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Tangents to subsolutions: existence and uniqueness, Part I (*)

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Abstract. — There is an interesting potential theory associated to each degenerate elliptic, fully nonlinear equation $f(D^2u) = 0$. These include all the potential theories attached to calibrated geometries. This paper begins the study of tangents to the subsolutions in these theories, a topic inspired by the results of Kiselman in the classical plurisubharmonic case. Fundamental to this study is a new invariant of the equation, called the *Riesz characteristic*, which governs asymptotic structures. The existence of tangents to subsolutions is established in general, as is the existence of an upper semi-continuous density function. Two theorems establishing the strong uniqueness of tangents (which means tangents are always unique and are Riesz kernels) are proved. They cover all $\mathrm{O}(n)$-invariant convex cone equations and their complex and quaternionic analogues, with the exception of the homogeneous Monge–Ampère equations, where uniqueness fails. They also cover a large class of geometrically defined subequations which includes those coming from calibrations. A discreteness result for the sets where the density is $\geqslant c > 0$ is also established in any case where strong uniqueness holds. A further result (which is sharp) asserts the Hölder continuity of subsolutions when the Riesz characteristic p satisfies $1 \leqslant p < 2$. Many explicit examples are examined.

The second part of this paper [23] is devoted to the "geometric cases". A Homogeneity Theorem and an additional Strong Uniqueness Theorem are proved, and the tangents in the Monge–Ampère cases are completely classified.

Résumé. — Il existe une théorie du potentiel intéressante associée à chaque équation, nonlinéaire et élliptique dégénérée, de la forme $f(D^2u) = 0$. Ceci inclut toutes les théories du potentiel associées aux calibrations. Cet article commence l'étude des tangents aux sous-solutions dans ces théories, un sujet inspiré par l'oeuvre de Kiselman dans le cas pluri-potentiel classique. Fondamentale à notre étude est une nouvelle invariante, la caractéristique de Riesz, qui gouverne les structures asymptotiques. L'existence de tangents aux sous-solutions est établie en général; on démontre aussi l'existence générale d'une fonction de densité, semi-continue supérieurement.

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Article proposé par Vincent Guedj.

Deux théorèmes qui établissent l'unicité forte de tangents (i.e., tangents sont toujours unique et sont noyaux de Riesz) sont démontrés. Ils comprennent toutes les sous-équations qui sont des cones convexes et $\mathrm{O}(n)$-invariants, ainsi que leurs analogues complexes et quaternioniques, avec l'exception de l'équation de Monge–Ampère, pour laquelle l'unicité forte ne tient pas. Ils s'appliquent aussi à une grande classe de sous-équations définies géométriquement. Parmi elles sont toutes celles qui proviennent de calibrations. Un résultat de finitude locale, pour les ensembles de densité $\geqslant c > 0$, est établi dans chaque cas où régit l'unicité forte. Selon un autre résultat, quand la caractéristique de Riesz p satisfait $1 \leqslant p < 2$, alors toutes les functions sous-harmoniques sont Hölder-continues. On considère beaucoup d'exemples explicites.

La deuxième partie de cet article [23] concerne les "cas géométriques". On y établit un Théorème d'Homogénéité et un Théorème d'Unicité Forte. Aussi, les espaces tangents pour les équations de Monge–Ampère (réelles, complexes et quaternioniques) sont classifiés complètement.

1. Introduction

The point of this paper is to introduce and study tangents for a wide class of degenerate elliptic, fully nonlinear equations of the form $\mathbf{F}(D^2u) = 0$ in $\mathbb{R}^n$. It was inspired by Kieselman's study [29] (cf. [30]) of tangents to plurisubharmonic functions in classical pluripotential theory. The aim is to develop techniques for studying the behavior, in particular the singular behavior, of subsolutions: the upper semi-continuous functions u which satisfy $\mathbf{F}(D^2u) \geqslant 0$ in the viscosity sense. A number of quite general results are obtained. These include existence, uniqueness and "harmonicity" of tangents for a wide range of equations. Densities for subsolutions are defined and shown to be upper semi-continuous, and a structure theorem is proved for the sets where the density is $\geqslant c > 0$. A key to the analysis is the notion of the Riesz characteristic of the equation. This invariant is a real number $p \geqslant 1$ which governs the asymptotic behavior of singularities, and is easily computed in all of the examples, no matter how degenerate (see Sections 3 and 4).

For this study we focus on the closed set $F = \{A \in \mathrm{Sym}^2(\mathbb{R}^n) : \mathbf{F}(A) \geqslant 0\}$ (cf. [14, 32]), and the operator $\mathbf{F}$ will play no role. This set is always assumed to have the following three properties:

(1) (Positivity) $F + \mathcal{P} \subset F$ where $\mathcal{P} \equiv \{A \geqslant 0\}$.
(2) (ST-Invariance) F is invariant under a subgroup $G \subset \mathrm{O}(n)$ which acts transitively on the sphere $S^{n-1} \subset \mathbb{R}^n$.
(3) (Cone Property) $tF \subset F$ for all $t \geqslant 0$.

A closed set F satisfying Positivity is called a subequation, and the viscosity F-subsolutions are called F-subharmonic functions. Each subequation F has its own potential theory ([14, 17]). For some of the results here, in addition to these three conditions, F is also assumed to be convex. In this case distribution theory provides an alternate but equivalent foundation (Theorem 9.5) for subsolutions, which is helpful.

The equations covered here include many classical examples coming from real, complex and calibrated geometry, such as the Monge–Ampère and Hessian equations. The reader is encouraged to glance at Section 4 for some basic examples.

At the time of the first writing of this paper the authors were unaware of its connections to the important work of Armstrong, Sirakov and Smart [1]. They also studied conical subequations $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ with the additional assumption that F is uniformly elliptic. This is a stringent assumption which eliminates many of the examples arising from geometry. They also studied only solutions (as opposed to the much more general subsolutions considered here). On the other hand they do not assume invariance or convexity, which is extremely nice. There are also connections of our work to that of Labutin [33] who, like Armstrong, Sirakov and Smart, studied uniformly elliptic equations. At the end of this introduction the overlap / lack of overlap is discussed in more detail.

We begin the paper by introducing the algebraically defined and easily computable *Riesz characteristic* p_F for F, which determines much of the behavior of subsolutions examined here. The name comes from the fact that when $p \equiv p_F$ is finite, the classical p^{th} *Riesz kernel* $K_p(|x|)$, where

$$K_p(t) = \begin{cases} t^{2-p} & \text{if } 1 \leqslant p < 2 \\ \log t & \text{if } p = 2 \\ -\frac{1}{t^{p-2}} & \text{if } 2 < p < \infty, \end{cases} \tag{1.1}$$

is a solution of the non-linear equation F. In fact, every increasing radial solution is of the form $\Theta K_p(|x|) + C$ for constants $\Theta \geqslant 0$ and $C \in \mathbb{R}$. The signs in (1.1) have been chosen so that $K_p(t)$ is always increasing.

When p is finite, there is an associated tangential p-flow on F-subharmonic functions u at each point x_0, given for $x_0 = 0$ by

$$u_r(x) = \begin{cases} r^{p-2}u(rx) & \text{if } p \neq 2 \\ u(rx) - M(u,r) & \text{if } p = 2, \end{cases} \tag{1.2}$$

where

$$M(u,r) \equiv \sup_{|x| \leqslant r} u. \tag{1.3}$$

The *tangents* to u at $0 \in \mathbb{R}^n$ are defined to be the set $T_0(u)$ cluster points of the flow (1.2). When F is convex, these cluster points are taken in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. When $1 \leqslant p_F < 2$ (but F not necessarily convex), they can be taken in the local β-Hölder norm for $\beta < 2-p$. In either case, $U \in T_0(u)$ if and only if there exists a sequence $r_j \downarrow 0$ such that $u_{r_j} \to U$ (in the appropriate space). It is a basic result that tangents are always entire F-subharmonic functions on $\mathbb{R}^n$. In particular, the L^1_{loc}-limits have unique upper semi-continuous representatives which are viscosity F-subsolutions (see Theorem 9.5 (2)). A fundamental result, which is proved in Sections 11 and 15, is the following.

Theorem 1.1 (Existence of Tangents). — *If F is convex or if $p_F < 2$, then tangents always exist.*

A natural question is whether tangents are actually solutions (as opposed to subsolutions). The answer is no (if $p_F \geqslant 2$). Classical pluripotential theory provides (self) tangent examples with large singular sets. It also provides the remedy: an appropriate concept enlarging the space of (viscosity) solutions.

An F-subharmonic function on $X^{\mathrm{open}} \subset \mathbb{R}^n$ is called *F-maximal* if for each F-subharmonic function v on X and each compact subset $K \subset X$,

$$v \leqslant u \ \text{ on } X - K \quad \Rightarrow \quad v \leqslant u \ \text{ on } X.$$

If u is F-maximal on X, then on any subdomain $Y \subset X$ where u is continuous, it is a viscosity solution (or "F-harmonic"). In particular, it is always the Perron function for its boundary values on any ball. A second fundamental result is the following (see Theorem 10.2 and Corollary 10.3).

Theorem 1.2 (Maximality of Tangents). — *If F is convex, then tangents are always maximal outside the origin in $\mathbb{R}^n$. If $p_F < 2$, then tangents are F-harmonic (maximal and continuous) outside the origin.*

Existence and regularity (in the weakened form of maximality) for tangents brings us to the natural question of uniqueness. Here there are several versions.

We say that *uniqueness of tangents* holds for the subequation F if for every F-subharmonic function u defined in a neighborhood of 0, there is exactly one tangent to u at 0.

We say that *strong uniqueness of tangents* holds for F if for every such u, the unique tangent is $\Theta(u)K_p(|x|)$, with $\Theta(u) \geqslant 0$.

We say that *homogeneity of tangents* holds for F if every tangent to an F-subharmonic is fixed by the tangential p-flow (1.2).

Since the flow takes a tangent to u to another tangent to u, uniqueness of tangents implies homogeneity of tangents.

Several important special cases where uniqueness holds are discussed in Section 12 (Propositions 12.2, 12.4 and 12.5).

One of the main results of this paper is the Strong Uniqueness Theorem in Section 13. Note that there is a natural action of the group $\mathrm{O}(n)$ on $\mathrm{Sym}^2(\mathbb{R}^n)$. The subequations $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ which are $\mathrm{O}(n)$-invariant are exactly those which are defined in terms of the eigenvalues of the matrices $A \in \mathrm{Sym}^2(\mathbb{R}^n)$. Every such subequation has a complex and quaternionic counterpart defined on $\mathbb{C}^n$ and $\mathbb{H}^n$ by applying the same eigenvalue constraints to the complex or quaternionic hermitian symmetric part of A.

Theorem 1.3.I (Strong Uniqueness of Tangents I). — *Suppose F is a convex $O(n)$-invariant subequation, or the complex or quaternionic counterpart of such an equation. Then, except for the three basic cases $\mathcal{P}, \mathcal{P}^{\mathbb{C}}, \mathcal{P}^{\mathbb{H}}$, strong uniqueness of tangents holds for F.*

There do exist *non-convex* $\mathrm{O}(n)$-invariant subequations of every Riesz characteristic for which strong uniqueness fails. See Example 13.15.

Theorem 1.3.I establishes strong uniqueness for a wide range of equations. These include the k^{th} Hessian equations ($k < n$) and p-convexity equations (p real, $1 \leqslant p \leqslant n$), the trace powers of the Hessian, equations coming from Gårding polynomials, and much more. Each of these has a complex and a quaternionic counterpart to which Theorem 1.3.I applies. However, there are many $\mathrm{U}(n)$- and $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$-invariant subequations, arising from calibrations and Lagrangian geometry, which have no $\mathrm{O}(n)$-invariant counterpart, so that Theorem 1.3.I does not apply. Results in these cases are provided by Theorems 1.3.II and 1.3.III below, which require a completely different method of proof.

Suppose $F = F(\mathbb{G})$ is a subequation defined by a compact subset $\mathbb{G} \subset G(p, \mathbb{R}^n)$ of the Grassmannian of p-planes in $\mathbb{R}^n$ (see Example 4.4).

Theorem 1.3.II (Strong Uniqueness II). — *Fix $p \geqslant 2$ and $n \geqslant 3$. Then strong uniqueness of tangents to $F(\mathbb{G})$-subharmonic functions holds for:*

(1) *Every compact $SU(n)$-invariant subset $\mathbb{G} \subset G^{\mathbb{R}}(p, \mathbb{C}^n)$ with the one exception $\mathbb{G} = G^{\mathbb{C}}(1, \mathbb{C}^n)$,*
(2) *Every compact $Sp(n) \cdot \mathrm{Sp}(1)$-invariant subset $\mathbb{G} \subset G^{\mathbb{R}}(p, \mathbb{H}^n)$ with three exceptions, namely the sets of real p-planes which lie in a quaternion line for $p = 2, 3, 4$ (when $p = 4$ this is $G^{\mathbb{H}}(1, \mathbb{H}^n)$),*
(3) *For $p \geqslant 5$, every compact $Sp(n)$-invariant subset $\mathbb{G} \subset G^{\mathbb{R}}(p, \mathbb{H}^n)$.*

This result is based on a companion theorem which has further applications. Given $\mathbb{G} \subset G(p, \mathbb{R}^n)$ as above, we say that $\mathbb{G}$ has the *transitivity property* if for any two vectors $x, y \in \mathbb{R}^n$ there exist $W_1, \ldots, W_k \in \mathbb{G}$

with $x \in W_1, y \in W_k$ and $\dim(W_i \cap W_{i+1}) > 0$ for all $i = 1, \ldots, k-1$. The subequations attached to Lagrangian, Special Lagrangian, Associative, Coassociative, and Cayley geometries all have this property.

Theorem 1.3.III (Strong Uniqueness III). — *If $\mathbb{G}$ has the transitivity property, then strong uniqueness of tangents holds for all $F(\mathbb{G})$-subharmonic functions.*

Theorems 1.3.II and 1.3.III will be proved in Part II ([23]) of this paper. There homogeneity of tangents is proved first, and then strong uniqueness is established. This method makes no use of uniform ellipticity, and has its roots in pluripotential theory, not viscosity theory.

It is important to note that uniqueness of tangents does not always hold. In the basic case of convex functions ($F = \mathcal{P}$) we have uniqueness, but strong uniqueness fails. For classical plurisubharmonic functions (the complex counterpart: $F = \mathcal{P}^{\mathbb{C}}$), the uniqueness question was raised in [11] and answered in the negative by Kiselman [29], who actually characterized the sets which can arise as $T_0(u)$ for a plurisubharmonic function u in $\mathbb{C}^n$. In Part II of this paper a similar result is obtained for the quaternionic counterpart $\mathcal{P}^{\mathbb{H}}$.

The proof of Theorem 1.3.I involves several steps. The first step is of a classical nature going back to standard potential theory for the Laplacian and used by Labutin and Armstrong–Sirakov–Smart in viscosity theory. In our formulation it involves various characterizations of radial F-harmonics. For example, a result (Theorems 2.4 and 2.7), straightforward in the smooth case, but which fills a gap in the literature, characterizes the radial viscosity subsolutions $u(x) = \psi(|x|)$ as the subsolutions of the one-variable subequation

$$R_F \ : \ \psi''(r) + \frac{p_F - 1}{r}\psi'(r) \geqslant 0 \tag{1.4}$$

This classical subequaton is reviewed in detail in Section 5. Several important facts are derived. For example, all subsolutions of (1.4) are continuous, which has the important consequence that if a radial function is F-maximal, then it is F-harmonic (a solution), and hence of the form $\Theta K_p(|x|) + c$. Another consequence of (1.4) is that quotients $\frac{\psi(r)-\psi(t)}{K(r)-K(t)}$ are jointly (or "doubly") monotone. This can be applied to a general non-radial F-subsolution u by associating to u several radial functions which are also F-subharmonic (Lemmas 6.1 and 6.2). The simplest is the maximum $M(u, |x|)$ defined by (1.3), which is a basic tool in [33, 35, 34] and [1]. We choose the following formulation (see Section 6).

Theorem 1.4 (Double Monotonicity). — *Let u be F-subharmonic in a neighborhood of the origin in $\mathbb{R}^n$. Then*

$$\frac{M(u,r)-M(u,s)}{K(r)-K(s)} \quad \text{is increasing in } r \text{ and } s. \tag{1.5}$$

for all $0<s<r$ where M is defined.

Furthermore, if F is convex, the same statement holds with $M(u,r)$ replaced by either

$$S(u,r) \equiv \fint_S u(r\sigma)\,\mathrm{d}\sigma \qquad \text{or} \qquad V(u,r) \equiv \fint_B u(rx)\,\mathrm{d}x \tag{1.6}$$

(the spherical or volume average) where $B \equiv \{|x| \leqslant 1\}$ is the unit ball, $S \equiv \partial B$ is the unit sphere, and $\fint_S = \frac{1}{|S|}\int_S$ denotes the average or "normalized" integral.

This theorem has several immediate consequences for the functions $\Psi(u,r)$ for $\Psi = M, S, V$. In particular, it leads to the concept of densities (see Corollary 5.4).

Definition 1.5. — *Suppose u is F-subharmonic in a neighborhood of $0 \in \mathbb{R}^n$. Then the M-*density *of u at 0 is the decreasing limit*

$$\Theta^M(u,0) \equiv \lim_{s<r\downarrow 0} \frac{M(u,r)-M(u,s)}{K(r)-K(s)}.$$

When F is convex, there are also Ψ-densities

$$\Theta^\Psi(u,0) \equiv \lim_{s<r\downarrow 0} \frac{\Psi(u,r)-\Psi(u,s)}{K(r)-K(s)}.$$

for $\Psi = S$ and V as in (1.6).

Elementary results concerning these densities are established in Lemma 5.5.

When F is convex, each F subharmonic function is classically Δ-subharmonic, and so $\Delta u = \mu \geqslant 0$ (a positive measure). Thus we also have the standard "mass density"

$$\Theta^q(\mu,0) \equiv \lim_{r\downarrow 0} \frac{\mu\left(B_r(0)\right)}{\alpha(q)r^q} \qquad \text{where } q = n-p.$$

In this convex case all of the densities for M, S, V and μ are universally related, and when $p=2$ we have the further result that $\Theta^M = \Theta^S = \Theta^V$ (see Propositions 7.1 and 7.2).

As noted, tangents need not be unique. However, the averages of tangents are uniquely determined by the density alone, even in the most degenerate cases. This is step two in the proof of the Stong Uniqueness Theorem 1.3.I. It is also the key step in the proof of existence (Theorem 1.1) and maximality (Theorem 1.2).

In the classical case of pluripotential theory the Riesz characteristic is 2, and our next result, when $p = 2$, is an extension of the work of Kiselman [29].

Theorem 1.6 (Averages of Tangents). — *Suppose F is convex and u is an F-subharmonic function defined in a neighborhood of the origin in $\mathbb{R}^n$. Let $p = p_F$ be the Riesz characteristic of F. If $p \neq 2$, then each tangent U to u at 0 has averages*

$$\begin{aligned} M(r) &= \sup_S U(r\sigma) = \Theta^M(u)K(r)\,, \\ S(r) &= \fint_S U(r\sigma)\,\mathrm{d}\sigma = \Theta^S(u)K(r)\,, \\ \text{and} \quad V(r) &= \fint_B U(rx)\,\mathrm{d}x = \Theta^V(u)K(r)\,. \end{aligned} \tag{1.7}$$

In particular,

$$\Theta^\Psi(U) = \Theta^\Psi(u) \quad \text{for } \ \Psi = M, S \text{ or } V. \tag{1.8}$$

When $p = 2$, all the densities of u and any tangent U to u at 0, agree, and will be simply denoted by $\Theta = \Theta(u)$. Specifically, we have

$$\Theta(u) = \Theta^M(U) = \Theta^S(U) = \Theta^V(U) = \Theta^M(u) = \Theta^S(u) = \Theta^V(u). \tag{1.9}$$

Moreover, the averages of a tangent U to u are given by

$$M(r) = \Theta\log r, \quad S(r) = \Theta\log r + \fint_S U, \quad \text{and } V(r) = \Theta\log r + \fint_B U. \tag{1.10}$$

This result about spherical averages of tangents has many applications, for example it is enough to prove maximality of tangents (see Theorem 8.2).

Theorem 1.7 (Maximality Criterion). — *Suppose F is convex and U is an F-subsolution on an annular region A about 0. If the spherical average $S(U, |x|)$ is an increasing F-solution on A, then U is maximal on A.*

Some of the remaining steps in the proof of Theorem 1.3.I, which are given in detail in Section 12, can be outlined as follows. By applying the maximality criterion we conclude in Theorem 10.2 that all tangents are F-maximal. Now if F' is any subequation which contains F and has the same Riesz characteristic, then an F-tangent U to u is also an F'-tangent to u. In the $\mathrm{O}(n)$-invariant (and the other cases of Theorem 1.3.III) it is somewhat surprising that there is a simple convex subequation of characteristic p which contains all the others (Proposition 13.10). This largest subequation is very nice; in particular, it is uniformly elliptic. This, together with Theorem 8.7, shows that tangents are harmonic for this largest subequation, and that they are C^1. One completes the proof of Theorem 1.3.I by showing that for each tangent U and rotation g, we must have $U = g^*U$ or otherwise one can produce a tangent which is not C^1.

As with most notions of density in analysis, we have the following.

Theorem 1.8 (Upper Semi-Continuity of Density). — *Suppose u is F-subharmonic on an open set $X \subset \mathbb{R}^n$. Then each of the densities*

$$\Theta^M(u,x), \quad \Theta^S(u,x), \quad \Theta^V(u,x)$$

considered above is an upper semi-continuous function of x. Equivalently, for all $c \geqslant 0$ and each Θ as above, the sets

$$E_c \equiv \{x : \Theta(u,x) \geqslant c\} \textit{ are closed.}$$

We also note that by standard geometric measure theory

$$c\mathcal{H}^{n-p}(E_c) \leqslant \mu(X).$$

In many cases one can say much more about these high density sets E_c for $c > 0$.

For classical plurisubharmonic functions in $\mathbb{C}^n$ a deep theorem, due to L. Hörmander, E. Bombieri and in its final form by Siu ([3, 25, 40]), states that E_c is a complex analytic subvariety. One straightforwardly deduces from this result that for the 2-convexity subequation $\mathcal{P}_2$ in $\mathbb{R}^{2n}$ the set E_c is discrete, since $\mathcal{P}^{\mathbb{C}}(J) \subset \mathcal{P}_2$ for all orthogonal (parallel) complex structures J on $\mathbb{R}^{2n}$. This very restrictive corollary has a quite general extension.

Theorem 1.9 (Structure of High Density Sets). — *Suppose strong uniqueness of tangents holds for F. Then for any F-subharmonic function u, the set $E_c(u)$ is discrete.*

Theorem 1.9 is essentially sharp. Suppose Ω is a domain with strictly convex boundary. Given any finite subset $E = \{x_j\}_{j=1}^N \subset \Omega$, any set of numbers $\Theta_j > 0$, $j = 1, \ldots, N$, and any $\varphi \in C(\partial\Omega)$, there exists a unique continuous $u : \overline{\Omega} \to [-\infty, \infty)$ such that

(1) u is F-harmonic on $\Omega - E$,
(2) $u\big|_{\partial\Omega} = \varphi$, and
(3) $\Theta(u, x_j) = \Theta_j$ for $j = 1, \ldots, N$.

See Remark 14.2 for more details.

The subequations with characteristic $1 \leqslant p < 2$ are very different in nature from those where $p \geqslant 2$. They are discussed in detail in Section 15. In particular, the following is proved.

Theorem 1.10 (Hölder Continuity $1 \leqslant p < 2$). — *Suppose F is a (not necessarily convex) subequation with Riesz characteristic $1 \leqslant p < 2$. Then each F-subharmonic function is locally Hölder continuous with exponent $\alpha \equiv 2 - p$.*

Furthermore, if u is an F-subharmonic defined in a neighborhood of $0 \in \mathbb{R}^n$, then every sequence $\{u_{r_j}\}_{j=1}^{\infty}$ with $r_j \downarrow 0$, has a subsequence which converges locally uniformly to an F-subharmonic function U on $\mathbb{R}^n$. In fact for each $0 < \beta < 2 - p$ there exists a subsequence which converges locally in β-Hölder norm. Finally, when F is convex, this limit U is F-harmonic on $\mathbb{R}^n - \{0\}$.

For the k^{th} Hessian equation the Riesz characteristic is $p = n/k$. For $k > n/2$, the Hölder continuity result for this subequation is a fundamental theorem of Trudinger and Wang [42], and their proof can be carried over to more general convex equations. However, we do not require convexity in Theorem 1.10.

In Appendix A we examine the radial subequation for the "subaffine" case $\widetilde{\mathcal{P}} \equiv \{\lambda_{\max} \geqslant 0\}$ and establish a basic dichotomy: the Increasing/Decreasing Lemma.

In Appendix B we show that the subequation $\mathcal{P}(\delta) \equiv \{A + \delta \mathrm{tr}(A) \geqslant 0\}$ is uniformly elliptic in the conventional sense.

