 3,1 / 2]$, we say that a bounded, measurable function $u=u(t, x ; \mathbf{W})=u_{t}(x ; \mathbf{W})$ is an analytically weak solution to (2.5), if for all functions $\varphi \in C_{\exp }^{3}\left(\mathbb{R}^{d}\right)$, we have $\left(Y^{\varphi},\left(Y^{\varphi}\right)^{\prime}\right) \in$ $\mathscr{D}_{W}^{2 \alpha}$ with

$$
\left(Y_{t}\right)_{i}^{\varphi}:=\left\langle u_{t}, \Gamma_{i}^{*} \varphi\right\rangle, \quad\left(Y_{t}^{\varphi}\right)_{i j}^{\prime}:=-\left\langle u_{t}, \Gamma_{j}^{*} \Gamma_{i}^{*} \varphi\right\rangle
$$

that is

$$
\begin{equation*}
\left\|Y^{\varphi},\left(Y^{\varphi}\right)^{\prime}\right\|_{W, 2 \alpha}<\infty \tag{2.6}
\end{equation*}
$$

and the following equation is satisfied:

$$
\begin{equation*}
\left\langle u_{t}, \varphi\right\rangle=\langle g, \varphi\rangle+\int_{t}^{T}\left\langle u_{r}, L^{*} \varphi\right\rangle \mathrm{d} r+\int_{t}^{T}\left\langle u_{r}, \Gamma^{*} \varphi\right\rangle \mathrm{d} \mathbf{W}_{r}, \quad 0 \leqslant t \leqslant T \tag{2.7}
\end{equation*}
$$

Here, $\int Y \mathrm{~d} \mathbf{W}$ is the rough integral against $\left(Y, Y^{\prime}\right)$.
Remark 2.4. - Different from the smooth case, Lemma 2.2, we work with test functions in the larger class $C_{\exp }^{3}$ here. This is necessary, since with presence of the rough integral we were not able to automatically enlarge the space of functions for which the integral equation holds, as was done in the proof of Lemma 2.2.

Remark 2.5. - Heuristically, the origin of the compensator term $Y_{t}^{\prime}=$ $\left\langle u_{t}, \Gamma^{*} \Gamma^{*} \varphi\right\rangle$ can be seen as follows. One certainly expects that

$$
\int_{s}^{t}\left\langle u_{r}, \Gamma^{*} \varphi\right\rangle \mathrm{d} \mathbf{W}_{r} \approx\left\langle u_{s}, \Gamma^{*} \varphi\right\rangle W_{s, t}
$$

where $a \approx b$ means $a-b=O\left(|t-s|^{2 \alpha}\right)$. Hence, in view of (2.7),

$$
\left\langle u_{t}, \varphi\right\rangle-\left\langle u_{s}, \varphi\right\rangle \approx-\int_{s}^{t}\left\langle u, \Gamma^{*} \varphi\right\rangle \mathrm{d} \mathbf{W} \approx-\left\langle u_{s}, \Gamma^{*} \varphi\right\rangle W_{s, t}
$$

Replacing $\varphi$ by $\Gamma^{*} \varphi$ (note that the latter is not in $C_{\exp }^{3}$ though) gives

$$
\left\langle u_{t}, \Gamma^{*} \varphi\right\rangle-\left\langle u_{s}, \Gamma^{*} \varphi\right\rangle=-\left\langle u_{s}, \Gamma^{*} \Gamma^{*} \varphi\right\rangle W_{s, t}+O\left(|v-u|^{2 \alpha}\right)
$$

so that $t \mapsto\left\langle u_{t}, \Gamma^{*} \varphi\right\rangle$ is controlled by $W$, with Gubinelli derivative $-\left\langle u_{t}, \Gamma^{*} \Gamma^{*} \varphi\right\rangle$.

Definition 2.6 (Regular backward RPDE solution). - Given an $\alpha$ Hölder rough path $\mathbf{W}=(W, \mathbb{W}), \alpha \in(1 / 3,1 / 2]$, we say that a function $u=u(t, x ; \mathbf{W}) \in C^{0,2}$ (with respect to $t, x$ ) is a solution to (2.5) if, for all $x \in \mathbb{R}^{d},(\Gamma u(\cdot, x), \Gamma \Gamma u(\cdot, x))$ is controlled by $W$ (Definition 4.1) and

$$
\begin{equation*}
u(t, x)=g(x)+\int_{t}^{T} L u(r, x) \mathrm{d} r+\int_{t}^{T} \Gamma u(r, x) \mathrm{d} \mathbf{W}_{r} \tag{2.8}
\end{equation*}
$$

Remark 2.7. - If a regular solution in the sense of Definition 2.6 possesses a uniform bound on the control (see for example (2.10) below) then it is also a weak solution in the sense of Definition 2.3.

Recall that geometric rough paths are limits of smooth paths under the appropriate rough path metric. While rough path integration does not rely on this assumption (indeed, (2.7), resp. (2.8) were formulated for a general
$\alpha$-Hölder rough path), it is a very natural assumption when it comes to stability results.

ThEOREM 2.8. - Throughout, $\mathbf{W}$ is a geometric $\alpha$-Hölder rough path, $\alpha \in(1 / 3,1 / 2]$. Assume $\sigma_{i}, \beta_{j} \in C_{b}^{3}\left(\mathbb{R}^{d}\right), b \in C_{b}^{1}\left(\mathbb{R}^{d}\right), c \in C_{b}^{1}\left(\mathbb{R}^{d}\right), \gamma_{k} \in$ $C_{b}^{2}\left(\mathbb{R}^{d}\right)$. Consider $g \in C_{b}^{0}\left(\mathbb{R}^{d}\right)$.
(1) Stability. Let $u=u^{W}$ be the solution to (2.1) as given by the Feynman-Kac representation (2.2), whenever $W \in C^{1}$. Pick $W^{\epsilon} \in C^{1}$ convergent in rough path sense to $\mathbf{W}$. Then there exists a bounded, continuous function $u^{\mathbf{W}}$, independent of the choice of the approximating sequence, so that $u^{W^{\epsilon}} \rightarrow u^{\mathbf{W}}$ uniformly. The resulting map $\mathbf{W} \mapsto u^{\mathbf{W}}$ is continuous. Moreover, the following Feynman-Kac representation holds:

$$
u^{\mathbf{W}}(t, x)=\mathbb{E}^{t, x}\left[g\left(X_{T}\right) \exp \left(\int_{t}^{T} c\left(X_{r}\right) \mathrm{d} r+\int_{t}^{T} \gamma\left(X_{r}\right) \mathrm{d} \mathbf{W}_{r}\right)\right]
$$

where $X$ solves the rough SDE (see Appendix, Lemma 4.19)

$$
\begin{equation*}
\mathrm{d} X_{t}=\sigma\left(X_{t}\right) \mathrm{d} B(\omega)+b\left(X_{t}\right) \mathrm{d} t+\beta\left(X_{t}\right) \mathrm{d} \mathbf{W}_{t} \tag{2.9}
\end{equation*}
$$

where $B$ is a $d_{B}$-dimensional Brownian motion.
(2) Analytically weak backward RPDE solution. Let $u=u^{\mathbf{W}}$ be the function constructed in (1). Then $u=u^{\mathbf{W}} \in C_{b}\left([0, T] \times \mathbb{R}^{d}\right)$ is a bounded solution to (2.5) in the sense of Definition 2.3. Moreover, (2.6) is bounded, uniformly over bounded sets of $\varphi$ in $C_{\exp }^{3}\left(\mathbb{R}^{d}\right)$, and $u^{\mathbf{W}}$ is the only solution in the class of $C_{b}$ functions $u$ satisfying this uniform bound on (2.6).
(3) Regular backward RPDE solution. Assume $\sigma_{i}, \beta_{j} \in C_{b}^{6}\left(\mathbb{R}^{d}\right), b \in$ $C_{b}^{4}\left(\mathbb{R}^{d}\right), c \in C_{b}^{4}\left(\mathbb{R}^{d}\right), \gamma_{k} \in C_{b}^{6}\left(\mathbb{R}^{d}\right)$ and $g \in C_{b}^{4}\left(\mathbb{R}^{d}\right)$. Then $u=$ $u^{\mathbf{W}} \in C_{b}^{0,4}\left([0, T] \times \mathbb{R}^{d}\right)$ is a bounded solution to (2.5) in the sense of Definition 2.6. It is the only solution in the class of functions in $C_{b}^{0,4}\left([0, T] \times \mathbb{R}^{d}\right)$ that satisfies

$$
\begin{equation*}
\sup _{x}\|\Gamma u(\cdot, x), \Gamma \Gamma u(\cdot, x)\|_{W, 2 \alpha}<\infty . \tag{2.10}
\end{equation*}
$$

$$
\text { If moreover } g \in C_{\exp }^{4}\left(\mathbb{R}^{d}\right) \text {, then } u \in C_{\exp }^{0,4}\left([0, T] \times \mathbb{R}^{d}\right)
$$

Remark 2.9. - We consider solutions in $C_{b}^{0,4}$, instead of the obvious choice $C_{b}^{0,2}$, because of two reasons. First, in order to show that $u$ is controlled by $W$ we need $g \in C_{b}^{4}\left(\mathbb{R}^{d}\right)$ which automatically gives us $u \in$ $C_{b}^{0,4}\left([0, T] \times \mathbb{R}^{d}\right)$. Second, this additional regularity is needed for the uniqueness proof via duality.

Remark 2.10. - Results of the type in Theorem 2.8(1), even in nonlinear situations, were obtained in $[4,5,9,7,14]$. However, in all these references, the only intrinsic meaning of these equations was given in terms of a transformed equation, somewhat in the spirit of the Lions-Souganidis [23] theory of stochastic viscosity solutions. On the contrary, parts (2) and (3) of the above theorem present a direct intrinsic characterization. See also [3, Ch. 3].

Proof. - (1): This follows from stability of "rough SDEs", see Lemma 4.19.
(2): Existence. - For simplicity only, we take $c=\gamma=b=0$ so that

$$
\begin{aligned}
u(s, x) & =\mathbb{E}\left[g\left(X_{T}^{s, x}\right)\right] \\
\mathrm{d} X_{t} & =\sigma\left(X_{t}\right) \mathrm{d} B_{t}(\omega)+\beta\left(X_{t}\right) \mathrm{d} \mathbf{W}_{t}
\end{aligned}
$$

(By $X^{s, x}$ we mean the unique solution started at $X_{s}=x$.) In the following we consider the above SDE as an RDE w.r.t. the joint lift $\mathbf{Z}=(Z, \mathbb{Z})$ of W and the Brownian motion $B$ (see Lemma 4.18 and Lemma 4.19 below). Denote with $\Phi$ its associated flow.

Recall $Y_{t}=\left\langle u_{t}, \bar{\varphi}\right\rangle, Y_{t}^{\prime}=-\left\langle u_{t}, \Gamma^{*} \bar{\varphi}\right\rangle$, where $\bar{\varphi}:=\Gamma^{*} \varphi$. Since $\varphi \in C_{\exp }^{3}$ and $\beta_{j} \in C_{b}^{2}, j=1, \ldots, e$, we have that $\bar{\varphi} \in C_{\exp }^{2}$. Then

$$
\begin{aligned}
& Y_{t}-Y_{s}-Y_{t}^{\prime} W_{s, t} \\
& =\mathbb{E}\left[\int_{\mathbb{R}^{d}}\left\{g\left(\Phi_{t, T}(x)\right)-g\left(\Phi_{s, T}(x)\right)\right\} \bar{\varphi}(x)+g\left(\Phi_{t, T}(x)\right) \Gamma^{*} \bar{\varphi}(x) W_{s, t} \mathrm{~d} x\right] \\
& =\mathbb{E}\left[\int _ { \mathbb { R } ^ { d } } g ( y ) \left\{\bar{\varphi}\left(\Phi_{t, T}^{-1}(y)\right) \operatorname{det}\left(D \Phi_{t, T}^{-1}(y)\right)-\bar{\varphi}\left(\Phi_{s, T}^{-1}(y)\right) \operatorname{det}\left(D \Phi_{s, T}^{-1}(y)\right)\right.\right. \\
& \\
& \left.\left.+\Gamma^{*} \bar{\varphi}\left(\Phi_{t, T}^{-1}(y)\right) \operatorname{det}\left(D \Phi_{t, T}^{-1}(y)\right) W_{s, t}\right\} \mathrm{~d} y\right] \\
& =\mathbb{E}\left[\int _ { \mathbb { R } ^ { d } } g ( y ) \left\{\bar{\varphi}\left(\Phi_{t, T}^{-1}(y)\right) \operatorname{det}\left(D \Phi_{t, T}^{-1}(y)\right)-\bar{\varphi}\left(\Phi_{s, T}^{-1}(y)\right) \operatorname{det}\left(D \Phi_{s, T}^{-1}(y)\right)\right.\right. \\
& \\
& \quad+\Gamma^{*} \bar{\varphi}\left(\Phi_{t, T}^{-1}(y)\right) \operatorname{det}\left(D \Phi_{t, T}^{-1}(y)\right) W_{s, t} \\
& \\
& \\
& \left.\left.+\Gamma^{*} \bar{\varphi}\left(\Phi_{t, T}^{-1}(y)\right) \operatorname{det}\left(D \Phi_{t, T}^{-1}(y)\right) B_{s, t}\right\} \mathrm{~d} y\right]
\end{aligned}
$$

Note that $\Gamma^{*} \varphi=-\operatorname{div}(b \varphi)$. Hence the term in curly brackets is bounded in absolute value, using Lemma 4.9, by a constant times

$$
\|\bar{\varphi}\|_{C_{b}^{3}(M(y))} \exp \left(C N_{1 ;[0, T]}(\mathbf{Z})\right)\left(\|\mathbf{Z}\|_{\alpha}+1\right)^{17+3 d}|t-s|^{2 \alpha}
$$

