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The Curie-Weiss Model of SOC in Higher Dimension (*)

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Abstract. — We build and study a multidimensional version of the Curie-Weiss model of self-organized criticality we have designed in [2]. For symmetric distributions satisfying some integrability condition, we prove that the sum S_n of the randoms vectors in the model has a typical critical asymptotic behaviour. The fluctuations are of order $n^{3/4}$ and the limiting law has a density proportional to the exponential of a fourth-degree polynomial.

Résumé. — Nous construisons et étudions une version multi-dimensionnelle du modèle d'Ising Curie-Weiss de criticalité auto-organisée que nous avons introduit dans [2]. Pour des distributions vérifiant une certaine condition d'intégrabilité, nous montrons que la somme S_n des variables aléatoires du modèle a un comportement asymptotique critique typique. Les fluctuations sont d'ordre $n^{3/4}$ et la loi limite admet une densité proportionnelle à l'exponentielle d'un polynôme de degré quatre.

1. Introduction

In [2] and [5], we introduced a *Curie-Weiss model of self-organized criticality* (SOC): we transformed the distribution associated to the generalized Ising Curie-Weiss model by implementing an automatic control of the inverse temperature which forces the model to evolve towards a critical state. It is the model given by an infinite triangular array of real-valued random variables $(X_n^k)_{1\leqslant k\leqslant n}$ such that, for all $n\geqslant 1$, $(X_n^1,\dots,X_n^n)$ has the distribution

$$\frac{1}{Z_n}\exp\left(\frac{1}{2}\frac{(x_1+\cdots+x_n)^2}{x_1^2+\cdots+x_n^2}\right)\mathbb{1}_{\{x_1^2+\cdots+x_n^2>0\}}\prod_{i=1}^n d\rho(x_i),$$

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where ρ is a probability measure on $\mathbb{R}$ which is not the Dirac mass at 0, and where Z_n is the normalization constant. We extended the study of this model in [8], [7], [6] and [9]. For symmetric distributions satisfying some exponential moments condition, we proved that the sum S_n of the random variables behaves as in the typical critical generalized Ising Curie-Weiss model: the fluctuations are of order $n^{3/4}$ and the limiting law is $C\exp(-\lambda x^4)\,dx$ where C and λ are suitable positive constants. Moreover, by construction, the model does not depend on any external parameter. That is why we can conclude it exhibits the phenomenon of self-organized criticality (SOC). Our motivations for studying such a model are detailed in [2].

Let $d \geqslant 1$. In this paper we define a d-dimensional version of the Curie-Weiss model of SOC, i.e, such that the X_n^k, $1 \leqslant k \leqslant n$, are random vectors in $\mathbb{R}^d$. Let us start by defining the *d-dimensional generalized Ising Curie-Weiss model*. Let ρ be a symmetric probability measure on $\mathbb{R}^d$ such that

$$\forall\, v \geqslant 0 \qquad \int_{\mathbb{R}^d} \exp(v\|z\|^2)\,d\rho(z) < \infty\,.$$

Assume that its covariance matrix

$$\Sigma = \int_{\mathbb{R}^d} z\,{}^t z\,d\rho(z)$$

is invertible. It is known to be equivalent to non-degeneracy of ρ, *i.e.* that there no hyperplane has full measure. The d-dimensional generalized Ising Curie-Weiss model associated to ρ and to the temperature field T (which is here a $d\times d$ symmetric positive definite matrix) is defined through an infinite triangular array of random vectors $(X_n^k)_{1\leqslant k\leqslant n}$ such that, for all $n \geqslant 1$, $(X_n^1,\dots,X_n^n)$ has the distribution

$$\frac{1}{Z_n(T)}\exp\left(\frac{1}{2n}\big\langle T^{-1}(x_1+\cdots+x_n),(x_1+\cdots+x_n)\big\rangle\right)\prod_{i=1}^n d\rho(x_i)\,,$$

where $Z_n(T)$ is a normalization. When $d=1$ and $\rho=(\delta_{-1}+\delta_1)/2$, we recover the classical Ising Curie-Weiss model. Let $S_n = X_n^1+\cdots+X_n^n$ for any $n\geqslant 1$. By extending the methods of Ellis and Newmann (see [4]) to the higher dimension, we obtain that, under some « sub-Gaussian » hypothesis on ρ, if $T-\Sigma$ is a symmetric positive definite matrix, then

$$\frac{S_n}{\sqrt{n}} \underset{n\to+\infty}{\overset{\mathscr{L}}{\longrightarrow}} \mathcal{N}_d\big(0, T(T-\Sigma)^{-1}\Sigma\big),$$

the centered d-dimensional Gaussian distribution with covariance matrix $T(T-\Sigma)^{-1}\Sigma$. If $T=\Sigma$ (critical case) then

$$\frac{S_n}{n^{3/4}} \underset{n\to+\infty}{\overset{\mathscr{L}}{\longrightarrow}} C_\rho \exp\left(-\phi_\rho(s_1,\dots,s_d)\right)\,ds_1\,\cdots\,ds_d,$$

where C_ρ is a normalization constant and ϕ_ρ is an homogeneous polynomial of degree four in $\mathbb{R}[X_1, \dots, X_d]$ such that $\exp(-\phi_\rho)$ is integrable with respect to the Lebesgue measure on $\mathbb{R}^d$. Detailed proofs of these results are given in section 23 of [6]. These results highlight that the non-critical fluctuations are normal (in the Gaussian sense) while the critical fluctuations are of order $n^{3/4}$.