While in Section 4 we give a number of examples to which our theory applies, many more examples are given in the appendix to Part II. That appendix also constructs the maximal and minimal subequations of Riesz characteristic p (showing, in particular, that these largest and smallest subequations exist). There is a companion result describing the largest and smallest *convex* subequations of characteristic p. The largest is given in Proposition 13.10. The smallest is given in Lemma A.1 of Part II.

It is worth noting that the main results in this paper (existence, strong uniqueness, maximality, etc.) apply to any subequation obtained by a linear change of variables, i.e. of the form $g^t F g$ for $g \in \mathrm{GL}_n(\mathbb{R})$ (where F is as assumed herein). This means for cone subequations F which are invariant under a conjugate subgroup $g^{-1}Gg$ where $G \subset \mathrm{O}(n)$ acts transitively on S^{n-1}. Of course the notion of Riesz characteristic must be reformulated in this case, and the Riesz kernel $K_p(|x|)$ must be replaced by its transform $K_p(|gx|)$.

The Work of Armstrong, Sirakov and Smart

In the very interesting paper [1] the authors also study conical subequations $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ with the added assumption that F is uniformly elliptic. However, they do not assume invariance or convexity. An important part of their work (which is "automatic" in our case) proves the existence and uniqueness of *fundamental solutions*: F-harmonic functions Φ on $\mathbb{R}^n - \{0\}$, which are invariant under the flow $\Phi_r(x) = r^{p-2}\Phi(rx)$ for some $p \geqslant 1, p \neq 2$ and bounded from above or below. (When $p = 2$ the log enters as it does here.) They show the existence and uniqueness of two families of such solutions (up to positive scalars and additive constants) among all entire punctured F-harmonics with a one-sided bound. In our degenerate cases two fundamental solutions are not always available. In fact, they are if and only if both F and the dual $\widetilde{F}$ have finite Riesz characteristics. (See Proposition 3.16 for a description of all such subequations.)

One of the results in [1] is closely related to the work here. They prove existence and strong uniqueness of tangents to solutions of uniformly elliptic equations. That is, under their assumptions that F is conical and uniformly elliptic, they prove that: Any *F-harmonic* function defined on $B_\epsilon - \{0\}$ and bounded above (or below), has a unique tangent of the form $\Theta\Phi$ for some $\Theta \geqslant 0$ (see [1, (5.13), ff.]).

This paper addresses a much broader class of functions, namely subsolutions to degenerate elliptic equations. Naturally the equations must be in some ways restricted, but the results apply to a wide range of geometrically interesting cases. Here it is shown that tangents exist and are maximal, and that maximal plus continuous implies F-harmonic. However, it is not true that maximal implies continuous in this general case. It fails for example for $\mathcal{P}_{\mathbb{C}}$, as does uniqueness of tangents (not just strong uniqueness, see Kiselman [29]).

Said differently, the step from maximal to F-harmonic does not always hold in the degenerate subharmonic case, and it is somewhat surprising that strong uniqueness of tangents can actually be established for such a broad spectrum of interesting subequations with $p \geqslant 2$.

We should add that the techniques used in proving strong uniqueness in the non-$\mathrm{O}(n)$-invariant cases are substantially different from those in the $\mathrm{O}(n)$-case, and they appear in the sequel (Part II) to this paper.

For the question of existence we need to assume convexity or that $1 \leqslant p < 2$. This is quite reasonable since we are dealing with subsolutions and the equations are only degenerate elliptic. One needs a function space in which to extract convergent subsequences just to get off the ground. These assumptions provide such spaces, namely L^1_{loc} and Hölder.

The work in [1] is related to earlier results of Labutin [33, 34, 35] who studied the Pucci extremal equations. He established among other things a removable singularity result and an extension of a classical result of Bôcher. In this work the classical Riesz kernels also play a prominent role. There is a careful account of the relationship to the work of Armstrong–Sirakov–Smart given in [1].

Historical Reflections

In 1982 the authors showed that for each calibration on a riemannian manifold there is an associated family of minimal subvarieties, forming a *calibrated geometry* [13]. More recently [16] it was discovered that the calibration also determines a potential theory of functions whose restrictions to each of the distinguished submanifolds are subharmonic. Although there is an analoguein this setting of the $i\partial\overline{\partial}$ operator from complex geometry, that operator does not play a critical role in the development of the potential theory [14]. In fact, somewhat surprisingly, a corresponding potential theory can be established for any collection of submanifolds determined by requiring their tangent spaces to be in an arbitrary given closed subset of the grassmannian. Even more generally one has the potential theory associated to an elliptic (possibly degenerate) nonlinear inequality $F(D^2u) \geqslant 0$, provided by viscosity subsolutions ([6]).

This raises the possibility of cross-fertilization between two well established and deep fields, pluripotential theory (in several complex variables) and nonlinear elliptic theory. This paper, although not the first, can be viewed as an example of this phenomenon. The authors believe there are many more to come.

2. The Radial Subequations Associated to a Subequation F

In this section we first describe the ordinary differential inequality which governs C^2 radial (i.e., spherically symmetric) F-subharmonic functions. Our main result fills an apparent gap in the literature by extending this characterization to general upper semi-continuous radial F-subharmonics. Somewhat surprisingly this extension requires the attention of Lemma 2.10 below.

Suppose $\psi(t)$ is of class C^2 on an interval contained in the positive real numbers. We also consider ψ as the function $\psi(|x|)$ of x on the corresponding annular region in $\mathbb{R}^n$.

Lemma 2.1.
$$D_x^2\psi = \frac{\psi'(|x|)}{|x|}P_{[x]^\perp} + \psi''(|x|)P_{[x]}\,, \tag{2.1}$$
where $P_{[x]} = \frac{x\circ x}{|x|^2}$ denotes orthogonal projection onto the line $[x]$ through $x \neq 0$ and $P_{[x]^\perp} = I - P_{[x]}$ denotes orthogonal projection onto the hyperplane with normal $[x]$.

Proof. — First note that $D(|x|) = \frac{x}{|x|}$, and therefore $D^2(|x|) = D(\frac{x}{|x|}) = \frac{1}{|x|}I - \frac{x}{|x|^2}\circ\frac{x}{|x|} = \frac{1}{|x|}(I - P_{[x]}) = \frac{1}{|x|}P_{[x]^\perp}$. Hence,
$$D_x\psi = \psi'(|x|)\frac{x}{|x|} \qquad \text{and}$$
$$D_x^2\psi = \psi'(|x|)D\left(\frac{x}{|x|}\right) + \psi''(|x|)\frac{x}{|x|}\circ\frac{x}{|x|} = \frac{\psi'(|x|)}{|x|}P_{[x]^\perp} + \psi''(|x|)P_{[x]}\,. \ \square$$

Corollary 2.2. — *The second derivative $D_x^2\psi$ has eigenvalues $\frac{\psi'(|x|)}{|x|}$ with multiplicity $n-1$ and $\psi''(|x|)$ with multiplicity 1.*

Let $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ be a pure second-order constant coefficient subequation. Then by Lemma 2.1 a radial C^2-function $u(x) = \psi(|x|)$ is F-subharmonic on an annular region in $\mathbb{R}^n$ if and only if
$$D_x^2 u = \frac{\psi'(t)}{t}P_{e^\perp} + \psi''(t)P_e \in F\,, \tag{2.2}$$
for $t = |x|$ in the corresponding interval in $(0,\infty)$. We use $\lambda = \psi'(t)$ and $a = \psi''(t)$ as one-variable jet coordinates. Then the basic one-variable subequation associated with F is defined as follows.

Definition 2.3. — *The* radial subequation associated with F *is the reduced variable coefficient subequation R_F on $(0,\infty)$ whose fibre at t is*
$$(R_F)_t \equiv \left\{(\lambda, a) \in \mathbb{R}^2 : \frac{\lambda}{t}P_{e^\perp} + aP_e \in F, \forall\,|e| = 1\right\}.$$
Thus for C^2-functions we have that
$$u(x) \equiv \psi(|x|) \text{ is } F \text{ subharmonic} \iff \psi(t) \text{ is } R_F \text{ subharmonic}. \tag{2.3}$$

We extend this to the viscosity setting where F-subharmonic functions are just upper semi-continuous (see [5, 6, 14, 17] for definitions). The proof given below of the implication $\Rightarrow$ is elementary, whereas the proof of $\Leftarrow$ will require a lemma. Note that the equivalence: $u(x) = \psi(|x|)$ is upper semicontinuous $\iff$ $\psi(t)$ is upper semicontinuous, is obvious.

Theorem 2.4 (Radial Subharmonics). — *The function* $u(x) \equiv \psi(|x|)$ *is* F*-subharmonic on an annular region in* $\mathbb{R}^n$ *if and only if* $\psi(t)$ *is* R_F*-subharmonic on the corresponding open sub-interval of* $(0,\infty)$.

Remark 2.5. — In all but this section of the paper, the subequations F will be assumed to be cones, unless explicitly stated to the contrary. For such subequations the maximum principle holds, i.e., it holds for each F-subharmonic function $u(x)$ (see Theorem A.2). Consequently, if $u(x) = \psi(|x|)$ is a radial F-subharmonic on a ball about 0, then $\psi(t)$ must be increasing in t. This motivates focusing on an "increasing" version of Theorem 2.4.

We will use the fact, which is elementary to establish, that for an upper semi-continuous function $\psi(t)$,

$$\psi(t) \text{ is increasing} \quad \iff \quad \psi \text{ is } \{\lambda \geqslant 0\}\text{-subharmonic}. \tag{2.4}$$

(See [12] for a proof.)

Definition 2.6. — *The* increasing radial subharmonic equation $R_F^{\uparrow}$ on $(0,\infty)$ *is defined by*

$$R_F^{\uparrow} = R_F \cap \{\lambda \geqslant 0\}. \tag{2.5}$$

In light of (2.3), it is obvious that for C^2-functions $\psi(t)$:

$$\psi(t) \text{ is } R_F^{\uparrow}\text{-subharmonic} \quad \iff \quad \psi(|x|) \text{ is } F\cap\{x\cdot p \geqslant 0\}\text{-subharmonic} \tag{2.6}$$

where the variable coefficient first-order subequation $\{x \cdot p \geqslant 0\}$ is the constraint $x \cdot D_x u \geqslant 0$ on C^2-functions. The equivalence (2.6) can be extended as in Theorem 2.4.

Theorem 2.7 (Increasing Radial Subharmonics). — *The function* $u(x) \equiv \psi(|x|)$ *is an increasing, radial* F*-subharmonic function if and only if* $\psi(t)$ *is* $R_F^{\uparrow}$*-subharmonic.*

Remark 2.8. — We will sometimes blur the distinction between $\psi(t)$ and $u(x) = \psi(|x|)$ by calling $\psi(t)$ a radial (or increasing radial) F-subharmonic.

Remark 2.9. — The statement and proof of a theorem analogous to 2.7 for decreasing radial subharmonics is left to the reader.

Proof of Theorem 2.4.

($\Rightarrow$): Suppose $u(x) \equiv \psi(|x|)$ is F-subharmonic. If $\varphi(t)$ is a test function for $\psi(t)$ at t_0, then $\varphi(|x|)$ is a test function for $\psi(|x|)$ at any point on the t_0-sphere in $\mathbb{R}^n$. Therefore $D^2_{x_0}\varphi \in F$. Applying the formula for $D^2_{x_0}\varphi$ in terms of $\varphi'(t_0)$ and $\varphi''(t_0)$, the equivalence (2.3), and the definition of $(R_F)_{t_0}$, we have $J^2_{t_0}\varphi \in R_F$. This proves that $\psi(t)$ is R_F-subharmonic.

($\Leftarrow$): Suppose that $\psi(t)$ is R_F-subharmonic. We must show that $u(x) \equiv \psi(|x|)$ is F-subharmonic. That is, given a test function $\varphi(x)$ for $u(x)$ at a point x_0, we must show that $D^2_{x_0}\varphi \in F$.

Suppose that there exists a *smooth* function $\overline{\psi}(t)$, defined near $t_0 = |x_0|$, such that $\overline{\varphi}(x) \equiv \overline{\psi}(|x|)$ satisfies

$$u(x) \leqslant \overline{\varphi}(x) \leqslant \varphi(x) \tag{2.7}$$

near x_0. Then $\overline{\psi}(t)$ is a test function for $\psi(t)$ at t_0. Hence, the 2-jet of $\overline{\psi}$ at t_0 belongs to R_F. By Lemma 2.1 and the discussion above, this implies that $D^2_{x_0}\overline{\varphi} \in F$. The inequality $\overline{\varphi}(x) \leqslant \varphi(x)$ (with equality at x_0) implies that $D^2_{x_0}\varphi = D^2_{x_0}\overline{\varphi} + P$ for some $P \geqslant 0$, which proves that $D^2_{x_0}\varphi \in F$ as desired.

To complete this argument by finding $\overline{\psi}(t)$ there is some flexibility given by Lemma 2.4 in [17] so that not all test functions $\varphi(x)$ need be considered. First we may choose new coordinates $z = (t, y)$ near x_0 so that $t \equiv |x|$. (Thus $t =$ constant defines the sphere of radius t near x_0.) Furthermore, we may assume that $\varphi(z)$ is a polynomial of degree $\leqslant 2$ in $z = (t, y)$ and that it is a *strict* local test function, i.e., $u(z) < \varphi(z)$ for $z \neq z_0$. Now Lemma 2.10 below ensures the existence of $\overline{\varphi}(x) = \overline{\psi}(|x|)$ satisfying (2.7). $\square$

Let $z = (t, y)$ denote standard coordinates on $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^\ell$. Fix a point $z_0 = (t_0, y_0)$ and let $u(t)$ be an upper semi-continuous function (of t alone) and $\varphi(z)$ a C^2-function, both defined in a neighborhood of z_0.

Lemma 2.10. — *Suppose $u(t) < \varphi(z)$ for $z \neq z_0$ with equality at z_0. If $\varphi(z)$ is a polynomial of degree $\leqslant 2$, then there exists a polynomial $\overline{\varphi}(t)$ of degree $\leqslant 2$ with*

$$u(t) \leqslant \overline{\varphi}(t) \leqslant \varphi(z) \qquad \textit{near } z_0\,. \tag{2.8}$$

Proof. — We may assume $z_0 = 0$ and $u(0) = \varphi(0) = 0$. Then

$$\varphi(z) = \langle p, t\rangle + \langle q, y\rangle + \langle At, t\rangle + 2\langle Bt, y\rangle + \langle Cy, y\rangle\,.$$

We assume $u(t) < \varphi(t, y)$ for $|t| \leqslant \epsilon$ and $|y| \leqslant \delta$ with $(t, y) \neq (0, 0)$.

Setting $t = 0$, we have $0 = u(0) < \langle q, y\rangle + \langle Cy, y\rangle$ for $y \neq 0$ sufficiently small. Therefore, $q = 0$ and $C > 0$ (positive definite). Now define

$$\overline{\varphi}(t) \equiv \langle p, t\rangle + \langle (A - B^t C^{-1} B)t, t\rangle\,. \tag{2.9}$$

The inequalities in (2.8) follow from the fact that for t sufficiently small,

$$\overline{\varphi}(t) = \inf_{|y|\leqslant\delta} \varphi(z) = \langle p, t\rangle + \langle At, t\rangle + \inf_{|y|\leqslant\delta}\{2\langle Bt, y\rangle + \langle Cy, y\rangle\}. \tag{2.10}$$

To prove (2.10) fix t and consider the function $2\langle Bt, y\rangle + \langle Cy, y\rangle$. Since $C > 0$, it has a unique minimum point at the critical point $y = -C^{-1}Bt$. The minimum value is $-\langle B^t C^{-1} Bt, t\rangle$. If t is sufficiently small, the critical point y satisfies $|y| < \delta$, which proves (2.7). $\square$

Proof of Theorem 2.7. — The arguments given for Theorem 2.4 along with the following missing steps provide the proof. If $\varphi(t)$ is a test function for $\psi(t)$ at a point t_0, then $\varphi(|x|)$ is a test function for $\psi(|x|)$ at x_0 whenever $|x_0| = t_0$. Now

$$D_{x_0}\varphi = \varphi'(|x_0|)\frac{x_0}{|x_0|} \quad \text{and hence} \quad x_0 \cdot D_{x_0}\varphi = |x_0|\varphi'(|x_0|). \tag{2.11}$$

Hence, if $\psi(|x|)$ is $\{p{\cdot}x \geqslant 0\}$-subharmonic, then $\psi(t)$ is $\{\lambda \geqslant 0\}$-subharmonic, and thus increasing. Conversely, if $\psi(t)$ is increasing and $\varphi(x)$ is a test function for $\psi(|x|)$ at x_0, then $\overline{\varphi}(t) \equiv \varphi(\frac{tx_0}{|x_0|})$ is a test function for $\psi(t)$ at $t_0 = |x_0|$. Hence, $\overline{\varphi}'(t_0) \geqslant 0$. However, $\overline{\varphi}'(t_0) = (D_{x_0}\varphi) \cdot x_0$. $\square$

3. ST-Invariant Cone Subequations: The Riesz Characteristic

This section is devoted to investigating the cone subequations which satisfy a weak form of invariance which will be referred to as *spherical transitivity (ST)*. Two characteristic numbers (p, q) will be associated with each such subequation F. They uniquely determine the radial subequation for F and, as we shall show in this and the following sections, can be easily computed in any example. Moreover, we give a complete description of all possible examples (of ST-invariant subequations with characteristics (p, q)) in the second subsection here. Most readers will prefer to come back to this subsection. Although it adds important perspective to the scope of ST-cone subequations, it is not used in the subsequent results of the paper.

Recall from the introduction that a subequation $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is said to be *ST-invariant* if there exists a subgroup $G \subset \mathrm{O}(n)$ which acts transitively on the sphere $S^{n-1} \subset \mathbb{R}^n$ and leaves F invariant (under the induced action of G on $\mathrm{Sym}^2(\mathbb{R}^n)$).

For an ST-invariant cone subequation F,

$$\text{the slices } F \cap \mathrm{span}\{P_{e^\perp}, P_e\} \text{ for } e \in S^{n-1} \text{ are all isomorphic.} \tag{3.1}$$

Note that $\mathrm{span}\{P_{e^\perp}, P_e\} = \mathrm{span}\{I, P_e\}$ and that the induced action on $\mathrm{Sym}^2(\mathbb{R}^n)$ sends P_e to $P_{g(e)}$. In particular,

$$\lambda P_{e^\perp} + \mu P_e \in F \text{ for one } e \in S^{n-1} \iff \lambda P_{e^\perp} + \mu P_e \in F \text{ for all } e \in S^{n-1}. \tag{3.2}$$

This weakening of ST-invariance will be referred to as *weak invariance*.

The Riesz Characteristics

We begin by focusing on the first of the two characteristics (p, q). Although there is an abundance of interesting ST-invariant cone subequations in dimensions $\geqslant 3$, there are not many increasing radial subequations. In fact they are described by a single "characteristic" number p between 1 and ∞, which determines a one-variable subequation as follows.

Definition 3.1. — *For each p with $1 \leqslant p < \infty$, the* increasing radial subequation $R_p^{\uparrow}$ *is defined by*

$$R_p^{\uparrow} : a + \frac{(p-1)}{t}\lambda \geqslant 0 \quad \textit{and} \quad \lambda \geqslant 0\,, \tag{3.3}$$

while for $p = \infty$, the subequation $R_\infty^{\uparrow}$ is first-order and defined by $R_\infty^{\uparrow} = \{\lambda \geqslant 0\}$.

Definition 3.2 (The Increasing Riesz Characteristic). — *Suppose F is an ST-invariant cone subequation. The* increasing characteristic p_F of F *is defined to be*

$$p_F \equiv \sup\{\overline{p} : P_{e^\perp} - (\overline{p}-1)P_e \in F\}. \tag{3.4a}$$

Equivalently, for finite Riesz characteristic, p_F is the unique number p such that

$$P_{e^\perp} - (p-1)P_e \in \partial F. \tag{3.4b}$$

Proposition 3.3 (Increasing). — *Suppose that F is an ST-invariant cone subequation. Then the increasing radial subequation $R_F^{\uparrow}$ equals $R_p^{\uparrow}$ where $p = p_F$ is the increasing Riesz characteristic of F.*

Proof. — Using Definitions 2.3, 2.6, 3.1 and 3.2, we must show that for $\lambda \geqslant 0$

$$\frac{\lambda}{t}P_{e^\perp} + aP_e \in F \quad \iff \quad a + \frac{p-1}{t}\lambda \geqslant 0\,.$$

Set $-(\overline{p}-1) \equiv at/\lambda$, so that $\frac{\lambda}{t}P_{e^\perp} + aP_e \in F \iff P_{e^\perp} - (\overline{p}-1)P_e \in F$. Then $\overline{p} \leqslant p \iff -\frac{at}{\lambda} \leqslant p-1 \iff a + \frac{p-1}{t}\lambda \geqslant 0$. $\square$

Note that by Definition 3.2, the positivity condition for F, and the fact that $0 \in F$, we have that $p_F \geqslant 1$. Thus $1 \leqslant p_F \leqslant \infty$.

The only equation with $p_F = 1$ is $\mathcal{P}$. At the other extreme we have $p_F = \infty$. Here there is a test which is very simple to apply in all the ST-invariant examples, namely: $p_F = \infty$ iff $-P_e \in F$. Hence, determining when $p_F < \infty$ is also simple, namely: $p_F < \infty$ iff $-P_e \notin F$.

We recall the fact that for a subequation F, the *dual subequation* $\widetilde{F}$ is defined as

$$\widetilde{F} = -(\sim \operatorname{Int} F) = \sim(-\operatorname{Int} F)\,. \tag{DE}$$

Lemma 3.4. — *For ST-invariant cone subequations F*

(1) $p_F = 1 \iff P_{e^\perp} \in \partial F \iff F = \mathcal{P}$
(2) $p_F = \infty \iff -P_e \in F \iff -P_e \in \partial F.$
(3) $p_F < \infty \iff -P_e \notin F \iff P_e \in \operatorname{Int} \widetilde{F} \iff \mathcal{P} - \{0\} \subset \operatorname{Int} \widetilde{F}.$

Actually, as noted above, it is easy to compute the exact value of p_F in all the examples.

Proof of (1). — Note first that $p_F > 1 \iff P_{e^\perp} - \epsilon P_e \in F$ for all small $\epsilon > 0$. Now if F contains an element A with at least one eigenvalue strictly negative, then by positivity and the cone property there is an element $A' = P_{e^\perp} - \epsilon P_e \in F$. Hence $F \neq \mathcal{P} \Rightarrow p_F > 1$.

Proof of (2). — Note first that $-P_e \in F \Rightarrow \alpha P_{e^\perp} - P_e \in F \ \forall \alpha \geqslant 0 \Rightarrow P_{e^\perp} - (p-1)P_e \in F \ \forall p \geqslant 1 \Rightarrow p_F = \infty$. On the other hand $-P_e \notin F \Rightarrow \epsilon P_{e^\perp} - P_e \notin F \ \forall \epsilon \geqslant 0$ small $\Rightarrow P_{e^\perp} - (p-1)P_e \notin F \ \forall p$ large $\Rightarrow p_F < \infty$. To complete the proof of (2), note that $-P_e \in \operatorname{Int} F$ cannot occur unless $F = \operatorname{Sym}^2(\mathbb{R}^n)$ since $-P_e \in \operatorname{Int} F \Rightarrow 0 \in \operatorname{Int} F$.

Proof of (3). — By (DE) above we have $\sim(-F) = \operatorname{Int} \widetilde{F}$, and the first part of (3) follows from the first part of (2). For the last $\Rightarrow$ note that $\mathcal{P}$ is the convex cone hull of the P_e's. The last $\Leftarrow$ is obvious. ☐

The primary application of the Riesz characteristics (and the reason for choosing the name) is the fact that the solutions of the associated increasing radial equation $R_p^{\uparrow}$ are given by the Riesz kernels.

Proposition 3.5. — *An ST-invariant cone subequation F has finite Riesz characteristic $p = p_F$ if and only if the increasing radial harmonics for F are:*

$$\Theta K_p(|x|) + C \tag{3.5}$$

where $\Theta \geqslant 0$, $C \in \mathbb{R}$, and $K_p(t)$ is the p^{th} Riesz function defined on $0 < t < \infty$ by

$$K_p(t) = \begin{cases} t^{2-p} & \text{if } 1 \leqslant p < 2 \\ \log t & \text{if } p = 2 \\ -\frac{1}{t^{p-2}} & \text{if } 2 < p < \infty. \end{cases} \tag{3.6}$$

Proof. — From (3.4b) it is easy to see that $u(x) \equiv \psi(|x|)$ is F-subharmonic if and only if $\psi(t)$ is $R_p^{\uparrow}$-subharmonic. The ordinary differential equation given by equality in (3.3) is easily solved, and $\Theta K_p(t) + C$ are the viscosity solutions. One can check directly using Lemma 2.1 that

$$D^2 \overline{K}_p(|x|) = \frac{1}{|x|^p}\left(P_{[x]^\perp} - (p-1)P_{[x]}\right) \quad \text{and} \quad D\overline{K}_p = \frac{x}{|x|^p} \tag{3.7}$$

where K_p has been renormalized to

$$\overline{K}_p \equiv \frac{1}{|p-2|} K_p \quad \text{if } p \neq 2 \text{ and } \overline{K}_2 = K_2\,. \tag{3.8}$$

□

The sign of $K_p(t)$ has been chosen so that $K_p(|x|)$ is a *increasing* or *downward-pointing* F-harmonic on $\mathbb{R}^n - \{0\}$. The actual normalization in (3.6) is simpler when the focus is on the function u, while the normalization in (3.8) is simpler when the focus is on the first and second derivatives of u.