Hence

$$
\begin{aligned}
\mid Y_{t} & -Y_{s}-Y_{t}^{\prime} W_{s, t} \mid \\
& \lesssim \int_{\mathbb{R}^{d}} \mathbb{E}\left[\|\varphi\|_{C_{b}^{3}(M(y))} \exp \left(C N_{1 ;[0, T]}(\mathbf{Z})\right)\left(\|\mathbf{Z}\|_{\alpha}+1\right)^{17+3 d}\right] \mathrm{d} y|t-s|^{2 \alpha} .
\end{aligned}
$$

Next observe that

$$
\begin{aligned}
& \mathbb{E}\left[\|\varphi\|_{C_{b}^{3}(M(y))} \exp \left(C N_{1 ;[0, T]}(\mathbf{Z})\right)\left(\|\mathbf{Z}\|_{\alpha}+1\right)^{17+3 d}\right] \\
& \quad \leqslant \mathbb{E}\left[\|\varphi\|_{C_{b}^{3}(M(y))}^{2}\right]^{1 / 2} \mathbb{E}\left[\exp \left(2 C N_{1 ;[0, T]}(\mathbf{Z})\right)\left(\|\mathbf{Z}\|_{\alpha}+1\right)^{34+6 d}\right]^{1 / 2}
\end{aligned}
$$

and Lemma 4.18 now implies that the last term is bounded and Lemma 4.21 implies that the first term decays exponentially in $y$. Therefore

$$
\left|Y_{t}-Y_{s}-Y_{t}^{\prime} W_{s, t}\right| \lesssim|t-s|^{2 \alpha}
$$

as desired.
The estimate $\left|Y_{t}^{\prime}-Y_{s}^{\prime}\right| \leqslant C|t-s|^{\alpha}$ is shown analogously, and then

$$
\left|Y_{t}-Y_{s}-Y_{s}^{\prime} W_{s, t}\right| \leqslant\left|Y_{t}-Y_{s}-Y_{t}^{\prime} W_{s, t}\right|+\left|Y_{s, t}^{\prime} W_{s, t}\right|=O\left(|t-s|^{2 \alpha}\right),
$$

as desired.
It remains to show that the integral equation (2.7) is satisfied. For this let $W^{n}$ be a sequence of smooth paths converging to $\mathbf{W}$ in $\alpha$-rough path metric. Let $u^{n}$ be the solution to (2.3) as given by Lemma $2.2(2)$.

Part (1) of the theorem now implies that $u^{n}$ converges locally uniformly to $u$, hence the convergence of all the terms in (2.7) except the rough integral is immediate. For the rough integral, in view of Theorem 4.16 in [12], it is enough to show that

$$
\begin{array}{ll}
\sup _{n}\left\|Y^{\prime n}\right\|_{\alpha}<\infty & \sup _{n}\left\|Y^{\prime n}-Y^{\prime}\right\|_{\infty} \rightarrow 0 \\
\sup _{n}\left\|R^{n}\right\|_{2 \alpha}<\infty & \sup _{n}\left\|R^{n}-R\right\|_{\infty} \rightarrow 0,
\end{array}
$$

with $Y_{t}^{n}:=\left\langle u_{t}^{n}, \Gamma^{*} \varphi\right\rangle, Y^{\prime n}:=\left\langle u_{t}^{n}, \Gamma^{*} \Gamma^{*} \varphi\right\rangle$ and

$$
R_{s, t}^{n}=\left\langle u_{t}^{n}-u_{s}^{n}, \Gamma^{*} \varphi\right\rangle-\left\langle u_{s}^{n}, \Gamma^{*} \Gamma^{*} \varphi\right\rangle W_{s, t}^{n} .
$$

The first two statements follow from the fact that the preceding considerations were uniform for $\mathbf{W}$ bounded in rough path norm. Finally, convergence in supremum norm of $Y_{t}^{\prime n}-Y_{t}^{\prime}=\left\langle u_{t}^{n}-u_{t}, \Gamma^{*} \Gamma^{*} \varphi\right\rangle$ and $R_{s, t}^{n}-R_{s, t}$ follows from local uniform convergence of $u^{n}$.

Uniqueness. - Let $\phi \in C_{\exp }^{0,3}\left([t, T], \mathbb{R}^{d}\right)$ be such that ${ }^{(5)}$

$$
\phi(t, x)=\varphi(x)+\int_{t}^{T} \alpha(r, x) \mathrm{d} r+\int_{t}^{T} \eta_{i}(r, x) \mathrm{d} \mathbf{W}_{r}^{i}
$$

with $\alpha \in C_{\text {exp }}^{0,3}\left([0, T] \times \mathbb{R}^{d}\right)$, and $\left(\eta_{i=1, \ldots, e}, \eta_{i, j=1, \ldots, e}^{\prime}\right)$ controlled by $W$. Assume moreover for some $\delta>0$

$$
\begin{aligned}
\left\|\eta(x), \eta^{\prime}(x)\right\|_{W, 2 \alpha} & \lesssim e^{-\delta|x|} \\
\left\|D \eta(x), D \eta^{\prime}(x)\right\|_{W, 2 \alpha} & \lesssim e^{-\delta|x|}
\end{aligned}
$$

Then by Lemma 4.17

$$
\begin{aligned}
\left\langle u_{T}, \phi_{T}\right\rangle=\left\langle u_{t}, \phi_{t}\right\rangle-\int_{t}^{T}\left\langle u_{r},\right. & \left.L^{*} \phi_{r}\right\rangle \mathrm{d} r-\int_{t}^{T}\left\langle u_{r}, \Gamma_{k}^{*} \phi_{r}\right\rangle \mathrm{d} \mathbf{W}_{r}^{k} \\
& +\int_{t}^{T}\left\langle u_{r}, \alpha(r)\right\rangle \mathrm{d} r+\int_{t}^{T}\left\langle u_{r}, \eta_{k}(r)\right\rangle \mathrm{d} \mathbf{W}_{r}^{k}
\end{aligned}
$$

So it remains to find, for given $\varphi$, such a $\phi$ with $\alpha(r)=L^{*} \phi(r), \eta_{i}(r)=$ $\Gamma_{i}^{*} \phi(r)$ and $\eta_{i, j}^{\prime}(r)=\Gamma_{j}^{*} \Gamma_{i}^{*} \phi(r)$. But this is exactly what Theorem 3.5(3) gives us for $\varphi \in C_{\exp }^{4}\left(\mathbb{R}^{d}\right)$. Then

$$
\left\langle g, \phi_{T}\right\rangle=\left\langle u_{T}, \phi_{T}\right\rangle=\left\langle u_{t}, \phi_{t}\right\rangle=\left\langle u_{t}, \varphi\right\rangle,
$$

which gives uniqueness of $u_{t}$. This holds for all $t \in[0, T]$, which gives uniqueness of $u$.
(3): Again, for simplicity only, we take $c=\gamma=b=0$ so that

$$
u(t, x)=\mathbb{E}\left[g\left(X_{T}^{t, x}\right)\right]
$$

Then

$$
D u(t, x)=\mathbb{E}\left[D g\left(X_{T}^{t, x}\right) D X_{T}^{t, x}\right]
$$

Indeed, using the integrability of $D X$ given by Lemma 4.19 and the fact that $D g$ is bounded, the statement follows from interchanging differentiation and integration, see for example Theorem 8.1.2 in [11].

[^4]Then by Lemma 4.14

$$
\begin{aligned}
& D u(t, x)-D u(s, x) \\
& =\mathbb{E}\left[D g\left(X_{T}^{t, x}\right) D X_{T}^{t, x}-D g\left(X_{T}^{s, x}\right) D X_{T}^{s, x}\right] \\
& =\mathbb{E}\left[\int_{s}^{t} D g\left(X_{T}^{r, x}\right) \mathrm{d} D X_{T}^{r, x}+\int_{s}^{t} D^{2} g\left(X_{T}^{r, x}\right)\left\langle\mathrm{d} X_{T}^{r, x}, D X_{T}^{r, x} \cdot\right\rangle\right] \\
& =\mathbb{E}\left[\int_{s}^{t} D g\left(X_{T}^{r, x}\right) D^{2} X_{T}^{r, x}\left\langle V(x) \mathrm{d} Z_{r}, \cdot\right\rangle+\int_{s}^{t} D g\left(X_{T}^{r, x}\right) D V(x) D X_{T}^{r, x} \mathrm{~d} Z_{r}\right. \\
& \left.\quad+\int_{s}^{t} D^{2} g\left(X_{T}^{r, x}\right)\left\langle D X_{T}^{r, x} V(x) \mathrm{d} Z_{r}, D X_{T}^{r, x} \cdot\right\rangle\right]+O(|t-s|) \\
& \begin{array}{r}
=\mathbb{E}\left[D g\left(X_{T}^{s, x}\right) D^{2} X_{T}^{s, x}\left\langle\beta(x) W_{s, t} \cdot \cdot\right\rangle\right]+\mathbb{E}\left[D g\left(X_{T}^{s, x}\right) D X_{T}^{s, x} D \beta(x)\right] W_{s, t} \\
\quad+\mathbb{E}\left[D^{2} g\left(X_{T}^{s, x}\right)\left\langle D X_{T}^{s, x} V(x) W_{s, t}, D X_{T}^{s, x} \cdot\right\rangle\right]+O\left(|t-s|^{2 \alpha}\right) \\
= \\
\partial_{x}[\beta(x) D u(s, x)] W_{s, t}+O\left(|t-s|^{2 \alpha}\right) .
\end{array}
\end{aligned}
$$

So $\Gamma u$ is controlled as claimed and (2.10) is satisfied. Showing that $u \in C_{b}^{0,4}$ also follows from differentiation under the expectation and the proof that the integral equation is satisfied now follows by using smooth approximations to $\mathbf{W}$, as in part (2).

Uniqueness follows from existence of the measure-valued forward equation. The argument is dual to the one that will be used in the proof of Theorem $3.5(2)$, so we omit the proof here.

Finally, the exponential decay of $u$, if $g \in C_{\exp }^{4}$, follows from Lemma 4.20.

## 3. The forward equation

We now consider the forward equation

$$
\left\{\begin{align*}
\partial_{t} \rho_{t} & =L^{*} \rho_{t}+\sum_{k=1}^{e} \Gamma_{k}^{*} \rho_{t} \dot{W}_{t}^{k}\left(\equiv L^{*} \rho_{t}+\Gamma^{*} \rho_{t} \dot{W}_{t}\right)  \tag{3.1}\\
\rho_{0} & =p_{0}
\end{align*}\right.
$$

on the space $\mathcal{M}\left(\mathbb{R}^{d}\right)$ of finite measures on $\mathbb{R}^{d}$.
Equation (3.1) is dual to the backward equation - considered in the previous section - in a sense that will be made precise in the following (see in particular Corollary 3.7 below).

The space $\mathcal{M}\left(\mathbb{R}^{d}\right)$ is endowed with the weak topology; that is $\mu_{n} \rightarrow \mu$ if $\mu_{n}(f) \rightarrow \mu(f)$ for all $f \in C_{b}\left(\mathbb{R}^{d}\right)$. It is metrizable with compatible metric
given by the Kantorovich-Rubinstein metric $d$, defined as

$$
d(\mu, \nu):=\sup _{\|f\|_{C_{b}^{1}\left(\mathbb{R}^{d}\right)} \leqslant 1}\left|\int_{\mathbb{R}^{d}} f(x) \nu(\mathrm{d} x)-\int_{\mathbb{R}^{d}} f(x) \mu(\mathrm{d} x)\right|
$$

(see Chapter 8.3 in [2]). A compatible metric on the space of continuous finite-measure-valued paths is then given by $d(\mu ., \rho):.=\sup _{t \leqslant T} d\left(\mu_{t}, \rho_{t}\right)$.

Lemma 3.1. - Assume $c, b, \sigma_{i}, \gamma_{j}, \beta_{k} \in C_{b}^{2}, i=1, \ldots, d_{B}, j, k=1, \ldots, e$. Define for $W \in C^{1}$ the measure valued process $\rho$ via its action on $f \in C_{b}\left(\mathbb{R}^{d}\right)$ as

$$
\begin{equation*}
\rho_{t}(f):=\mathbb{E}^{0, \nu}\left[f\left(X_{t}\right) \exp \left(\int_{0}^{t} c\left(X_{s}\right) \mathrm{d} s+\int_{0}^{t} \gamma\left(X_{s}\right) \dot{W}_{s} \mathrm{~d} s\right)\right] \tag{3.2}
\end{equation*}
$$

where $\nu \in \mathcal{M}\left(\mathbb{R}^{d}\right)$ is the law of the initial condition of the diffusion $X$ with dynamics

$$
\mathrm{d} X_{t}=\sigma\left(X_{t}\right) \mathrm{d} B(\omega)+b\left(X_{t}\right) \mathrm{d} t+\beta\left(X_{t}\right) \dot{W}_{t} \mathrm{~d} t
$$

where $B$ is a $d_{B}$-dimensional Brownian motion.
(1) Then $\rho$ is the unique, continuous $\mathcal{M}\left(\mathbb{R}^{d}\right)$-valued path which satisfies, for all $f \in C_{b}^{2}\left(\mathbb{R}^{d}\right)$,

$$
\begin{equation*}
\rho_{t}(f)=\nu(f)+\int_{0}^{t} \rho_{s}(L f) \mathrm{d} s+\int_{0}^{t} \rho_{s}\left(\Gamma_{k} f\right) \mathrm{d} W_{s}^{k} \tag{3.3}
\end{equation*}
$$

(2) Assume moreover $\sigma_{i} \in C_{b}^{4}\left(\mathbb{R}^{d}\right), i=1, \ldots, d_{B}, b, \beta_{k} \in C_{b}^{3}\left(\mathbb{R}^{d}\right), k=$ $1, \ldots, e$.