Now we try to modify this model in order to construct a d-dimensional SOC model. As in [2], we search an automatic control of the temperature field T, which would be a function of the random variables in the model, so that, when n goes to $+\infty$, T converges towards the critical value Σ of the model. We start with the following observation: if $(Y_n)_{n\geqslant 1}$ is a sequence of independent random vectors with identical distribution ρ, then, by the law of large numbers,

$$\frac{\widehat{\Sigma}_n}{n} \underset{n\to+\infty}{\overset{\text{a.s}}{\longrightarrow}} \Sigma,$$

where

$$\forall\, n \geqslant 1 \qquad \widehat{\Sigma}_n = X_n^1\, {}^t(X_n^1) + \cdots + X_n^n\, {}^t(X_n^n).$$

This convergence provides us with an estimator of Σ. If we believe that a similar convergence holds in the d-dimensional generalized Ising Curie-Weiss model, then we are tempted to « replace T by $\widehat{\Sigma}_n/n$ » in the previous distribution. Hence, in this paper, we consider the following model:

The model. Let $(X_n^k)_{n\geqslant d,\, 1\leqslant k\leqslant n}$ be an infinite triangular array of random vectors in $\mathbb{R}^d$ such that, for any $n \geqslant d$, $(X_n^1, \dots, X_n^n)$ has the distribution $\widetilde{\mu}_{n,\rho}$, the probability measure on $(\mathbb{R}^d)^n$ with density

$$(x_1, \dots, x_n) \longmapsto \frac{1}{Z_n} \exp\left(\frac{1}{2}\left\langle \left(\sum_{i=1}^n x_i\, {}^t x_i\right)^{-1} \left(\sum_{i=1}^n x_i\right), \left(\sum_{i=1}^n x_i\right)\right\rangle\right)$$

with respect to $\rho^{\otimes n}$ on the set

$$D_n^+ = \left\{ (x_1, \dots, x_n) \in (\mathbb{R}^d)^n : \det\left(\sum_{i=1}^n x_i\, {}^t x_i\right) > 0 \right\},$$

where

$$Z_n = \int_{D_n^+} \exp\left(\frac{1}{2}\left\langle \left(\sum_{i=1}^n x_i\, {}^t x_i\right)^{-1} \left(\sum_{i=1}^n x_i\right), \left(\sum_{i=1}^n x_i\right)\right\rangle\right) \prod_{i=1}^n d\rho(x_i).$$

For any $n \geqslant d$, we denote $S_n = X_n^1 + \cdots + X_n^n \in \mathbb{R}^d$ and

$$T_n = X_n^1\, {}^t(X_n^1) + \cdots + X_n^n\, {}^t(X_n^n).$$

According to the construction of this model and according to our results in one dimension, we expect that the fluctuations are of order $n^{3/4}$. Our main theorem states that they are indeed:

Theorem 1.1. — *Let ρ be a symmetric probability measure on $\mathbb{R}^d$ satisfying the two following hypothesis:*

(H1) *there exists $v_0 > 0$ such that $\displaystyle\int_{\mathbb{R}^d} e^{v_0\|z\|^2}\,d\rho(z) < \infty$,*

(H2) *the ρ-measure of any vector hyperplane of $\mathbb{R}^d$ is less than $1/\sqrt{e}$.*

Let Σ be the covariance matrix of ρ and let M_4 be the function defined on $\mathbb{R}^d$ by

$$\forall\, z \in \mathbb{R}^d \qquad M_4(z) = \int_{\mathbb{R}^d} \langle z, y\rangle^4\, d\rho(y).$$

Law of large numbers: *Under $\widetilde{\mu}_{n,\rho}$, $(S_n/n, T_n/n)$ converges in probability to $(0, \Sigma)$.*

Fluctuation result: *Under $\widetilde{\mu}_{n,\rho}$,*

$$\frac{S_n}{n^{3/4}} \overset{\mathscr{L}}{\underset{n\to\infty}{\longrightarrow}} \frac{\exp\left(-\dfrac{1}{12}M_4\big(\Sigma^{-1}z\big)\right)\,dz}{\displaystyle\int_{\mathbb{R}^d}\exp\left(-\dfrac{1}{12}M_4\big(\Sigma^{-1}u\big)\right)\,du}.$$

We prove that the matrix Σ is invertible in subsection 2.2.1. In section 2.2.2, we prove rigorously that this model is well-defined, *i.e.* $Z_n \in]0, +\infty[$ for any $n \geqslant d$. After giving large deviation results in subsection 2.2.3, we show the law of large numbers in section 3. Finally, in section 4, we prove that the function

$$z \longmapsto \exp\left(-M_4\left(\Sigma^{-1/2}z\right)/12\right)$$

is integrable on $\mathbb{R}^d$ and that $S_n/n^{3/4}$ converges in distribution to the announced limiting distribution.

Remark : in the case where $d = 1$, we have already proved this theorem in [2], [7] and [9]. Moreover we succeeded to remove hypothesis $(H2)$ – which turns out to be simply $\rho(\{0\}) < 1/\sqrt{e}$ when $d = 1$ – with a conditioning argument. It seems not immediate that such arguments could extend in the case where $d \geqslant 2$. However this assumption together with hypothesis $(H1)$ are technical hypothesis and we believe that the result should be true if ρ is only a non-degenerate symmetric probability measure on $\mathbb{R}^d$ having a finite fourth moment.

2. Preliminaries

In this section, we suppose that ρ is a symmetric probability measure on $\mathbb{R}^d$ satisfying hypothesis $(H1)$ and $(H2)$.

2.1. Σ is a symmetric positive definite matrix

Since ρ satisfies hypothesis $(H1)$, the covariance matrix Σ is well-defined. It is of course a symmetric positive semi-definite matrix. Let $\mathcal{H}$ be a hyperplane of $\mathbb{R}^d$. If $\mathcal{H}$ is a vector hyperplane then, by hypothesis, $\rho(\mathcal{H}) < 1/\sqrt{e} < 1$. If $\mathcal{H}$ is an affine (but not vector) hyperplane then,

$$\rho(\mathcal{H}) = \rho(-\mathcal{H}) = \frac{1}{2}(\rho(\mathcal{H}) + \rho(-\mathcal{H})) \leqslant \frac{1}{2} < 1,$$

since ρ is symmetric and $\mathcal{H} \cap (-\mathcal{H}) = \varnothing$. In both cases $\rho(\mathcal{H}) < 1$ thus ρ is a non-degenerate probability measure on $\mathbb{R}^d$. As a consequence Σ is positive definite.