The second of the two numbers (p, q) can also be defined in several equivalent ways.

Definition 3.6 (The Decreasing Riesz Characteristic). — *For each ST-invariant cone subequation F, this characteristic, denoted q_F, is defined by*

$$q_F = \sup\{\bar{q} : -P_{e^\perp} + (\bar{q}-1)P_e \notin F\}, \tag{3.9a}$$

or equivalently q_F is the unique number q such that

$$-P_{e^\perp} + (q-1)P_e \in \partial F, \tag{3.9b}$$

or finally, q_F can be defined to be the increasing characteristic of the dual subequation, i.e.

$$q_F = p_{\widetilde{F}}\,. \tag{3.9c}$$

Since $\partial\widetilde{F} = -\partial F$, the equivalence of (3.9c) follows easily from (3.4b). Thus the decreasing characteristic of F might also be called the *dual characteristic* of F.

For each $1 \leqslant q < \infty$ set

$$R_q^{\downarrow} : a + \frac{q-1}{t}\lambda \geqslant 0 \quad \text{and} \quad \lambda \leqslant 0 \tag{3.10}$$

while for $q = \infty$ the subequation $R_q^{\downarrow}$ is first-order and defined by $R_\infty^{\downarrow} = \{\lambda \leqslant 0\}$.

Then the decreasing versions of Propositions 3.3, Lemma 3.4(3) and Proposition 3.5 state the following.

Proposition 3.7 (Decreasing).

$$R_F^{\downarrow} = R_q^{\downarrow} \qquad \textit{with } q \equiv q_F\,. \tag{3.11a}$$

$$F \textit{ has finite decreasing characteristic } q_F \quad \iff \quad P_e \in \operatorname{Int} F, \tag{3.11b}$$

which in turn holds if and only if the decreasing radial F-harmonics are

$$-\Theta K_q(|x|) + C \qquad \text{where } \Theta \geqslant 0 \text{ and } C \in \mathbb{R}, \text{ and } q = q_F\,. \tag{3.11c}$$

Remark. — In summary we have that:

(1) For some p finite, $K_p(|x|)$ is an increasing (or downward-pointing) F-harmonic on $\mathbb{R}^n - \{0\}$ $\iff$ $-P_e \notin F$ $\iff$ F has finite increasing characteristic.
(2) For some q finite, $-K_q(|x|)$ is an decreasing (or upward-pointing) F-harmonic on $\mathbb{R}^n - \{0\}$ $\iff$ $P_e \in \operatorname{Int} F$ $\iff$ F has finite decreasing characteristic.
(3) Both $K_p(|x|)$ and $-K_q(|x|)$ are F-harmonic on $\mathbb{R}^n - \{0\}$ $\iff$ F has both characteristics (p,q) finite $\iff$ $-P_e \notin F$ and $P_e \in \operatorname{Int} F$.

These criteria hold for a significant number of degenerate (non uniformly elliptic) subequations. (See the next section and Appendix A in Part II.) However, in case (3) if either F or $\widetilde{F}$ is convex, then both are uniformly elliptic. Conversely, uniform ellipticity always implies that (p,q) are both finite even in the non-convex case.

Finally, combining both characteristics we have

Proposition 3.8. — *If F has characteristics (p,q), then the radial subequation for F is*

$$R_F = R_p^{\uparrow} \cup R_q^{\downarrow}. \tag{3.12}$$

Remark 3.9 (Boundary Convexity and the Riesz Characteristic). — The finiteness of the two characteristics of F, which is so easy to ascertain, is equivalent to automatic boundary convexity for all domains.

Proposition 3.10. — *The boundary $\partial\Omega$ of every smoothly bounded domain $\Omega \subset\subset \mathbb{R}^n$ is*

(1) *strictly F-convex* $\iff$ $p_{\widetilde{F}} = q_F < \infty$ $\iff$ $P_e \in \operatorname{Int} F$,
(2) *strictly $\widetilde{F}$-convex* $\iff$ $p_F = q_{\widetilde{F}} < \infty$ $\iff$ $-P_e \notin F$,
(3) *both strictly F- and $\widetilde{F}$-convex* $\iff$ (p_F, q_F) *is finite* $\iff$ $P_e \in \operatorname{Int} F$ *and* $-P_e \notin F$.

Proof. — We first prove (2). By Lemma 5.3 (ii′) in [14], $\partial\Omega$ is strictly $\widetilde{F}$-convex at $x \in \partial\Omega$ for all domains Ω if and only if

$$\forall\, B \in \operatorname{Sym}^2(W), B + tP_e \in \operatorname{Int} \widetilde{F} \quad \text{for all } t \geqslant \text{some } t_0\,. \tag{3.13}$$

where $|e| = 1$ and $W = e^{\perp}$. Now (3.13) $\Rightarrow$ $P_e \in \operatorname{Int} \widetilde{F} \Rightarrow \frac{1}{t}B + P_e \in \operatorname{Int} \widetilde{F}$ for all $t \geqslant$ some t_0 $\Rightarrow$ (3.13). Thus (3.13) is equivalent to $p_F < \infty$ by Lemma 3.4 (3). The proof of (1) follows by duality, and (1) and (2) together imply (3). ☐

Results in [14] immediately imply the following.

THEOREM 3.11 (Universal Solvability of the Dirichlet Problem). — *Suppose that F is an ST-invariant cone subequation for which both Riesz characterstics p_F and q_F are finite (or equivalently for which the simple condition $P_e \in \operatorname{Int} F$ and $-P_e \notin F$ holds). Then for every domain $\Omega \subset\subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, and for every $\varphi \in C(\partial\Omega)$, there exists a unique $h \in C(\overline{\Omega})$ such that*

(1) *h is F-harmonic on Ω, and*
(2) *$h\big|_{\partial\Omega} = \varphi$.*

Remark 3.12. — In fact Theorem 3.11 holds for any constant coefficient second-order subequation F if and only if its asymptotic cone subequation $\overrightarrow{F}$ satisfies $P_e \in \operatorname{Int} F$ and $-P_e \notin F$ for all $|e| = 1$.

A Description of all ST-Invariant Cone Subequations

Although it is always easy to compute the characteristics (p, q) of a given F, it is still enlightening to give a description (or construction) of all the possible ST-invariant cone subequations with characteristics (p, q).

The following specific examples are instrumental in this description. For $A \in \operatorname{Sym}^2(\mathbb{R}^n)$ let $\lambda_1(A) \leqslant \cdots \leqslant \lambda_n(A)$ denote the ordered eigenvalues of A, and set $\lambda_{\min}(A) \equiv \lambda_1(A)$ and $\lambda_{\max}(A) \equiv \lambda_n(A)$. We then define

$$\mathcal{P}_p^{\min/\max} \equiv \{A : \lambda_{\min}(A) + (p-1)\lambda_{\max}(A) \geqslant 0\} \tag{3.14}$$

$$\mathcal{P}_p^{\min/2} \equiv \{A : \lambda_{\min}(A) + (p-1)\lambda_2(A) \geqslant 0\} \tag{3.15}$$

It is clear that both of these are $\mathrm{O}(n)$-invariant cone subequations. Both $A \equiv P_{e^\perp} - (p-1)P_e$ and $B \equiv -P_{e^\perp} + \frac{1}{p-1}P_e$ have the property that $\lambda_{\min} + (p-1)\lambda_{\max} = 0$, which shows that $A, B \in \partial\mathcal{P}_p^{\min/\max}$ and hence $\mathcal{P}_p^{\min/\max}$ has characteristics (p, q) where q satisfies $(p-1)(q-1) = 1$. Similarly, $\mathcal{P}_p^{\min/2}$ has characteristics (p, ∞) if $n \geqslant 3$.

Our general discussion is a characterization in terms of these two examples and their duals.

PROPOSITION 3.13. — *Suppose that F is an ST-invariant (not necessarily convex) cone subequation. Then F has a finite (increasing) Riesz characteristic p if and only if*

$$\mathcal{P}_p^{\min/2} \subset F \subset \mathcal{P}_p^{\min/\max}. \tag{3.16}$$

Equivalently, $K_p(|x|)$ is an increasing (or downward-pointing) radial F-harmonic. In particular, both the "smallest" and the "largest" subequations, $\mathcal{P}_p^{\min/2}$ and $\mathcal{P}_p^{\min/\max}$, have Riesz characteristic p.

Proof. — Let $A(p) \equiv P_{e^\perp} - (p-1)P_e$. If F satisfies (3.16), then $A(p) \in \mathcal{P}_p^{\min/2} \Rightarrow A(p) \in F$, and $A(p) \notin \operatorname{Int} \mathcal{P}_p^{\min/\max} \Rightarrow A(p) \notin \operatorname{Int} F$, which proves that $A(p) \in \partial F$, and hence F has characteristic p.

Each $A \in \mathrm{Sym}^2(\mathbb{R}^n)$ can be written as a sum $A = \lambda_1 P_{e_1} + \cdots + \lambda_n P_{e_n}$ using the ordered eigenvalues of A. Set $B_0 \equiv \lambda_1 P_{e_1} + \lambda_2 P_{e_1^\perp}$, and $B_1 \equiv \lambda_1 P_{e_1} + \lambda_n P_{e_1^\perp}$, and note that $B_0 \leqslant A \leqslant B_1$.

If $A \in \mathcal{P}_p^{\min/2}$, then $\lambda_1 + (p-1)\lambda_2 \geqslant 0$. Thus, $B_0 \in \mathcal{P}_p^{\min/2}$. Since $\mathcal{P}_p^{\min/2}$ and F have the same profile given by (3.1) (and $\lambda_2 \geqslant 0$), we conclude that $B_0 \in F$. However, $B_0 \leqslant A$ proving that $A \in F$.

For the other inclusion, pick $A \in F$. Since $F \subset \widetilde{\mathcal{P}}$, we have $\lambda_{\max} \geqslant 0$. Now $A \leqslant B_1$ implies $B_1 \in F$. Again F and $\mathcal{P}_p^{\min/\max}$ have the same profile given by (3.1). Therefore, $B_1 \in \mathcal{P}_p^{\min/\max}$. This implies by definition that $A \in \mathcal{P}_p^{\min/\max}$. $\square$

This imposes a constraint on the decreasing characteristic q of F.

Corollary 3.14. — *The characteristics of F satisfy*

$$(p-1)(q-1) \geqslant 1. \tag{3.17}$$

Proof. — It follows from Definition 3.6 (3.9a) that if one shrinks a subequation, then its decreasing characteristic goes up. Thus if F has characteristic p, we have $\mathcal{P}_p^{\min/\max} \supset F$ and so the decreasing characteristic q of F satisfies $q - 1 \geqslant q_{\mathcal{P}_p^{\min/\max}} - 1 = 1/(p-1)$. $\square$

Remark. — The only ST-invariant cone subequation with given characteristics (p, q) satisfying equality in (3.17) is $\mathcal{P}_p^{\min/\max}$. This can be proved by using Proposition 3.15 below, but details are omitted here.

It is just as easy to describe all examples with dual characteristic q. First note that the duals of the two subequations in (3.16) are given by

$$\widetilde{\mathcal{P}}_p^{\min/2} : \lambda_{\max}(A) + (p-1)\lambda_{n-1}(A) \geqslant 0, \tag{3.18}$$

$$\widetilde{\mathcal{P}}_p^{\min/\max} : \lambda_{\max}(A) + (p-1)\lambda_{\min}(A) \geqslant 0. \tag{3.19}$$

Note that the increasing characteristics of these two subequations are both ∞, and the decreasing characteristics are p by (3.9c).

Applying Proposition 3.13 to $\widetilde{F}$ now yields the following result.

Proposition 3.15. — *Suppose that F is an ST-invariant (not necessarily convex) cone subequation. Then F has a finite (decreasing) Riesz characteristic q if and only if*

$$\widetilde{\mathcal{P}}_q^{\min/\max} \subset F \subset \widetilde{\mathcal{P}}_q^{\min/2}. \tag{3.20}$$

Proof. — $\mathcal{P}_q^{\min/2} \subset \widetilde{F} \subset \mathcal{P}_q^{\min/\max} \iff \widetilde{\mathcal{P}}_q^{\min/\max} \subset F \subset \widetilde{\mathcal{P}}_q^{\min/2}$. □

Finally, it is possible to describe all the ST-invariant cone subequations with both characteristics finite.

Proposition 3.16. — *Suppose that F is an ST-invariant cone subequation. Then F has both Riesz characteristics (p,q) finite if and only if*

$$\mathcal{P}_p^{\min/2} \cup \widetilde{\mathcal{P}}_q^{\min/\max} \subset F \subset \mathcal{P}_p^{\min/\max} \cap \widetilde{\mathcal{P}}_q^{\min/2}. \tag{3.21}$$

Such subequations exist if and only if

$$(p-1)(q-1) \geqslant 1, \tag{3.22}$$

and so in particular if this constraint holds for (p,q), then both

$$\mathcal{P}_p^{\min/2} \cup \widetilde{\mathcal{P}}_q^{\min/\max} \text{ and } \mathcal{P}_p^{\min/\max} \cap \widetilde{\mathcal{P}}_q^{\min/2} \text{ have characteristics } (p,q). \tag{3.23}$$

Proof. — Note that (3.21) holds if and only if both (3.16) and (3.20) hold. Thus by Propositions 3.13 and 3.15, F has finite Riesz characteristics (p,q) if and only if (3.21) holds.

Corollary 3.14 states that if F has characteristics (p,q), then (3.22) must hold. Now suppose that (3.22) holds. Then

$$\widetilde{\mathcal{P}}_q^{\min/\max} \subset \mathcal{P}_p^{\min/\max} \quad \text{and} \quad \mathcal{P}_p^{\min/2} \subset \widetilde{\mathcal{P}}_q^{\min/2} \tag{3.24}$$

because $\lambda_{\max} + (q-1)\lambda_{\min} \geqslant 0 \Rightarrow \lambda_{\min} + (p-1)\lambda_{\max} \geqslant 0$ if $p-1 \geqslant 1/(q-1)$; and $\lambda_{\min} + (p-1)\lambda_2 \geqslant 0 \Rightarrow \lambda_{n-1} + (p-1)\lambda_{\max} \geqslant 0 \Rightarrow \lambda_{\max} + (q-1)\lambda_{n-1} \geqslant 0$ if $q-1 \geqslant 1/(p-1)$. Finally, (3.24) implies that $\mathcal{P}_p^{\min/2} \cup \widetilde{\mathcal{P}}_q^{\min/\max} \subset \mathcal{P}_p^{\min/\max} \cap \widetilde{\mathcal{P}}_q^{\min/2}$ so that both of these subequations have characteristics (p,q). □

4. Some Illustrative Examples

For the basic subequations the Riesz characteristic is quite easy to compute. We shall illustrate this with a selection of examples of differing types. A detailed discussion of subequations of characteristic p, and further results, are given in Appendix A of Part II.

For $A \in \mathrm{Sym}^2(\mathbb{R}^n)$ we let

$$\lambda_1(A) \leqslant \lambda_2(A) \leqslant \cdots \leqslant \lambda_n(A) \tag{4.1}$$

denote the *ordered eigenvalues* of A.

Example 4.1 (The p-Convexity Equation). — For each real number p with $1 \leqslant p \leqslant n$, the smallest (see Lemma A.2 in Part II) convex cone subequation with characteristic p is also one of the most basic:

$$\mathcal{P}_p \equiv \{A : \lambda_1(A) + \cdots + \lambda_{[p]}(A) + (p-[p])\lambda_{[p]+1}(A) \geqslant 0\}. \tag{4.2}$$

For p an integer the $\mathcal{P}_p$-subharmonic functions are characterized by the fact that their restrictions to minimal submanifolds of dimension p are intrinsically subharmonic. For this and a discussion of the geometry associated with this equation, see [22]. (Results for integer p go back to H. Wu [37, 45].) Note, by the way, that $\mathcal{P}_1 = \mathcal{P}$ is the homogeneous Monge–Ampère subequation and $\mathcal{P}_n = \Delta$ is the standard Laplacian.

There are complex and quaternionic analogues $\mathcal{P}_p^{\mathbb{C}}$ and $\mathcal{P}_p^{\mathbb{H}}$ defined by (4.2) but using the eigenvalues of the complex (respectively quaternionic) hermitian symmetric part of $A = D^2u$. When $p = 1$ this yields the homogeneous complex and quaternionic Monge–Ampère subequations. The $\mathcal{P}_p^{\mathbb{C}}$-subharmonic functions are characterized by the fact that their restrictions to complex p-dimensional submanifolds are Δ-subharmonic. The Riesz characteristics of $\mathcal{P}_p^{\mathbb{C}}$ and $\mathcal{P}_p^{\mathbb{H}}$ are $2p$ and $4p$ respectively. See Lemma 4.8 below.

Example 4.2 (The Elementary Symmetric or Hessian Equations). — For each integer k, $1 \leqslant k \leqslant n$, let $\sigma_k(A)$ denote the k^{th} elementary symmetric function of the eigenvalues of $A \in \operatorname{Sym}^2(\mathbb{R}^n)$. The convex cone subequation

$$\Sigma_k = \{A : \sigma_1(A) \geqslant 0, \sigma_2(A) \geqslant 0, \ldots, \sigma_k(A) \geqslant 0\} \tag{4.3}$$

has Riesz characteristic

$$p_{\Sigma_k} \equiv \frac{n}{k}. \tag{4.4}$$

These subequations, often called *hessian equations*, have been the focus of much study (e.g., [33, 34, 35, 42, 43, 44]). There are again complex and quaternionic analogues $\Sigma_k^{\mathbb{C}}$ and $\Sigma_k^{\mathbb{H}}$ with Riesz characteristics $2n/k$ and $4n/k$ respectively.

Example 4.3 (The δ-Uniformly Elliptic Equations). — The δ-uniformly elliptic regularization of the basic subequation $\mathcal{P} \equiv \{A \geqslant 0\}$ (cf. Example 4.10) is

$$\mathcal{P}(\delta) \equiv \left\{A : A + \tfrac{\delta}{n}\operatorname{tr}(A)I \geqslant 0\right\}. \tag{4.5}$$

These are convex cone subequations with Riesz characteristic $p = n(1+\delta)/(n+\delta)$. Given p with $1 \leqslant p \leqslant n$ and setting

$$\delta = \frac{n(p-1)}{n-p}. \tag{4.6}$$

Lemma A.2 of Part II states that $\mathcal{P}(\delta)$ is the largest $\mathrm{O}(n)$-invariant convex cone subequation with Riesz characteristic p. There are again complex and quaternionic analogues described in Example 4.7 below.

Example 4.4 (Geometrically Defined Subequations). — These important examples account for our choice of normalization in defining the Riesz characteristic. Fix a compact subset $\mathbb{G} \subset G(p, \mathbb{R}^n)$ in the Grassmannian of p-planes in $\mathbb{R}^n$, and define

$$F(\mathbb{G}) \equiv \{A : \operatorname{tr}_W(A) \geqslant 0 \text{ for all } W \in \mathbb{G}\} \tag{4.7}$$

where $\mathrm{tr}_W(A)$ denotes the trace of $A\big|_W$. Assuming the ST-invariance of $\mathbb{G}$, the Riesz characteristic is easily seen to be

$$p_{F(\mathbb{G})} = p. \tag{4.8}$$

Many interesting subequations arise this way. When $\mathbb{G} = G(p, \mathbb{R}^n), G^{\mathbb{C}}(p, \mathbb{C}^n)$ and $G^{\mathbb{H}}(p, \mathbb{H}^n)$ we retrieve the integer cases in Example 4.1 above. There are many other interesting examples. One such is LAG $\subset G^{\mathbb{R}}(n, \mathbb{C}^n)$, the set of Lagrangian n-planes in $\mathbb{C}^n$. Closely related are the sets of isotropic p-planes, and p-planes satisfying certain CR (Cauchy–Riemann) conditions. Also of interest is SLAG $\subset$ LAG, the special Lagrangian planes (cf. [13]). This latter is an example of a subequation associated to a calibration [16]. Other particularly interesting examples come from the associative and coassociative calibrations in $\mathbb{R}^7$ and the Cayley calibration in $\mathbb{R}^8$. All the specific subequations in this paragraph have the property that they are ST-invariant, i.e., invariant under a subgroup $G \subset \mathrm{O}(n)$ which acts transitively on the sphere $S^{n-1} \subset \mathbb{R}^n$.

These geometrically defined subequations will be the sole focus of Part II of this paper.

Example 4.5 (Branches of Gårding Operators). — In many of the cases above, one can associate a homogeneous polynomial operator $\Phi(D^2u)$. When the polynomial Φ is Gårding hyperbolic with respect to the identity I (which is typically the case), the equation has many branches [10, 15, 21].

The simplest case is $\mathcal{P} = \mathcal{P}_1$ where the operator is $\Phi(A) = \det_{\mathbb{R}}(A)$. Here the branches are given by $\{\lambda_k(A) \geqslant 0\}$ (see (4.1)). Unfortunately, in this case the branches for $k > 1$ have infinite characteristic.

For the general Gårding polynomial $\Phi(A)$ of degree m, there are ordered eigenvalues,

$$\Lambda_1(A) \leqslant \Lambda_2(A) \leqslant \cdots \leqslant \Lambda_m(A), \quad \text{and} \quad \Phi(A) = \Lambda_1(A) \cdots \Lambda_m(A). \tag{4.1$'$}$$

Just as with $\det_{\mathbb{R}}(A)$, the k^{th} branch is defined by $\{\Lambda_k(A) \geqslant 0\}$ for $k = 1, \ldots, m$. The Riesz characteristics $p_1 \leqslant \cdots \leqslant p_m$ of these respective branches are determined by the eigenvalues of P_e (assuming ST-invariance). They are exactly the numbers $1/\Lambda_j(P_e)$, $j = 1, \ldots, m$ arranged in increasing order (see Proposition A.10 in Part II). Therefore the number of branches with finite Riesz characteristic equals the number of non-zero eigenvalues of P_e. Only the first and smallest branch is convex, and it is uniformly elliptic $\iff$ all branches are uniformly elliptic $\iff$ $\Phi(P_e) > 0$.

Gårding operators are plentiful. For instance, in each of our first three examples there is an associated Gårding operator, and hence each comes equipped with branches. To illustrate, for the case where p is an integer in

Example 4.1, we have

$$\Phi(A) = \prod_{i_1<\cdots<i_p} (\lambda_{i_1}(A)+\cdots+\lambda_{i_p}(A)) = \det\left(D_A : \Lambda^p\mathbb{R}^n \longrightarrow \Lambda^p\mathbb{R}^n\right). \quad (4.9)$$

Said differently, $\Lambda_I(A) = \lambda_{i_1}(A) + \cdots + \lambda_{i_p}(A)$ are the eigenvalues. Here D_A is the extension of A as a derivation. The k^{th} branch is given by requiring that the k^{th} ordered p-fold sum of the λ_i's be $\geqslant 0$. One easily computes that *the first* $\binom{n-1}{p-1}$ *branches have Riesz characteristic* p and the remaining branches have infinite characteristic.

In Example 4.2 the Gårding operator is $\Phi(A) = \sigma_k(A)$. Although the eigenvalues $\Lambda_j(A)$ of Φ do not have an explicit formula in terms of the standard eigenvalues of A, the eigenvalues of $A = P_e$ are all zero except for one which equals k/n. Hence, Σ_k has characteristic n/k and all other branches have characteristic ∞.

In Example 4.3 the eigenvalues are

$$\Lambda_k(A) = \lambda_k(A) + \frac{\delta}{n(1+\delta)}\mathrm{tr}(A), \quad k = 1, \ldots, n.$$

Hence, each of the k^{th} branches $\{\Lambda_k(A) \geqslant 0\}$, for $k \geqslant 2$, has the same Riesz characteristic $p = n(1 + \frac{1}{\delta})$, which is finite but larger than n, while as noted above, the first branch $\mathcal{P}(\delta)$ has characteristic $n(1+\delta)/(n+\delta)$.