If $\nu$ has a density $p_{0} \in C_{\exp }^{2}\left(\mathbb{R}^{d}\right)$ then $\rho_{t}$ has a density $p \in$ $C_{\text {exp }}^{1,2}\left([0, T] \times \mathbb{R}^{d}\right)$ which is the unique bounded classical solution to (3.1).
Remark 3.2. - We choose $p_{0} \in C_{\exp }^{2}\left(\mathbb{R}^{d}\right)$ in part (2) since this is what we shall work with in the rough case. In the smooth case, the assumptions on the density $p_{0}$ can be weakened. Assume for example that $\nu$ has a density $p_{0} \in C_{b}^{2} \cap L^{1}$. Then $\rho_{t}$ has a $C_{b}^{2}$ density $p_{t}$ for all $t \geqslant 0$ and $p \in C_{b}^{1,2}$ is the unique bounded classical solution to (3.1). Moreover,

$$
\begin{equation*}
\left\|p_{t}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)}=\rho_{t}(1)=\mathbb{E}^{0, \nu}\left[\exp \left(\int_{0}^{t} c\left(X_{s}\right) \mathrm{d} s+\int_{0}^{t} \gamma\left(X_{s}\right) \dot{W}_{s} \mathrm{~d} s\right)\right] \tag{3.4}
\end{equation*}
$$

Indeed, by the smoothness assumptions on the coefficients, (3.1) has a unique solution in $C_{b}^{1,2}$ (this can again be seen by a Feynman-Kac argument, as in Lemma 2.2).

We have to show that the unique classical solution $p_{t} \in C_{b}^{2}$ of (3.1) with non-negative initial condition $p_{0} \in C_{b}^{2} \cap L^{1}$ is integrable. First recall that from the maximum principle, $p_{t} \geqslant 0$ for all $t \geqslant 0$ (see for example

Theorem 8.1.4 in [20]). Note that (3.1) implies that $\frac{\mathrm{d}}{\mathrm{d} t} \int \varphi p_{t} \mathrm{~d} x=\int \tilde{L}_{t} \varphi p_{t} \mathrm{~d} x$, where $\tilde{L}_{t} \phi:=L \phi+\Gamma_{k} \phi \dot{W}_{t}^{k}$, hence

$$
\begin{equation*}
\int \varphi p_{t} \mathrm{~d} x=\int \varphi p_{0} \mathrm{~d} x+\int_{0}^{t} \int \tilde{L}_{s} \varphi p_{s} \mathrm{~d} x \mathrm{~d} s \tag{3.5}
\end{equation*}
$$

for any smooth and compactly supported function $\varphi$. Our aim now is to extend this equality to the constant function $\varphi \equiv 1$. To this end consider for $\varepsilon>0$ the function

$$
\varphi_{\varepsilon}(x):=\varphi\left(\varepsilon\|x\|^{2}\right),
$$

where $\varphi(r)=(1+r)^{-\frac{d+1}{2}}, r \geqslant 0$. It is easy to check that both $\varphi_{\varepsilon}$ and

$$
\begin{aligned}
& L \varphi_{\varepsilon}(x) \\
& \begin{aligned}
=-\varepsilon(d+1)(1+ & \left.\varepsilon\|x\|^{2}\right)^{-\frac{d+3}{2}}\left(\sum_{i j}\left(\sigma \sigma^{T}\right)_{i j}(x)+\sum_{i}\left(\tilde{b}_{i}\right)_{t}(x) x_{i}\right)+c \varphi_{\varepsilon}(x) \\
& +\varepsilon^{2}(d+1)(d+3)\left(1+\varepsilon\|x\|^{2}\right)^{-\frac{d+5}{2}}\left(\sum_{i j}\left(\sigma \sigma^{T}\right)_{i j}(x) x_{i} x_{j}\right)
\end{aligned}
\end{aligned}
$$

are integrable. Since the coefficients $\sigma \sigma^{T}$ and $\tilde{b}_{t}:=b+\Gamma_{k} \dot{W}_{t}^{k}$ have at most linear growth and $c$ is bounded, there exists a finite constant $M$, independent of $\varepsilon$, such that

$$
\tilde{L}_{s} \varphi_{\varepsilon} \leqslant M \varphi_{\varepsilon}
$$

Next fix a smooth compactly supported test function $\chi$ on $\mathbb{R}$ satisfying $1_{[-1,1]} \leqslant \chi \leqslant 1_{[-2,2]}$ and let $\chi_{N}(x):=\chi\left(\frac{\|x\|^{2}}{N^{2}}\right)$. Then $\chi_{N} \varphi_{\varepsilon}$ is compactly supported and

$$
\begin{aligned}
& \tilde{L}_{t}\left(\chi_{N} \varphi_{\varepsilon}\right) \\
& \begin{aligned}
=\chi_{N} \tilde{L}_{t} \varphi_{\varepsilon}-4 \varepsilon \frac{e+1}{N^{2}} \chi^{\prime}\left(\frac{\|x\|^{2}}{N^{2}}\right)\left(1+\varepsilon\|x\|^{2}\right)^{-\frac{d+3}{2}} \sum_{i j} & \left(\sigma \sigma^{T}\right)(x) x_{i} x_{j} \\
& +\left(\left(L_{0}\right)_{t} \chi_{N}\right) \varphi_{\varepsilon}
\end{aligned}
\end{aligned}
$$

where $\left(\tilde{L}_{0}\right)_{t} u=\tilde{L}_{t} u-c u$. Again due to the assumptions on the coefficients of $L$ (resp. $L_{0}$ ) we obtain that $L_{0} \chi_{N}$ is uniformly bounded in $N$, so that $\left|\tilde{L}_{t}\left(\chi_{N} \varphi_{\varepsilon}\right)\right|$ is uniformly bounded in $N$ in terms of $\varphi_{\varepsilon}$ and $\left|\tilde{L}_{t} \varphi_{\varepsilon}\right|$. Since $\tilde{L}_{t}\left(\chi_{N} \varphi_{\varepsilon}\right) \rightarrow \tilde{L}_{t} \varphi_{\varepsilon}$ pointwise, Lebesgue's dominated convergence now implies that (3.5) extends to the limit $N \rightarrow \infty$, hence

$$
\begin{align*}
\int \varphi_{\varepsilon} p_{t} \mathrm{~d} x & =\int \varphi_{\varepsilon} p_{0} \mathrm{~d} x+\int_{0}^{t} \int \tilde{L}_{s} \varphi_{\varepsilon} p_{s} \mathrm{~d} x \mathrm{~d} s  \tag{3.6}\\
& \leqslant \int \varphi_{\varepsilon} p_{0} \mathrm{~d} x+M \int_{0}^{t} \int \varphi_{\varepsilon} p_{s} \mathrm{~d} x \mathrm{~d} s
\end{align*}
$$

Gronwall's lemma now implies that

$$
\int \varphi_{\varepsilon} p_{t} \mathrm{~d} x \leqslant e^{M t} \int \varphi_{\varepsilon} p_{0} \mathrm{~d} x
$$

Since $p_{0}$ is integrable, we can now take the limit $\varepsilon \downarrow 0$ to conclude with Fatou's lemma that

$$
\int p_{t} \mathrm{~d} x \leqslant e^{M t} \int p_{0} \mathrm{~d} x<\infty
$$

Hence $\nu_{t}(f):=\int p_{t}(x) f(x) \mathrm{d} x$ defines a finite-measure valued path and it satisfies (3.3). By uniqueness it hence coincides with $\rho$. The expression for the $L^{1}$-norm of $p_{t}$ then follows from (1).

Proof. - (1): Equation (3.3) is satisfied by an application of Itô's formula, see for example Theorem 3.24 in [1] for a similar argument. Uniqueness follows as in Theorem 4.16 in [1]. Let us sketch the argument. ${ }^{(6)}$ First one shows that every solution to (3.3) also satisfies for $\varphi \in C_{b}^{1,2}\left([0, T], \mathbb{R}^{d}\right)$

$$
\begin{equation*}
\rho_{t}(\varphi)=\nu(\varphi)+\int_{0}^{t} \rho_{s}\left(\partial_{t} \varphi+L \varphi\right) \mathrm{d} s+\int_{0}^{t} \rho_{s}\left(\Gamma_{k} \varphi\right) \mathrm{d} W_{s}^{k} \tag{3.7}
\end{equation*}
$$

(Here and below, omit summation over $k=1, \ldots, e$ ). Given now $\Phi \in$ $C_{b}^{\infty}\left(\mathbb{R}^{d}\right)$ and $t \leqslant T$ consider any solution $v \in C_{b}^{1,2}\left([0, t], \mathbb{R}^{d}\right)$ to the backward equation

$$
\begin{aligned}
-\partial_{t} v & =L v+\Gamma_{k} v \dot{W}_{t}^{k} \\
v(t, \cdot) & =\Phi
\end{aligned}
$$

the existence of which follows from Lemma 2.2.
Given two solutions $\rho, \bar{\rho}$ to (3.3), we then have, by (3.7),

$$
\rho_{t}(\Phi)=\rho_{t}\left(v_{t}\right)=\rho_{0}\left(v_{0}\right)=\bar{\rho}_{0}\left(v_{0}\right)=\bar{\rho}_{t}\left(v_{t}\right)=\bar{\rho}_{t}(\Phi),
$$

so $\rho_{t}$ and $\bar{\rho}_{t}$ coincide on $C_{b}^{\infty}$. By pointwise uniformly bounded convergence they then also coincide on $C_{b}$, and hence $\rho_{t}=\bar{\rho}_{t}$ as desired.
(2): In this case we can classically solve the equation

$$
\left\{\begin{align*}
\partial_{t} \rho_{t} & =L^{*} \rho_{t}+\Gamma_{k}^{*} \rho_{t} \dot{W}_{t}^{k}  \tag{3.8}\\
\rho_{0} & =p_{0}
\end{align*}\right.
$$

Indeed, the cofficients, see (1.4), all are in $C_{b}^{2}$. Hence using Lemma 2.2 (after a trivial time inversion) we get a unique bounded classical solution $p \in$ $C_{\text {exp }}^{1,2}\left([0, T] \times \mathbb{R}^{d}\right)$ to this PDE. It is in particular integrable and hence defines

[^5]a measure-valued function $\mu$ on $[0, T]$ which satisfies (3.3). By uniqueness for this equation it coincides with $\rho$.

When replacing $W$ by a rough path $\mathbf{W}$, we are interested in the following equation

$$
\left\{\begin{align*}
\mathrm{d} \rho_{t} & =L^{*} \rho_{t} \mathrm{~d} t+\Gamma^{*} \rho_{t} \mathrm{~d} \mathbf{W}_{t}  \tag{3.9}\\
\rho_{0} & =\nu
\end{align*}\right.
$$

Two ways to make sense of this equation are given in the following definitions.

Definition 3.3 (Measure-valued forward RPDE solution). - Given an $\alpha$-Hölder rough path $\mathbf{W}=(W, \mathbb{W})$, $\alpha \in(1 / 3,1 / 2]$, and $\nu \in \mathcal{M}\left(\mathbb{R}^{d}\right)$, we say that a continuous finite-measure-valued path $\rho_{t}$ is a weak solution to (3.9) if for all $f \in C_{b}^{3}\left(\mathbb{R}^{d}\right), \rho_{t}\left(\Gamma_{k} f\right)_{k=1, \ldots, e}$ is controlled by $W$ with Gubinelli derivative $\rho_{t}\left(\Gamma_{j} \Gamma_{k} f\right)_{k, j=1, \ldots, e}$, that is

$$
\begin{equation*}
\|\rho .(\Gamma f), \rho .(\Gamma \Gamma f)\|_{W, 2 \alpha}<\infty \tag{3.10}
\end{equation*}
$$

and the integral equation

$$
\begin{equation*}
\rho_{t}(f)=\nu(f)+\int_{0}^{t} \rho_{s}(L f) \mathrm{d} s+\int_{0}^{t} \rho_{s}(\Gamma f) \mathrm{d} \mathbf{W}_{s}, \tag{3.11}
\end{equation*}
$$

holds.
Definition 3.4 (Regular forward RPDE solution). - Given an $\alpha$-Hölder rough path $\mathbf{W}=(W, \mathbb{W}), \alpha \in(1 / 3,1 / 2]$, and $\phi \in C_{b}^{2}\left(\mathbb{R}^{d}\right)$ we say that $p \in C_{b}^{0,2}\left([0, T] \times \mathbb{R}^{d}\right)$ is a regular solution to

$$
\left\{\begin{align*}
d p_{t} & =L^{*} p_{t} \mathrm{~d} t+\Gamma^{*} p_{t} \mathrm{~d} \mathbf{W}_{t}  \tag{3.12}\\
p_{0} & =\phi
\end{align*}\right.
$$

if, for all $x \in \mathbb{R}^{d},\left(\Gamma^{*} u(\cdot, x), \Gamma^{*} \Gamma^{*} u(\cdot, x)\right)$ is controlled by $W$ (Definition 4.1) and

$$
p_{t}(x)=\phi+\int_{0}^{t} L^{*} p_{s}(x) \mathrm{d} s+\int_{0}^{t} \Gamma^{*} p_{s}(x) \mathrm{d} \mathbf{W}_{s}
$$

Theorem 3.5. - Throughout, $\mathbf{W}$ is a geometric $\alpha$-Hölder rough path, $\alpha \in(1 / 3,1 / 2]$. Assume $\sigma_{i}, \beta_{j} \in C_{b}^{3}\left(\mathbb{R}^{d}\right), b \in C_{b}^{1}\left(\mathbb{R}^{d}\right), c \in C_{b}^{1}\left(\mathbb{R}^{d}\right), \gamma_{k} \in$ $C_{b}^{2}\left(\mathbb{R}^{d}\right)$. Let $\nu$ be a finite measure.
(1) Stability. Let $\rho=\rho^{W}$ be the solution to (3.3) as given by the Feynman-Kac representation (3.2), whenever $W \in C^{1}$. Pick $W^{\epsilon} \in$ $C^{1}$ convergent in rough path sense to $\mathbf{W}$. Then there exists a continuous finite-measure-valued function $\rho^{\mathbf{W}}$, independent of the choice
of the approximating sequence, so that $d\left(\rho^{W^{\epsilon}}, \rho^{\mathbf{W}}\right) \rightarrow 0$. The resulting map $\mathbf{W} \mapsto \rho^{\mathbf{W}}$ is continuous. Moreover, the following FeynmanKac representation holds for $f \in C_{b}\left(\mathbb{R}^{d}\right)$

$$
\rho_{t}^{\mathbf{W}}(f):=\mathbb{E}^{0, \rho_{0}}\left[f\left(X_{t}\right) \exp \left(\int_{0}^{t} c\left(X_{s}\right) \mathrm{d} s+\int_{0}^{t} \gamma\left(X_{s}\right) \mathrm{d} \mathbf{W}_{s}\right)\right]
$$

where $X$ solves the same rough $S D E$ as in Theorem 2.8.
(2) Measure-valued forward RPDE solution. The measure-valued path $\rho^{\mathbf{W}}$ constructed in part (1) is a solution to (3.9) in the sense of Definition 3.3. Moreover, (3.10) is bounded, uniformly over bounded sets of $f$ in $C_{b}^{3}\left(\mathbb{R}^{d}\right)$. If the coefficients satisfy the stronger conditions of Theorem 2.8(3) then $\rho^{\mathbf{W}}$ is the only solution in the class of measure-valued functions $\rho$ satisfying this uniform bound on (3.10).
(3) Regular forward RPDE solution. Assume $\sigma_{i} \in C_{b}^{6}\left(\mathbb{R}^{d}\right), \beta_{j} \in C_{b}^{7}\left(\mathbb{R}^{d}\right)$, $b \in C_{b}^{5}\left(\mathbb{R}^{d}\right), \gamma_{k} \in C_{b}^{6}\left(\mathbb{R}^{d}\right), c \in C_{b}^{4}\left(\mathbb{R}^{d}\right)$.