Notice that the hypothesis that $\rho(\mathcal{H}) < 1/\sqrt{e}$ is not involved on this point. We only need that ρ is non-degenerate.

2.2. The model is well-defined

Let us prove that the model is well defined, *i.e.* $Z_n \in]0, +\infty[$ for any $n \geqslant d$.

LEMMA 2.1. — *Let $n \geqslant 1$ and let $x_1, \dots, x_n$ be vectors in $\mathbb{R}^d$. We denote*

$$A_n = x_1\, {}^t x_1 + \cdots + x_n\, {}^t x_n.$$

⋆ *If $n < d$, then A_n is non-invertible.*
⋆ *If $n = d$, then A_n is invertible if and only if $(x_1, \dots, x_n)$ is a basis of $\mathbb{R}^d$.*
⋆ *If $n > d$ and if the vectors $x_1, \dots, x_n$ span $\mathbb{R}^d$, then A_n is invertible.*

Proof. ⋆ Let $n \leqslant d$. If $n < d$, we put $x_{n+1} = \cdots = x_d = 0$. We denote by B the $d \times d$ matrix such that its columns are $x_1, \dots, x_d$. We have then, for any $1 \leqslant k, l \leqslant d$,

$$(B\, {}^t B)_{k,l} = \sum_{i=1}^{d} B_{k,i} B_{l,i} = \sum_{i=1}^{d} x_i(k) x_i(l) = \sum_{i=1}^{d} (x_i\, {}^t x_i)_{k,l} = (A_n)_{k,l}.$$

Therefore $A_n = B\, {}^t B$ and thus A_n is invertible if and only if B is invertible. As a consequence A_n is invertible if and only if $(x_1, \dots, x_d)$ is a basis of

$\mathbb{R}^d$. In the case where $n < d$, B has at least a null column and thus is not invertible.

$\star$ Let $n > d$ and assume that the vectors $x_1, \dots, x_n$ span $\mathbb{R}^d$. Then there exists then $1 \leqslant i_1 < \dots < i_d \leqslant n$ such that $(x_{i_1}, \dots, x_{i_d})$ is a basis of $\mathbb{R}^d$. As a consequence, by the previous case, A_n is the sum of a symmetric positive definite matrix and $n-d$ other symmetric positive semi-definite matrices. Therefore A_n is definite thus invertible. $\square$

Let $n \geqslant d$. The non-degeneracy of ρ implies that its support is not included in a hyperplane of $\mathbb{R}^d$. As a consequence

$$\rho^{\otimes n}\big(\{\, (x_1, \dots, x_n) \in (\mathbb{R}^d)^n : (x_1, \dots, x_d) \text{ is a basis of } \mathbb{R}^d \,\}\big) > 0.$$

The previous lemma yields

$$\rho^{\otimes n}\big(\{\, (x_1, \dots, x_n) \in (\mathbb{R}^d)^n : x_1 \, {}^t\!x_1 + \dots + x_n \, {}^t\!x_n \text{ is invertible} \,\}\big) > 0,$$

i.e. $\rho^{\otimes n}(D_n^+) > 0$. Therefore $Z_n > 0$.

Let $\langle \cdot\,,\,\cdot \rangle$ be the usual scalar product on $\mathbb{R}^d$ and $\|\cdot\|$ be the Euclidean norm. We denote:

- $\mathcal{S}_d$ the space of $d \times d$ symmetric matrices.
- $\mathcal{S}_d^+$ the space of all matrices in $\mathcal{S}_d$ which are positive semi-definite.
- $\mathcal{S}_d^{++}$ the space of all matrices in $\mathcal{S}_d$ which are positive definite.

We introduce the sets

$$\Delta = \{\, (x, M) \in \mathbb{R}^d \times \mathcal{S}_d^+ : M - x \, {}^t\!x \in \mathcal{S}_d^+ \,\}.$$

and

$$\Delta^* = \{\, (x, M) \in \mathbb{R}^d \times \mathcal{S}_d^{++} : M - x \, {}^t\!x \in \mathcal{S}_d^+ \,\}.$$

The two following lemmas guarantee that $Z_n < +\infty$ pour tout $n \geqslant 1$.

Lemma 2.2. — *If $(x, M) \in \Delta^*$ then $\langle M^{-1}x, x \rangle \leqslant 1$.*

Proof. The matrix $M - x \, {}^t\!x$ is symmetric positive semi-definite. Hence

$$\forall\, y \in \mathbb{R}^d \qquad \langle x, y \rangle^2 = \langle x \, {}^t\!x \, y, y \rangle \leqslant \langle My, y \rangle.$$

Applying this inequality to $y = M^{-1}x$, we get

$$\langle x, M^{-1}x \rangle^2 \leqslant \langle M^{-1}x, x \rangle.$$

If $x = 0$ then $\langle M^{-1}x, x \rangle = 0 \leqslant 1$. If $x \neq 0$, since $M \in \mathcal{S}_d^{++}$, we have $\langle M^{-1}x, x \rangle > 0$ and thus $\langle M^{-1}x, x \rangle \leqslant 1$. $\square$

Let $n \geqslant 1$. For any $(x_1, \dots, x_n) \in (\mathbb{R}^d)^n$,

$$m = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \Longrightarrow \quad \frac{1}{n}\sum_{i=1}^{n} x_i\,{}^t\!x_i - m\,{}^t\!m = \frac{1}{n}\sum_{i=1}^{n}(x_i - m)\,{}^t(x_i - m) \in \mathcal{S}_d^+.$$

Therefore, for any $(x_1, \dots, x_n) \in D_n^+$,

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i, \frac{1}{n}\sum_{i=1}^{n} x_i\,{}^t\!x_i\right) \in \Delta^*$$

and thus

$$\frac{1}{2}\left\langle \left(\sum_{i=1}^{n} x_i\,{}^t\!x_i\right)^{-1} \left(\sum_{i=1}^{n} x_i\right), \left(\sum_{i=1}^{n} x_i\right)\right\rangle \leqslant \frac{n}{2}.$$

Hence $Z_n \leqslant e^{n/2} < +\infty$ and the model is well-defined for any $n \geqslant d$.