Example 4.6 (Trace Powers of the Hessian). — Consider the non-convex cone subequation

$$F \equiv \{A : \mathrm{tr}\,(A^q) \geqslant 0\}$$

where $q > 1$ is an odd integer. More generally one could define A^q for any $q > 0$ by using the function t^q for $t \geqslant 0$ and $-|t|^q$ for $t < 0$. In all cases one computes that the Riesz characteristic is

$$p_F = 1 + (n-1)^{\frac{1}{q}}.$$

More generally, for $k \in [1, n]$ and $q > 0$ real numbers, there is the subequation

$$F \equiv \{A : \lambda_1^q(A) + \cdots + \lambda_{[k]}^q(A) + (k - [k])\lambda_{[k]+1}^q \geqslant 0\}$$

with t^q defined as above. Here the Riesz characteristic is

$$p_F = 1 + (k-1)^{\frac{1}{q}}.$$

Example 4.7 (Complex and Quaternionic Analogues). — Suppose $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is an $\mathrm{O}(n)$-invariant subequation. Then F can be defined by the constraint set $E \subset \mathbb{R}^n$ imposed by F on the eigenvalues $\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))$. Thus $A \in F \iff \lambda(A) \in E$. The equation F has *complex and quaternionic analogues* $F^{\mathbb{C}}$ and $F^{\mathbb{H}}$, defined on $\mathbb{C}^n = (\mathbb{R}^{2n}, J)$ and

$\mathbb{H}^n = (\mathbb{R}^{4n}, I, J, K)$ respectively, as follows. For $A \in \mathrm{Sym}^2(\mathbb{R}^{2n})$ consider the hermitian symmetric part

$$A_{\mathbb{C}} \equiv \frac{1}{2}(A - JAJ)$$

whose eigenspaces are complex lines with ordered eigenvalues $\lambda_1(A_{\mathbb{C}}) \leqslant \cdots \leqslant \lambda_n(A_{\mathbb{C}})$. One now defines $F^{\mathbb{C}}$ by applying the eigenvalue constraints E of F to these eigenvalues of $A_{\mathbb{C}}$. The story in the quaternionic case is parallel and uses the quaternionic hermitian symmetric part $A_{\mathbb{H}} \equiv \frac{1}{4}(A - IAI - JAJ - KAK)$ and eigenvalues $\lambda_k(A_{\mathbb{H}})$.

Lemma 4.8. — *If F is an $O(n)$-invariant cone subequation with Riesz characteristic p, then the Riesz characteristics of $F^{\mathbb{C}}$ and $F^{\mathbb{H}}$ are*

$$p_{F^{\mathbb{C}}} = 2p \quad \textit{and} \quad p_{F^{\mathbb{H}}} = 4p\,.$$

Proof. — We consider the complex case. If $A = P_{e^\perp} - (p-1)P_e \in \mathrm{Sym}^2(\mathbb{R}^{2n})$, then one computes that

$$A_{\mathbb{C}} = P_{\mathbb{C}e^\perp} - \left(\frac{p}{2} - 1\right) P_{\mathbb{C}e} \quad \text{and} \quad A_{\mathbb{H}} = P_{\mathbb{H}e^\perp} - \left(\frac{p}{4} - 1\right) P_{\mathbb{H}e} \tag{4.10}$$

which displays the eigenvalues of $A_{\mathbb{C}}$ and $A_{\mathbb{H}}$. □

Example 4.9 (The Subequation Determined by a Gårding Operator and a Universal Eigenvalue Constraint). — The procedures above can be greatly generalized. Note, to begin, that given an $\mathrm{O}(m)$-invariant subequation F, the eigenvalue set $E \equiv \lambda(F)$ is closed, invariant under permutation of coordinates and $\mathbb{R}^m_+$-monotone. Conversely, any such eigenvalue set E determines an $\mathrm{O}(m)$-invariant subequation $F = \lambda^{-1}(E)$. Each such E is a *universal eigenvalue subequation* in the sense that, for each degree-m Gårding operator Φ on $\mathrm{Sym}^2(\mathbb{R}^n)$, the set $F \equiv \lambda_\Phi^{-1}(E)$ is a subequation on $\mathbb{R}^n$, where $\lambda_\Phi : \mathrm{Sym}^2(\mathbb{R}^n) \to \mathbb{R}^m$ is the eigenvalue map associated to Φ. See Proposition A.8 in [23] for the details and further discussion.

Example 4.10 (The δ-Uniformly Elliptic Regularization of a Subequation). Given a cone subequation $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ and $\delta > 0$, define

$$F(\delta) \equiv \left\{A : A + \tfrac{\delta}{n}\mathrm{tr}(A)I \in F\right\}. \tag{4.11}$$

This equation satisfies the uniformly elliptic condition:

$$F(\delta) + \mathcal{P}(\delta) \subset F(\delta)\,. \tag{4.12}$$

One computes that

$$F \text{ has Riesz characteristic } p \iff F(\delta) \text{ has Riesz characteristic } \frac{pn(1+\delta)}{n+\delta p}.$$

5. K_p-Convexity and Monotonicity

In this section we give a fairly complete discussion of the classical one-variable results that underlie this paper. They concern the properties of subsolutions to the one-variable subequation R_p introduced below.

In the following section we will prove that associated to each F-subharmonic function u there are three functions of r (denoted $M(r)$, $S(r)$ and $V(r)$), which are subsolutions of R_p, and which capture much of the asymptotic behavior of u. By Lemma 5.1 (3) below this will imply the key double monotonicity result, Theorem 6.4, which is needed for defining the notion of density and for proving our main theorems.

Fix a real number p with $1 \leqslant p < \infty$, and for $r > 0$ consider the *one-variable Riesz kernel*

$$K_p(r) \equiv \frac{1}{(2-p)} r^{2-p} \quad \text{if } p \neq 2 \text{ and } K_2(r) = \log r\,. \tag{5.1}$$

With this normalization

$$K_p'(r) = \frac{1}{r^{p-1}} \qquad \text{for all } 1 \leqslant p < \infty\,.$$

Note that $K_p(r)$ is a strictly increasing solution to the subequation

$$R_p : \psi''(r) + \frac{p-1}{r}\psi'(r) \geqslant 0 \quad \text{on } (0,\infty)\,. \tag{5.2a}$$

Alternatively,

$$R_p : \frac{d}{dr}\left(r^{p-1}\psi'(r)\right) = \frac{d}{dr}\left(\frac{\psi'(r)}{K_p'(r)}\right) \geqslant 0 \quad \text{on } (0,\infty)\,. \tag{5.2b}$$

All solutions of R_p are of the form

$$h(r) \equiv CK_p(r) + k \quad \text{with } C, k \in \mathbb{R}\,. \qquad \text{(Riesz Harmonics)} \tag{5.3}$$

Note that $h(r)$ is increasing if and only if $C \geqslant 0$.

The change of variables

$$s = K_p(r) \text{ along with its inverse } r = K_p^{-1}(s) \tag{5.4}$$

play an important role. The inverse $r(s) = K_p^{-1}(s)$ is defined on the range of K_p which is the interval $(0,\infty)$ when $1 \leqslant p < 2$, all of $\mathbb{R}$ when $p = 2$, and $(-\infty, 0)$ for $2 < p < \infty$.

Lemma 5.1 (The Equivalences). — *The following conditions on an upper semi-continuous function $\psi(r)$, defined on a subinterval of $(0,\infty)$, are equivalent.*

(1) (R_p-Subharmonic) $\psi(r)$ *satisfies the subequation R_p defined by* (5.2).

(2) (K_p-Convexity) *$\psi(r)$ is K_p-convex, meaning that under the change of variables* (5.4)*), the function $f(s) \equiv \psi(r(s))$ is a convex function of s.*
(3) (K_p-Monotonicity) $\frac{\psi(r)-\psi(t)}{K_p(r)-K_p(t)}$ *is non-decreasing in r and t ($r \neq t$).*
(4) (R_p-Comparison) *If $\psi(r) \leqslant CK_p(r)+k$ for $r = s$ and $r = t$, then the inequality holds for all r between s and t.*

Proof. — Now $f(s) \equiv \psi(r(s))$ implies $\psi(r) = f(K_p(r))$. First, assume ψ is smooth. Then

$$\psi'(r) = f'(s)K_p'(r) \quad \text{and hence} \quad \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\psi'(r)}{K_p'(r)}\right) = f''(s)K_p'(r). \tag{5.5}$$

For general ψ, the fact that: $\varphi(r)$ is a test function for ψ at r_0 if and only if $\varphi(r(s))$ is a test function for $\psi(r(s))$ at $s_0 \equiv s(r_0)$, reduces the proof to the smooth case. Since viscosity convexity $f''(s) \geqslant 0$ is equivalent to classical convexity (see for example, [14, Prop. 2.6]), this proves that (1) $\iff$ (2).

Now (3) is just monotonicity of the slopes of secant lines to the function $f(s) \equiv \psi(r(s))$, and hence it is equivalent to the convexity of $f(s)$. Assertion (4) is just the statement that $f(s)$ is convex if and only if f satisfies comparison with affine functions $Cs+k$. □

Corollary 5.2. — *Let $\psi(r)$ satisfy the equivalent conditions in Lemma 5.1. Then*

(1) *the function $\psi(r)$ is locally Lipschitz continuous,*
(2) *the left and right hand derivatives $\psi'_{\pm}(r)$ exist.*

Proof. — The corresponding statements for the function $f(s) \equiv \psi(r(s))$ with $r(s) = K^{-1}(s)$ are standard classical facts about the convex function f. □

Densities

The remainder of this section is devoted to describing properties of a function $\psi(r)$, defined on an interval $(0, r_0)$ (with $r_0 = \infty$ possible), under the

Hypothesis 5.3. — *ψ satisfies the equivalent conditions* (1)–(4) *of Lemma 5.1.*

The Properties (1) and/or (3) enable us to introduce the following.

Corollary 5.4 (Existence of Densities). — *The decreasing limits*

$$\Theta^\psi = \lim_{\substack{r,t\to 0\\ t>r>0}} \frac{\psi(t)-\psi(r)}{K(t)-K(r)} = \lim_{r\to 0} \frac{\psi'_\pm(r)}{K'(r)} \tag{5.6}$$

exist and define the density Θ^ψ. *Moreover, if ψ is increasing, then* $0 \leqslant \Theta^\psi < \infty$.

Proof. — To see that the two decreasing limits in (5.6) agree divide the numerator and denominator of $\frac{\psi(r+\delta)-\psi(r)}{K(r+\delta)-K(r)}$ by δ and let $\delta \to 0$. $\square$

In this one-variable context, rather than in its later applications, it might be better to call this "the derivative of $\psi(r(s))$ at $r=0$".

Note that the monotonicity quotient in (3) remains unchanged if ψ is replaced by a translate $\psi - c$ with $c \in \mathbb{R}$. In particular, the densities of u and $u-c$ are the same. This point is critical in establishing the following.

Lemma 5.5. — *If ψ is increasing, then there exists $c \in \mathbb{R}$ and r_0 such that*

$$\frac{\psi(r)-c}{K(r)} \quad \textit{decreases to } \Theta^\psi \ \textit{as } 0<r<r_0 \ \textit{decreases down to } 0.$$

Moreover,

$$\frac{\psi(r)-\psi(0)}{K(r)} \qquad \textit{decreases to } \Theta^\psi \ \textit{if } 1 \leqslant p < 2, \ \textit{and} \tag{5.7a}$$

$$\lim_{r\to 0} \frac{\psi(r)}{K(r)} = \Theta^\psi \quad \textit{if } 2 \leqslant p < \infty. \tag{5.7b}$$

(Note: if we set $\psi(0)=0$ when $1 \leqslant p < 2$, then we have in all cases ($1 \leqslant p < \infty$) that $\lim_{r\to 0} \psi(r)/K(r) = \Theta^\psi$*.)*

Proof. — For any value of p, $1 \leqslant p < \infty$, there is exactly one point in$[0,\infty]$ where K vanishes. However there are three cases: $K(0)=0$ if $1 \leqslant p < 2$, $K(1)=0$ if $p=2$, and $K(\infty)=0$ if $2<p$. First let us suppose that the function ψ is defined and finite on an interval containing the point where K vanishes. Then one can take r in (5.6) to be that point, and the Proposition follows immediately from the double monotonicity in (3) of Lemma 5.1. Specifically, in the three cases we obtain that:

$$(1 \leqslant p < 2) \ \frac{\psi(t)-\psi(0)}{K(r)} \ \text{decreases to } \Theta^\psi \text{ as } 0<t<r_0 \text{ decreases to } 0, \tag{5.7a$'$}$$

$$(p=2) \ \frac{\psi(t)-\psi(1)}{K(t)} \ \text{decreases to } \Theta^\psi \text{ as } 0<t<r_0 \text{ decreases to } 0, \tag{5.7b$'$}$$

$$(2<p) \ \frac{\psi(t)-\psi(\infty)}{K(t)} \ \text{decreases to } \Theta^\psi \text{ as } 0<t<r_0 \text{ decreases to } 0. \tag{5.7c$'$}$$

This leaves us with an extension problem in the last two cases. Namely we must prove that there exists an $r_0 > 0$ such that the restriction of the given ψ to $(0, r_0)$

$$(p=2) \quad \text{has an extension } \overline{\psi} \text{ to } (0,1] \text{ satisfying Lemma 5.1(3), and} \tag{5.8b}$$

$$(2<p) \quad \begin{array}{l}\text{has an extension } \overline{\psi} \text{ to } (0,\infty) \text{ satisfying Lemma 5.1(3)} \\ \text{with } \psi(\infty)<\infty.\end{array} \tag{5.8c}$$

Suppose that ψ has domain containing $(0, r_0]$ (and if $p = 2$ that $r_0 < 1$ since if $r_0 \geqslant 1$ in this case we are finished.) Make the change of variables $s_0 \equiv K^{-1}(r_0)$ as in Lemma 5.1(2). Since $2 \leqslant p < \infty$, we have $s_0 < 0$. The convex increasing function $f(s)$ on $(-\infty, s_0)$ can be extended to a convex increasing function $\overline{f}$ on $(-\infty, 0]$ by defining $\overline{f}$ to be the affine function

$$a(s) \equiv f'_-(s_0)(s - s_0) + f(s_0) \tag{5.9}$$

on $s_0 \leqslant s \leqslant 0$. Since the graph of $a(s)$ is a supporting line for the epigraph of f over $(-\infty, s_0)$, this extension $\overline{f}$ is convex and increasing on $(-\infty, 0]$.

Observe now that by translating our original ψ by a suitable additive constant, we can insure that $\overline{f} < 0$ on $(-\infty, 0]$. Now set $\overline{\psi}(t) \equiv \overline{f}(K(t))$, where $0 < t \leqslant 1$ if $p = 2$ and $0 < t \leqslant \infty$ if $2 < p$. Finally, by (5.7b$'$) where $\psi(1)$ is finite, and (5.7c$'$) where $\psi(\infty)$ is finite, the fact that $K(0) = -\infty$ implies that $\lim_{r\to 0} \psi(r)/K(r) = \Theta^{\psi}$. $\square$

Remark 5.6. — The subequation $R_p : \psi'' + \frac{p-1}{r}\psi'(r) \geqslant 0$ is linear and could have been interpreted in the distributional sense as well as the viscosity sense.

6. Monotonicity and Stability of Averages for F-Subharmonic Functions

In this section we discuss three of the basic ways of taking an average of an F-subharmonic function, and show that each average produces a radial F-subharmonic. Since the radial F-subharmonics are just one-variable R_p-subharmonics (Proposition 3.3), they are well understood and enjoy all the properties of Lemma 5.1. In particular, they satisfy the double monotonicity described in Theorem 6.4 below, which provides the vehicle for defining the densities explored in the next section. Finally, the stability of these averages under the tangential flow is established in Lemma 6.5.

We assume as always that the subequation F is an ST-invariant cone with invariance group $G \subset \mathrm{O}(n)$. We further assume that the Riesz characteristic p of F is finite. This is because when $p = \infty$, the increasing radial subequation $R_F^{\uparrow}$ is simply $g'(t) \geqslant 0$ (Proposition 3.3 and Definition 3.1). Thus, when $p = \infty$, all increasing functions $g(t)$ determine increasing radial subharmonics $g(|x|)$, and no sensible notion of density is possible.

To begin we set some notation. Let $B_r(x_0) = \{x : |x - x_0| \leqslant r\}$ denote the ball of radius r about x_0, and set $S_r(x_0) \equiv \partial B_r(x_0)$. Let $A(a, b; x_0) \equiv \{x : a < |x - x_0| < b\}$ denote an annular region centered at x_0. Here and elsewhere, when $x_0 = 0$, reference to it will be dropped from the notation. Thus, $B_r = B_r(0)$ and $S_r = \partial B_r$. Similarly we set $B = B_1$ and $S = \partial B$.

The first average only requires that F be an ST-invariant cone (not necessarily convex). We denote the *(spherical) maximum* for an F-subharmonic function u defined on a region containing $S_r(x_0)$ by

$$M(u, x_0; r) \equiv \sup_S u(x_0 + rx)\,, \tag{6.1a}$$

Note that if u is F-subharmonic on $B_R(x_0)$, then by the maximum principle

$$\begin{gathered} M(u, x_0; r) = \sup_B u(x_0 + rx) = \sup_{B_r(x_0)} u \\ \text{and hence is increasing for } 0 \leqslant r \leqslant R. \end{gathered} \tag{6.1b}$$

By the ST-invariance of F

$$M(u, x_0; |x|) \equiv \sup_{g \in G} u(x_0 + gx)\,. \tag{6.2}$$

We now simplify by setting $x_0 = 0$ and using the abbreviated notation $M(r) \equiv M(u; r) = M(u, 0; r)$ when the meaning is obvious.

Lemma 6.1. — *If u is F-subharmonic on an annular region $A(a, b)$, then $M(|x|)$ is a radial F-subharmonic function on $A(a, b)$. If u is F-subharmonic on B_R, then $M(r)$ is also increasing in r.*

Proof. — Let $u_g(x) \equiv u(gx)$ with $g \in G$. Then $M(|x|) = \sup_{g \in G} u_g(x)$. Since F is G-invariant, each u_g is F-subharmonic. Therefore, by the standard "families locally bounded above" property for F, it suffices to prove that $M(t)$ is upper semi-continuous. This is done as follows. For each $\delta > 0$, $N_\delta \equiv \{x : u(x) < M(t) + \delta\}$ is an open set containing ∂B_t. Hence the annulus $A(t - \epsilon, t + \epsilon)$ is contained in N_δ for $\epsilon > 0$ small. Thus $M(r) < M(t) + \delta$ if $t - \epsilon \leqslant r \leqslant t + \epsilon$, proving that $M(t)$ is upper semi-continuous, and hence $M(|x|)$ is F-subharmonic. □

For the other averages we make the further standing assumption that F is convex. In this case we note the following.

$$F \text{ is an ST-invariant convex cone} \quad \Rightarrow \quad F \subset \Delta \equiv \{\operatorname{tr}(A) \geqslant 0\}. \tag{6.3}$$

Proof. — If $F \cap \{\mathrm{tr}(A) = c < 0\}$ is non-empty, then invariance plus convexity implies that $-\frac{c}{n}I \in F$. Now by the cone property, $-\lambda I \in F$ for all $\lambda > 0$. This along with positivity implies that $F = \mathrm{Sym}^2(\mathbb{R}^n)$. Since $\mathrm{tr}(P_{e^\perp} - (p-1)P_e) = n - p$, the condition $F \subset \{\mathrm{tr}\, A \geqslant 0\}$ implies $p_F \leqslant n$. Therefore,

$$F \text{ is an ST-invariant convex cone} \qquad \Rightarrow \qquad 1 \leqslant p_F \leqslant n\,. \tag{6.4}$$

We now define the *spherical* and *volume averages* of u at x_0 by

$$S(u, x_0; r) \equiv \frac{1}{|S|}\int_{\sigma\in S} u(x_0 + r\sigma)\,\mathrm{d}\sigma \equiv \fint_S u(x_0 + r\sigma)\,\mathrm{d}\sigma, \tag{6.5a}$$

$$V(u, x_0; r) \equiv \frac{1}{|B|}\int_{x\in B} u(x_0 + rx)\,\mathrm{d}x \equiv \fint_B u(x_0 + rx)\,\mathrm{d}x. \tag{6.5b}$$

Note that for any upper semi-continuous function u, each of these functions is jointly upper semi-continuous in (x_0, r) since $u(x_0 + rx)$ is the infimum of $\varphi(x_0 + rx)$ taken over continuous functions $\varphi \geqslant u$. ☐

Lemma 6.2. — *Suppose that u is F-subharmonic on the annulus $A(a,b)$. Then $S(u; |x|)$ is a radial F-subharmonic on $A(a,b)$. If u is F-subharmonic on the ball B_R, then both $S(u;|x|)$ and $V(u;|x|)$ are increasing radial F-subharmonic functions on B_R (with limiting values $S(u,0) = V(u,0) = u(0)$ at $x = 0$).*

Proof. — As noted above $S(u;r)$ and $V(u;r)$ are upper semi-continuous in r, and hence so are the functions $S(u;|x|)$ and $V(u;|x|)$ of x defined on B_R. The statement about their limiting values at $x = 0$ is a standard fact about Δ-subharmonic functions. It remains to show that $S(u;|x|)$ and $V(u;|x|)$ are F-subharmonic on B_R. Note that

$$S(|x|) = \int_G u(gx)\,\mathrm{d}g \tag{6.6}$$

for a suitably normalized invariant measure $\mathrm{d}g$ on G, and that

$$V(|x|) = n\int_0^1 S(\rho|x|)\rho^{n-1}\,\mathrm{d}\rho \qquad \text{since} \qquad |B| = \frac{1}{n}|S|. \tag{6.7}$$

To prove (6.7), set $|x| = r$ and compute $V(r) = \frac{1}{|B|}\int_B u(ry)\,\mathrm{d}y$ using polar coordinates. Now since F is a convex cone subequation, averages such as (6.6) and (6.7) preserve F-subharmonicity. This is explained further by Theorem 9.5 and Remark 9.6. ☐

Remark 6.3. — By Theorem 2.4 and Theorem 2.7, the lemmas above could have been restated by concluding that the functions $M(r)$, $S(r)$ and $V(r)$ are $R_F^{\uparrow}$-subharmonic on $(0,R)$, or R_p-subharmonic on (a,b) in the annular cases.

The properties of upper semi-continuous functions $\psi(r)$ satisfying R_p have been presented in detail in Section 5. We make full use of those results by applying them to the three functions $M(u, x_0, r), S(u, x_0, r)$ and $V(u, x_0, r)$, where u is an F-subharmonic function. This includes the K_p-convexity, the K_p-monotonicity and the R_p-comparison properties of Lemma 5.1.

In particular, the K_p-monotonicity, part (3) of Lemma 5.1, gives the following basic result.

Theorem 6.4 (Double Monotonicity). — *Let u be F-subharmonic in an annular region about the origin in $\mathbb{R}^n$. Then*

$$\frac{M(u,r)-M(u,s)}{K(r)-K(s)} \quad \textit{is increasing in } r \textit{ and } s. \tag{1.5}$$

for all $0 < s < r$ where M is defined.

Furthermore, if F is convex, the same statement holds with $M(u,r)$ replaced by $S(u,r)$; or by $V(u,r)$ provided that u is F-subharmonic on a ball about the origin.

It is an important fact that each of these averages is stable under limits in L^1. This basic classical fact can be found in [26, §III.3.2]. We state it here in slightly different form needed later for tangents.

Lemma 6.5 (Stability of Averages). — *Suppose u_j is a sequence of Δ-subharmonic functions on B_R converging in $L^1(B_R)$ to a Δ-subharmonic function U. Then for $0 < r < R$,*

(1) $M(U,r) = \lim_{j\to\infty} M(u_j, r)$,
(2) $S(U,r) = \lim_{j\to\infty} S(u_j, r)$,
(3) $V(U,r) = \lim_{j\to\infty} V(u_j, r)$,

Proof. — Taking $K \equiv B_r$ in (3.2.7) of Theorem 3.2.1 in [26] gives us that

$$\limsup_{j\to\infty} M(u_j, r) \leqslant M(u,r).$$

Suppose there exists $C < M(u,r)$ such that $M(u_j,r) \leqslant C$ for all j sufficiently large. Then in the L^1-limit we would have $u - C \leqslant 0$ a.e. on B_r. However, for Δ-subharmonic functions, this implies that $u - C \leqslant 0$ everywhere on B_r, contrary to the definition of $M(u,r)$. We conclude that $\limsup_{j\to\infty} M(u_j,r) = M(u,r)$. The fact that this is also true for all subsequences proves (1).

As discussed in the paragraph prior to Proposition 3.2.14 in [26], the Theorem 3.2.13 can be applied to spherical measure σ_r on ∂B_r. Thus $u_j\sigma_r$

converges to $U\sigma_r$ in the weak topology of measures, yielding (2). Finally, (3) is implied directly by the hypothesis of $L^1(B_r)$-convergence. □

7. Densities for F-Subharmonic Functions: Upper Semi-Continuity

From the results of the last section and Corollary 5.4 we have three densities,

$$\Theta^M(u,x), \quad \Theta^S(u,x) \quad \text{and} \quad \Theta^V(u,x)$$

associated to an F-subharmonic function u defined in a neighborhood of the origin. For the second two densities, we must assume that F is convex. Under this convexity assumption there exists a fourth , even more classical density.

The Mass Density

Note that by (6.3) u is classically Δ-subharmonic. Thus Δu is a measure $\mu \geqslant 0$, which means Δu has a "mass density". Given a measure $\mu \geqslant 0$ defined in a neighborhood of a point $x_0 \in \mathbb{R}^n$, and $0 < k \leqslant n$, the limit

$$\Theta^k(\mu, x_0) \equiv \lim_{r \downarrow 0} \frac{\mu\left(B_r(x_0)\right)}{\alpha(k) r^k}, \tag{7.1}$$

if it exists, is called the *k-dimensional mass density of μ at x_0*. (See, for example, [9, 2.10.19] for discussion and definition of the constants $\alpha(k)$.) When k is an integer, $\alpha(k) = |B^k|$, the volume of the unit ball in $\mathbb{R}^k$. Suppose $\Theta^k(\mu, x)$ exists everywhere or replace lim by lim sup in (7.1). Fix an open set X, a constant $c > 0$, and define $E_c \equiv \{x \in X : \Theta^k(\mu, x) \geqslant c\}$. Then the Hausdorff k-measure satisfies (cf. [39, p. 11])

$$c\mathcal{H}^k(E_c) \leqslant \mu(X).$$

Comparing Densities

The next proposition states that: *All densities but Θ^M "agree"*, where "agree" means "are equal up to universal factors".