If $\rho_{0}$ has a density $p_{0} \in C_{\exp }^{4}\left(\mathbb{R}^{d}\right)$, then $\rho_{t}$ has a density $p_{t}$ for all times, and $p \in C_{\exp }^{0,4}\left([0, T] \times \mathbb{R}^{d}\right)$ is a solution to (3.12) in the sense of Definition 3.4.

It is the only solution that in addition satisfies for some $\delta>0$

$$
\left\|\Gamma^{*} u(\cdot, x), \Gamma^{*} \Gamma^{*} u(\cdot, x)\right\|_{W, 2 \alpha} \lesssim e^{-\delta|x|}
$$

Proof. - (1): First of all we note that for fixed $f \in C_{b}\left(\mathbb{R}^{d}\right)$ and fixed $t$ we have that

$$
W \mapsto \rho_{t}^{W}(f)
$$

is continuous in rough path topology. Indeed, this follows from Lemma 4.19 and is also seen to hold uniformly in $t$ and in bounded sets of $f$ in $C_{b}\left(\mathbb{R}^{d}\right)$. This also immediately gives the stated Feynman-Kac representation.
(2): Fix $f \in C_{b}^{3}\left(\mathbb{R}^{d}\right)$ and for simplicity take $b=\gamma=c=0$. Then note that

$$
\begin{aligned}
& f\left(X_{t}\right)=f\left(X_{s}\right)+\int_{s}^{t} L f\left(X_{r}\right) \mathrm{d} r+\int_{s}^{t}\left\langle\sigma_{i}\left(X_{r}\right), D f\left(X_{r}\right)\right\rangle \mathrm{d} B_{r}^{i} \\
&+\int_{s}^{t} \Gamma_{i} f\left(X_{r}\right) \mathrm{d} \mathbf{W}_{r}^{i}
\end{aligned}
$$

Taking expectation and applying Lemma 3.6 we get

$$
\rho_{t}(f)=\rho_{s}(f)+\int_{s}^{t} \rho_{r}(L f) \mathrm{d} r+\int_{s}^{t} \rho_{r}\left(\Gamma_{i} f\right) \mathrm{d} \mathbf{W}_{r}^{i}
$$

as well as the desired uniform bound on (3.10).

To show uniqueness in part (2), let $\phi \in C_{b}^{0,3}\left([0, t], \mathbb{R}^{d}\right)$ be such that

$$
\phi(s, x)=\varphi(x)+\int_{s}^{t} \alpha(r, x) \mathrm{d} r+\int_{s}^{t} \eta_{i}(r, x) \mathrm{d} \mathbf{W}_{r}^{i}
$$

for some $\left(\eta_{i=1, \ldots, e}, \eta_{i, j=1, \ldots, e}^{\prime}\right)$ controlled by $W$, uniformly over $x$, i.e.

$$
\begin{aligned}
\sup _{x}\left[\left\|\eta(x), \eta^{\prime}(x)\right\|_{W, 2 \alpha}\right]<\infty, \\
\sup _{x}\left[\left\|\Gamma_{i} \phi(x), \Gamma_{i} \eta(x)\right\|_{W, 2 \alpha}\right]<\infty, \quad i=1, \ldots, e .
\end{aligned}
$$

Moreover assume that $\eta \in C_{b}^{0,3}\left([0, T] \times \mathbb{R}^{d}\right)$. Then by Lemma 4.15

$$
\begin{aligned}
\rho_{t}\left(\phi_{t}\right)=\rho_{0}\left(\phi_{0}\right)+\int_{0}^{t} \rho_{r}\left(L \phi_{r}\right) \mathrm{d} r & +\int_{0}^{t} \rho_{r}\left(\Gamma_{k} \phi_{r}\right) \mathrm{d} \mathbf{W}_{r}^{k} \\
& -\int_{0}^{t} \rho_{r}(\alpha(r)) \mathrm{d} r-\int_{0}^{t} \rho_{r}\left(\eta_{k}(r)\right) \mathrm{d} \mathbf{W}_{r}^{k}
\end{aligned}
$$

So it remains to find, for given $\varphi$, such a $\phi$ with $\alpha(r)=L \phi(r), \eta_{i}(r)=$ $\Gamma_{i} \phi(r)$ and $\eta_{i, j}^{\prime}(r)=\Gamma_{j} \Gamma_{i} \phi(r)$. But this is exactly what Theorem 2.8 (3) gives us for $\varphi \in C_{b}^{4}\left(\mathbb{R}^{d}\right)$. Then

$$
\rho_{t}(\varphi)=\rho_{t}\left(\phi_{t}\right)=\rho_{0}\left(\phi_{0}\right)
$$

which gives uniqueness of $\rho$.
(3): The coefficients of the adjoint equation are given in (1.4). In particular $\tilde{\sigma}_{i} \in C_{b}^{6}\left(\mathbb{R}^{d}\right), \tilde{\beta}_{j} \in C_{b}^{6}\left(\mathbb{R}^{d}\right), \tilde{b} \in C_{b}^{4}\left(\mathbb{R}^{d}\right), \tilde{c} \in C_{b}^{4}\left(\mathbb{R}^{d}\right), \tilde{\gamma}_{k} \in C_{b}^{6}\left(\mathbb{R}^{d}\right)$. Hence the adjoint equation fits into the setting of Theorem 2.8(3). In particular there exists a $C_{b}^{0,4}$ solution to (3.12) and we can represent it as

$$
\begin{aligned}
& p_{t}(x) \\
& =\mathbb{E}\left[p_{0}\left(\tilde{X}_{T}^{T-t, x, \mathbf{W}}\right) \exp \left(\int_{T-t}^{T} \tilde{c}\left(X_{r}^{T-t, x, \mathbf{W}}\right) \mathrm{d} r+\int_{T-t}^{T} \tilde{\gamma}\left(X_{r}^{T-t, x, \mathbf{W}}\right) \mathrm{d} \mathbf{W}_{r}\right)\right],
\end{aligned}
$$

for a rough SDE $\tilde{X}$. The exponential estimates on the control then follow by a similar argument as in the proof of Theorem 2.8(3), using Lemma 4.21 on the integrands $D g, D^{2} g$. Finally, $p_{t}$ is the density of $\rho_{t}$ of part (1) due to the following reason: $p_{t}$ is integrable because of the exponential decay, the corresponding measure satisfies (3.12), which by uniqueness for that equation then coincides with $\rho$.

The following lemma was needed in the previous proof.

Lemma 3.6. - Assume $\sigma_{i}, \beta_{j} \in C_{b}^{3}\left(\mathbb{R}^{d}\right), b \in C_{b}^{1}\left(\mathbb{R}^{d}\right)$. Let $\mathbf{Z}$ be the joint lift of a Brownian motion $B$ with a (deterministic) geometric $\alpha$-Hölder rough path $\mathbf{W}, \alpha \in(1 / 3,1 / 2]$ (see Lemma 4.18). Let $X$ be the random $R D E$ solution to

$$
\begin{aligned}
X_{t} & =X_{0}+\int_{0}^{t} b\left(X_{r}\right) \mathrm{d} r+\int_{0}^{t}(\sigma, \beta) \mathrm{d} \mathbf{Z} \\
" & =" X_{0}+\int_{0}^{t} b\left(X_{r}\right) \mathrm{d} r+\int_{0}^{t} \sigma_{i}\left(X_{r}\right) \mathrm{d} B_{r}^{i}+\int_{0}^{t} \beta_{i}\left(X_{r}\right) \mathrm{d} \mathbf{W}_{r}^{i}
\end{aligned}
$$

Let $f \in C_{b}^{3}\left(\mathbb{R}^{d}\right)$ and define

$$
\begin{aligned}
\left(Y_{t}\right)_{i} & :=\mathbb{E}\left[\beta_{i}^{z}\left(X_{t}\right) \partial_{k} f\left(X_{t}\right)\right] \\
\left(Y_{t}^{\prime}\right)_{i, j} & :=\mathbb{E}\left[\partial_{k}\left[\partial_{z} f \beta_{i}^{z}\right]\left(X_{r}\right) \beta_{j}^{k}\left(X_{r}\right)\right] .
\end{aligned}
$$

Then $\left(Y, Y^{\prime}\right) \in \mathcal{D}_{W}^{2 \alpha}$ and

$$
\begin{equation*}
\mathbb{E}\left[f\left(X_{t}\right)\right]=\mathbb{E}\left[f\left(X_{0}\right)\right]+\int_{0}^{t} \mathbb{E}\left[b f\left(X_{r}\right)\right] \mathrm{d} r+\int_{0}^{t}\left(Y, Y^{\prime}\right) \mathrm{d} \mathbf{W}_{r}^{i} \tag{3.13}
\end{equation*}
$$

Moreover for all $R>0$,

$$
\sup _{\|f\|_{C_{b}^{3}} \leqslant R}\left\|\left(Y, Y^{\prime}\right)\right\|_{W, 2 \alpha}<\infty
$$

Proof. - For simplicity take $b=0$. First

$$
X_{s, t}=\sigma_{i}\left(X_{s}\right) B_{s, t}^{i}+\beta_{i}\left(X_{s}\right) W_{s, t}^{i}+R_{s, t}
$$

where $\|R\|_{2 \alpha} \leqslant C\left(1+\|\mathbf{Z}\|_{\alpha}\right)^{3}$ (see Lemma 4.7).
Then, with $g_{i}:=\beta_{i}^{z} \partial_{z} f$

$$
g_{i}(X)_{s, t}=\sigma_{j}^{k}\left(X_{s}\right) \partial_{k} g_{i}\left(X_{s}\right) B_{s, t}^{j}+\beta_{j}^{k}\left(X_{s}\right) \partial_{k} g_{i}\left(X_{s}\right) W_{s, t}^{j}+R_{s, t}
$$

with $\left\|\beta_{j}^{k}\left(X_{s}\right) \partial_{k} g\left(X_{s}\right)\right\|_{\alpha}+\|R\|_{2 \alpha} \leqslant C\left(1+\|\mathbf{Z}\|_{\alpha}\right)^{3}$ (see Lemma 4.13).
Taking expectation and using integrability of $\mathbf{Z}$ (Lemma 4.18), we get

$$
Y_{s, t}=Y_{s}^{\prime} W_{s, t}+\bar{R}_{s, t}
$$

with $\left\|Y^{\prime}\right\|_{\alpha}+\|\bar{R}\|_{2 \alpha} \leqslant C<\infty$, as desired.
Now for $W$ smooth, equation (3.13) is satisfied by Fubini's theorem. Showing it for $\mathbf{W}$ a geometric rough path then follows via smooth approximations. This has for example already been done - in a similar setting - in the proof of Theorem 2.8, so we omit the details here.

The following result in the proof of the previous theorem is worth to be formulated separately.

Corollary 3.7 (Duality). - Assume the conditions of Theorem 2.8(3). Let $u$ be the unique solution to the backward equation (2.7) and $\rho$ be the unique solution to the forward equation (3.11). Then

$$
\rho_{t}\left(u_{t}\right)=\rho_{0}\left(u_{0}\right) \quad \forall t \in[0, T] .
$$

## 4. Appendix

### 4.1. Rough differential equations

We recall the basic notions from rough path theory, for more details we refer to [12]. Let $V$ be a finite dimensional vector space. For a function $f:[0, T] \times[0, T] \rightarrow V$ and $\alpha>0$ we define

$$
\|f\|_{\alpha}:=\|f\|_{\alpha ;[0, T]}:=\sup _{0 \leqslant s \neq t \leqslant T} \frac{\left|f_{s, t}\right|}{|t-s|^{\alpha}} .
$$

If $g:[0, T] \rightarrow V$, we can and will consider it as a two parameter function via $g_{s, t}:=g_{t}-g_{s}$. The space $C^{\alpha}(V)$ of $\alpha$-Hölder continuous functions, on some fixed interval $[0, T]$, is then the space of all functions $g$ that satisfy $\|g\|_{\alpha}<\infty$..