2.3. Large deviations for $(S_n/n, T_n/n)$

As in the one-dimensional case (see [2]), we introduce

$$F : (x, M) \in \Delta^* \longmapsto \frac{\langle M^{-1}x, x\rangle}{2}.$$

For any $n \geqslant d$, the distribution of $(S_n/n, T_n/n)$ under $\widetilde{\mu}_{n,\rho}$ is

$$\frac{\exp(nF(x, M))\mathbb{1}_{\{(x,M)\in\Delta^*\}}\, d\widetilde{\nu}_{n,\rho}(x, M)}{\displaystyle\int_{\Delta^*} \exp(nF(s, N))\, d\widetilde{\nu}_{n,\rho}(s, N)},$$

where $\widetilde{\nu}_{n,\rho}$ is the law of

$$\left(\frac{S_n}{n}, \frac{T_n}{n}\right) = \frac{1}{n}\sum_{i=1}^{n}\left(Y_i, Y_i\,{}^t Y_i\right)$$

when $Y_1, \dots, Y_n$ are independent random vectors with common law ρ.

We endow $\mathbb{R}^d \times \mathcal{S}_d$ with the scalar product given by

$$((x, M), (y, N)) \longmapsto \langle x, y\rangle + \mathrm{tr}(MN) = \sum_{i=1}^{d} x_i y_i + \sum_{i=1}^{d}\sum_{j=1}^{d} m_{i,j} n_{i,j}.$$

We denote by $\|\cdot\|_d$ the associated norm. Notice that

$$\forall\, z \in \mathbb{R}^d \quad \forall\, A \in \mathcal{S}_d \qquad \mathrm{tr}(z\,{}^t\!zA) = \sum_{i=1}^{d}\sum_{j=1}^{d} z_i z_j a_{i,j} = \langle Az, z\rangle.$$

Let ν_ρ be the law of $(Z, Z\,{}^tZ)$ when Z is a random vector with distribution ρ. We define its Log-Laplace Λ, by

$$\forall\ (u,A) \in \mathbb{R}^d \times \mathcal{S}_d \qquad \Lambda(u,A) = \ln \int_{\mathbb{R}^d \times \mathcal{S}_d} \exp\left(\langle z,u\rangle + \mathrm{tr}(MA)\right)\, d\nu_\rho(z,M)$$
$$= \ln \int_{\mathbb{R}^d} \exp\left(\langle u,z\rangle + \langle Az,z\rangle\right)\, d\rho(z),$$

and its Cramér transform I by

$$\forall\ (x,M) \in \mathbb{R}^d \times \mathcal{S}_d \qquad I(x,M) = \sup_{(u,A)\in\mathbb{R}^d\times\mathcal{S}_d} \big(\, \langle x,u\rangle + \mathrm{tr}(MA) - \Lambda(u,A)\,\big).$$

Let D_Λ and D_I be the domains of $\mathbb{R}^d \times \mathcal{S}_d$ where Λ and I are respectively finite. All these definitions generalize the case where $d = 1$, treated in [2] and [7].

For any $(u,A) \in \mathbb{R}^d \times \mathcal{S}_d$, we have

$$\exp\, \Lambda(u,A) \leqslant \int_{\mathbb{R}^d} \exp\left(\|u\|\,\|z\| + \sqrt{\mathrm{tr}(A^2)}\,\|z\|^2\right)\, d\rho(z)$$
$$\leqslant \int_{\mathbb{R}^d} \exp\left(\|(u,A)\|_d\, \max(\|z\|, \|z\|^2)\right)\, d\rho(z)$$
$$\leqslant \exp\left(\|(u,A)\|_d\right) + \int_{\mathbb{R}^d} \exp\left(\|(u,A)\|_d\, \|z\|^2\right)\, d\rho(z).$$

Therefore hypothesis $(H1)$ is sufficient to ensure that $(0, O_d)$ belongs to $\overset{o}{D}_\Lambda$, where O_d denotes the $d \times d$ matrix whose coefficients are all zero. As a consequence Cramér's theorem (cf. [3]) implies that $(\widetilde{\nu}_{n,\rho})_{n\geqslant 1}$ satisfies the large deviation principle with speed n and governed by the good rate function I.

3. Convergence in probability of $(S_n/n, T_n/n)$

We saw in the previous section that, under the hypothesis of theorem 1.1, the sequence $(\widetilde{\nu}_{n,\rho})_{n\geqslant 1}$ satisfies the large deviation principle with speed n and governed by the good rate function I. This result and Varadhan's lemma (see [3]) suggest that, asymptotically, $(S_n/n, T_n/n)$ concentrates on the minima of the function $I - F$. In subsection 3.3.1, we prove that $I - F$ has a unique minimum at $(0, \Sigma)$ on Δ^* and we extend F on the entire closed set Δ so that it remains true on Δ. This is the key ingredient for the proof of the law of large numbers in theorem 1.1, given in subsection 3.3.2.

3.1. Minimum de $I-F$

Proposition 3.1. — *If ρ is a symmetric non-degenerate probability measure on $\mathbb{R}^d$, then*

$$\forall\, x \in \mathbb{R}^d \backslash \{0\} \quad \forall\, M \in \mathcal{S}_d^{++} \qquad I(x,M) > \frac{\langle M^{-1}x, x\rangle}{2}.$$

Moreover, if Λ is finite in a neighbourhood of $(0, O_d)$, then the function $I-F$ has a unique minimum at $(0,\Sigma)$ on Δ^*.