Proposition 7.1. — *Suppose that u is F-subharmonic near x_0 where F is convex with characteristic p, and set $\mu = \Delta u$. Then when $p \neq 2$,*

$$\Theta^S(u, x_0) = \frac{n-p+2}{n}\Theta^V(u, x_0) = \frac{\alpha(n-p)}{n|p-2|\alpha(n)}\Theta^{n-p}(\mu, x_0), \tag{7.2}$$

and when $p = 2$ we have that

$$\Theta^S(u, x_0) = \Theta^V(u, x_0) = \frac{\alpha(n-2)}{n\alpha(n)}\Theta^{n-2}(\mu, x_0). \tag{7.3}$$

The discussion of all densities is completed by showing that the maximum density and the spherical density are in general "comparable", and in fact equal when $p = 2$.

Proposition 7.2. — *Suppose that u is F-subharmonic near x_0 where F is convex and of characteristic p. Then there exists a constant $C = C(p, n) > 1$ such that*

$$\Theta^M(u, x_0) \leqslant \Theta^S(u, x_0) \leqslant C\Theta^M(u, x_0) \qquad \text{if } 2 < p < \infty, \text{ and} \tag{7.4}$$

$$\Theta^S(u, x_0) \leqslant \Theta^M(u, x_0) \leqslant C\Theta^S(u, x_0) \qquad \text{if } 1 < p < 2, \text{ while} \tag{7.5}$$

$$\Theta^M(u, x_0) = \Theta^S(u, x_0) \qquad \text{if } p = 2. \tag{7.6}$$

Remark 7.3. — Kiselman proved the equality in (7.6) in the plurisubharmonic case where $F = \mathcal{P}^{\mathbb{C}}$ on $\mathbb{C}^n$ (see [29, p. 161, line 2ff.]) by using Harnack's Inequality for Δ-subharmonic functions. The same proof works for any convex F of characteristic $p = 2$. Note that for $p = 1$ the left inequality in (7.5) holds but the right inequality fails, even for linear functions.

Proof of Proposition 7.1. — We give the proof of the first equality for all p using (6.7). Taking $x_0 = 0$ and dropping u and x_0 from the notation, it says that

$$V(r) = n\int_0^1 S(rt)t^{n-1}\,\mathrm{d}t. \tag{7.7}$$

Hence, we have

$$\frac{V(r)}{K(r)} = n\int_0^1 \frac{S(rt)}{K(rt)}\frac{K(rt)}{K(r)}t^{n-1}\,\mathrm{d}t.$$

When $p \neq 2$, $K(rt)/K(r) = 1/t^{p-2}$, so that letting $r \downarrow 0$ and integrating yields the first equality in (7.2). When $p = 2$,

$$\frac{K(rt)}{K(r)} = 1 + \frac{\log t}{\log r},$$

so letting $r \downarrow 0$ and integrating yields the first equality $\Theta^V(u) = \Theta^S(u)$ in (7.3).

For the proof of the second equalities we show that the mass density $\Theta^{n-p}(\mu)$ ($\mu = \Delta u$) is the desired multiple of the spherical density $\Theta^S(u)$. Recall the classical fact that

$$\mu(B_r) = (n-2)|S|\frac{S'_-(r)}{K'_n(r)}. \tag{7.8}$$

(See in [26, Thm. 3.2.16, (3.2.13)′] for a proof.) Since

$$\frac{n-2}{K_n'(r)} = r^{n-1} = \frac{|p-2|r^{n-p}}{K_p'(r)} \quad \text{when } p \neq 2,$$

we have

$$r^{p-n}\mu(B_r) = |p-2||S|\frac{S_-'(r)}{K_n'(r)} \quad \text{when } p \neq 2. \tag{7.8′}$$

If $p = 2$, this holds with $|p-2|$ replaced by 1. Finally, letting $r \downarrow 0$ and using (5.6) completes the proof. □

Proof of Proposition (7.2). — For simplicity let $x_0 = 0$. Note that for all p and r we have $S(u,r) \leqslant M(u,r)$. On the other hand, $K(r) < 0$ when $p \geqslant 2$ and $K(r) > 0$ when $p < 2$. Dividing by $K(r)$ and letting $r \downarrow 0$ then gives the inequalities on the left as well as the inequality $\Theta^M(u) \leqslant \Theta^S(u)$ when $p = 2$ (since u and $u + c$ have the same density, we can assume that $u(0) = 0$ when $p < 2$.)

The remainder of the proof is a consequence of Harnack's inequality. The standard form of this inequality is for a function $v \leqslant 0$ which is Δ-subharmonic on B_ρ. It says, with φ defined by

$$\varphi(\lambda) \equiv \frac{1-\lambda}{(1+\lambda)^{n-1}} \qquad \text{for } 0 < \lambda < 1,$$

that

$$M(v, \lambda r) \leqslant \varphi(\lambda) S(v, r) \quad \text{for all } 0 < r \leqslant \rho. \tag{7.9}$$

(See, for example [7, Prop. 4.2.2].) For an arbitrary Δ-subharmonic function v, the function $v - M(v,r)$ is $\leqslant 0$ on B_r. Hence, (7.9) gives the following more general form of Harnack's inequality

$$M(v,\lambda r) - M(v,r) \leqslant \varphi(\lambda)\big(S(v,r) - M(v,r)\big) \quad \text{for all } 0 < r \leqslant \rho. \tag{7.10}$$

for functions not necessarily $\leqslant 0$.

Suppose first that $p > 2$. We may assume $u(0) = -\infty$ since otherwise the assertion is trivial. Then u is negative near 0, and we can apply the standard form (7.9)of Harnack's inequality to obtain

$$\frac{M(u,\lambda r)}{K(\lambda r)} \geqslant \lambda^{p-2}\varphi(\lambda)\frac{S(u,r)}{K(r)}.$$

Letting $r \downarrow 0$ gives $\Theta^M(u,0) \geqslant c\Theta^S(u,0)$ where $c = \lambda^{p-2}\varphi(\lambda) > 0$. This gives (7.4) with $C = 1/c$. (Note that $c \equiv \sup_\lambda \lambda^{p-2}\varphi(\lambda)$ provides the best constant C.)

Suppose now that $1 < p < 2$. Replace u by $u(x) - u(0)$ so that $u(0) = 0$. Since densities are unchanged by adding a constant, we have $\Theta^M(u,0) = \lim_{r\downarrow 0} M(u,r)/K(r)$ and $\Theta^S(u,0) = \lim_{r\downarrow 0} S(u,r)/K(r)$ by Corollary 5.4.

Since u may not be $\leqslant 0$, we must use the general form (7.10) of Harnack. Dividing by $K(r)$ gives

$$\frac{(1+\lambda)^{n-1}}{1-\lambda}\left(\frac{M(u,\lambda r)}{K(r)} - \frac{M(u,r)}{K(r)}\right) \leqslant \frac{S(u,r)}{K(r)} - \frac{M(u,r)}{K(r)}. \tag{7.11}$$

Using the fact that $K(\lambda r) = \lambda^{2-p}K(r)$ and letting $r \downarrow 0$ gives

$$\psi(\lambda)\Theta^M(u,0) \leqslant \Theta^S(u,0) \qquad \text{with } \psi(\lambda) = 1 + \frac{(1+\lambda)^{n-1}}{1-\lambda}(\lambda^{2-p}-1).$$

Now direct calculation shows that $\lim_{\lambda\downarrow 0}\psi'(\lambda) = \infty$, and so $c \equiv \sup_{0<\lambda<1}\psi(\lambda) > 0$. This gives the desired result with $C = 1/c$.

It remains to prove that $\Theta^S(u) \leqslant \Theta^M(u)$ when $p = 2$. Set $\lambda = 1/e$ in (7.11) and note the fact that $K(r) = \log r = \log\frac{r}{e} + 1 = K(\lambda r) + 1 = K(\lambda r)(1 + o(r))$. Then taking the limit as $r \to 0$ in (7.11) yields $0 = \Theta^M(u) - \Theta^M(u) \leqslant \Theta^S(u) - \Theta^M(u)$ by Corollary 5.4. This completes the proof of Proposition 7.2. □

The Upper Semi-Continuity of Density

Theorem 7.4. — *Each of the densities $\Theta^M(u,x)$, $\Theta^S(u,x)$, and $\Theta^V(u,x)$ considered above is an upper semi-continuous function of x.*

Proof. — Because of Proposition 7.1 there are only two cases to consider. We must show that

$$\limsup_{\substack{x\to x_0\\ x\neq x_0}} \Theta(u,x) \leqslant \Theta(u,x_0). \tag{7.12}$$

Set $x_0 = 0$. Assume $0 < |x| < r < t$. Then

$$\Theta^\psi(u,x) \leqslant \frac{\psi(u,x,t) - \psi(u,x,r)}{K(t)-K(r)}. \tag{7.13}$$

Case 1. $\psi = M$. — By using the facts that $B_t(x) \subset B_{t+|x|}(0)$ and $B_{r-|x|}(0) \subset B_r(x)$, we see that the last quantity above is

$$\leqslant \frac{\sup_{B_{t+|x|}(0)} u - \sup_{B_{r-|x|}(0)} u}{K(t)-K(r)}.$$

The function $M(u,0,r) \equiv \sup_{B_r(0)} u$ is continuous (see Corollary 5.2 (1)) and increasing. Therefore,

$$\limsup_{\substack{x\to 0\\ x\neq 0}} \Theta^M(u,x) \leqslant \frac{\sup_{B_t(0)} u - \sup_{B_r(0)} u}{K(t)-K(r)}, \qquad 0<r<t.$$

Finally, the limit of the RHS as $r,t \to 0$ equals $\Theta^M(u,0)$. This proves the first case.

Case 2. $\psi = V$. — It suffices to note that $\lim_{x\to 0} V(u,x,t) = V(u,0,t)$, which follows since $V(u,x,t) = ⨍_B u(x+ty)\,dy$ and u converges in $L^1(B)$ to $u(ty)$ as $x \to 0$. ☐

Remark. — By using Theorem 3.2.13 in [26], one can show that $u(x+t\sigma)\,\mathrm{d}\sigma$ converges weakly in measure to $u(t\sigma)\,\mathrm{d}\sigma$ as $x \to 0$. This gives a direct proof that $S(u,x,t)$ is continuous in x at $x = 0$ without using Proposition 7.1

Corollary 7.5. — *For all $c > 0$, the set*

$$E_c \equiv \{x : \Theta(u,x) \geqslant c\} \text{ is closed.}$$

Remark. — When $p = 1$ the set where $\Theta(u) = 0$ is just the set of differentiability points of u (see (5.5) in Part II).

8. Maximality of Subharmonics with Harmonic Averages

In this section we extend the standard notion of maximality in pluripotential theory to each F-potential theory. This notion extends the notion of being F-harmonic, but is still very close to it. In fact, a maximal function is harmonic if and only if it is continuous. Our main result, Theorem 8.2, is key for the study of tangents. It provides a new criterion for an F-subharmonic function to be F-maximal. An excellent reference for pluripotential theory is [31].

Definition 8.1. — *An F-subharmonic function u on an open set $X \subset \mathbb{R}^n$ is said to be* F-maximal *on X if for each F-subharmonic function v on X and each compact subset $K \subset X$,*

$$v \leqslant u \quad \text{on } X - K \quad \Rightarrow \quad v \leqslant u \quad \text{on } X. \tag{8.1}$$

Note that by replacing v with $\max\{u,v\}$*, condition (8.1) is equivalent to*

$$v \geqslant u \text{ on } X \quad \text{and} \quad v = u \text{ on } X - K \quad \Rightarrow \quad v = u \quad \text{on } X. \tag{8.1$'$}$$

Most of the previous results come together in the proof of the next result.

Theorem 8.2 (The Maximality Criterion). — *Suppose that F is an ST-invariant convex cone subequation, and U is an F-subharmonic function on the annulus $A = \{x : a < |x| < b\}$. If the spherical average*

$$\begin{gathered} S(U,t) \equiv ⨍_S U(t\sigma)\,\mathrm{d}\sigma \quad \textit{determines an} \\ \textit{increasing } F \textit{ harmonic } S(U,|x|) \text{ on } A(a,b), \end{gathered} \tag{8.2}$$

then the function

$$U \textit{ is } F\textit{-maximal on } A. \tag{8.3}$$

Proof. — The hypothesis on U can be restated as the condition

$$S(U,t) \textit{is } R_F^{\uparrow} \textit{harmonic on } (a,b), \tag{8.2$'$}$$

by Theorem 2.7. By Proposition 3.3, $R_F^{\uparrow} = R_p^{\uparrow}$, so by Proposition 3.5 this proves that (8.2′) is equivalent to

$$S(U,t) = \Theta K(t) + c \quad \text{on } (a,b) \tag{8.2$''$}$$

for constants $\Theta \geqslant 0$ and $c \in \mathbb{R}$. Now by the homogeneity of S and K, this is equivalent to

$$\frac{S(U,t) - S(U,r)}{K(t) - K(r)} = \Theta \geqslant 0 \text{ for all } r < t \text{ in } (a,b) \tag{8.2$'''$}$$

for some constant $\Theta \geqslant 0$.

As in (8.1′) assume that v is F-subharmonic on A with $v \geqslant U$ and that outside a compact subset $K \subset A$ we have $v = U$. By the fundamental double monotonicity Theorem 6.4 we have that for $a < r < t < b$,

$$\frac{S(v,t) - S(v,r)}{K(t) - K(r)} \quad \text{is increasing in } r \text{ and } t. \tag{8.4}$$

Since $v = U$ outside K, this quotient equals Θ if both r and t are sufficiently close to a or sufficiently close to b. Hence, this quotient equals Θ for all $r < t$ in (a,b). That is, $S(v,t)$ satisfies (8.2‴). It follows that $S(v,t)$, in addition to $S(U,t)$, satisfies (8.2″). Therefore,

$$S(v,t) = S(U,t) + c \quad \forall\, t \in (a,b). \tag{8.5}$$

Taking t close to a shows that $c = 0$. Now the fact that $S(v,t) = S(U,t)$ for all $t \in (a,b)$ combined with the inequality $U \leqslant v$ implies that $U = v$ on A, thus proving that U is F-maximal on A. □

The following additional facts about F-maximal functions are standard in pluripotential theory, where $F = \mathcal{P}^{\mathbb{C}}$. The proofs easily adapt to the more general subequation F, but since these results are not part of the viscosity literature, we inlcude them for the convenience of the reader. Throughout the remainder of this section F is an arbitrary subequation, i.e., a closed set $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ which satisfies $F + \mathcal{P} \subset F$.

Proposition 8.3. — *If u is F-harmonic on X, then u is F-maximal on X.*

This is immediate since comparison holds for F (cf. [19, Thm. 6.4]). The only thing standing in the way of the converse is the continuity of u.

Example 8.4. — The subequation $F = \mathcal{P}^{\mathbb{C}}$ of pluripotential theory has many functions, such as $\log|z_1|$ on $\mathbb{C}^2$, which are maximal but not F-harmonic. In fact any function $u(z_1)$, which is Δ-subharmonic on a domain

$X_0 \subset \mathbb{C}$, when considered as a function $\overline{u}(z) \equiv u(z_1)$ on $X = X_0 \times \mathbb{C}^{n-1}$ with $n \geqslant 2$, is $\mathcal{P}^{\mathbb{C}}$-maximal. (If $v(z) \leqslant u(z_1)$ on $X - K$, then by applying the maximum principle to v on slices $z_1 =$ constant, we get $v(z) \leqslant \overline{u}(z)$ on X.) Now $\overline{u}(z) \equiv u(z_1)$ is $\mathcal{P}^{\mathbb{C}}$-harmonic if and only if u is continuous, however, u is not necessarily continuous even if it is bounded.

Proposition 8.5. — *If u is F-maximal and continuous on X, then u is F-harmonic on X.*

Proof. — This is the standard "bump-function" argument which occurs for example as far back as [2] or in [28]. It goes as follows. Suppose u is not F-harmonic but is F-maximal, and therefore F-subharmonic. Then $v \equiv -u$ is not $\widetilde{F}$-subharmonic. Therefore, by Lemma 2.4 in [17], there exist $x \in X$, $\epsilon > 0$ and a quadratic polynomial $Q(y)$ such that $v(y) < Q(y) - \epsilon|y-x|^2$ on $\overline{B_r(x)} - \{x\}$ with equality at $y = x$, but $D_x^2 Q \notin \widetilde{F}$, i.e., $-D_x^2 Q \in \operatorname{Int} F$. Thus, $w \equiv -Q + \delta$ is strictly F-subharmonic at x, and hence in a neighborhood $B_r(x)$. Pick $\delta > 0$ sufficiently small that $v(y) < Q(y) - \delta = -w(y)$ on $\partial B_r(x)$. Then $w(y) < u(y)$ on $\partial B_r(x)$, but $w(x) = u(x) + \delta$. This proves that u is not maximal. ☐

F-harmonic functions may not be closed under decreasing limits. For instance in Example 8.4 each $u(z_1)$ which is Δ-subharmonic is the decreasing limit of functions $u_j(z_1)$ which are smooth and Δ-subharmonic. The extensions $\overline{u}_j \to \overline{u}$ to $\mathbb{C}^n$ give an example for the case $F = \mathcal{P}^{\mathbb{C}}$.

This defect is corrected by enlarging the space of F-harmonic functions to the space of F-maximal functions. (This is the smallest such enlargement by Theorem 8.7 below.)

Proposition 8.6. — *If u is the decreasing limit of a sequence of F-maximal functions, then u is F-maximal.*

Proof. — Suppose $\{u_j\}$ are F-maximal and $u_j \downarrow u$ on an open set X. Fix a compact set $K \subset X$. Then $v \leqslant u$ on $X - K \Rightarrow v \leqslant u_j$ on $X - K \Rightarrow v \leqslant u_j$ on $X \Rightarrow v \leqslant u$ on X. ☐

This fact has a strong converse.

Theorem 8.7. — *If u is locally F-maximal, then u is locally the decreasing limit $u = \lim_{j\to\infty} u_j$ of F-harmonic functions u_j.*

The proof of this fact requires a lemma.

Lemma 8.8. — *Suppose u is F-subharmonic on X, $\Omega^{open} \subset\subset X$, and $v \in \operatorname{USC}(\overline{\Omega})$ is F-subharmonic on Ω. If $v \leqslant u$ on $\partial\Omega$, then*

$$\overline{v} \equiv \begin{cases} \max\{u, v\} & \text{on } \overline{\Omega} \\ u & \text{on } X - \overline{\Omega} \end{cases}$$

is F-subharmonic on X.

Proof. — Sup-convolution provides a decreasing sequence $u^\epsilon \downarrow u$ of continuous F-subharmonic functions which are defined on subdomains which contain $\overline{\Omega}$ and increase to X. Set

$$v^\epsilon_\delta \equiv \begin{cases} \max\{u^\epsilon + \delta, v\} & \text{on } \overline{\Omega} \\ u^\epsilon + \delta & \text{on } X - \overline{\Omega}. \end{cases}$$

Since $\{v < u^\epsilon + \delta\}$ is a relatively open subset of $\overline{\Omega}$ containing $\partial\Omega$, the function v^ϵ_δ is F-subharmonic on domains containing $\overline{\Omega}$ which increase to X as $\epsilon \downarrow 0$. Finally, $v^\epsilon_\delta \downarrow \overline{v}$ as $\epsilon, \delta \downarrow 0$, proving that $\overline{v}$ is F-subharmonic on X. $\square$

Using this Lemma 8.8, the definitions (8.1) and (8.1′) of F-maximality on X can be further refined as follows:

For each domain $\Omega \subset\subset X$ and $v \in \mathrm{USC}(\overline{\Omega})$ which is F-subharmonic on Ω,

$$v \leqslant u \ \text{ on } \partial\Omega \quad \Rightarrow \quad v \leqslant u \ \text{ on } \overline{\Omega}. \tag{8.1''}$$

Using this definition of F-maximality together with the fact that on balls $B \subset \mathbb{R}^n$ the Dirichlet problem is uniquely solvable by the Perron function, it is easy to prove Theorem 8.7.

Proof of Theorem 8.7. — Suppose u is maximal on X and $\overline{B} \subset X$ is a closed ball. Choose $\varphi_j \in C(\partial B)$ such that $\varphi_j \downarrow u\big|_{\partial\Omega}$. Let $u_j \in C(\overline{\Omega})$ denote the solution to the Dirichlet Problem on $\overline{B}$ with $u_j\big|_{\partial\Omega} = \varphi_j$ and u_j F-harmonic on B. Since u_j is the Perron function for boundary values φ_j, we have $u \leqslant u_j$ for all j and $u_j \downarrow v \equiv \lim_j u_j$ is decreasing. Thus $u \leqslant v$. Also $v\big|_{\partial B} = \lim u_j\big|_{\partial B} = \lim \varphi_j = u\big|_{\partial B}$, and v is F-subharmonic on B. Thus, by (8.1″) above, $v \leqslant u$ on $\overline{B}$. Hence, $u = v = \lim u_j$. $\square$

9. Tangents to Subharmonics

Now we come to the main topic of the paper: introducing the notion of tangents to F-subharmonics. In this section the ST-invariant cone subequation F on $\mathbb{R}^n$ is assumed to be convex. We shall work at a fixed point, which for simplicity is assumed to be the origin. That is, given an F-subharmonic function u defined in a neighborhood of 0, we define the notion of tangent functions to u at 0. A required clarification is given by Proposition 9.4. The basic properties of a tangent U to u at 0 are then established in Theorems 10.4 and 11.2.

Definition 9.1. — *Suppose that u is F-subharmonic on the ball about the origin of radius ρ. The* tangential p-flow *(or p-*homothety*) determined by the Riesz characteristic $p = p_F$ of F is defined as follows.*

(1) $u_r(x) = r^{p-2}u(rx)$ *if $p > 2$,*
(2) $u_r(x) = \frac{1}{r^{2-p}}\left[u(rx) - u(0)\right]$ *if $1 \leqslant p < 2$, and*
(3) $u_r(x) = u(rx) - M(u,r)$ *if $p = 2$.*

Remark 9.2. — Suppose $1 \leqslant p < 2$. Since $u(0) = M(u,0)$ is finite, some readers may prefer to assume once and for all in part (2) that $u(0) = 0$ so that the p-flows for all $p \neq 2$ are the same, namely that

$$u_r(x) = r^{p-2}u(rx) \qquad \text{if } p \neq 2. \tag{9.1}$$

Others may wish to make this assumption in the proofs.

Note that the functions u_r are F-subharmonic on $B_{\rho/r}$, and as $r \to 0$, these balls expand to $\mathbb{R}^n$.

An upper semi-continuous function $U(x)$ on $\mathbb{R}^n$ taking values in $[-\infty, \infty)$ is invariant under this flow if and only if there exists an u.s.c. function g on the unit sphere S such that

$$U(x) = |x|^{p-2} g\left(\frac{x}{|x|}\right) \quad \text{in the cases where } p \neq 2,$$

while in the case where $p = 2$, we leave it to the reader to prove that

$$U(x) = \Theta \log|x| + g\left(\frac{x}{|x|}\right) \quad \text{with } \sup_{S^{n-1}} g = 0 \text{ and } \Theta \geqslant 0 \text{ a constant.}$$

Functions of this form will be said to have *Riesz homogeneity p.*

Under our assumptions on F each F-subharmonic function u is L^1_{loc} since it is Δ-subharmonic by (6.3).

Definition 9.3 (Tangents). — *Suppose that u is an F-subharmonic function defined in a neighborhood of the origin. For each sequence $r_j \searrow 0$ such that*

$$\overline{U} \equiv \lim_{j\to\infty} u_{r_j} \ \textit{converges in } L^1_{\mathrm{loc}}(\mathbb{R}^n), \tag{9.2}$$

the point-wise defined function

$$U(x) \equiv \lim_{r\to 0} \operatorname*{ess\,sup}_{B_r(x)} \overline{U} \tag{9.3}$$

is called a tangent to U at 0. *We let $T_0(u)$ denote the set of all such tangents U. (We will refer to $\overline{U}$, satisfying* (9.2)*, as an* L^1_{loc}-tangent *when the distinction between the function U and the equivalence class of functions $\overline{U}$ is important.)*

Our first result clarifies this Definition.

Proposition 9.4. — *Each tangent U to u at 0 is an entire F-subharmonic function on $\mathbb{R}^n$. Moreover, U belongs to the L^1_{loc}-class $\overline{U} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and is the unique F-subharmonic function in this L^1_{loc}-class.*

To prove Proposition 9.4 we use the following result established in [18, Cor. 5.4] (see [20] for generalizations). We say that a subequation F *can be defined using fewer of the variables in* $\mathbb{R}^n$ if there exist an $(n-1)$-dimensional subspace $W \subset \mathbb{R}^n$ and a subequation $F' \subset \mathrm{Sym}^2(W)$ which determines F by: $A \in F \iff A\big|_W \in F'$.