Definition 4.1 (Controlled path). - Let $W$ be an e-dimensional $\alpha$ Hölder continuous path, $\alpha \in(1 / 3,1 / 2]$. Let $V$ be a finite dimensional vector space, $A$ path $Y \in C^{\alpha}(V)$ is controlled by $W$ with derivative $Y^{\prime} \in$ $C^{\alpha}\left(L\left(\mathbb{R}^{e}, V\right)\right)$ (in short $\left.\left(Y, Y^{\prime}\right) \in \mathscr{D}_{W}^{2 \alpha}\right)$, if

$$
Y_{s, t}=Y_{s}^{\prime} W_{s, t}+R_{s, t}^{Y}
$$

with $R^{Y}=O\left(|t-s|^{2 \alpha}\right)$. Also say that $\left(Y, Y^{\prime}\right)$ is controlled by $W$. We use the following semi-norm on $\mathscr{D}_{W}^{2 \alpha}$ :

$$
\left\|Y, Y^{\prime}\right\|_{W, 2 \alpha}:=\left\|Y^{\prime}\right\|_{\alpha}+\left\|R^{Y}\right\|_{2 \alpha} .
$$

Definition 4.2 (Rough path). - Let $\alpha \in(1 / 3,1 / 2]$. An $\alpha$-Hölder rough path in $\mathbb{R}^{e}$ is a tuple $\mathbf{W}=(W, \mathbb{W})$, such that $W$ is a $\mathbb{R}^{e}$ valued $\alpha$-Hölder path and $\mathbb{W}:[0, T] \times[0, T] \rightarrow \mathbb{R}^{e \times e}$ satisfies

$$
\|\mathbb{W}\|_{2 \alpha}<\infty \quad \text { and } \quad \mathbb{W}_{s, t}-\mathbb{W}_{s, u}-\mathbb{W}_{u, t}=W_{s, u} \otimes W_{u, t}
$$

for all $s, t, u \in[0, T]$. Define the $\alpha$-Hölder rough path "norm"

$$
\|\mathbf{W}\|_{\alpha}:=\|W\|_{\alpha}+\|\mathbb{W}\|_{2 \alpha} .
$$

When the time-interval is not obvious from context we also write $\|\mathbf{W}\|_{\alpha ;[0, T]}$ etc. The metric space of all $\alpha$-Hölder rough paths is denoted $\mathscr{C}^{\alpha}$.

A rough path is called geometric if it is the limit of smooth functions under the rough path distance, given by $\|W-\tilde{W}\|_{\alpha}+\|\mathbb{W}-\tilde{\mathbb{W}}\|_{2 \alpha}$, with $\mathbb{W}_{s, t}=\int_{s}^{t} W_{s, r} \otimes \mathrm{~d} W_{r}$, similar for $\tilde{\mathbb{W}}$.

We shall sometimes need the weaker p-variation rough path "norm" (usually $p=1 / \alpha$ )

$$
\|\mathbf{W}\|_{p-v a r}:=\|W\|_{p}+\|\mathbb{W}\|_{p / 2-v a r}
$$

where

$$
\|W\|_{p}:=\left(\sup _{\mathcal{P}} \sum_{[u, v] \in \mathcal{P}}\left|W_{u, v}\right|^{p}\right)^{1 / p}
$$

where the supremum is over all finite partitions of $[0, T]$ (or the interval under consideration) and the definition of $\|\mathbb{W}\|_{p / 2-v a r}$ is analogously.

Definition 4.3 (Rough integral). - For $\alpha \in(1 / 3,1 / 2], \mathbf{W}=(W, \mathbb{W})$ an $\alpha$-Hölder rough path (in $\left.\mathbb{R}^{e}\right)$ and $\left(Y, Y^{\prime}\right)$ a controlled path, with $Y$ and $Y^{\prime}$ taking values in $V=L\left(\mathbb{R}^{e}, \mathbb{R}^{d}\right)$ and $L\left(\mathbb{R}^{e \times e}, \mathbb{R}^{d}\right)$ respectively, the following $\mathbb{R}^{d}$-valued integral is well-defined

$$
\int_{0}^{T} Y \mathrm{~d} \mathbf{W}:=\int_{0}^{T}\left(Y_{r}, Y_{r}^{\prime}\right) \mathrm{d} \mathbf{W}_{r}:=\lim _{|\mathcal{P}| \rightarrow 0} \sum_{[u, v] \in \mathcal{P}} Y_{u} W_{u, v}+Y_{u}^{\prime} \mathbb{W}_{u, v}
$$

where the limit is over finite partitions of $[0, T]$ with mesh-size going to zero.
Remark 4.4. - (Rough integral, abusive notation) In most of this paper, $d=1$. In coordinates, $Y=\left(Y_{1}, \ldots, Y_{e}\right)$, and we sometimes find it convenient to abuse notation and write

$$
\int_{0}^{T} Y \mathrm{~d} \mathbf{W}=\int_{0}^{T} Y_{i} \mathrm{~d} \mathbf{W}^{i}
$$

suggesting an implicit summation over $i=1, \ldots, e$. Note that this abusive notation hides the important contribution of $Y^{\prime}$ and $\mathbb{W}=\left(\mathbb{W}^{i, j}: 1 \leqslant i, j \leqslant d\right)$.

Definition 4.5. - Let $\omega$ be a control function (see [16, Def. 1.6]). For $a>0$ and $[s, t] \subset[0, T]$ we set

$$
\begin{aligned}
\tau_{0}(a) & =s \\
\tau_{i+1}(a) & =\inf \left\{u: \omega\left(\tau_{i}, u\right) \geqslant a, \tau_{i}(a)<u \leqslant t\right\} \wedge t
\end{aligned}
$$

and define

$$
N_{a,[s, t]}(\omega)=\sup \left\{n \in \mathbb{N} \cup\{0\}: \tau_{n}(a)<t\right\}
$$

When $\omega$ arises from a (homogeneous) p-variation norm of a (p-rough) path, such as $\omega_{\mathbf{W}}=\|\mathbf{W}\|_{p-v a r ;[\cdot, \cdot]}^{p}$, we shall also write

$$
N_{a,[s, t]}(\mathbf{W}):=N_{a,[s, t]}\left(\omega_{\mathbf{W}}\right)
$$

Remark 4.6. - The importance of $N_{a ;[0, T]}(\mathbf{W})$ stems from the fact that it has - contrary to $\|\mathbf{W}\|_{p-v a r}^{p}$ - Gaussian integrability if $\mathbf{W}=\mathbf{B}$, the lift of Brownian motion (see $[6,15]$ ), or if $\mathbf{W}=\mathbf{Z}$, the joint lift of Brownian motion and a deterministic rough path used in the proof of Theorem 2.8 (see Lemma 4.18 (3)).

Lemma 4.7 (Bounded vector fields). - Let $\mathbf{W}$ be a geometric $\alpha$-Hölder rough path, $\alpha \in(1 / 3,1 / 2]$. Let

$$
\mathrm{d} Y=V(Y) \mathrm{d} \mathbf{W}
$$

where $V=\left(V_{i}\right)_{i=1, \ldots, e}$ is a collection of $C_{b}^{3}\left(\mathbb{R}^{d}\right)$ vector fields.
Then with $\left(Y, Y^{\prime}\right):=(Y, V(Y))$ we have

$$
\begin{equation*}
\left\|Y, Y^{\prime}\right\|_{W, \alpha} \leqslant C\left(1+\|\mathbf{W}\|_{\alpha}\right)^{3} \tag{4.1}
\end{equation*}
$$

Also

$$
\begin{equation*}
\|Y\|_{1 / \alpha-\operatorname{var}} \leqslant C\left(1+N_{1 ;[0, T]}(\mathbf{W})\right) . \tag{4.2}
\end{equation*}
$$

Proof. - (4.1) follows from [12, Prop. 8.3] and (4.2) follows from [15, Lem. 4, Cor. 3].

Lemma 4.8 (Linear vector fields). - Let $\mathbf{W}$ be a geometric $\alpha$-Hölder rough path, $\alpha \in(1 / 3,1 / 2]$. Let

$$
\mathrm{d} Y=V(Y) \mathrm{d} \mathbf{W}
$$

where $V=\left(V_{i}\right)_{i=1, \ldots, e}$ is a collection of linear vector fields of the form $V_{i}(z)=A_{i} z+b_{i}$, where $A_{i}$ are $d \times d$ matrices and $b_{i} \in \mathbb{R}^{d}$. Then for $0 \leqslant s \leqslant t \leqslant T$ :
(1) $\|Y\|_{p-v a r ;[s, t]} \leqslant C\left(1+\left|y_{0}\right|\right)\|\mathbf{W}\|_{p-v a r ;[s, t]} \exp \left(C N_{1 ;[0, T]}(\mathbf{W})\right)$.
with $p:=1 / \alpha$, which implies

$$
\|Y\|_{\alpha ;[0, T]} \leqslant C\left(1+\left|y_{0}\right|\right)\|\mathbf{W}\|_{\alpha ;[0, T]} \exp \left(C N_{1 ;[0, T]}(\mathbf{W})\right)
$$

(2) $\left|Y_{s, t}-V\left(Y_{s}\right) W_{s, t}-D V\left(Y_{s}\right) V\left(Y_{s}\right) \mathbb{W}_{s,}\right|$

$$
\leqslant C \exp \left(C N_{1 ;[0, T]}(\mathbf{W})\right)\|\mathbf{W}\|_{p-v a r ;[s, t]}^{3},
$$

which means that with $\left(Y, Y^{\prime}\right):=(Y, V(Y))$ we have

$$
\left\|Y, Y^{\prime}\right\|_{W, \alpha} \leqslant C \exp \left(C N_{1}\right)\left(\|\mathbf{W}\|_{\alpha ;[0, T]}^{2} \vee\|\mathbf{W}\|_{\alpha ;[0, T]}^{3}\right)
$$

Proof. - (1): In what follows $C$ is a constant that can change from line to line.

From [16, Thm. 10.53] we have for any $s \leqslant u \leqslant v \leqslant t$ :

$$
\left\|Y_{u, v}\right\| \leqslant C\left(1+\left|Y_{u}\right|\right)\|W\|_{p-v a r ;[u, v]} \exp \left(c\|W\|_{p-v a r ;[u, v]}^{p}\right)
$$

Then, using $\left\|Y_{u, v}\right\|=d\left(Y_{u}, Y_{v}\right) \geqslant d\left(Y_{s}, Y_{v}\right)-d\left(Y_{s}, Y_{u}\right)=\left\|Y_{s, v}\right\|-\left\|Y_{s, u}\right\|$, we have

$$
\begin{align*}
\left\|Y_{s, v}\right\| & \leqslant C\left(1+\left|Y_{u}\right|\right)\|W\|_{p-v a r ;[u, v]} \exp \left(C\|W\|_{p-v a r ;[u, v]}^{p}\right)+\left\|Y_{s, u}\right\| \\
& \leqslant C\left(1+\left|Y_{s}\right|+\left\|Y_{s, u}\right\|\right)\|W\|_{p-v a r ;[u, v]} \exp \left(C\|W\|_{p-v a r ;[u, v]}^{p}\right)+\left\|Y_{s, u}\right\| \tag{4.3}
\end{align*}
$$

On the one hand, this gives

$$
\left\|Y_{s, v}\right\| \leqslant C\left(1+\left|Y_{s}\right|+\left\|Y_{s, u}\right\|\right) \exp \left(C\|W\|_{p-v a r ;[u, v]}^{p}\right)
$$

Now letting $s=\tau_{0}<\cdots<\tau_{M}<\tau_{M+1}=v \leqslant t$, by induction,

$$
\begin{aligned}
\left\|Y_{s, v}\right\| & \leqslant C^{M+1}\left((M+1)\left(1+\left|Y_{s}\right|\right)\right) \exp \left(C \sum_{i=0}^{M}\|W\|_{p-v a r ;\left[\tau_{i}, \tau_{i+1}\right]}^{p}\right) \\
& \leqslant C^{M+1}\left(1+\left|Y_{s}\right|\right) \exp \left(C \sum_{i=0}^{M}\|W\|_{p-v a r ;\left[\tau_{i}, \tau_{i+1}\right]}^{p}\right)
\end{aligned}
$$

Hence

$$
\sup _{u \in[s, t]}\left\|Y_{s, u}\right\| \leqslant C\left(1+\left|Y_{s}\right|\right) \exp \left(C N_{1 ;[s, t]}\right)
$$

Then, using again (4.3),

$$
\begin{aligned}
& \left\|Y_{s, v}\right\| \\
& \leqslant C\left(1+\left|Y_{s}\right|+\left\|Y_{s, u}\right\|\right)\|W\|_{p-v a r ;[u, v]} \exp \left(C\|W\|_{p-v a r ;[u, v]}^{p}\right)+\left\|Y_{s, u}\right\| \\
& \leqslant C\left(1+\left|Y_{s}\right|+C\left(1+\left|Y_{s}\right|\right) \exp \left(C N_{1 ;[s, t]}\right)\right) \\
& \quad \times\|W\|_{p-v a r ;[u, v]} \exp \left(C\|W\|_{p-v a r ;[u, v]}^{p}\right)+\left\|Y_{s, u}\right\| \\
& \leqslant C^{2} 2\left(1+\left|Y_{s}\right|\right)\|W\|_{p-v a r ;[u, v]} \exp \left(C\|W\|_{p-v a r ;[u, v]}^{p}\right) \exp \left(C N_{1 ;[s, t]}\right) \\
& +\left\|Y_{s, u}\right\| .
\end{aligned}
$$

Then letting $s=\tau_{0}<\cdots<\tau_{M}<\tau_{M+1}=v \leqslant t$, by induction,

$$
\begin{aligned}
& \left\|Y_{s, v}\right\| \\
& \begin{aligned}
& \leqslant \sum_{i=0}^{M} C^{2} 2\left(1+\left|Y_{s}\right|\right)\|W\|_{p-v a r ;\left[\tau_{i}, \tau_{i+1}\right]} \\
& \times \exp \left(C\|W\|_{p-v a r ;\left[\tau_{i}, \tau_{i+1}\right]}^{p}\right) \exp \left(C N_{1 ;[s, t]}\right) \\
& \leqslant \sum_{i=0}^{M} C^{2} 2\left(1+\left|Y_{s}\right|\right)\|W\|_{p-v a r ;[s, v]} \exp \left(C\|W\|_{p-v a r ;\left[\tau_{i}, \tau_{i+1}\right]}^{p}\right) \exp \left(C N_{1 ;[s, t]}\right) \\
& \leqslant(M+1) C\left(1+\left|Y_{s}\right|\right)\|W\|_{p-v a r ;[s, v]} \\
& \quad \times \exp \left(C \max _{i}\|W\|_{p-v a r ;\left[\tau_{i}, \tau_{i+1}\right]}^{p}\right) \exp \left(C N_{1 ;[s, t]}\right)
\end{aligned}
\end{aligned}
$$

Then

$$
\begin{aligned}
\left\|Y_{s, t}\right\| & \leqslant\left(N_{1 ;[s, t]}+1\right) C\left(1+\left|Y_{s}\right|\right) \exp \left(C N_{1 ;[s, t]}\right)\|W\|_{p-v a r ;[s, t]} \\
& \leqslant C\left(1+\left|Y_{0}\right|\right) \exp \left(C N_{1 ;[0, T]}\right)\|W\|_{p-v a r ;[s, t]}
\end{aligned}
$$

as desired.
(2): It is straightforward to construct $C_{b}^{3}$ vector fields $\tilde{V}_{i}$, that coincide with $V_{i}$ on an open neighborhood of $Y$ and they can be chosen in such a way that

$$
\|\tilde{V}\|_{C_{b}^{3}} \leqslant \max _{i}\left(\left|A_{i}\right|+\left|b_{i}\right|\right)\left(|Y|_{\infty}+2\right) \leqslant C \exp \left(C N_{1}\right)
$$

The first statement then follows from [16, Cor. 10.15], which also yields the desired bound on $\left\|R^{Y}\right\|_{2 \alpha}$. The bound on $\left\|Y^{\prime}\right\|_{\alpha}$ follows from Step 1.