Proof. Let $x \in \mathbb{R}^d \backslash \{0\}$ and $M \in \mathcal{S}_d^{++}$. By taking $A = -M^{-1}x\,{}^t\!xM^{-1}/2$ and $u = M^{-1}x$, we get

$$\langle u, x\rangle + \mathrm{tr}(AM) = \langle M^{-1}x, x\rangle - \frac{1}{2}\mathrm{tr}(M^{-1}x\,{}^t\!x) = \frac{\langle M^{-1}x, x\rangle}{2}.$$

As a consequence

$$I(x,M) \geqslant \frac{\langle M^{-1}x, x\rangle}{2} - \Lambda\left(M^{-1}x, -\frac{1}{2}M^{-1}x\,{}^t\!xM^{-1}\right).$$

For any $z \in \mathbb{R}^d$, we have ${}^t\!zM^{-1}x = \langle M^{-1}x, z\rangle = \mathrm{tr}(z\,{}^t(M^{-1}x)) \in \mathbb{R}$ thus

$$-\frac{1}{2}\mathrm{tr}(z\,{}^t\!zM^{-1}x\,{}^t\!xM^{-1}) = -\frac{\langle M^{-1}x, z\rangle}{2}\mathrm{tr}(z\,{}^t\!xM^{-1}) = -\frac{\langle M^{-1}x, z\rangle^2}{2}.$$

Therefore

$$\Lambda\left(M^{-1}x, -\frac{1}{2}M^{-1}x\,{}^t\!xM^{-1}\right) = \ln\int_{\mathbb{R}^d}\exp\left(\langle M^{-1}x, z\rangle - \frac{\langle M^{-1}x, z\rangle^2}{2}\right)d\rho(z).$$

By symmetry of ρ, we have, for any $s \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d}\exp\left(\langle s, z\rangle - \frac{\langle s, z\rangle^2}{2}\right)d\rho(z) = \int_{\mathbb{R}^d}\exp\left(-\langle s, z\rangle - \frac{\langle s, z\rangle^2}{2}\right)d\rho(z)$$
$$= \frac{1}{2}\left(\int_{\mathbb{R}^d}\exp\left(\langle s, z\rangle - \frac{\langle s, z\rangle^2}{2}\right)d\rho(z) + \int_{\mathbb{R}^d}\exp\left(-\langle s, z\rangle - \frac{\langle s, z\rangle^2}{2}\right)d\rho(z)\right)$$
$$= \int_{\mathbb{R}^d}\cosh(\langle s, z\rangle)\,\exp\left(-\frac{\langle s, z\rangle^2}{2}\right)d\rho(z).$$

As a consequence

$$\Lambda\left(M^{-1}x, -\frac{1}{2}M^{-1}x\,{}^t\!xM^{-1}\right) =$$
$$\ln\int_{\mathbb{R}^d}\cosh\left(\langle M^{-1}x, z\rangle\right)\,\exp\left(-\frac{\langle M^{-1}x, z\rangle^2}{2}\right)d\rho(z).$$

It is straightforward to see that the function $y \longmapsto 1 - \cosh(y)\exp(-y^2/2)$ is non-negative on $\mathbb{R}$ and vanishes only at 0. Hence, for any $z \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \cosh\left(\langle s, z\rangle\right) \exp\left(-\frac{\langle s, z\rangle^2}{2}\right) d\rho(z) \leqslant 1,$$

and equality holds if and only if $\rho(\{z : \langle s, z\rangle = 0\}) = 1$. The non-degeneracy of ρ implied that the equality case only holds if $s = 0$. Applying this to $s = M^{-1}x \neq 0$, we obtain

$$\Lambda\left(M^{-1}x, -\frac{1}{2}M^{-1}x\,{}^t\!xM^{-1}\right) < 0,$$

and thus $I(x, M) > \langle M^{-1}x, x\rangle/2$.

Suppose now that $x = 0$ and $M \in \mathcal{S}_d^{++}$. Then

$$I(x, M) - \frac{\langle M^{-1}x, x\rangle}{2} = I(0, M).$$

If we assume that Λ is finite in a neighbourhood of $(0, \dots, 0, O_d)$, then $I(0, M) = 0$ if and only if $M = \Sigma$ (see proposition III.4 of [6]). This ends the proof of the proposition. □

However, in order to apply Varadhan's lemma, F must be extended to an upper semi-continuous function on the entire closed set Δ. To this end, we put

$$\forall\,(x, M) \in \Delta\backslash\Delta^* \qquad F(x, M) = \frac{1}{2},$$

and it is easy to check that F is indeed an upper semi-continuous function on Δ.

Now we prove the inequality in proposition 3.1 holds on Δ.

Let $(x, M) \in \mathbb{R}^d \times \mathcal{S}_d^+$. We denote by $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \dots, \leqslant \lambda_d$ the eigenvalues (not necessary distinct) of M. There exists an orthogonal matrix P such that $M = PD\,{}^tP$, where D is the diagonal matrix such that $D_{i,i} = \lambda_i$ for any $i \in \{1, \dots, d\}$. We have

$$\begin{aligned} I(x, M) &= \sup_{(u,A)\in\mathbb{R}^d\times\mathcal{S}_d} \big(\langle x, u\rangle + \operatorname{tr}(PD\,{}^tPA) - \Lambda(u, A)\big) \\ &= \sup_{(u,A)\in\mathbb{R}^d\times\mathcal{S}_d} \big(\langle x, u\rangle + \operatorname{tr}(DA) - \Lambda(u, PA\,{}^tP)\big). \end{aligned}$$

Assume that $M \notin \mathcal{S}_d^{++}$ and denote by $k = k_M \geqslant 1$ the dimension of the kernel of M. Let $a \in]-\infty, 0[$. By taking $u = 0$ and A the symmetric matrix such that

$$\forall\,(i, j) \in \{1, \dots, d\} \qquad A_{i,j} = \begin{cases} a & \text{if} \quad i = j \in \{1, \dots, k\}, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain

$$I(x,M) \geqslant -\Lambda(u, PA\,{}^tP) = -\ln \int_{\mathbb{R}^d} \exp \langle PA\,{}^tPz, z\rangle \, d\rho(z),$$

i.e.