An important point in the following result is that the same representative u of the L^1_{loc}-class $\overline{u}$ (given by (9.4)) is the correct representative, no matter which subequation F is being considered.

Theorem 9.5 (Distributional versus Viscosity Subharmonics). — *Suppose F is a convex cone subequation which cannot be defined using fewer of the variables in $\mathbb{R}^n$.*

(1) *If u is F-subharmonic in the viscosity sense, then u is L^1_{loc} and F-subharmonic in the distributional sense.*
(2) *If $\overline{u}$ is an F-subharmonic distribution, then $\overline{u} \in L^1_{\mathrm{loc}}$ and the limit*

$$u(x) = \lim_{r\to 0} \operatorname*{ess\,sup}_{B_r(x)} \overline{u} \qquad \textit{exists at each point} \tag{9.4}$$

and defines an upper semi-continuous function u in the L^1_{loc}-class $\overline{u}$ which is F-subharmonic in the viscosity sense. Moreover, u is the unique such representative of $\overline{u}$.

Remark 9.6. — We refer the reader to Sections 3, 4 and 5 of [18] for a fuller discussion of this result and the definition of an F-subharmonic distribution (Definition 4.1 and Proposition 4.3). However, the terminology used in [18] is somewhat different. Here we use the terminology employed in [20]. In [18] a convex cone subequation F is called a “positive cone” and denoted $\mathcal{P}^+$. The polar cone is denoted by $\mathcal{P}_+$. A convex cone subequation which cannot be defined using fewer of the variables in $\mathbb{R}^n$ is called an “elliptic cone”.

From the distributional point of view it is straightforward to see that averages, or more generally convolution, of an F-subharmonic function u with any non-negative measure is again F-subharmonic.

Proof of Proposition 9.4. — We use these facts about the ST-invariant convex cone subequation F:

(1) $F \subset \Delta$;
(2) $1 \leqslant p_F \leqslant n$;
(3) F cannot be defined using fewer of the variables in $\mathbb{R}^n$.

Properties (1) and (2) have already been noted in (6.3) and (6.4). For Property (3) note that the ST-invariance of F rules out the possibility that F could be defined using fewer of the variables in $\mathbb{R}^n$. Because of (3) one can apply Theorem 9.5.

Suppose $\overline{U} = \lim_{j\to\infty} u_{r_j}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is an L^1_{loc} tangent to u at 0. Since F is a cone, each u_r is viscosity F-subharmonic, and hence in L^1_{loc} and distributionally F-subharmonic by Part (1) of Theorem 9.5. Hence, in the limit, $\overline{U}$ is distributionally F-subharmonic. Now apply Part (2) of Theorem 9.5 to $\overline{U}$ to complete the proof. □

In light of Proposition 9.4 we frequently drop the distinction between U and $\overline{U}$.

10. Uniqueness of Averages of Both Tangents and of Flows

Most of the properties of tangents can be deduced from the following result, which proves that averages of tangents are always unique by showing that they are radial harmonics.

Theorem 10.1 (Averages of Tangents). — *Suppose that u is an F-subharmonic function defined in a neighborhood of the origin in $\mathbb{R}^n$. Let $p = p_F$ be the Riesz characteristic of F.*

If $p \neq 2$, then each tangent U to u at 0 has averages

$$\begin{aligned} M(r) &\equiv \sup_S U(r\sigma) = \Theta^M(u)K(r), \\ S(r) &\equiv \fint_S U(r\sigma)\,\mathrm{d}\sigma = \Theta^S(u)K(r), \\ \text{and } V(r) &\equiv \fint_B U(rx)\,\mathrm{d}x = \Theta^V(u)K(r). \end{aligned} \tag{10.1}$$

In particular,

$$\Theta^\Psi(U) = \Theta^\Psi(u) \qquad \text{for } \Psi = M, S, \text{ or } V. \tag{10.2}$$

When $p = 2$, all the densities of u and any tangent U to u at 0, agree, and will be simply denoted by $\Theta = \Theta(u)$. Specifically, we have

$$\Theta(u) = \Theta^M(U) = \Theta^S(U) = \Theta^V(U) = \Theta^M(u) = \Theta^S(u) = \Theta^V(u). \quad (10.3)$$

Moreover, the averages of a tangent U to u are given by

$$M(r) = \Theta \log r, \quad S(r) = \Theta \log r + \fint_S U, \quad \text{and } V(r) = \Theta \log r + \fint_B U, \quad (10.4)$$

with

$$-C\Theta \leqslant \fint_S U \leqslant 0 \ \text{ and } \ -(C+1)\Theta \leqslant \fint_B U, \ \text{ where } C = \frac{1}{\varphi\left(\frac{1}{e}\right)} > 1, \quad (10.5)$$

and where $\varphi(\lambda) = (1-\lambda)/(1+\lambda)^{n-1}$.

When $p \neq 2$, these formulas show that any two tangents have the same maxima $M(r)$ and the same spherical averages $S(r)$ and volume averages $V(r)$, all being the appropriate density times $K(r)$. When $p = 2$, $M(r)$, $S(r)$ and $V(r)$ all agree with $\Theta \log r$ modulo an additive constant, but the constant depends on the tangent U, not just on u.

In all cases, for each tangent U, the function $S(U, |x|)$ is F-harmonic on $\mathbb{R}^n - \{0\}$ since $\Theta K(|x|) + C$ is F-harmonic there (Proposition 3.5).

Combining Theorem 8.2 and Theorem 10.1 is one of the main ingredients of the paper and has the following immediate consequence.

Theorem 10.2. — *Every tangent to an F-subharmonic function is F-maximal.*

Applying Proposition 8.5 yields the following.

Corollary 10.3. — *Every continuous tangent to an F-subharmonic function is F-harmonic.*

Theorem 10.1, the uniqueness of averages of tangents, follows from the stability of averages (Lemma 6.5) and the uniqueness of the averages of a flow. Its proof is given at the end of this section.

We may assume that $u(0) = 0$ if $1 \leqslant p < 2$ (Remark 9.2), and that $u(0) = -\infty$ if $2 \leqslant p < \infty$.

Theorem 10.4 (Averages of Flows). — *For $p \neq 2$ and $\Psi = M, S$ or V,*

$$\lim_{s \downarrow 0} \Psi(u_s, r) = \Theta^\Psi(u) K(r). \quad (10.6)$$

For $p=2$, if $\Psi = M$ we also have

$$\lim_{s\downarrow 0} M(u_s, r) = \Theta^M(u)K(r) = \Theta^M(u)\log r. \tag{10.6a}$$

In this case the limit is decreasing and uniform in $r \leqslant R$. For $\Psi = S$ or V we have

$$\liminf_{s\downarrow 0} S(u_s, r) \geqslant \Theta^M(u)(\log r - C), \tag{10.6b}$$

$$\text{and} \quad \liminf_{s\downarrow 0} V(u_s, r) \geqslant \Theta^M(u)(\log r - C - 1) \tag{10.6c}$$

with C as in (10.5).

Direct calculations from the definitions of the flow and the averages establish the next result.

LEMMA 10.5. — *For $\Psi = M, S,$ or V:*

$$\Psi(u_s, r) = s^{p-2}\Psi(u, sr) = \frac{\Psi(u, sr)}{K(sr)}K(r) \qquad \text{if } p \neq 2, \tag{10.7}$$

$$\Psi(u_s, r) = \Psi(u, sr) - M(u, s) = \frac{\Psi(u, sr) - M(u, s)}{K(sr) - K(s)}K(r) \quad \text{if } p = 2. \tag{10.8}$$

Proof. — For example, when Ψ is the volume average V and $p \neq 2$, we have

$$V(u_s, r) = \frac{1}{|B|}\int_B u_s(rx)\,\mathrm{d}x = \frac{s^{p-2}}{|B|}\int_B u(rsx)\,\mathrm{d}x = s^{p-2}V(u, rs).$$

The remaining calculations are left to the reader. $\square$

Proof of Theorem 10.4. — By Lemma 5.5, the identity (10.7) implies (10.6) for $p \neq 2$. In the case where $p = 2$ the limit (10.6a) for the maximum follows from (10.8) by the double monotonicity Theorem 6.4. The limit (10.6c) for V follows from the limit (10.6c) for S since $V(u_s, r) = n\int_0^1 S(u_s, t)\, t^{n-1}\,\mathrm{d}t$ by (6.7), and $n\int_0^1 (\log rt - C)\, t^{n-1}\, dt = \log r - C - 1$.

It remains to prove (10.6b). Harnack's inequality in the form (7.10) with $v = u_s$ and $\lambda = 1/e$ states that

$$C\left(M\left(u_s, \tfrac{r}{e}\right) - M(u_s, r)\right) + M(u_s, r) \leqslant S(u_s, r).$$

We know the limit of the terms involving M as $s \downarrow 0$. This gives

$$C\Theta^M(u)\left(\log \tfrac{r}{e} - \log r\right) + \Theta^M(u)\log r \leqslant \liminf_{s\downarrow 0} S(u_s, r)$$

as desired. $\square$

Proof of Theorem 10.1. — The density statements for u are contained in Propositions 7.1 and 7.2. The density statements for U follow from the formulas in Theorem 10.1 and the density statements for u. The formulas in Theorem 10.1 follow immediately from the formulas in Theorem 10.4 for the averages of flows and the stability of averages (Lemma 6.5), with the exception of (10.4) for S and V, and the estimates in (10.5).

The estimates (10.6b) and (10.6c) and the Stability Lemma 6.5 show that for any tangent U to u at 0,

$$\Theta^M(u)\,(\log r - C) \leqslant S(U,r) \quad \text{and} \quad \Theta^M(u)\,(\log r - C - 1) \leqslant V(U,r)$$

for all $0 < r < \infty$. Also we have that $V(U,r) \leqslant S(U,r) \leqslant M(U,r) = \Theta^M(u)\log r$.

Since $V(U,e^t)$ and $S(U,e^t)$ are entire convex functions of t, the linear inequalities

$$\Theta(t-C) \leqslant S(U,e^t) \leqslant \Theta t \quad \text{and} \quad \Theta(t-C-1) \leqslant V(U,e^t) \leqslant \Theta t$$

imply that $S(U,e^t) = \Theta(t+k)$ and $V(U,e^t) = \Theta(t+k'-1)$ where k and k' satisfy $-C \leqslant k, k' \leqslant 0$. $\square$

11. Existence of Tangents

We now address the basic existence question. Again F is assumed here to be convex. However, in the case where $1 \leqslant p < 2$ much stronger results are true even if F is just a cone and not necessarily convex. These stronger results are established in Section 15.

Theorem 11.1 (Existence of Tangents). — *Suppose that u is F-subharmonic on a ball B_ρ. For each $R > 0$ there exists $\delta > 0$ such that the family $\{u_r\}_{0<r\leqslant\delta}$ is unformly bounded above and bounded in norm in $L^1(B_R)$. In particular, the set $\{u_r\}_{0<r\leqslant\delta}$ is precompact in $L^1(B_R)$.*

Proof. — An upper bound for u can be chosen to be any number greater than $\Theta^M(u)K(R)$ by (10.6) if $p \neq 2$ and by (10.6a) if $p = 2$. Consequently the boundedness in $L^1(B_R)$ is equivalent to a lower bound for $V(u_s, R)$ which is uniform in s. This lower bound can be chosen to be any number less than $\Theta^V(u)K(R)$ if $p \neq 2$, or $\Theta^M(u)(\log R - C - 1)$ if $p = 2$, by (10.6) and (10.6c) respectively in Theorem 10.4. $\square$

The basic properties of the tangent set $T_0(u)$ are contained in the following theorem. Again see Section 15 for the stronger versions of parts (2) and (4) using the Hölder topology instead of the L^1_{loc}-topology when $1 \leqslant p < 2$.

THEOREM 11.2. — *Suppose that* u *is an* F*-subharmonic function defined in a neighborhood of the origin in* $\mathbb{R}^n$*. Then the tangent set* $T_0(u)$ *to* u *at* 0 *satisfies:*

(1) $T_0(u)$ *is non-empty.*
(2) $T_0(u)$ *is a compact subset of* $L^1_{\rm loc}(\mathbb{R}^n)$*.*
(3) $T_0(u)$ *is invariant under the homothety* $U \to U_r$*.*
(4) $T_0(u)$ *is a connected subset of* $L^1_{\rm loc}(\mathbb{R}^n)$*.*

Proof. — Parts (1) and (2) are immediate from Theorem 11.1. The arguments for parts (3) and (4) are given in [38, Prop. 1.1.1]. We include them here for completeness. To prove (3) note that $U(x) = \lim_{r_j \downarrow 0} u_{r_j}(x)$ implies $U_r(x) = \lim_{s_j \downarrow 0} u_{s_j}(x)$ with $s_j = rr_j$. To prove (4) suppose $u_{r_j} \to U_0$ and $u_{t_j} \to U_1$ with U_0 and U_1 elements of disjoint open sets N_0 and N_1 which cover $T_0(u)$. We can assume $r_j < t_j$ for all j and choose s_j between r_j and t_j with $u_{s_j} \notin N_0 \cup N_1$. (Note that $s \mapsto u_s$ is a continuous map into $L^1_{\rm loc}$.) By Theorem 11.1 the sequence u_{s_j} has a convergent subsequence, and its limit is in neither N_0 nor N_1, a contradiction. $\square$

12. Uniqueness of Tangents

In this section we discuss some basic situations where tangents are unique. Our main uniqueness results are are stated and proved in subsequent sections. As in Sections 9–11 we assume that F is convex with finite Riesz characteristic p.

DEFINITION 12.1. — *Suppose* u *is an* F*-subharmonic function defined in a neighborhood of the origin.*

(1) *If* $T_0(u) = \{U\}$ *is a singleton, then we say that* uniqueness of tangents holds for u*. If uniqueness of tangents holds for all such* u*, we say the that* uniqueness of tangents holds for F.
(2) *If* $T_0(u) = \{\Theta K(|x|)\}$ *with* $\Theta \geqslant 0$ *a constant, then we say that* strong uniqueness of tangents holds for u*. If strong uniqueness of tangents holds for all such* u*, then we say that* strong uniqueness of tangents holds for F.
(3) *If every tangent* U *to* u *satisfies* $U_r = U\ \forall\, r$*, then we say that* homogeneity of tangents holds for u*. If homogeneity of tangents holds for all such* u*, then we say that* homogeneity of tangents holds for F.

Now (2) $\Rightarrow$ (1) $\Rightarrow$ (3). The first implication is obvious. For the second, note that (1) can be rephrased since

$$T_0(u) = \{U\} \quad \iff \quad \lim_{r \to 0} u_r \text{ exists in } L^1_{\rm loc}(\mathbb{R}^n) \text{ and equals } U. \tag{12.1}$$

Thus by (1), u_{r_j} and u_{rr_j} have the same limit U, but u_{rr_j} has limit U_r, which proves (3).

In general, $S(u,r) \leqslant M(u,r)$. Therefore,

$$\text{For } 2 \leqslant p \leqslant n, \quad \Theta^M(u) \leqslant \Theta^S(u), \qquad \text{and for } 1 \leqslant p < 2, \quad \Theta^S(u) \leqslant \Theta^M(u) \quad (12.2)$$

by (5.7) since $K > 0$ in the first case and $K < 0$ in the second case. However, if strong uniqueness holds for u, then all densities "agree" because of Proposition 7.1 and the following.

$$\text{If for some } \Theta \geqslant 0, T_0(u) = \{\Theta K\}, \text{ then } \Theta^M(u) = \Theta^S(u) = \Theta. \qquad (12.3)$$

This follows from (10.2) and the fact that $\Theta^M(K) = \Theta^S(K) = 1$.

There are two classical cases where strong uniqueness holds, that will prove useful later. For the sake of completeness we include proofs.

Proposition 12.2 (Radial Subharmonics). — *Suppose that $u(x)=f(|x|)$ is a radial F-subharmonic function defined on a neighborhood of 0. Then*

$$\lim_{r\to 0} u_r = \Theta(u) K_p(|x|)$$

in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and uniformly on compact subsets in $\mathbb{R}^n - \{0\}$. Thus, $T_0(u) = \{\Theta K_p\}$.

Proof. — Since u is radial, we have that $u_r(x) = M(u_r, |x|)$, but by Theorem 10.4 we know that $\lim_{r\downarrow 0} M(u_r, |x|) = \Theta K_p(|x|)$ uniformly in $0 < |x| \leqslant R$. □

Remark 12.3. — The conclusion of convergence in $C(\mathbb{R}^n - \{0\})$ only requires F to be an ST-invariant cone subequation with finite characteristic. It does not require convexity.

Proposition 12.4 (Newtonian Case). — *Suppose u is a Δ-subharmonic function defined on a neighborhood of 0. Then*

$$\lim_{r\to 0} u_r(x) = -\frac{\Theta(u)}{|x|^{n-2}} \qquad \textit{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \ \textit{ when } n \geqslant 3,$$

$$\textit{and} \quad \lim_{r\to 0} u_r(x) = \Theta(u) \log|x| \qquad \textit{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \ \textit{ when } n = 2.$$

Proof. — Each such u is of the form $u = v + h$ where $v = K * v$ is a Newtonian potential and h is harmonic near the origin. (Take the measure ν to be a cut-off of the measure $\mu = \Delta u$ in a small ball about the origin.) This reduces the proof to the case $v \equiv K * \nu$. (In the $n = 2$ case u_r and $v_r + h_r$ differ by $M(v,r) + M(h,r) - M(u,r)$, but this error has limit zero.)

Now one checks that: for $n \geqslant 3$, $(K * \nu)_r = K * ((\frac{1}{r})_* \nu)$ and for $n = 2$, $(K * \nu)(rx) = K * ((\frac{1}{r})_* \nu)(x) + \nu(1) \log r$, so that $M(K * \nu, r) = M(K * (\frac{1}{r})_* \nu, 1) + \nu(1) \log r$. Now $\lim_{r \to 0} (\frac{1}{r})_* \nu$ always exists weakly in the space of measures and equals $\Theta[0]$, where $\Theta = \lim_{r \to 0} \nu(B_r)$ is the zero-dimensional density of ν at 0. Since $K \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the limit of $(K * \nu)_r$ exists in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and equals $K * (\Theta[0]) = \Theta K$. (Note that for $n = 2$, $M(K * (\frac{1}{r})_* \nu, 1)$ has limit $M(\Theta \log |x|, 1) = 0$.)

In the $n = 2$ case there is a different proof following Kiselman [29]. Note that by (10.4) we have $M(U, r) = \Theta \log r$ for any tangent U to u at 0. In particular, $U(x) - \Theta \log |x|$ is $\leqslant 0$ on $\mathbb{R}^2$ and Δ-subharmonic on $\mathbb{R}^2 - \{0\}$. Hence, it can be extended to $\mathbb{R}^2$ as a subharmonic function, and then by Liouville's Theorem it must be constant. Since $M(u_r, 1) = 0$ for all r small, $M(U, r) = 0$, proving that the constant is zero. □

Proposition 12.4 can be partly generalized.

Proposition 12.4′ (Riesz Potentials, $p > 2$). — *Suppose $u = K_p * \nu$ where $\nu \geqslant 0$ is a compactly supported measure. Then*

$$\lim_{r \to 0} u_r = -\frac{\Theta(\nu)}{|x|^{p-2}} \qquad \text{in } \ L^1_{\mathrm{loc}}(\mathbb{R}^n)$$

where, up to a universal constant, $\Theta(\nu) = \lim_{r \to 0} \nu(B_r)$.

Proof. — Ignoring constants, we have (cf. [36])

$$\Delta u = (\Delta K_p) * \nu = K_{p+2} * \nu \equiv \mu .$$

Note that

$$K_n * \mu = K_n * K_{p+2} * \nu = K_p * \nu = u .$$

We compute that

$$u_r(x) = r^{p-2} u(rx) = r^{p-2} (K_p * \nu)(rx) \text{ is equal to } K_p * \left\{ \left(\tfrac{1}{r}\right)_* \nu \right\},$$

and observe that $\lim_{r \downarrow 0} \left(\frac{1}{r}\right)_* \nu = \Theta(\nu)[0]$. □

We complete this section with a final case where strong uniqueness holds.

Proposition 12.5 (Zero Density). — *Suppose that u is F-subharmonic in a neighborhood of the origin and F is convex with $p > 1$. If any of the densities of u is zero at 0, then all the densities of u vanish at 0, and in this case*

$$\lim_{r \to 0} u_r = 0 \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n). \tag{12.4}$$

If F is not convex but $1 \leqslant p < 2$, then $\Theta^M(u, 0) = 0$ implies that

$$\lim_{r \to 0} u_r = 0 \quad \text{locally in } \alpha \text{ Holder norm}, \ \alpha = 2 - p. \tag{12.5}$$

Proof. — The equality of zero densities is a direct consequence of Propositions 7.1 and 7.2, while (12.4) follows from Theorem 10.4.

The proof of the final assertion of Proposition 12.5 is postponed as it follows immediately from (15.11). ☐

13. The Strong Uniqueness Theorem I

In this section we give two proofs of one of our two main results concerning strong uniqueness. Recall that every $\mathrm{O}(n)$-invariant subequation F has complex and quaternionic analogues $F^{\mathbb{C}}$ and $F^{\mathbb{H}}$, which are invariant under $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ respectively (see Example 4.7).

Theorem 13.1. — *Suppose that F is $O(n)$-invariant and convex with finite Riesz characteristic p. Then, except for the case $F = \mathcal{P}$, strong uniqueness of tangents holds for F. Furthermore, except for the cases $\mathcal{P}^{\mathbb{C}}$ and $\mathcal{P}^{\mathbb{H}}$, strong uniqueness of tangents also holds for the complex and quaternionic analogues $F^{\mathbb{C}}$ and $F^{\mathbb{H}}$ of F.*

Remark 13.2. — For the subequations $\mathcal{P}, \mathcal{P}^{\mathbb{C}}$ and $\mathcal{P}^{\mathbb{H}}$, strong uniqueness fails dramatically. Nonetheless, tangents are classified in these cases. This is discussed in Part II of this paper.

Proof. — Let u be F-subharmonic in a neighborhood of the origin and choose $U \in T_0(u)$. Then

$$U(x) = \lim_{j\to\infty} u_{r_j}(x)$$

for a sequence $r_j \downarrow 0$, where the flow $u_{r_j}(x)$, given in Definition 9.1, depends on p.

Theorem 10.2 states that

$$U \in T_0(u) \quad \Rightarrow \quad U \text{ is } F\text{-maximal on } \mathbb{R}^n - \{0\}, \tag{13.1}$$

and

$$U \in T_0(u) \text{ and } U \in C(\mathbb{R}^n - \{0\}) \;\Rightarrow\; U \text{ is } F\text{-harmonic on } \mathbb{R}^n - \{0\}. \tag{13.2}$$

We first prove the theorem under the additional assumption that F is uniformly elliptic. (Note, however, from Section 4 that there many examples of subequations F which are not uniformly elliptic, but for which the theorem still applies.)

Proposition 13.3. — *If, in addition to the hypotheses of Theorem 13.1, F is uniformly elliptic, then strong uniqueness of tangents holds for F.*

Proof of Proposition 13.3. — Two regularity results are needed for F. They can be stated as follows.

Fact 13.4. — A sequence $\{u_j\}$ of F-harmonics on $X^{\text{open}} \subset \mathbb{R}^n$, which is bounded in $L^\infty(K)$ for each compact $K \subset X$, is precompact in $C(X)$.

Fact 13.5. — Each F-harmonic function is C^1.

The reader is referred to [4] and [41] for these results. Also, for Fact 13.4, one can use the Krylov–Safanov Hölder Estimate 4 in [8] which holds with $f = 0$ because of the First Linearization on p. 107.

Recall that F is assumed to be invariant under a subgroup $G \subseteq \mathrm{O}(n)$ which acts transitively on S^n.

Lemma 13.6.

(1) *Suppose $U \in T_0(u)$. Then $g^*U \in T_0(g^*u)$ for each $g \in G$, and the densities $\Theta^S(g^*U) = \Theta^S(U) = \Theta^S(u) = \Theta^S(g^*u)$ are all equal.*
(2) *If $U \in T_0(u)$ and $V \in T_0(v)$, then* $\max\{U, V\} \in T_0(\max\{u, v\})$.
(3) *If $U \in T_0(u)$ and $g \in G$, then* $\max\{U, g^*U\} \in T_0(\max\{u, g^*u\})$.

The straightforward proofs are omitted.

The proof of Proposition 13.3 will progress in three stages. First we establish strong uniqueness for continuous tangents, then for tangents which are locally bounded, and finally for general tangents.

The proof that $U = \Theta K_p$ for $U \in C(\mathbb{R}^n - \{0\})$ is as follows. Note that for $g \in G$, $\max\{U, g^*U\} \in C(\mathbb{R}^n - \{0\})$, and therefore by Lemma 13.6 and (13.2),

$$\max\{U, g^*U\} \text{ is } F\text{-harmonic on } \mathbb{R}^n - \{0\} \text{ for each } g \in G. \tag{13.3}$$

By the C^1-regularity result Fact 13.5 we have that

$$\max\{U, g^*U\} \text{ is } C^1 \text{ on } \mathbb{R}^n - \{0\} \text{ for each } g \in G. \tag{13.4}$$

Although the maximum of two F-subharmonics is always subharmonic, it is unusual for the maximum of two distinct F-harmonics to be F-harmonic. In fact we have the following.