Lemma 4.9. - Let $\varphi \in C_{b}^{2}\left(\mathbb{R}^{d}, \mathbb{R}\right)$, $\mathbf{W}$ a geometric $\beta$-Hölder rough path in $\mathbb{R}^{e}, \beta \in(1 / 3,1 / 2], V_{1}, \ldots, V_{e} \in C_{b}^{3}\left(\mathbb{R}^{d}\right)$, and $\Psi$ the flow to the $R D E$

$$
\mathrm{d} Y=V(Y) \mathrm{d} \mathbf{W}
$$

Then

$$
\begin{align*}
& \left|\varphi\left(\Psi_{t, T}^{-1}\right) \operatorname{det}\left(D \Psi_{t, T}^{-1}\right)-\varphi\left(\Psi_{s, T}^{-1}\right) \operatorname{det}\left(D \Psi_{s, T}^{-1}\right)\right| \\
& \quad \leqslant C\|\varphi\|_{C_{b}^{2}(M(y))} \exp \left(C N_{1 ;[0, T]}(\mathbf{W})\right)\left(\|\mathbf{W}\|_{\beta}+1\right)^{17+3 d}|t-s|^{\beta} \\
& \left|\varphi\left(\Psi_{t, T}^{-1}\right) \operatorname{det}\left(D \Psi_{t, T}^{-1}\right)-\varphi\left(\Psi_{s, T}^{-1}\right) \operatorname{det}\left(D \Psi_{s, T}^{-1}\right)-\operatorname{div}(\varphi V)\left(\Psi_{t, T}^{-1}\right) \operatorname{det}\left(D \Psi_{t, T}^{-1}\right) W_{s, t}\right| \\
& \quad \leqslant C\|\varphi\|_{C_{b}^{2}(M(y))} \exp \left(C N_{1 ;[0, T]}(\mathbf{W})\right)\left(\|\mathbf{W}\|_{\beta}+1\right)^{17+3 d}|t-s|^{2 \beta} \tag{4.4}
\end{align*}
$$

with $C=C(\beta, V, \varphi)$. Here the inverse flow and its Jacobian are evaluated at $y \in \mathbb{R}^{d}$. Moreover we used

$$
M(y):=\left\{x: \inf _{r \in[0, T]}\left|\Psi_{T-r, T}^{-1}(y)\right|-1 \leqslant|x| \leqslant \sup _{r \in[0, T]}\left|\Psi_{T-r, T}^{-1}(y)\right|+1\right\}
$$

Proof. - We shall need the fact that the inverse flow and its Jacobian satisfy the following RDEs (see for example $[16, \S 11]$ ):

$$
\begin{array}{ll}
\mathrm{d} \Psi_{T-\cdot, T}^{-1}(y)=V\left(\Psi_{T-\cdot, T}^{-1}(y)\right) \mathrm{d} W_{T-\cdot}, & \Psi_{T, T}^{-1}(y)=y, \\
\mathrm{~d} D \Psi_{T-\cdot, T}^{-1}(y)=D V_{i}\left(\Psi_{T-\cdot, T}^{-1}(y)\right) D \Psi_{T-, T}^{-1}(y) \mathrm{d} W_{T-\cdot}^{i}, & D \Psi_{T, T}^{-1}(y)=I \tag{4.6}
\end{array}
$$

We proceed to show the second inequality of the statement, as the first one follows analogously. In what follows $C$ will denote a constant changing from line to line, only depending on $\beta, V, \varphi$ (but not on $W$ or $y$ ).

Let $A_{r}:=\varphi\left(\Psi_{T-r, T}^{-1}(y)\right), B_{r}:=\operatorname{det}\left(D \Psi_{T-r, T}^{-1}(y)\right)$. Using (4.5) we have that $\left(A, A^{\prime}\right) \in \mathscr{D}_{\overleftarrow{W}}^{2 \beta}$, with

$$
A_{r}^{\prime}=\left\langle D \varphi\left(\Psi_{T-r, T}^{-1}(y)\right), V\left(\Psi_{T-r, T}^{-1}(y)\right)\right\rangle
$$

and $\overleftarrow{W}_{r}:=W_{T-r}$. More specifically, by Lemma 4.7 together with Lemma 4.13

$$
\|A\|_{\overleftarrow{W}, \beta} \leqslant C\|\varphi\|_{C_{b}^{2}(M(y))}\left(\|\mathbf{W}\|_{\alpha}+1\right)^{8}
$$

where

$$
M(y):=\left\{x: \inf _{r \in[0, T]}\left|\Psi_{T-r, T}^{-1}(y)\right|-1 \leqslant|x| \leqslant \sup _{r \in[0, T]}\left|\Psi_{T-r, T}^{-1}(y)\right|+1\right\}
$$

Moreover using (4.6) and the derivative of the determinant,

$$
\begin{equation*}
\left.D \operatorname{det}\right|_{A} \cdot M=\operatorname{det}(A) \operatorname{Tr}\left[A^{-1} M\right] \tag{4.7}
\end{equation*}
$$

we get that $\left(B, B^{\prime}\right) \in \mathscr{D} \frac{2 \beta}{\overleftarrow{W}}$, with

$$
B_{r}^{\prime}=(\operatorname{div} V)\left(\Psi_{T-r, T}^{-1}(y)\right) \operatorname{det}\left(D \Psi_{T-r, T}^{-1}(y)\right)
$$

More specifically, by Lemma 4.8.2 together with Lemma 4.13

$$
\begin{aligned}
\|B\|_{\overleftarrow{W}, \beta} & \leqslant C\|\operatorname{det}\|_{C_{b}^{2}(N(y))} \exp \left(C N_{1 ;[0, T]}\right)\left(\|\mathbf{W}\|_{\alpha}+1\right)^{8} \\
& \leqslant C\left(1+\sup _{r \in[0, T]}\left|D \Psi_{T-r, T}^{-1}(y)\right|\right)^{d} \exp \left(C N_{1 ;[0, T]}\right)\left(\|\mathbf{W}\|_{\alpha}+1\right)^{8}
\end{aligned}
$$

where $N(y):=\left\{A:|A| \leqslant \sup _{r \in[0, T]}\left|D \Psi_{T-r, T}^{-1}(y)\right|+1\right\}$.

Applying Lemma 4.11, we get for $0 \leqslant u<v \leqslant T$ :

$$
\begin{aligned}
& \left|A_{v} B_{v}-A_{u} B_{u}-\left(A_{u}^{\prime} B_{u}-A_{u} B_{u}^{\prime}\right) \overleftarrow{W}_{u, v}\right| \\
& \leqslant C\left(1+\|\mathbf{W}\|_{\alpha}\right)\left(|\varphi(y)|+\|\varphi\|_{C_{b}^{2}(M(y))}\left(\|\mathbf{W}\|_{\alpha}+1\right)^{8}\right) \\
& \quad \times\left(1+\left(1+\sup _{r \in[0, T]}\left|D \Psi_{T-r, T}^{-1}(y)\right|\right)^{d} \exp \left(C N_{1 ;[0, T]}\right)\left(\|\mathbf{W}\|_{\alpha}+1\right)^{8}\right)
\end{aligned}
$$

Finally noting

$$
A_{r}^{\prime} B_{r}+A_{r} B_{r}^{\prime}=-\operatorname{div}(\varphi V)\left(\Psi_{T-r, T}^{-1}(y)\right) \operatorname{det}\left(D \varphi\left(\Psi_{T-r, T}^{-1}(y)\right)\right.
$$

and using $t=T-u, s=T-t$, the desired result follows from Lemma 4.12.

The following result from the previous proof is worth noting separately.
Lemma 4.10 (Liouville's formula for RDEs). - Let $\mathbf{Z}$ be a matrix-valued, geometric $\alpha$-Hölder rough path, $\alpha \in(1 / 3,1 / 2]$, and consider the matrixvalued linear equation

$$
\begin{aligned}
\mathrm{d} M_{t} & =\sum_{i=1}^{e} \mathrm{~d} \mathbf{Z}_{t}^{i} \cdot M_{t} \\
M_{0} & =I \in \mathbb{R}^{d \times d}
\end{aligned}
$$

Denote $D_{t}:=\operatorname{det}\left(M_{t}\right)$, then

$$
\begin{aligned}
\mathrm{d} D_{t} & =\sum_{i=1}^{e} D_{t} \operatorname{tr}\left[\mathrm{~d} \mathbf{Z}_{t}^{i}\right] \\
D_{0} & =1
\end{aligned}
$$

which is explicitly solved as

$$
D_{t}=\exp \left(\sum_{i=1}^{e} \operatorname{tr}\left[Z_{t}^{i}\right]-\operatorname{tr}\left[Z_{0}^{i}\right]\right)
$$

Lemma 4.11. - Let $W \in \mathscr{C}^{\alpha}$, and $\left(A, A^{\prime}\right),\left(B, B^{\prime}\right) \in \mathscr{D}_{W}^{2 \alpha}$. Then $\left(Y, Y^{\prime}\right):=\left(A B, A^{\prime} B+A B^{\prime}\right) \in \mathscr{D}_{W}^{2 \alpha}$ and $\left\|Y^{\prime}\right\|_{\alpha}+\left\|R^{Y}\right\|_{2 \alpha} \leqslant C\left(1+\|W\|_{\alpha}\right)\left(\left|A_{0}\right|+\left\|A, A^{\prime}\right\|_{W, \alpha}\right)\left(\left|B_{0}\right|+\left\|B, B^{\prime}\right\|_{W, \alpha}\right)$.

Proof. - Straightforward calculation.
Lemma 4.12. - Let $\tilde{Y}_{t}:=Y_{T-t}, \tilde{Y}_{t}^{\prime}:=Y_{T-t}^{\prime}, \tilde{W}_{t}:=W_{T-t}$. If $\left(\tilde{Y}, \tilde{Y}^{\prime}\right) \in$ $\mathscr{D}_{\tilde{W}}^{2 \alpha}$, then $\left(Y, Y^{\prime}\right) \in \mathscr{D}_{W}^{2 \alpha}$ and

$$
\left\|Y^{\prime}\right\|_{\alpha}+\left\|R^{Y}\right\|_{2 \alpha} \leqslant\left\|\tilde{Y}^{\prime}\right\|_{\alpha}+\left\|R^{\tilde{Y}}\right\|_{2 \alpha}+\left\|\tilde{Y}^{\prime}\right\|_{\alpha}\|W\|_{\alpha}
$$

Proof. - This follows from

$$
\begin{aligned}
Y_{t}-Y_{s}-Y_{s}^{\prime} W_{s, t} & =Y_{t}-Y_{s}-Y_{t}^{\prime} W_{s, t}+Y_{s, t}^{\prime} W_{s, t} \\
& =\tilde{Y}_{u}-\tilde{Y}_{v}-\tilde{Y}_{u}^{\prime} W_{v, u}+Y_{s, t}^{\prime} W_{s, t} \\
& =-\left(\tilde{Y}_{v}-\tilde{Y}_{u}-\tilde{Y}_{u}^{\prime} W_{u, v}\right)+Y_{s, t}^{\prime} W_{s, t}
\end{aligned}
$$

where $v:=T-s, u:=T-t$.
Lemma 4.13. - Let $W$ be an $\alpha$-Hölder path, $\alpha \in(1 / 3,1 / 2]$. Let $\left(Y, Y^{\prime}\right) \in$ $\mathscr{D}_{W}^{2 \alpha}, \phi \in C_{b}^{2}$. Then $\left(\phi(Y), D \phi(Y) Y^{\prime}\right) \in \mathscr{D}_{W}^{2 \alpha}$ with
$\left\|\phi(Y), D \phi(Y) Y^{\prime}\right\|_{W, \alpha}$

$$
\leqslant C(\alpha, T)\|\phi\|_{C_{b}^{2}}\left(1+\|W\|_{\alpha}\right)^{2}\left(1+\left|Y_{0}^{\prime}\right|+\left\|Y, Y^{\prime}\right\|_{W, \alpha}\right)^{2}
$$