$$\forall\, a \in \mathbb{R} \qquad I(x,M) \geqslant -\ln \int_{\mathbb{R}^d} \exp\left(a \sum_{j=1}^{k} \left({}^tPz\right)_j^2 \right) \, d\rho(z).$$

For any $z \in \mathbb{R}^d$, we have

$$\sum_{j=1}^{k} \left({}^tPz\right)_j^2 = 0 \quad \Longleftrightarrow \quad z \in \mathrm{Ker}(M)^{\perp},$$

since $(Pe_1, \ldots, Pe_k)$ is a basis of $\mathrm{Ker}(M)$ (they are the eigenvectors of M associated to the eigenvalue 0). As a consequence

$$\forall\, z \in \mathbb{R}^d \qquad \exp\left(a \sum_{j=1}^{k} \left({}^tPz\right)_j^2 \right) \underset{a \to -\infty}{\longrightarrow} \mathbb{1}_{\mathrm{Ker}(M)^{\perp}}(z).$$

Moreover the left term defines a function which is bounded above by 1. Therefore the dominated convergence theorem implies that

$$\int_{\mathbb{R}^d} \exp\left(a \sum_{j=1}^{k} \left({}^tPz\right)_j^2 \right) \, d\rho(z) \underset{a \to -\infty}{\longrightarrow} \rho\big(\mathrm{Ker}(M)^{\perp}\big),$$

hence

$$I(x,M) \geqslant -\ln \rho\big(\mathrm{Ker}(M)^{\perp}\big),$$

so that $I(x,M) > 1/2$ as soon as $\rho\big(\mathrm{Ker}(M)^{\perp}\big) < e^{-1/2}$. Since $\mathrm{Ker}(M)^{\perp}$ is included in some vector hyperplane of $\mathbb{R}^d$, we obtain the following proposition:

Proposition 3.2. — *If ρ is a symmetric probability measure on $\mathbb{R}^d$ satisfying hypothesis $(H1)$ and $(H2)$, then $I - F$ has a unique minimum at $(0,\Sigma)$ on Δ.*

3.2. Convergence of $(S_n/n, T_n/n)$ under $\widetilde{\mu}_{n,\rho}$

Let us first prove the following proposition, which is a consequence of Varadhan's lemma.

Proposition 3.3. — *Let ρ be a symmetric probability measure on $\mathbb{R}^d$ with a positive definite covariance matrix Σ. We have*

$$\liminf_{n \to +\infty} \frac{1}{n} \ln Z_n \geqslant 0.$$

Suppose that ρ satisfies hypothesis $(H1)$ and $(H2)$. If $\mathcal{A}$ is a closed subset of $\mathbb{R}^d \times \mathcal{S}_d$ which does not contain $(0,\Sigma)$, then

$$\limsup_{n\to+\infty} \frac{1}{n} \ln \int_{\Delta^*\cap\mathcal{A}} \exp\left(\frac{n\langle M^{-1}x,x\rangle}{2}\right) d\widetilde{\nu}_{n,\rho}(x,M) < 0.$$

Proof. The set $\overset{o}{\Delta}$, the interior of Δ^*, contains $(0,\Sigma)$ thus Cramér's theorem (cf. [3]) implies that

$$\begin{aligned}\liminf_{n\to+\infty} \frac{1}{n} \ln Z_n &= \liminf_{n\to+\infty} \frac{1}{n} \ln \int_{\Delta^*} \exp\left(\frac{n\langle M^{-1}x,x\rangle}{2}\right) d\widetilde{\nu}_{n,\rho}(x,M) \\ &\geqslant \liminf_{n\to+\infty} \frac{1}{n} \ln \widetilde{\nu}_{n,\rho}(\Delta^*) \geqslant -\inf\left\{ I(x,M) : (x,M) \in \overset{o}{\Delta}\right\} = 0.\end{aligned}$$

We prove now the second inequality. Since ρ verifies hypothesis $(H1)$, we have that $(0,O_d) \in \overset{o}{D}_\Lambda$. Cramér's theorem implies then that $(\widetilde{\nu}_{n,\rho})_{n\geqslant 1}$ satisfies the large deviation principle with speed n and the good rate function I. Since F is upper semi-continuous on the closed set Δ, a variant of Varadhan's lemma (see Lemma 4.3.6 of [3]) yields

$$\begin{aligned}&\limsup_{n\to+\infty} \frac{1}{n} \ln \int_{\Delta^*\cap\mathcal{A}} \exp\left(\frac{n\langle M^{-1}x,x\rangle}{2}\right) d\widetilde{\nu}_{n,\rho}(x,M) \\ &\qquad\leqslant \limsup_{n\to+\infty} \frac{1}{n} \ln \int_{\Delta\cap\mathcal{A}} \exp\left(nF(x,M)\right) d\widetilde{\nu}_{n,\rho}(x,M) \leqslant \sup_{\Delta\cap\mathcal{A}} \left(F - I \right).\end{aligned}$$

Since ρ satisfies hypothesis $(H2)$, proposition 3.2 implies that $I - F$ has a unique minimum at $(0,\Sigma)$ on Δ. Since the closed subset $\Delta \cap \mathcal{A}$ does not contain $(0,\Sigma)$ and since F is upper semi-continuous and I is a good rate function, we have

$$\sup_{\Delta\cap\mathcal{A}} \left(F - I \right) < 0.$$

This proves the second inequality of the proposition. □

Proof of the law of large numbers in theorem 1.1. Suppose that ρ is symmetric and satisfies hypothesis $(H1)$ and $(H2)$. Let us denote by $\theta_{n,\rho}$ the law of $(S_n/n, T_n/n)$ under $\widetilde{\mu}_{n,\rho}$. Let U be an open neighbourhood of $(0,\Sigma)$ in $\mathbb{R}^d \times \mathcal{S}_d$. Proposition 3.3 implies that

$$\begin{aligned}\limsup_{n\to+\infty} \frac{1}{n} \ln \theta_{n,\rho}(U^c) = \limsup_{n\to+\infty} \frac{1}{n} \ln \int_{\Delta^*\cap U^c} \exp\left(\frac{n\langle M^{-1}x,x\rangle}{2}\right) d\widetilde{\nu}_{n,\rho}(x,M) \\ - \liminf_{n\to+\infty} \frac{1}{n} \ln Z_n < 0.\end{aligned}$$