Lemma 13.7. — *Let f be a function on the unit sphere in $S \subset \mathbb{R}^n$ with the property that $\max\{f, g^*f\} \in C^1(S)$ for all $g \in G$. Then $f =$ constant.*

Proof of Lemma 13.7. — We begin with the case $G = \mathrm{O}(n)$. If we can prove constancy on every great circle in S^{n-1}, we are done. So we are immediately reduced to the case $n = 2$. Lifting to the covering $\mathbb{R} \to S^1$ we are then reduced to the following elementary fact:

Fact 13.8. — Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function with the property that for all $a \in \mathbb{R}$, the function $F_a(x) \equiv \max\{f(x), f(x+a)\}$ is differentiable. Then $f \equiv$ constant.

We see this as follows. If f is not constant, there exists a point x with $f'(x) > 0$. Since it is periodic, there must also exist a point y with $f'(y) < 0$. Set $a = y - x$. Then the left hand derivative of F_a at x is < 0 (if it exists), and the right hand one is > 0. This completes the argument for $G = \mathrm{O}(n)$.

Consider now the general case of a closed subgroup $G \subset \mathrm{O}(n)$. Fix $x \in S^{n-1}$ and decompose the Lie algebra as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}$ (orthogonal with respect to the Killing form of $\mathfrak{so}(n)$), where $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{so}(n-1)$ is the Lie algebra of the subgroup $K \equiv \{g \in G : g(x) = x\}$. Now the differential of the G-action at x gives an isomorphism $\mathfrak{g} \cong T_x(S^{n-1})$, and for every 1-parameter subgroup $\varphi_t \subset G$ generated by an element of $\mathfrak{g}$, the orbit is a great circle. The argument made above for $\mathrm{O}(n)$ now applies, and Lemma 13.7 is proved. □

Taken together, these two lemmas prove that the punctured harmonic $U(x)$ is radial (constant on spheres about the origin). Therefore, by Proposition 3.5, $U = \Theta K + C$, and by (10.1), $C = 0$. This completes the proof of Proposition 13.3 if $U \in C(\mathbb{R}^n - \{0\})$.

For the next step we establish the following strengthening of Proposition 8.5 which reduces the case $U \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n - \{0\})$ to the case $U \in C(\mathbb{R}^n - \{0\})$.

Proposition 13.9. — *Suppose F is uniformly elliptic. Then each locally bounded F-maximal function is F-harmonic.*

Proof of Proposition 13.9. — Suppose u is an F-maximal L^∞_{loc}-function on a domain $X \subset \mathbb{R}^n$. By Theorem 8.7 for any compact set $K \subset X$, u is the decreasing limit of a sequence $\{u_j\}_j$ of F-harmonic functions on a neighborhood of K. By Fact 13.4, the limit u is continuous, and hence F-harmonic by Proposition 8.5. □

This completes the second stage of the proof of Proposition 13.3 where $U \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n - \{0\})$. It remains to prove the last stage where U is a general tangent.

By Lemma 13.6 (2), for each $N > 0$ we have $U^N \equiv \max\{U, NK_p\} \in T_0(\max\{u, NK_p\})$. Since $U^N \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n - \{0\})$, U^N is a multiple of K_p. We now observe that U^N decreases down to U as $N \to \infty$. Hence, if each U^N is a multiple of the Riesz kernel, then so is U. This completes the proof of Proposition 13.3 □

The last result needed for the proof of Theorem 13.1 in the $\mathrm{O}(n)$-invariant case is the following proposition, which reduces the case of our general F

of characteristic p, to a specific maximal such equation, which is uniformly elliptic.

Proposition 13.10. — *The subequation*

$$\mathcal{P}_p^{largest} \stackrel{\text{def}}{=} \left\{ A : A + \frac{p-1}{n-p}(\mathrm{tr}A)I \geqslant 0 \right\}$$

contains all the $O(n)$-invariant convex *cone subequations F of Riesz characteristic p, and has Riesz characteristic p itself. Since*

$$\mathcal{P}_p^{\mathrm{largest}} = \mathcal{P}(\delta) \qquad \textit{with } \delta = \frac{(p-1)n}{n-p}$$

(see Example 4.3), the subequation $\mathcal{P}_p^{\mathrm{largest}}$ is uniformly elliptic when $p > 1$.

Proof. — Suppose $A = \lambda_1 P_{e_1} + \cdots + \lambda_n P_{e_n}$ is in diagonal form with $\lambda_1 \leqslant \cdots \leqslant \lambda_n$. Then by definition (4.5) we know that

$$A \notin \mathcal{P}(\delta) \quad \iff \quad \langle A, P_{e_1} + \tfrac{\delta}{n} I \rangle = \lambda_1 + \tfrac{\delta}{n}\,(\lambda_1 + \cdots + \lambda_n) < 0\,.$$

If $\mu' = \pi(\lambda')$ is a permutation of $\lambda' = (\lambda_2, \ldots, \lambda_n)$, then $A_\pi \equiv \lambda_1 P_{e_1} + \mu_2 P_{e_2} + \cdots + \mu_n P_{e_n}$ also belongs to the open half-space H defined by $\langle A, P_{e_1} + \frac{\delta}{n} I \rangle < 0$, and H is disjoint from $\mathcal{P}(\delta)$. Averaging A over these permutations yields $B \equiv \lambda_1 P_{e_1} + \frac{\Sigma}{n-1} p_{e_1^\perp}$ where $\Sigma \equiv \lambda_2 + \cdots + \lambda_n$. Since $B \in H$ we have $B \notin \mathcal{P}(\delta)$. Hence setting $e \equiv e_1$ and using the fact that $\mathcal{P}(\delta)$ is a cone, we can rescale to obtain $B' \equiv P_{e^\perp} - (p'-1)P_e \notin \mathcal{P}(\delta)$. Since the characteristic of $\mathcal{P}(\delta)$ is equal to p, this proves that $p' > p$.

Now if $A \in F$, then since F is $\mathrm{O}(n)$-invariant and convex, the average $B \in F$. Finally since F is a cone, $B' \in F$. Since $p' > p$, this proves that F has Riesz characteristic $> p$, contrary to assumption. $\square$

Proposition 13.10 says that if U is a tangent to an F-subharmonic function, where F satisfies the hypotheses, then U is $\mathcal{P}_p^{\mathrm{largest}}$-tangent. Since the subequation $\mathcal{P}_p^{\mathrm{largest}}$ is uniformly elliptic, Proposition 13.3 applies, which completes the proof of Theorem 13.1 in the orthogonally invariant case.

Remark 13.11. — Some (in fact, many) readers may be uncomfortable with the assertion that $\mathcal{P}(\delta)$-harmonics have the regularity of viscosity solutions to equations which are convex and uniformly elliptic in the conventional sense. A discussion of this point is given in Appendix B.

Consider now the complex analogue $F^{\mathbb{C}}$ of F on $\mathbb{C}^n$. Then we have $F^{\mathbb{C}} \subset \mathcal{P}^{\mathbb{C}}(\delta)$, the complex analogue of the subequation defined in Proposition 13.10. Now for any $A \in \mathrm{Sym}^2_{\mathbb{R}}(\mathbb{C}^n)$ one has that $\mathrm{tr}(A) = 2\mathrm{tr}_{\mathbb{C}}(A_{\mathbb{C}})$ and $\lambda_1(A) \leqslant \lambda_1^{\mathbb{C}}(A_{\mathbb{C}})$. Hence, $\mathcal{P}(\frac{\delta}{2}) \subset \mathcal{P}^{\mathbb{C}}(\delta)$ as subsets of $\mathrm{Sym}^2(\mathbb{R}^{2n}) = \mathrm{Sym}^2_{\mathbb{R}}(\mathbb{C}^n)$. It follows that $\mathcal{P}^{\mathbb{C}}(\delta)$ is uniformly elliptic (for $p > 1$). The arguments given above now go through to establish the theorem in this case.

The case of the quaternionic analogue $F^{\mathbb{H}}$ is proved in exactly the same way. This completes the proof of Theorem 13.1. $\square$

For the interested reader we present a second argument for Theorem 13.1 where the passage from maximal to harmonic is based on regularization via the group G, a technique which is discussed, for example, in [27].

A Slightly Different Proof of Theorem 13.1. — Let u be F-subharmonic in a neighborhood of the origin and choose $U \in T_0(u)$. For clarity of exposition we work in the case $p > 2$. Then

$$U(x) = \lim_{j\to\infty} r_j^{p-2} u(r_j x)$$

for a sequence $r_j \downarrow 0$. Let $\chi = \chi_\epsilon : G \to [0, \infty)$ be a family of smooth functions converging to the δ-function at the identity in G, and for any function f which is L^1_{loc} in $\mathbb{R}^n - \{0\}$ and in $L^1(S^{n-1}(r))$ for all r, define

$$f^\epsilon(x) \equiv \int_G f(gx)\chi(g)\,\mathrm{d}g$$

where $\mathrm{d}g$ is Haar measure with unit volume on G. The following lemma is proved below.

Lemma 13.12.

$$U^\epsilon(x) = \lim_{j\to\infty} r_j^{p-2} u^\epsilon(r_j x).$$

Now by the Fubini Theorem, U^ϵ satisfies

$$\begin{aligned} S(U^\epsilon, r) &= \int_{|x|=1} U^\epsilon(rx)\,\mathrm{d}x = \int_{|x|=1} \left\{\int_G U(grx)\chi(g)\,\mathrm{d}g\right\} \mathrm{d}x \\ &= \int_G \left\{\int_{|x|=1} U(rgx)\,\mathrm{d}x\right\} \chi(g)\,\mathrm{d}g = \int_G S(U,r)\,\chi(g)\,\mathrm{d}g \\ &= S(U,r) = \Theta^S K(r). \end{aligned}$$

From this we conclude that U^ϵ is maximal by Theorem 8.2. The next lemma is also proved below.

Lemma 13.13. — *U^ϵ is continuous and converges to U in $L^1_{\mathrm{loc}}(\mathbb{R}^n - \{0\})$ as $\epsilon \to 0$.*

Note that the continuity of U^ϵ implies that it is F-harmonic (Proposition 8.5).

We now fix $g_0 \in G$ and define

$$V^\epsilon(x) \equiv U^\epsilon(g_0 x) = \lim_{j\to\infty} r_j^{p-2} u^\epsilon(r_j g_0 x)$$

where the second equality comes from Lemma 13.12. Clearly V^ϵ is a tangent, and it satisfies $S(V^\epsilon, r) = S(U^\epsilon, r) = \Theta^S K(r)$. In particular, V^ϵ is also maximal. Furthermore, note that

$$\max\{U^\epsilon(x), V^\epsilon(x)\} \equiv \lim_{j\to\infty} r_j^{p-2} \max\{u^\epsilon(r_j x), u^\epsilon(r_j g_0 x)\}$$

is also a tangent and hence maximal. We have proved the following.

Proposition 13.14. — *For all $g \in G$ and all $\epsilon > 0$ the function $\max\{U^\epsilon, g^* U^\epsilon\}$ is F-harmonic.*

As in the first proof we now apply elliptic regularity and Lemma 13.7 to conclude that each function $\max\{U^\epsilon, g^* U^\epsilon\}$ is C^1, and therefore that U^ϵ is constant on each sphere. Then by Corollary 10.3 U^ϵ is an increasing radial harmonic and therefore a multiple of the Riesz kernel. Since $U^\epsilon \to U$ in L^1_{loc}, we conclude that $U = \Theta^S(u) K(|x|)$. This completes our second proof in the orthogonally invariant case. Arguments for the complex and quaternionic analogous proceed as above. □

Proof of Lemma 13.12. — Let $U_j(x) \equiv r_j^{p-2} u(r_j x)$, so that $U_j \to U$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n - \{0\})$. Set $A = \{r \leqslant |x| \leqslant R\}$. Then

$$\begin{aligned}
\left\| U_j^\epsilon - U^\epsilon \right\|_{L^1(A)} &= \int_A \left| \int_G \{U_j(gx)\chi(g) - U(gx)\chi(g)\} \,\mathrm{d}g \right| \mathrm{d}x \\
&\leqslant \int_G \int_A |U_j(gx) - U(gx)| \,\mathrm{d}x \,\chi(g)\,\mathrm{d}g \\
&= \int_G \|g^* U_j - g^* U\|_{L^1(A)} \,\chi(g)\,\mathrm{d}g \\
&= \int_G \|U_j - U\|_{L^1(A)} \,\chi(g)\,\mathrm{d}g = \|U_j - U\|_{L^1(A)}\,.
\end{aligned}$$

Thus $\lim_{j\to\infty} U_j^\epsilon = \{\lim_{j\to\infty} U_j\}^\epsilon$ as claimed. □

Proof of Lemma 13.13. — It is standard that the restriction of U^ϵ to each sphere $\{|x| = r\}$ is continuous (in fact, smooth). We see this as follows. Suppose $x_j \to x$ in $\{|x| = r\}$. By transitivity we can write $x_j = g_j x$ where

$g_j \to 1$ in G. Then

$$\begin{aligned}
|U^\epsilon(x_j) - U^\epsilon(x)| &= \left| \int_G U(gx_j)\chi(g)\,\mathrm{d}g - \int_G U(gx)\chi(g)\,\mathrm{d}g \right| \\
&= \left| \int_G U(gg_jx)\chi(g)\,\mathrm{d}g - \int_G U(gx)\chi(g)\,\mathrm{d}g \right| \\
&= \left| \int_G U(hx)\chi(hg_j^{-1})\,dh - \int_G U(gx)\chi(g)\,\mathrm{d}g \right| \\
&= \left| \int_G U(gx)\left\{\chi(gg_j^{-1}) - \chi(g)\right\}\,\mathrm{d}g \right| \\
&\leqslant \int_G |U(gx)|\left|\chi(gg_j^{-1}) - \chi(g)\right|\,\mathrm{d}g \\
&\leqslant \left\{ \int_{\{|x|=r\}} |U(x)|\,\mathrm{d}g \right\} \sup_{g\in G} \left|\chi(gg_j^{-1}) - \chi(g)\right| \to 0\,.
\end{aligned}$$

We also know that U^ϵ is maximal, and in particular upper semi-continuous with $S(U^\epsilon, t) \equiv \Theta K(t)$ for all t.

Now for $|x| = r$, $g_0 \in G$, and any $r_1 < r < r_2$, the calculation above also shows that

$$|U^\epsilon(g_0x) - U^\epsilon(x)| \leqslant \sup_{r_1\leqslant t\leqslant r_2} \left\{ \int_{\{|x|=t\}} |U(x)|\,\mathrm{d}g \right\} \sup_{g\in G} \left|\chi(gg_0^{-1}) - \chi(g)\right|.$$

Now every y with $|y| = t$ and $|y - x| < \delta$ can be written as $y = g_0x$ with $d(g_0, 1) < \epsilon(\delta)$ where $\epsilon(\delta) \to 0$ as $\delta \to 0$. Thus we have

$$\begin{aligned}
&|U^\epsilon(y) - U^\epsilon(x)| \\
&\quad\leqslant \sup_{r_1\leqslant t\leqslant r_2} \left\{ \int_{\{|x|=t\}} |U(x)|\,\mathrm{d}g \right\} \sup_{d(g_0,1)<\epsilon(\delta)} \sup_{g\in G} \left|\chi(gg_0^{-1}) - \chi(g)\right| \\
&\quad\leqslant C\varphi(\delta)
\end{aligned}$$

for all $|x| = t, |y| = t$, $|y-x| < \delta$ and $r_1 \leqslant t \leqslant r_2$. This shows that the family of functions

$V_t \equiv U^\epsilon(tx)$is uniformly equicontinuous on the sphere $S^{n-1} = \{|x| = 1\}$.

Claim.

$$\lim_{t\to t_0} \sup_{S^{n-1}} |V_t - V_{t_0}| = 0\,.$$

Proof. — Let $t_j \to t_0$ be any sequence. Then by the equicontinuity above, there is a subsequence such that V_{t_j} converges uniformly to a limit $\widetilde{V}$ on S^{n-1}. We are done if we show that $\widetilde{V} = V_{t_0}$.

Now by the upper semi-continuity of U^ϵ we have

$$\widetilde{V}(x) = \lim_{j\to\infty} V_{t_j}(x) = \lim_{j\to\infty} U^\epsilon(t_j x) \leqslant U^\epsilon(t_0 x). \tag{13.5}$$

However, we also have that

$$\begin{aligned}\int_{S^{n-1}} \widetilde{V}(x)\,\mathrm{d}x = \lim_{j\to\infty}\int_{S^{n-1}} V_{t_j}(x)\,\mathrm{d}x &= \lim_{j\to\infty}\int_{S^{n-1}} U^\epsilon(t_j x)\,\mathrm{d}x \\ &= \int_{S^{n-1}} U^\epsilon(t_0 x)\,\mathrm{d}x.\end{aligned}$$

since the last two terms are just the averages $S(U^\epsilon, t_j) = \Theta K(t_j) \to \Theta K(t_0) = S(U^\epsilon, t_0)$. By the inequality (13.5) we conclude that $\widetilde{V}(x) = U^\epsilon(t_0 x) = V_{t_0}(x)$ for all $x \in S^{n-1}$. Thus we have shown that U^ϵ is continuous for all ϵ.

Now it is a general fact that $f^\epsilon \to f$ in L^1_{loc}. The proof is easy and the convergence is uniform when $f \in C_0^\infty$. The general case follows from the fact that C_0^∞ is dense in L^1 on compact domains. This completes the proof of Lemma 13.13. ☐

Example 13.15. — If one drops the convexity hypothesis in Theorem 13.1, then in dimensions $n \geqslant 3$ there are orthogonally invariant subequations of every finite Riesz characteristic for which strong uniqueness fails. To see this we consider the largest such subequation of characteristic p:

$$\mathcal{P}_p^{\min/\max} \equiv \{A : \lambda_{\min}(A) + (p-1)\lambda_{\max}(A) \geqslant 0\}.$$

(See Appendix A in Part II for a proof that there exists a largest and it is the one above.) To see that strong uniqueness fails for $\mathcal{P}_p^{\min/\max}$ we consider the following functions. Write $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}, m < n$ with coordinates $z = (x, y)$, and consider the function

$$u(x, y) \equiv \overline{K}_p(|x|)$$

where $\overline{K}_p$ is given by (3.8). Then $D_z^2 u = \frac{1}{|x|^p}(P_{x^\perp} - (p-1)P_x)$ has ordered eigenvalues

$$-\frac{(p-1)}{|x|^p}, 0, \dots, 0, \frac{1}{|x|^p}, \dots, \frac{1}{|x|^p},$$

from which it is clear that u is $\mathcal{P}_p^{\min/\max}$-subharmonic on $\mathbb{R}^n$ and, in fact, $\mathcal{P}_p^{\min/\max}$-harmonic for $x \neq 0$. Note that u has Riesz homogeneity p and is therefore its own tangent at points of the form $(0, y)$. Hence strong uniqueness fails for $\mathcal{P}_p^{\min/\max}$.

Straightforward calculation shows, however, that these "partial Riesz kernels" are not subharmonic for the largest *convex* subequation of characteristic p given in Proposition 13.10 above.

14. The Structure of the Sets E_c where the Density is $\geqslant c$

In this section we assume the subequation F on $\mathbb{R}^n$ is convex with finite Riesz characteristic $p \geqslant 2$. Fix $u \in F(X)$ where X is an open subset in $\mathbb{R}^n$. Let $\Theta = \Theta^V : X \to \mathbb{R}$ be the density function (for the volume function). For $c > 0$ define

$$E_c(u) \equiv \{x \in X : \Theta(x) \geqslant c\}.$$

For classical plurisubharmonic functions in $\mathbb{C}^n$ (where $F = \mathcal{P}^{\mathbb{C}}$), these sets have been of central importance. A deep theorem, due to L. Hörmander, E. Bombieri and in its final form by Siu ([25, 3, 40]), states that in this case E_c is a complex analytic subvariety. One straightforwardly deduces from this result that for the subequation $\mathcal{P}_2$ in $\mathbb{R}^{2n}$ the set E_c is discrete, since $\mathcal{P}^{\mathbb{C}}(J) \subset \mathcal{P}_2$ for all parallel complex structures J on $\mathbb{R}^{2n}$.

This strong corollary has a quite general extension.

Theorem 14.1. — *Suppose strong uniqueness of tangents holds for F (e.g., $F = \mathcal{P}_p$). Then for any F-subharmonic function u the set $E_c(u)$ is discrete.*

This result is essentially sharp. See Remark 14.2 below.

We will prove Theorem 14.1 in the following equivalent form. Consider an F-subharmonic function u where F has Riesz characteristic p with $2 < p < \infty$.

Theorem 14.1′. — *Suppose strong uniqueness of tangents holds for u at a point x_0, that is, suppose that the p-flow of u has limit*

$$\lim_{r \downarrow 0} u_r(x_0; x) = \Theta K(|x - x_0|) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n), \text{ for some } \Theta \geqslant 0. \tag{14.1}$$

Then

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} \Theta(u, x) = 0.$$

Proof. — Suppose the conclusion fails. Then there exists a sequence $x_j \to x_0$ with $\Theta(u, x_j) \geqslant c > 0$ for all j. Assume $x_0 = 0$, and set $x_j = r_j \sigma_j$ with $r_j = |x_j|$. Then $r_j \to 0$, and passing to a subsequence we can assume that $\sigma_j \to \sigma \in S^{n-1}$. The idea now is to apply the sequence of r_j-homotheties to u. This will give a sequence u_{r_j} of F-subharmonics with $\Theta(u_{r_j}, \sigma_j) \geqslant c$. With appropriate estimates from monotonicity, this will contradict (14.1).

To begin pick $\rho > 0$ small, and note that

$$\frac{V\left(u_{r_j}, \sigma_j, \rho\right)}{K(\rho)} = \frac{V\left(u, x_j, r_j \rho\right)}{K(r_j \rho)} \tag{14.2}$$

since

$$V\left(u_{r_j},\sigma_j,\rho\right)=\fint_B u_{r_j}\left(\sigma_j+\rho x\right)\,\mathrm{d}x=r_j^{p-2}\fint_B u\left(x_j+r_j\rho x\right)\,\mathrm{d}x$$

and

$$\frac{r_j^{p-2}}{K(\rho)}=\frac{1}{K(r_j\rho)}.$$

Next we show that for all j

$$\frac{V\left(u,x_j,r_j\rho\right)}{K(r_j\rho)}\geqslant\frac{c}{2}. \tag{14.3}$$

In fact, this uniform bound from below, on the convergence of $\frac{V(u,x_j,t)}{K(t)}$ to $\Theta(u,x_j)$, independent of x_j, is obtained from the monotonicity property (Theorem 6.4) as follows. Set $\alpha\equiv 2^{\frac{1}{p-2}}$. Fix x_j and abbreviate notation by setting $t=r_j\rho$ and $V(t)=V(u,x_j,t)=V(u,x_j,r_j\rho)$. We now apply the identity

$$\frac{V(t)}{K(t)}=\left[\frac{V(\alpha t)-V(t)}{K(\alpha t)-K(t)}\right]\frac{\left(1-\frac{K(\alpha t)}{K(t)}\right)}{\left(1-\frac{V(\alpha t)}{V(t)}\right)}, \tag{14.4}$$

with the constant $\alpha>0$ chosen so that $\frac{K(\alpha t)}{K(t)}=\alpha^{-(p-2)}=\frac{1}{2}$. We assume u and hence $V(t)$ is $\leqslant 0$ which can be obtained by subtracting a constant, or noting that $\lim_{x\to 0}u(x)=-\infty$ since $\Theta(u,0)\geqslant c$ by Theorem 7.4.

Then $V(t)\leqslant V(\alpha t)\leqslant 0$ since $V(t)$ is increasing in t, which implies that the reciprocal of $1-\frac{V(\alpha t)}{V(t)}$ is $\geqslant 1$.

By Theorem 6.4 this proves that, as desired,

$$\frac{V(t)}{K(t)}\geqslant\frac{c}{2}. \tag{14.3$'$}$$

Combining (14.2) and (14.3) we have

$$\frac{V\left(u_{r_j},\sigma_j,\rho\right)}{K(\rho)}\geqslant\frac{c}{2}. \tag{14.5}$$

By the hypothesis (14.1) we have

$$\lim_{r_j\downarrow 0}V\left(u_{r_j},\sigma_j,\rho\right)=\lim_{r_j\downarrow 0}\fint_{B_\rho(\sigma_j)}u_{r_j}=\Theta\fint_{B_\rho(\sigma)}K(|y|)\,\mathrm{d}y.$$

Therefore, by (14.5)

$$-\rho^{p-2}\Theta\fint_{B_\rho(\sigma)}K(|y|)\,\mathrm{d}y\geqslant\frac{c}{2}.$$

Since

$$\lim_{\rho\to 0} \fint_{B_\rho(\sigma)} K(|y|)\,\mathrm{d}y = K(1) = -1,$$

this implies that $c = 0$, a contradiction. □

Remark 14.2. — For F as above, any finite set can occur as the set E_c for an F-subharmonic function. In fact, more is true. In a separate paper [24] we construct F-subharmonics with prescribed asymptotics at a finite set of points and prescribed boundary values.