Proof. - See [12, Lem. 7.3].
Lemma 4.14 (Adjoint equation). - Let $\mathbf{W}$ be a geometric $\alpha$-Hölder rough path, $\alpha \in(1 / 3,1 / 2]$. Let $V=\left(V_{i}\right)_{i=1, \ldots, e}$ be a collection of $C_{b}^{3}\left(\mathbb{R}^{d}\right)$ vector fields. Let

$$
\begin{aligned}
\mathrm{d} Y_{s}^{t, x} & =V\left(Y_{s}^{t, x}\right) \mathrm{d} \mathbf{W}_{s} \\
Y_{t}^{t, x} & =x .
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathrm{d} Y_{T}^{t, x} & =-D Y_{T}^{t, x} V(x) \mathrm{d} \mathbf{W}_{t} \\
\mathrm{~d} D Y_{T}^{t, x} & =-D^{2} Y_{T}^{t, x}\left\langle V(x) \mathrm{d} \mathbf{W}_{t}, \cdot\right\rangle-D Y_{T}^{t, x} D V(x) \mathrm{d} \mathbf{W}_{t}
\end{aligned}
$$

Proof. - Take the time derivative of

$$
Y_{T}^{t, Y_{T}^{-1, t, x}}=x
$$

for the first identity and consider the enlarged equation

$$
\mathrm{d} Z=G(Z) \mathrm{d} \mathbf{W}
$$

with $G\left(x_{1}, x_{2}\right)=\left(V\left(x_{1}\right), D V\left(x_{1}\right) x_{2}\right)$ for the second identity.
Lemma 4.15. - Let $\alpha \in(1 / 3,1 / 2]$ and $\mathbf{W}$ a geometric $\alpha$-Hölder rough path. Let $\rho$ be a solution to the forward equation (3.9) in the sense of Definition 3.3; in particular $\left(\rho_{t}(f), \rho_{t}(\Gamma f)\right)$ is controlled for every $f \in C_{b}^{3}$. Assume moreover that for every $R>0$

$$
\sup _{\|f\|_{C_{b}^{3}\left(\mathbb{R}^{d}\right)}<R}\left\{\|\rho \cdot(f), \rho \cdot(\Gamma f)\|_{W, \alpha}\right\}<\infty .
$$

Let $\phi \in C_{b}^{0,3}\left([0, T] \times \mathbb{R}^{d}\right)$ be given, satisfiying for $s \leqslant t$

$$
\phi(t, x)=\phi(s, x)+\int_{s}^{t} \alpha_{r}(x) \mathrm{d} r+\int_{s}^{t}\left(\eta_{r}(x), \eta_{r}^{\prime}(x)\right) \mathrm{d} \mathbf{W}_{r}
$$

where $\left(\eta_{t}(x), \eta_{t}^{\prime}(x)\right) \in \mathscr{D}_{W}^{2 \alpha}$. Assume moreover

$$
\begin{aligned}
\sup _{x}\left[\left\|\eta(x), \eta^{\prime}(x)\right\|_{W, \alpha}\right]<\infty \\
\sup _{x}\left[\left\|\Gamma_{i} \phi(x), \Gamma_{i} \eta(x)\right\|_{W, \alpha}\right]<\infty, \quad i=1, \ldots, e
\end{aligned}
$$

and $\eta \in C_{b}^{0,3}\left([0, T] \times \mathbb{R}^{d}\right)$.
Then $\left(M, M^{\prime}\right),\left(N, N^{\prime}\right) \in \mathscr{D}_{W}^{2 \alpha}$ where

$$
\begin{aligned}
\left(M_{t}\right)_{i=1, \ldots, e} & :=\rho_{t}\left(\Gamma_{i} \phi_{t}\right), \\
\left(M_{t}^{\prime}\right)_{i, j=1, \ldots, e} & :=\rho_{t}\left(\Gamma_{j} \Gamma_{i} \phi_{t}\right)+\rho_{t}\left(\Gamma_{i}\left(\eta_{t}\right)_{j}\right), \\
\left(N_{t}\right)_{i=1, \ldots, e} & :=\rho_{t}\left(\left(\eta_{t}\right)_{i}\right), \\
\left(N_{t}^{\prime}\right)_{i, j=1, \ldots, e} & :=\rho_{t}\left(\left(\eta_{t}^{\prime}\right)_{i j}\right)+\rho_{t}\left(\Gamma_{j}\left(\eta_{t}\right)_{i}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
& \rho_{t}\left(\phi_{t}\right)=\rho_{s}\left(\phi_{s}\right)+\int_{s}^{t}\left(M, M^{\prime}\right)_{r} \mathrm{~d} \mathbf{W}_{r}+\int_{s}^{t}\left(N, N^{\prime}\right)_{r} \mathrm{~d} \mathbf{W}_{r} \\
&+\int_{s}^{t} \rho_{r}\left(\alpha_{r}+L \phi_{r}\right) \mathrm{d} r .
\end{aligned}
$$

Remark 4.16. - Note that with $\left(\eta, \eta^{\prime}\right) \equiv 0, \alpha \equiv 0, \phi(0, x)=f(x)$, this reduces to (3.11).

Proof. - First

$$
\begin{aligned}
\left(M_{t}\right)_{i}-\left(M_{s}\right)_{i} & =\rho_{t}\left(\Gamma_{i} \phi_{t}\right)-\rho_{s}\left(\Gamma_{i} \phi_{s}\right) \\
& =\rho_{s}\left(\Gamma_{i} \phi_{s, t}\right)-\rho_{s, t}\left(\Gamma_{i} \phi_{s}\right)+\rho_{s, t}\left(\Gamma_{i} \phi_{s, t}\right) \\
& =\rho_{s}\left(\Gamma_{i}\left(\eta_{s}\right)_{j}\right) W_{s, t}^{j}-\rho_{s}\left(\Gamma_{j} \Gamma_{i} \phi_{s}\right) W_{s, t}^{j}+O\left(|t-s|^{2 \alpha}\right)
\end{aligned}
$$

Here we used that by assumption $\Gamma_{i} \phi_{s, t}(x)=\Gamma_{i} \eta_{j}(x) W_{s, t}^{j}+O\left(|t-s|^{2 \alpha}\right)$, uniformly in $x$ and that $\rho_{s, t}\left(\Gamma_{i} f\right)=\rho_{s}\left(\Gamma_{j} \Gamma_{i} f\right) W_{s, t}^{j}+O\left(|t-s|^{2 \alpha}\right)$ uniformly over bounded sets of $f$ in $C_{b}^{3}$. It follows that $\left(M, M^{\prime}\right) \in \mathscr{D}_{W}^{2 \alpha}$. And analogously for $\left(N, N^{\prime}\right)$ :

$$
\begin{aligned}
\left(N_{t}\right)_{i}-\left(N_{s}\right)_{i} & =\rho_{t}\left(\left(\eta_{t}\right)_{i}\right)-\rho_{s}\left(\left(\eta_{s}\right)_{i}\right) \\
& =\rho_{s}\left(\left(\eta_{s, t}\right)_{i}\right)-\rho_{s, t}\left(\left(\eta_{s}\right)_{i}\right)+\rho_{s, t}\left(\left(\eta_{s, t}\right)_{i}\right) \\
& =\rho_{s}\left(\left(\eta_{s}^{\prime}\right)_{i, j} W_{s, t}^{j}+\rho_{s}\left(\Gamma_{j}\left(\eta_{s}\right)_{i}\right) W_{s, t}^{j}+O\left(|t-s|^{2 \alpha}\right),\right.
\end{aligned}
$$

since $\eta_{t} \in C_{b}^{3}\left(\mathbb{R}^{d}\right)$ uniformly in $t$.

Now for the integral equality, for simplicity take $L=0, \alpha=0$. Then

$$
\begin{aligned}
& \rho_{t}\left(\phi_{t}\right)-\rho_{s}\left(\phi_{s}\right) \\
& =\rho_{s}\left(\phi_{s, t}\right)+\rho_{s, t}\left(\phi_{s}\right)+\rho_{s, t}\left(\phi_{s, t}\right) \\
& =\rho_{s}\left(\left(\eta_{s}\right)_{i}\right) W_{s, t}^{i}+\rho_{t}\left(\left(\eta_{s}^{\prime}\right)_{i j}\right) \mathbb{W}_{s, t}^{i j}+\rho_{s}\left(\Gamma_{i} \phi_{s}\right) W_{s, t}^{i}+\rho_{s}\left(\Gamma_{j} \Gamma_{i} \phi_{s}\right) \mathbb{W}_{s, t}^{i j} \\
& +\rho_{s}\left(\Gamma_{i}\left(\eta_{s}\right)_{j}\right) W_{s, t}^{i} W_{s, t}^{j}+O\left(|t-s|^{3 \alpha}\right) \\
& =\rho_{s}\left(\left(\eta_{s}\right)_{i}\right) W_{s, t}^{i}+\left[\rho_{t}\left(\left(\eta_{s}^{\prime}\right)_{i j}\right)+\rho_{s}\left(\Gamma_{j}\left(\eta_{s}\right)_{i}\right)\right] \mathbb{W}_{s, t}^{i j}+\rho_{s}\left(\Gamma_{i} \phi_{s}\right) W_{s, t}^{i} \\
& +\left[\rho_{s}\left(\Gamma_{j} \Gamma_{i} \phi_{s}\right)+\rho_{s}\left(\Gamma_{i}\left(\eta_{s}\right)_{j}\right)\right] \mathbb{W}_{s, t}^{i j}+O\left(|t-s|^{3 \alpha}\right) .
\end{aligned}
$$

Now for every partition $\mathcal{P}$ of $[s, t]$ we have the trivial identity

$$
\rho_{t}\left(\phi_{t}\right)-\rho_{s}\left(\phi_{s}\right)=\sum_{[u, v] \in \mathcal{P}}\left[\rho_{v}\left(\phi_{v}\right)-\rho_{u}\left(\phi_{u}\right)\right] .
$$

The claimed equality then follows from taking the limit along partitions with mesh-size going to zero.

Lemma 4.17. - Let $\mathbf{W}$ be a geometric $\alpha$-Hölder rough path, $\alpha \in$ $(1 / 3,1 / 2]$. Let $u$ be a weak solution to the backward RPDE (2.5) in the sense of Definition 2.3; in particular $\left(\left\langle u ., \Gamma^{*} \phi\right\rangle,\left\langle u ., \Gamma^{*} \Gamma^{*} \phi\right\rangle\right)$ is controlled for every $\phi \in C_{\exp }^{3}\left(\mathbb{R}^{d}\right)$. Assume moreover that for every $R>0$

$$
\begin{equation*}
\sup _{\|f\|_{C_{\exp }^{3}\left(\mathbb{R}^{d}\right)}^{3}<R}\left\{\|(\langle u ., f\rangle,\langle u ., \Gamma f\rangle)\|_{W, \alpha}\right\}<\infty . \tag{4.8}
\end{equation*}
$$

Let $\phi \in C_{\exp }^{0,4}\left([0, T] \times \mathbb{R}^{d}\right)$ be given that satisfies for $s \leqslant t$

$$
\phi(t, x)=\phi(s, x)+\int_{s}^{t} \alpha_{r}(x) \mathrm{d} r+\int_{s}^{t}\left(\eta_{r}(x), \eta_{r}^{\prime}(x)\right) \mathrm{d} \mathbf{W}_{r}
$$

where $\left(\eta_{t}(x), \eta_{t}^{\prime}(x)\right) \in \mathscr{D}_{W}^{2 \alpha}$. Assume moreover for some $\delta>0$

$$
\begin{aligned}
\left\|\eta(x), \eta^{\prime}(x)\right\|_{W, \alpha} & \lesssim e^{-\delta|x|} \\
\left\|\Gamma_{i}^{*} \phi(x), \Gamma_{i}^{*} \eta(x)\right\|_{W, \alpha} & \lesssim e^{-\delta|x|}, \quad i=1, \ldots, e .
\end{aligned}
$$

In addition assume that $\eta \in C_{\exp }^{0,3}\left([0, T] \times \mathbb{R}^{d}\right)$.
Then $\left(M, M^{\prime}\right),\left(N, N^{\prime}\right) \in \mathscr{D}_{W}^{2 \alpha}$ where

$$
\begin{aligned}
\left(M_{t}\right)_{i=1, \ldots, e} & :=\left\langle u_{t}, \Gamma_{i}^{*} \phi_{t}\right\rangle \\
\left(M_{t}^{\prime}\right)_{i, j=1, \ldots, e} & :=\left\langle u_{t}, \Gamma_{j}^{*} \Gamma_{i}^{*} \phi_{t}\right\rangle+\left\langle u_{t}, \Gamma_{i}^{*}\left(\eta_{t}\right)_{j}\right\rangle, \\
\left(N_{t}\right)_{i=1, \ldots, e} & :=\left\langle u_{t},\left(\eta_{t}\right)_{i}\right\rangle \\
\left(N_{t}^{\prime}\right)_{i, j=1, \ldots, e} & :=\left\langle u_{t},\left(\eta_{t}^{\prime}\right)_{i j}\right\rangle+\left\langle u_{t}, \Gamma_{j}^{*}\left(\eta_{t}\right)_{i}\right\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
\left\langle u_{t}, \phi_{t}\right\rangle=\left\langle u_{s}, \phi_{s}\right\rangle-\int_{s}^{t}\left(M, M^{\prime}\right)_{r} \mathrm{~d} \mathbf{W}_{r}+\int_{s}^{t}(N, & \left.N^{\prime}\right)_{r} \mathrm{~d} \mathbf{W}_{r} \\
& +\int_{s}^{t}\left\langle u_{r}, \alpha_{r}-L^{*} \phi_{r}\right\rangle \mathrm{d} r
\end{aligned}
$$

Proof. - By assumption

$$
\left|\Gamma_{i}^{*} \phi_{t}(x)-\Gamma_{i}^{*} \phi_{s}(x)-\Gamma_{i}^{*} \eta_{s}(x) W_{s, t}\right| \lesssim e^{-\delta|x|}|t-s|^{2 \alpha} .
$$

Hence

$$
\begin{aligned}
& \left|\left\langle u_{s}, \Gamma_{i}^{*} \phi_{s, t}\right\rangle-\left\langle u_{s}, \Gamma_{i}^{*} \eta_{j}\right\rangle W_{s, t}^{j}\right| \\
& \quad \leqslant\|u\|_{\infty}\|\beta\|_{C_{b}^{2}\left(\mathbb{R}^{d}\right)}\left\|D \phi_{t}-D \phi_{s}-D \eta_{s} W_{s, t}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)} \lesssim|t-s|^{2 \alpha} .
\end{aligned}
$$

Moreover

$$
\left|\left\langle u_{s, t}, \Gamma_{i}^{*} \phi_{s}\right\rangle-\left\langle u_{s}, \Gamma_{j}^{*} \Gamma_{i}^{*} \phi_{s}\right\rangle W_{s, t}^{j}\right| \lesssim|t-s|^{2 \alpha},
$$

since $\phi \in C_{\exp }^{0,4}\left([0, T] \times \mathbb{R}^{d}\right)$ (and hence $\Gamma_{j}^{*} u_{t} \in C_{\exp }^{3}\left(\mathbb{R}^{d}\right)$ uniformly in $t$ ) and since $u$ satisfies the uniform bound (4.8). Then also $\left\langle u_{s, t}, \Gamma_{i}^{*} \phi_{s, t}\right\rangle \lesssim|t-s|^{2 \alpha}$ and hence

$$
\left(M_{t}\right)_{i}-\left(M_{s}\right)_{i}=\left\langle u_{t}, \Gamma_{j}^{*} \Gamma_{i}^{*} \phi_{t}\right\rangle W_{s, t}^{j}+\left\langle u_{t}, \Gamma_{i}^{*}\left(\eta_{t}\right)_{j}\right\rangle W_{s, t}^{j}+O\left(|t-s|^{2 \alpha}\right),
$$

hence $\left(M, M^{\prime}\right)$ is controlled. The argument for $\left(N, N^{\prime}\right)$ is similar and the proof now finishes as the preceding one.