Hence there exist $\varepsilon > 0$ and $n_0 \geqslant 1$ such that $\theta_{n,\rho}(U^c) \leqslant e^{-n\varepsilon}$ for any $n \geqslant n_0$. Therefore, for any neighbourhood U of $(0,\Sigma)$,

$$\lim_{n\to+\infty} \widetilde{\mu}_{n,\rho}\left(\left(\frac{S_n}{n},\frac{T_n}{n}\right)\in U^c\right)=0,$$

i.e. under $\widetilde{\mu}_{n,\rho}$, $(S_n/n, T_n/n)$ converges in probability to $(0,\Sigma)$. □

4. Convergence in distribution of $T_n^{-1/2}\, S_n/n^{1/4}$ under $\widetilde{\mu}_{n,\rho}$

In this section, we generalize theorem 1 of [9] to the higher dimension in order to prove our fluctuation result.

THEOREM 4.1. — *Let ρ be a symmetric non-degenerate probability measure on $\mathbb{R}^d$ such that*

$$\int_{\mathbb{R}^d} \|z\|^5 \, d\rho(z) < +\infty.$$

Let Σ the covariance matrix of ρ and let M_4 be the function defined in theorem 1.1. Then, under $\widetilde{\mu}_{n,\rho}$,

$$\frac{1}{n^{1/4}}\, T_n^{-1/2}\, S_n \xrightarrow[n\to\infty]{\mathscr{L}} \frac{\exp\left(-\dfrac{1}{12} M_4\big(\Sigma^{-1/2} z\big)\right)\, dz}{\displaystyle\int_{\mathbb{R}^d} \exp\left(-\dfrac{1}{12} M_4\big(\Sigma^{-1/2} u\big)\right)\, du}.$$

In the proof of this theorem, we show that the limiting law is well defined. Notice that, if $d=1$, then $\Sigma^{-1/2}=\sigma^{-1}$ and

$$\forall\, z\in\mathbb{R} \qquad M_4\big(\Sigma^{-1/2} z\big) = \frac{\mu_4 z^4}{\sigma^4}.$$

Hence theorem 4.1 is indeed a generalization of theorem 1 of [9]

4.1. Proof of theorem 4.1

Let $(X_n^k)_{n\geqslant d,\, 1\leqslant k\leqslant n}$ be an infinite triangular array of random variables such that, for any $n \geqslant d$, $(X_n^1,\dots,X_n^n)$ has the law $\widetilde{\mu}_{n,\rho}$. Let us recall that

$$\forall\, n\geqslant 1 \qquad S_n = X_n^1+\cdots+X_n^n \qquad \text{and} \qquad T_n = X_n^1\, {}^t(X_n^1)+\cdots+X_n^n\, {}^t(X_n^n).$$

and that $T_n \in \mathcal{S}_d^{++}$ almost surely. We use the Hubbard-Stratonovich transformation: let W be a random vector with standard multivariate Gaussian distribution and which is independent of $(X_n^k)_{n\geqslant d,\,1\leqslant k\leqslant n}$. Let $n\geqslant 1$ and f be a bounded continuous function on $\mathbb{R}^d$. We put

$$E_n=\mathbb{E}\left[f\left(\frac{W}{n^{1/4}}+\frac{1}{n^{1/4}}\,T_n^{-1/2}\,S_n\right)\right].$$

We introduce $(Y_i)_{i\geqslant 1}$ a sequence of independent random vectors with common distribution ρ. We denote

$$A_n=\sum_{i=1}^n Y_i,\qquad B_n=\left(\sum_{i=1}^n Y_i\,{}^tY_i\right)^{1/2}\qquad\text{and}\qquad \mathcal{B}_n=\big\{\det(B_n^2)>0\big\}.$$

We have

$$E_n=\frac{1}{Z_n(2\pi)^{d/2}}\,\mathbb{E}\Bigg[\mathbb{1}_{\mathcal{B}_n}\int_{\mathbb{R}^d}f\left(\frac{w}{n^{1/4}}+\frac{1}{n^{1/4}}B_n^{-1}A_n\right)\times\exp\left(\frac{1}{2}\Big\langle B_n^{-2}A_n,A_n\Big\rangle-\frac{\|w\|^2}{2}\right)dw\Bigg].$$

We make the change of variables $z=n^{-1/4}\left(w+B_n^{-1}A_n\right)$ in the integral and we get

$$E_n=C_n\,\mathbb{E}\left[\mathbb{1}_{\mathcal{B}_n}\int_{\mathbb{R}^d}f(z)\exp\left(-\frac{\sqrt{n}\|z\|^2}{2}+n^{1/4}\big\langle z,B_n^{-1}A_n\big\rangle\right)dz\right]$$

where $C_n=n^{d/4}Z_n^{-1}(2\pi)^{-d/2}$. Let $U_1,\dots,U_n,\varepsilon_1,\dots,\varepsilon_n$ be independent random variables such that the distribution of U_i is ρ and the distribution of ε_i is $(\delta_{-1}+\delta_1)/2$, for any $i\in\{1,\dots,n\}$. Since ρ is symmetric, the random variables $\varepsilon_1U_1,\dots,\varepsilon_nU_n$ are also independent with common distribution ρ. Therefore

$$E_n=C_n\,\mathbb{E}\left[\mathbb{1}_{\mathcal{B}_n}\int_{\mathbb{R}^d}f(z)\exp\left(-\frac{\sqrt{n}\|z\|^2}{2}+n^{1/4}\left\langle z,B_n^{-1}\left(\sum_{i=1}^n\varepsilon_iU_i\right)\right\rangle\right)dz\right].$$