THEOREM 14.3 ([24]). — *Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary $\partial\Omega$ which is strictly convex (or more generally strictly F-convex, cf.* [14]*). Let $E = \{x_j\}_{j=1}^N \subset \Omega$ be a finite subset, and $\{\Theta_j\}_{j=1}^N$ any set of positive real numbers. Then given any $\varphi \in C(\partial\Omega)$, there exists a unique $u \in \mathrm{USC}(\overline{\Omega})$ such that:*

(1) *u is F-harmonic in $\Omega - E$,*
(2) *$u\big|_{\partial\Omega} = \varphi$, and*
(3) *$\Theta(u, x_j) = \Theta_j$ for $j = 1, \ldots, N$.*

15. Subequations with Riesz characteristic $1 \leqslant p < 2$

When the Riesz characteristic satisfies $1 \leqslant p < 2$, the behavior and study of F-subharmonics differs greatly from the case $p \geqslant 2$.

$C^{0,\alpha}$ Regularity of Subharmonics

To begin, all F-subharmonics (not just the F-harmonics) are regular.

To be completely clear we formulate two hypotheses on a function u.

HYPOTHESIS A. — *$u \in F(X)$ where F is a (not necessarily convex) ST-invariant cone subequation with characteristic $1 \leqslant p < 2$.*

HYPOTHESIS B. — *$u \in \mathrm{USC}(X)$ satisfies the Maximum Principle (or (MP) for short) and K_p* double monotonicity*, that is, for all $y \in X$*

$$\frac{M(u,y,t) - M(u,y,s)}{K_p(t) - K_p(s)} \text{ is non decreasing in } s \text{ and } t \tag{15.1}$$

for all $0 \leqslant s < t < \mathrm{dist}(y, \partial X)$.

By Theorem 2.7 and Theorem 6.4

$$\text{Hypothesis A} \qquad \Rightarrow \qquad \text{Hypothesis B}. \tag{15.2}$$

Note that under Hypothesis B the density $\Theta(u, y)$ exists with $0 \leqslant \Theta(u, y) < \infty$ for each point $y \in X$. For an arbitrary function u, we abbreviate the Hölder norm on a compact set K (allowing the value $+\infty$) by

$$\|u\|_\alpha(K) \equiv \|u\|_{C^{0,\alpha}(K)}. \tag{15.3}$$

Theorem 15.1. — *Assume Hypothesis B. Then u is locally Hölder continuous on X with exponent $\alpha \equiv 2 - p$.*

More specifically, if $B_{3\rho}(x_0) \subset X$, then

$$\|u\|_\alpha \left(B_\rho(x_0)\right) \leqslant \left[\frac{R^\alpha}{(R-\rho)^\alpha - \rho^\alpha}\right] \frac{M(u, x_0, R) - u(x_0)}{R^\alpha} \tag{15.4}$$

for all $0 < 3\rho \leqslant R < \operatorname{dist}(x_0, \partial X)$. (In particular, $u(x_0) > -\infty$, i.e., u is finite-valued at each point $x_0 \in X$.)

Proof. — Assume $x, y \in B_\rho(x_0)$. Note that $x \in \partial B_{|x-y|}(y)$. Hence,

$$\frac{u(x) - u(y)}{|x-y|^\alpha} \leqslant \frac{M(u, y, |x-y|) - u(y)}{|x-y|^\alpha}.$$

Choose $R \geqslant 3\rho$. Since $x, y \in B_\rho(x_0)$, we have $|x - y| \leqslant 2\rho$ and hence $R \geqslant |x-y| + \rho$, or $R - \rho \geqslant |x-y|$. Therefore, by the monotonicity Hypothesis B

$$\frac{M(u, y, |x-y|) - u(y)}{|x-y|^\alpha} \leqslant \frac{M(u, y, R-\rho) - M(u, y, \rho)}{(R-\rho)^\alpha - \rho^\alpha}. \tag{15.5}$$

Now $B_{R-\rho}(y) \subset B_R(x_0)$ since $y \in B_\rho(x_0)$. This proves that

$$M(u, y, R-\rho) \leqslant M(u, x_0, R). \tag{15.6}$$

Also $x_0 \in B_\rho(y)$ and hence $u(x_0) \leqslant M(u, y, \rho)$, or equivalently

$$-M(u, y, \rho) \leqslant -u(x_0). \tag{15.7}$$

Now (15.6) and (15.7) imply that $M(u, y, R-\rho) - M(u, y, \rho) \leqslant M(u, x_0, R) - u(x_0)$ and (15.4) follows from (15.5). $\square$

Define the *infinitesimal Hölder norm of u at x_0* to be

$$\|u\|_\alpha(x_0) \equiv \lim_{\rho \to \infty} \|u\|_\alpha \left(B_\rho(x_0)\right). \tag{15.8}$$

Proposition 15.2. — *Under Hypothesis B,*

$$\|u\|_\alpha(x_0) \leqslant \frac{M(u, x_0, R) - u(x_0)}{R^\alpha} \leqslant \|u\|_\alpha \left(B_R(x_0)\right). \tag{15.9}$$

for all $0 < R < \operatorname{dist}(x_0, \partial X)$.

Proof. — For the first inequality, let $\rho \to 0$ on both sides of the inequality (15.4) in Theorem 15.1.

By the (MP) there exists $y \in \partial B_R(x_0)$ such that $M(u, x_0, R) = u(y)$, and hence

$$\frac{M(u,x_0,R) - u(x_0)}{R^\alpha} = \frac{u(y) - u(x_0)}{|y-x|^\alpha} \leqslant \|u\|_\alpha \left(B_R(x_0)\right).$$

Now it is easy to prove that the infinitesimal Hölder norm and the density are the same thing.

Corollary 15.3.

$$\|u\|_\alpha(x_0) = \Theta(u, x_0).$$

Proof. — Take the limit as $R \to 0$ in (15.9) and apply the definition of the density. □

Remark 15.4 (Hypothesis A). — Lemma A.1 in Part II states that $\mathcal{P}_p^{\min/\max} \equiv \{A : \lambda_{\min}(A) + (p-1)\lambda_{\max}(A) \geqslant 0\}$ is the maximal subequation of characteristic p, i.e. it contains every other subequation F of characteristic p. Thus the relevance of Theorem 15.1 for pure second-order subequations can be stated as follows.

Theorem 15.1 holds under

Hypothesis A′ ($0 < \alpha \leqslant 1$). — *The function u satisfies the subequation*

$$\lambda_{\min}(D^2u) + (1-\alpha)\lambda_{\max}(D^2u) \geqslant 0 \qquad \text{on } X$$

in the viscosity sense. Said differently, Hypothesis A and Hypothesis A′ are the same.

Remark 15.5. — The subequations $\mathcal{P}_p^{\min/\max}$ are never convex unless $p = 1$. In addition we have

$$\mathcal{P}_p^{\min/\max} \subset \Delta \quad \iff \quad p \leqslant 1 + \frac{1}{n-1} \quad \iff \quad \frac{n-2}{n-1} \leqslant \alpha \leqslant 1.$$

To see this, note that $\lambda_1 + (p-1)\lambda_n \geqslant 0 \Rightarrow \lambda_1 + \cdots + \lambda_n \geqslant 0$ if and only if $p - 1 \leqslant \frac{1}{n-1}$ since $\lambda_1 + \cdots + \lambda_n \geqslant (n-1)\lambda_1 + \lambda_n = (n-1)(\lambda_1 + \frac{1}{n-1}\lambda_n)$.

Existence of Tangents

In the range $1 \leqslant p < 2$ the arguments for the existence and structure of tangents have a different flavor from the case $p \geqslant 2$. Recall that in this range the *tangent flow*

$$u_r(x) = \frac{1}{r^\alpha}(u(rx) - u(0)) \qquad \text{where } \alpha = 2 - p,$$

is defined in Definition 9.1 (2).

Tangents to subharmonics have only been defined when F is convex (see Definition 9.3). However, because of the Hölder continuity when $1 \leqslant p < 2$, the definition can be extended to the more general cone case in Hypothesis A. In fact, Hypothesis B is enough. Give $C(\mathbb{R}^n)$ the topology of uniform convergence on compact subsets.

Definition 15.6 (Tangents). — *Suppose that u satisfies Hypothesis B in a neighborhood of the origin in $\mathbb{R}^n$. For each sequence $r_j \searrow 0$ such that*

$$U \equiv \lim_{j\to\infty} u_{r_j} \quad \textit{converges in } C(\mathbb{R}^n)\,, \tag{15.10}$$

the limit function U is called a tangent to u at *0, and $T_0(u)$ denotes the space of all such tangents.*

The version of Theorem 11.1 for $1 \leqslant p < 2$ is given as follows.

Theorem 15.7 (Existence of Tangents). — *Suppose u satisfies Hypothesis B on a ball about the origin. Then for each $\rho > 0$ there exists a $\delta > 0$ such that the family $\{u_r\}_{0<r\leqslant\delta}$ is bounded in norm in $C^{0,\alpha}(B_\rho)$. In fact,*

$$\limsup_{r\downarrow 0} \|u_r\|_\alpha\,(B_\rho) \leqslant \Theta^M(u,0) \qquad \forall\,\rho > 0\,. \tag{15.11}$$

In particular, the set $\{u_r\}_{0<r\leqslant\delta}$ is precompact in $C(\mathbb{R}^n)$.

Proof. — Note that $u_r(0) = 0$ so that Theorem 15.1 states that the α-Hölder norm of u_r on B_ρ satisfies

$$\|u_r\|_\alpha\,(B_\rho) \leqslant \frac{R^\alpha}{(R-\rho)^\alpha - \rho^\alpha}\,\frac{M(u_r,0,R)}{R^\alpha}$$

if rR is small and $0 < 3\rho \leqslant R$. Now by the definition of u_r

$$M(u_r,0,R) = \frac{M(u,0,rR) - u(0)}{r^\alpha}\,,$$

and therefore

$$\|u_r\|_\alpha\,(B_\rho) \leqslant \frac{R^\alpha}{(R-\rho)^\alpha - \rho^\alpha}\,\frac{M(u,0,rR) - u(0)}{(rR)^\alpha}\,.$$

Taking the $\limsup$ as $r \downarrow 0$ yields

$$\limsup_{r\downarrow 0} \|u_r\|_\alpha\,(B_\rho) \leqslant \frac{R^\alpha}{(R-\rho)^\alpha - \rho^\alpha}\,\Theta^M(u,0)\,.$$

Finally we can let $R \to \infty$, proving (15.11).

By the standard compact embedding theorem this proves that (taking the topology of Hölder norms on compact subsets)

$$\{u_r\}_{0<r\leqslant\delta} \text{ is precompact in } C^{0,\beta}(\mathbb{R}^n) \text{ for each } 0 \leqslant \beta < \alpha\,, \tag{15.12}$$

where $C^{0,\beta}(\mathbb{R}^n) = C(\mathbb{R}^n)$ when $\beta = 0$. $\square$

Remark. — If F is convex, then our previous L^1_{loc} Definition 9.3 of a tangent U to u at 0 is also applicable. It agrees with Definition 15.6 because of the precompactness.

The analogue of Theorem 11.2 is the same except that $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is replaced by $C(\mathbb{R}^n)$.

THEOREM 15.8. — *The tangent set $T_0(u)$ to an F-subharmonic function u satisfies:*

(1) *$T_0(u)$ is non-empty.*
(2) *$T_0(u)$ is a compact subset of $C(\mathbb{R}^n)$.*
(3) *$T_0(u)$ is invariant under the tangent flow $U \to U_r$.*
(4) *$T_0(u)$ is a connected subset of $C(\mathbb{R}^n)$.*

The proof is similar to that of Theorem 11.2 and is omitted.

As a consequence of Theorem 15.8 the Hölder norm of a tangent is finite on all of $\mathbb{R}^n$.

COROLLARY 15.9. — *If $U \in T_0(u)$, then*

$$\|U\|_\alpha(\mathbb{R}^n) = \Theta(u, 0) = \|u\|_\alpha(x_0)\,.$$

Uniqueness, Strong Uniqueness, and Homogeneity of Tangents

The three concepts are defined exactly as in Definition 12.1. For instance, uniqueness of tangents holds for u at 0 if $T_0(u) = \{U\}$ is a singleton, or equivalently (cf. (12.1))

$$\lim_{r\to 0} u_r \text{ exists in } C(\mathbb{R}^n) \text{ and equals } U. \tag{15.13}$$

Strong uniqueness holds for u at 0 if this limit $U = \Theta K_p$ where $\Theta = \Theta^M(u, 0)$. In this setting strong uniqueness for u is equivalent to the notion of asymptotic equivalence $u \sim \Theta|y|^\alpha$ defined by (15.3) below.

LEMMA 15.10. — *Strong uniqueness of tangents for u at 0 holds, i.e.,*

$$\lim_{r\to 0} u_r = \Theta K_p = \Theta|x|^\alpha \quad \textit{in } C(\mathbb{R}^n) \qquad \textit{with } \Theta \geqslant 0 \tag{15.14}$$

if and only if $u(y) \sim \Theta|y|^\alpha$, i.e.,

$$\lim_{y\to 0} \frac{u(y) - u(0)}{|y|^\alpha} = \Theta \geqslant 0\,. \tag{15.15}$$

Proof. — Actually, the equivalence of (15.14) and (15.15) is an elementary fact which holds for any continuous function defined in a neighborhood of the origin.

We can assume $u(0) = 0$. We first show that (15.15) $\Rightarrow$ (15.14). The inequality

$$\left|\frac{u(y)}{|y|^\alpha} - \Theta\right| \leqslant \epsilon$$

can be rewritten, with $y = rx$, as

$$|u_r(x) - \Theta|x|^\alpha| \leqslant \epsilon|x|^\alpha.$$

If the first holds for $|y| \leqslant \delta$, then the second holds for $|x| \leqslant R$ and $r \leqslant \delta/R$. Thus we have $|u_r(x) - \Theta|x|^\alpha| \leqslant \epsilon R^\alpha$, for all $|x| \leqslant R$ and $r \leqslant \delta/R$, which is enough to prove (15.13).

For the converse we need only assume that $u_r \to \Theta K$ uniformly on some sphere ∂B_R. The inequality

$$|u_r(x) - \Theta|x|^\alpha| \leqslant \epsilon$$

can be rewritten, with $y = rx$, as

$$\left|\frac{u(y)}{|y|^\alpha} - \Theta\right| \leqslant \frac{\epsilon}{|x|^\alpha}.$$

If the first holds for all $|x| = R$ and $r \leqslant \delta$, then the second holds for all $|y| \leqslant \delta R$ with the right-hand side replaced by ϵ/R^α. This is enough to prove that $\lim_{y\to 0} u(y)/|y|^\alpha = 0$. □

Remark 15.11. — We say that *strong uniqueness holds for a subequation F* if it holds for all F-subharmonics at 0. Recall that by Theorem 13.1 Strong Uniqueness of Tangents to subharmonics holds for every convex $\mathrm{O}(n)$-invariant subequation F with finite Riesz characteristic except $F = \mathcal{P}$. This section is only concerned with the cases $1 \leqslant p < 2$, or $1 < p < 2$ when $\mathcal{P}$ is excluded. This includes the subequations: $\mathcal{P}_p$ $(1 < p < 2)$, Σ_k $(p \equiv \frac{n}{k} < 2)$, $\mathcal{P}(\delta)$ $(\delta < \frac{n}{n-2})$, and others.

Harmonicity of Tangents when F is convex

If F is a convex cone ST-invariant subequation with finite characteristic, then by Theorem 10.2 every tangent to a subharmonic is maximal, and by Proposition 8.5, every continuous maximal function is F-harmonic. Thus the regularity result Theorem 15.1 implies the following for $1 \leqslant p < 2$.

Theorem 15.12. — *Let F be as above. Then for u F-subharmonic in a neighborhood of 0, every tangent $U \in T_0(u)$ is F-harmonic in $\mathbb{R}^n - \{0\}$.*

Removable Point Singularities

The next result should be compared with Theorem 1.9 (the case $\alpha^* < 0$) in [ASS], where F is assumed to be uniformly elliptic.

Theorem 15.13. — *Suppose that F is a cone subequation with a Riesz characteristic p and $1 < p < 2$. Suppose Strong Uniqueness of Tangents holds for F and $F + \mathcal{P}_p \subset F$ (i.e., F is $\mathcal{P}_p$-monotone). For each function H which is F-harmonic in a punctured neighborhood of x_0 and F-subharmonic across x_0, one has that*

$$H \text{ is } F\text{-harmonic across } x_0 \qquad \iff \qquad \text{the density } \Theta^M(H, x_0) = 0.$$

Proof. — Assume that $x_0 = 0$. By Proposition A.5 in [24], the strong uniqueness hypothesis can be restated as an asymptotic equivalence $\lim_{x\to 0} \frac{(H(x)-H(0))}{|x|^\alpha} = \Theta \geqslant 0$, which was denoted there as $H(x) \sim \Theta|x|^\alpha$, at $x_0 = 0$.

Suppose $\Theta = 0$. Then for all $\epsilon > 0$, $\exists\delta > 0$ such that $H(x) - H(0) \leqslant \epsilon|x|^\alpha$ if $|x| \leqslant \delta$. Set $V_\epsilon(x) \equiv -(H(x) - H(0)) + 2\epsilon|x|^\alpha$. Then $\epsilon|x|^\alpha \leqslant V_\epsilon(x)$ on $|x| \leqslant \delta$, which implies that V_ϵ has no test functions at 0. Since $\widetilde{F} + \mathcal{P}_p \subset \widetilde{F}$, the Addition Theorem (cf. [19]) implies that V_ϵ is $\widetilde{F}$-subharmonic on $B_\delta - \{0\}$. Thus V_ϵ is $\widetilde{F}$-subharmonic on B_δ. Since V_ϵ decreases to $-H(x) + H(0)$ as $\epsilon \to 0$, this proves that $-H$ is $\widetilde{F}$-subharmonic on B_δ, and hence H is F-harmonic.

Suppose $\Theta > 0$. Then for $0 < \epsilon < \Theta$ there exists $0 < \delta < 1$ with $\epsilon|x|^\alpha \leqslant H(x) - H(0)$ on B_δ. Therefore, $-(H(x) - H(0)) \leqslant -\epsilon|x|^\alpha \leqslant -\epsilon|x|^2$ if $|x| \leqslant \delta$, which proves that $-\epsilon|x|^2$ is a test function for $-H(x)$ at 0, and hence $-H$ is not subaffine. Finally, $0 \in F \Rightarrow \mathcal{P} \subset F \Rightarrow \widetilde{F} \subset \widetilde{\mathcal{P}}$, which proves that $-H$ is not $\widetilde{F}$-subharmonic. □

Appendix A. Subaffine Functions and a Dichotomy

For punctured radial subharmonics, i.e., a radial F-subharmonic function defined on a ball, there is a useful dichotomy between those which are increasing and those which are decreasing, which we now discuss. The subaffine equation $\widetilde{\mathcal{P}} = \{\lambda_{\max} \geqslant 0\}$ is an important special case, since it contains every subequation F (including itself) for which the maximum principle holds. It is also a special case in that the radial subequation $R_{\widetilde{\mathcal{P}}}$ on $(0, \infty)$ is constant coefficient. Using the jet variables (λ, a), we have

$$R_{\widetilde{\mathcal{P}}} = \widetilde{\mathbb{R}_+ \times \mathbb{R}_+} \equiv \{(\lambda, a) : \text{either } \lambda \geqslant 0 \text{ or } a \geqslant 0\}. \tag{A.1}$$

It is important to note that the maximum principle holds for this one-variable subequation.

This dual subequation $\widetilde{\mathbb{R}_+ \times \mathbb{R}_+}$ is more restrictive than one might guess. The next result shows that near the left endpoint of (a,b) there is a dichotomy for a subharmonic. It is either increasing or it is convex and decreasing.

Lemma A.1 (Increasing/Decreasing). — *Suppose that ψ is a general upper semi-continuous $\widetilde{\mathbb{R}_+ \times \mathbb{R}_+}$-subharmonic function on an open interval (a,b). Then either*

(1) *ψ is increasing on (a,b), or*
(2) *ψ is decreasing and convex on (a,b), or*
(3) *$\exists\ c \in (a,b)$ such that ψ is decreasing and convex on (a,c) and increasing on (c,b).*

Proof. — Suppose that ψ is not increasing on all of (a,b), that is, $\psi(r) > \psi(s)$ for some $a < r < s < b$. We claim that ψ is decreasing on (a,r). If not, there exist r_1, r_2 with $a < r_1 < r_2 < r$ and $\psi(r_1) < \psi(r_2)$. If $\psi(r_2) < \psi(r)$, then since $\psi(r) > \psi(s)$, ψ has a strict maximum on (r_2, s). Thus $\psi(r_2) \geqslant \psi(r) > \psi(s)$, and since $\psi(r_1) < \psi(r_2)$, we must have a strict maximum on (r_1, s).

Suppose further that ψ is not decreasing on all of (a,b), that is, $\psi(s) < \psi(t)$ for some $r < s < t < b$. The argument above shows that there exists a maximal $c \in (s,t)$ so that ψ is decreasing on (a,c). Now ψ must be increasing on (c,b) for if not, it would have a strict interior maximum on that interval.

When ψ is decreasing on (a,c), it must be convex there. To see this let φ be a test function for ψ at $t_0 \in (a,c)$. Then $0 \leqslant \psi(t) - \psi(t_0) \leqslant \varphi(t) - \varphi(t_0)$ for $t < t_0$. This implies that $\varphi'(t_0) \leqslant 0$. If $\varphi'(t_0) = 0$, then the same inequality implies that $\varphi''(t_0) \geqslant 0$. On the other hand, if $\varphi'(t_0) < 0$, then $\varphi''(t_0) \geqslant 0$ because ψ is $\widetilde{\mathbb{R}_+ \times \mathbb{R}_+}$-subharmonic. $\square$

We say that the maximum principle (MP) holds for a subequation F if it holds for all F-subharmonic functions.

Theorem A.2. — *The following conditions on a subequation $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ are equivalent.*

(1) *The maximum principle holds for F.*
(2) *$F \subset \widetilde{\mathcal{P}}$ (i.e., the subequation $\widetilde{\mathcal{P}}$ is universal for (MP)).*
(3) *$0 \notin \mathrm{Int}\, F$.*
(4) *$R_F \subseteq \widetilde{\mathbb{R}_+ \times \mathbb{R}_+}$.*

Proof. — Parts (1)–(3) were proved in [14, Lem. 2.2 and Prop. 4.8]. For part (4) note that $F \subset \widetilde{\mathcal{P}} \Rightarrow (R_F)_t \subset (R_{\widetilde{\mathcal{P}}})_t = \widetilde{\mathbb{R}_+ \times \mathbb{R}_+}$. If F is not contained in $\widetilde{\mathcal{P}} \equiv \{A : \lambda_{\max}(A) \geqslant 0\}$, then there exists $B < 0$ with $B \in F$. By positivity $-\epsilon I \in F$ for some $\epsilon > 0$, which implies that $(R_{\widetilde{\mathcal{P}}})_t$ is not contained in $\widetilde{\mathbb{R}_+ \times \mathbb{R}_+}$. □

These two results can be combined as follows.

Corollary A.3. — *If the (MP) holds for F, then the conclusions* (1), (2) *and* (3) *of the Increasing/Decreasing Lemma A.1 hold for any radial F-subharmonic function $u(x) = \psi(|x|)$ defined on an annulus. (In particular, if u is F-subharmonic on a ball, then $\psi(t)$ must be increasing.)*

Proof. — By Theorem 2.4 and Theorem A.2, ψ is $\widetilde{\mathbb{R}_+ \times \mathbb{R}_+}$-subharmonic, and hence Lemma A.1 applies to ψ. □

Appendix B. Uniform Ellipticity and $\mathcal{P}(\delta)$

The point of this section is to make clear that viscosity harmonics for the subequation

$$\mathcal{P}(\delta') = \left\{A \in \mathrm{Sym}^2(\mathbb{R}^n) : A + \delta\,\mathrm{tr}(A) \geqslant 0\right\}, \qquad \delta = \frac{\delta'}{n},$$

are solutions to a uniformly elliptic equation $F(D^2u) = 0$ as defined in [4], [6], [41], etc. We define the operator

$$F : \mathrm{Sym}^2(\mathbb{R}^n) \longrightarrow \mathbb{R} \qquad \text{by} \qquad F(A) \equiv \lambda_{\min}(A) + \delta \mathrm{tr}(A).$$

It is straightforward to verify that for all $P \geqslant 0$ one has

$$\delta\,\mathrm{tr}(P) \leqslant F(A+P) - F(A) \leqslant (1+\delta)\,\mathrm{tr}(P).$$

which is one of the standard equivalent versions of uniform ellipticity for the operator F appearing in the sources above.

Now since

$$\mathcal{P}(\delta') = \{A : F(A) \geqslant 0\} \quad \text{and} \quad \mathrm{Int}\,\mathcal{P}(\delta') = \{A : F(A) > 0\}$$

it is completely straightforward to verify that a continuous function u is a viscosity solution of $F(D^2u) = 0$ if and only if (in our terminology) u is $\mathcal{P}(\delta')$-harmonic.

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