### 4.2. Rough SDEs

Lemma 4.18. - For $\mathbf{W}$ a geometric $\alpha$-Hölder rough path, $\alpha \in(1 / 3,1 / 2]$, and a Brownian motion $B$, define $\mathbf{Z}=(Z, \mathbb{Z})$ as

$$
Z_{t}=\binom{B_{t}}{W_{t}}, \mathbb{Z}_{s, t}=\left(\begin{array}{ll}
\mathbb{B}_{s, t}^{I t o} & \int W \otimes \mathrm{~d} B \\
\int_{s}^{t} B_{s, t} \otimes \mathrm{~d} W & \mathbb{W}
\end{array}\right)
$$

Then
(1) $\mathbf{Z}$ is well-defined and, almost surely, an $\alpha$-Hölder rough path
(2) $\left|\|\mathbf{Z}(\mathbf{W})-\mathbf{Z}(\tilde{\mathbf{W}})\|_{\alpha}\right|_{L^{q}} \lesssim\|\mathbf{W}-\tilde{\mathbf{W}}\|_{\alpha}$
(3) $N_{1 ;[0, T]}\left(\|\mathbf{Z}\|_{p}^{p}\right)$ has Gaussian tails, uniformly over $\mathbf{W}$ bounded, for $p=\frac{1}{\alpha}$.

Proof. - This is proven in [10], the only difference being that there, $\mathbf{Z}$ is only shown to be an $\alpha^{\prime}$-Hölder rough path, for $\alpha^{\prime}<\alpha$. This stems from the fact, that there, a Kolmogorov-type argument is applied to the whole
rough path $\mathbf{Z}$, which in particular contains the deterministic path $W$, which explains the decay in perceived regularity.

Being more careful, and applying a Kolmogorov-type argument (e.g. [12, Thm. 3.1]) only to the second level, one sees that it is actually $\beta$-Hölder continuous, for $\beta<\alpha+1 / 2$. The first level is trivially $\alpha$-Hölder continuous. The claimed continuity in $\mathbf{W}$ is then improved similarly.

Lemma 4.19 (Rough SDE). - Let $\mathbf{W}$ be a geometric $\alpha$-Hölder rough path, $\alpha \in(1 / 3,1 / 2]$, and let $\mathbf{Z}=(Z, \mathbb{Z})$ be the joint lift of $\mathbf{W}$ and a Brownian motion $B$ given in the previous Lemma 4.18. Assume $\sigma_{i}, \beta_{j} \in C_{b}^{3}\left(\mathbb{R}^{d}\right), i=$ $1, \ldots, d_{B}, j=1, \ldots, e, b \in C_{b}^{1}\left(\mathbb{R}^{d}\right)$. Let $X=X(\omega)$ be the solution to the rough differential equation

$$
\mathrm{d} X=b(X) \mathrm{d} t+V(X) \mathrm{d} \mathbf{Z}
$$

where $V=(\sigma, \beta)$. Then $X$ formally solves the rough SDE

$$
\mathrm{d} X=b(X) \mathrm{d} t+\sigma(X) \mathrm{d} B+\beta(X) \mathrm{d} \mathbf{W}
$$

We have the following properties:

- For all $p \geqslant 1$, the mapping

$$
\begin{aligned}
\mathscr{C}^{\alpha} & \rightarrow \mathcal{S}^{p} \\
\mathbf{W} & \mapsto X
\end{aligned}
$$

is locally uniformly continuous. Here $\|X\|_{\mathcal{S}^{p}}^{p}:=\mathbb{E}\left[\sup _{t \leqslant T}\left|X_{t}\right|^{p}\right]$. Moreover for every $R>0$ there is $\delta>0$ such that

$$
\sup _{\|\mathbf{W}\|_{\alpha}<R} \mathbb{E}\left[\exp \left(\delta\left|X^{\mathbf{W}}\right|_{\infty}^{2}\right)\right]<\infty
$$

If in addition, for $n \geqslant 0, \sigma_{i}, \beta_{j} \in C_{b}^{3+n}, b \in C_{b}^{1+n}$, then the same holds true for $D^{n} X$.

- For $\mathbf{W}$ the canonical lift of a smooth path $W, X$ coincides with the classical SDE solution to

$$
\begin{equation*}
\mathrm{d} X_{t}=b\left(X_{t}\right) \mathrm{d} t+\sigma\left(X_{t}\right) \mathrm{d} B+\beta\left(X_{t}\right) \dot{W}_{t} \mathrm{~d} t \tag{4.9}
\end{equation*}
$$

- Let $c, g \in C_{b}^{1}\left(\mathbb{R}^{d}\right)$ and $\gamma \in C_{b}^{2}\left(\mathbb{R}^{d}\right)$, then $\int \gamma\left(X_{s}\right) \mathrm{d} \mathbf{W}_{\text {s }}$ is a welldefined rough integral, and moreover, for all $p \geqslant 1$,

$$
\begin{aligned}
{[0, T] \times \mathbb{R}^{d} \times \mathscr{C}^{\alpha} } & \rightarrow L^{p} \\
(t, x, \mathbf{W}) & \mapsto g\left(X_{T}^{t, x}\right) \exp \left(\int_{t}^{T} c\left(X_{r}^{t, x}\right) \mathrm{d} r+\int_{t}^{T} \gamma\left(X_{r}^{t, x}\right) \mathrm{d} \mathbf{W}_{r}\right)
\end{aligned}
$$

is continuous, uniformly in $t, x$ and locally uniformly in $\mathbf{W}$.

Proof. - If $b \in C_{b}^{1+\varepsilon}\left(\mathbb{R}^{d}\right)$ for some $\varepsilon>0$, this is shown in [10, Thm. 10]. Now, for $b \in C_{b}^{1}\left(\mathbb{R}^{d}\right)$ the same proof works, one just needs to use the improved result on RDEs with drift in [13, Prop. 3].

Lemma 4.20. - Let $\mathbf{W}$ be a geometric $\alpha$-Hölder rough path, $\alpha \in$ $(1 / 3,1 / 2]$. Let $X^{t, x}$ be the solution to the rough SDE (Lemma 4.19)

$$
\mathrm{d} X_{t}^{t, x}=b\left(X_{t}^{t, x}\right) \mathrm{d} t+\sigma\left(X_{t}^{t, x}\right) \mathrm{d} B+\beta\left(X_{t}^{t, x}\right) \mathrm{d} \mathbf{W}, \quad X_{t}^{t, x}=x
$$

Let $n \geqslant 0$ and assume $c \in C_{b}^{n}\left(\mathbb{R}^{d}\right), \gamma_{k} \in C_{b}^{2+n}\left(\mathbb{R}^{d}\right), \sigma_{i}, \beta_{j} \in C_{b}^{3+[(n-1) \vee 0]}\left(\mathbb{R}^{d}\right)$, $b \in C_{b}^{1+[(n-1) \vee 0]}$. Then for every $\phi \in C_{\exp }^{n}\left(\mathbb{R}^{d}\right)$ the function

$$
\psi(x):=\mathbb{E}\left[\phi\left(X_{T}^{t, x}\right) \exp \left(\int_{t}^{T} c\left(X_{r}^{t, x}\right) \mathrm{d} r+\int_{t}^{T} \gamma\left(X_{r}^{t, x}\right) \mathrm{d} \mathbf{W}_{r}\right)\right]
$$

is again in $C_{\exp }^{n}\left(\mathbb{R}^{d}\right)$, with $\|\psi\|_{C_{\exp }^{n}\left(\mathbb{R}^{d}\right)}$ bounded uniformly for $t \leqslant T$ and $\|\mathbf{W}\|_{\alpha}$ bounded.

Proof. - For $n=0$, let $C_{1}>0$ such that $|\psi(x)| \leqslant C_{1} \exp \left(-\frac{1}{C_{1}}|x|\right)$. Then

$$
\begin{aligned}
|\psi(x)| & =\mathbb{E}\left[\phi\left(X_{T}^{t, x}\right) \exp \left(\int_{t}^{T} c\left(X_{r}^{t, x}\right) \mathrm{d} r+\int_{t}^{T} \gamma\left(X_{r}^{t, x}\right) \mathrm{d} \mathbf{W}_{r}\right)\right] \\
& \leqslant C_{1} \mathbb{E}\left[\exp \left(-\frac{1}{C_{1}}\left|X_{T}^{t, x}\right|\right) \exp (\ldots)\right] \\
& \leqslant C_{1} \exp \left(-\frac{1}{C_{1}} x\right) \mathbb{E}\left[\exp \left(\frac{1}{C_{1}}\left|X_{T}^{t, x}-x\right|\right) \exp (\ldots)\right] \\
& \leqslant C_{1} \exp \left(-\frac{1}{C_{1}} x\right) \mathbb{E}\left[\exp \left(C_{2}\left(1+N_{1,[t, T]}(\mathbf{Z})\right)+T\|c\|_{\infty}\right)\right] \\
& \leqslant C_{1} \exp \left(-\frac{1}{C_{1}} x\right) \mathbb{E}\left[\exp \left(C_{3} N_{1,[0, T]}(\mathbf{Z})\right)\right]
\end{aligned}
$$

where we used (4.2) for the 4th line. This concludes the argument, since the expectation is finite by Lemma 4.18, uniformly for $\|\mathbf{W}\|_{\alpha}$ bounded. The case $n \geqslant 1$ follows similarly, by differentiating under the expectation.

Lemma 4.21. - Let $\mathbf{W}$ be a geometric $\alpha$-Hölder rough path, $\alpha \in$ $(1 / 3,1 / 2]$. Assume $\sigma_{i}, \beta_{j} \in C_{b}^{3}\left(\mathbb{R}^{d}\right), i=1, \ldots, d_{B}, j=1, \ldots, e, b \in C_{b}^{1}$. Let $X^{t, x}$ be the solution to the rough $S D E$ (Lemma 4.19)

$$
\mathrm{d} X=\sigma(X) \mathrm{d} B+\beta(X) \mathbf{W}
$$

Let $n \geqslant 0$. For every $\varphi \in C_{\exp }^{n}\left(\mathbb{R}^{d}\right)$, any $q \geqslant 1$, the function

$$
\psi(x):=\mathbb{E}\left[\|\varphi\|_{C_{b}^{n}(M(x))}^{q}\right]
$$

is in $C_{\mathrm{exp}}^{0}$. Here

$$
M(y):=\left\{x: \inf _{r \in[t, T]}\left|X_{r}^{t, y}\right|-1 \leqslant|x| \leqslant \sup _{r \in[t, T]}\left|X_{r}^{t, y}\right|+1\right\}
$$

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Proof. - This follows from

$$
\begin{aligned}
\|\varphi\|_{L^{\infty}(M(x))} & \leqslant C \exp \left(-\delta \inf _{r \in[t, T]}\left|X_{r}^{t, x}\right|\right) \\
& \leqslant C \exp \left(-\delta \inf _{r \in[t, T]}\left|X_{r}^{t, x}\right|\right) \\
& \leqslant C \exp (-\delta|x|) \exp \left(\delta \sup _{r \in[t, T]}\left|X_{r}^{t, x}-x\right|\right)
\end{aligned}
$$

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[^0]:    (1) Write $\Gamma[u] \circ \mathrm{d} W=\sum_{k=1}^{e} \Gamma_{k}[u] \circ \mathrm{d} W^{k}$.

[^1]:    ${ }^{(2)}$ There is no probability here, for $\mathbf{W}$ is a deterministic rough path. Nevertheless, with a view to later applications to SPDEs and to avoid misunderstandings, let us emphasize that in this paper "weak" is always understood as "analytically weak".

[^2]:    ${ }^{(3)}$ which we shall need in order to speak of bounded sets in $C_{\exp }^{n}\left(\mathbb{R}^{d}\right)$.

[^3]:    ${ }^{(4)}$ In [19] it is assumed that the term in the exponential is non-positive, but a term bounded from below poses no additional difficulty: just replace $u(t, x)$ by $u(t, x) e^{-c(T-t)}$ for $c$ sufficiently large.

[^4]:    ${ }^{(5)}$ Notation $\mathrm{d} \mathbf{W}^{i}, \mathrm{~d} \mathbf{W}^{k}, \ldots$ is somewhat abusive, see Remark 4.4.

[^5]:    ${ }^{(6)}$ Our setting here is simpler than in [1], since our coefficients (and their derivatives) are bounded.