In the case where the matrix $B_n^2=U_1{}^tU_1+\cdots+U_n{}^tU_n$ is invertible, we denote

$$\forall\,i\in\{1,\dots,n\}\qquad a_{i,n}=\left(\sum_{j=1}^n U_j\,{}^tU_j\right)^{-1/2}U_i.$$

By using Fubini's theorem and the independence of ε_i,U_i, $i\geqslant 1$, we obtain

$$E_n=C_n\,\mathbb{E}\Bigg[\mathbb{1}_{\mathcal{B}_n}\int_{\mathbb{R}^d}f(z)\exp\left(-\frac{\sqrt{n}\|z\|^2}{2}\right)\times\mathbb{E}\left(\prod_{i=1}^n\exp\left(n^{1/4}\varepsilon_i\langle z,a_{i,n}\rangle\right)\,\Bigg|\,(U_1,\dots,U_n)\right)dz\Bigg].$$

Therefore

$$E_n = C_n \mathbb{E}\left[\mathbb{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} f(z) \exp\left(-\frac{\sqrt{n}\|z\|^2}{2} \right) \exp\left(\sum_{i=1}^n \ln\cosh\left(n^{1/4} \langle z, a_{i,n} \rangle \right) \right) \, dz \right].$$

We define the function g by

$$\forall\, y \in \mathbb{R} \qquad g(y) = \ln\cosh y - \frac{y^2}{2}.$$

It is easy to see that $g(y) < 0$ if $y > 0$. Therefore

$$\sum_{i=1}^n \langle z, a_{i,n} \rangle^2 = \sum_{i=1}^n \langle z, (a_{i,n}\, {}^t a_{i,n}) z \rangle = \left\langle z, \left(\sum_{i=1}^n a_{i,n}\, {}^t a_{i,n} \right) z \right\rangle = \langle z, \mathrm{I}_d z \rangle = \|z\|^2.$$

As a consequence

$$E_n = C_n \, \mathbb{E}\left[\mathbb{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} f(z) \exp\left(\sum_{i=1}^n g(n^{1/4} \langle z, a_{i,n} \rangle) \right) \, dz \right].$$

Now we use Laplace's method. Let us examine the convergence of the term in the exponential: for any $z \in \mathbb{R}^d$ and $i \in \{1, \dots, n\}$, the Taylor-Lagrange formula states that there exists a random variable $\xi_{n,i}$ such that

$$g\big(n^{1/4} \langle z, a_{i,n} \rangle\big) = -\frac{n \langle z, a_{i,n} \rangle^4}{12} + \frac{n^{3/2} \langle z, a_{i,n} \rangle^5}{n^{1/4} 5!} g^{(5)}(\xi_{n,i}).$$

Let $z \in \mathbb{R}^d$. We have

$$n \sum_{i=1}^n \langle z, a_{i,n} \rangle^4 = n \sum_{i=1}^n \left\langle B_n^{-1} z, U_i \right\rangle^4 = \frac{1}{n} \sum_{i=1}^n \left\langle \sqrt{n} B_n^{-1} z, U_i \right\rangle^4 .$$

We denote $\zeta_n = \sqrt{n} B_n^{-1} z$. We have

$$n \sum_{i=1}^n \langle z, a_{i,n} \rangle^4 = \frac{1}{n} \sum_{i=1}^n \langle \zeta_n, U_i \rangle^4 = \frac{1}{n} \sum_{i=1}^n \left(\sum_{j=1}^d (\zeta_n)_j (U_i)_j \right)^4$$

$$= \sum_{1 \leqslant j_1, j_2, j_3, j_4 \leqslant d} (\zeta_n)_{j_1} (\zeta_n)_{j_2} (\zeta_n)_{j_3} (\zeta_n)_{j_4} \, \frac{1}{n} \sum_{i=1}^n (U_i)_{j_1} (U_i)_{j_2} (U_i)_{j_3} (U_i)_{j_4}.$$

Since ρ is non-degenerate, its covariance matrix Σ is invertible. Moreover ρ has a finite fourth moment thus the law of large numbers implies that

$$\zeta_n \xrightarrow[n \to +\infty]{\text{a.s}} \Sigma^{-1/2} z,$$

and that, for any $(j_1, j_2, j_3, j_4) \in \{1, \dots, d\}^4$,

$$\frac{1}{n} \sum_{i=1}^n (U_i)_{j_1} (U_i)_{j_2} (U_i)_{j_3} (U_i)_{j_4} \xrightarrow[n \to +\infty]{\text{a.s}} \int_{\mathbb{R}^d} y_{j_1} y_{j_2} y_{j_3} y_{j_4} \, d\rho(y).$$

As a consequence

$$n\sum_{i=1}^{n}\langle z,a_{i,n}\rangle^4 \underset{n\to+\infty}{\overset{\text{a.s}}{\longrightarrow}} M_4\left(\Sigma^{-1/2}z\right).$$

Since ρ has a finite fifth moment, we prove similarly that

$$n^{3/2}\sum_{i=1}^{n}\langle z,a_{i,n}\rangle^5 \underset{n\to+\infty}{\overset{\text{a.s}}{\longrightarrow}} M_5\left(\Sigma^{-1/2}z\right),$$

where $M_5(z)=\int_{\mathbb{R}^d}\langle z,y\rangle^5\,d\rho(y)$ for any $z\in\mathbb{R}^d$. Finally, by a simple computation, we see that $g^{(5)}$ is bounded over $\mathbb{R}$. Hence

$$\forall\, z\in\mathbb{R}^d\qquad \sum_{i=1}^{n} g\big(n^{1/4}\langle z,a_{i,n}\rangle\big) \underset{n\to+\infty}{\overset{\text{a.s}}{\longrightarrow}} -\frac{1}{12}M_4\left(\Sigma^{-1/2}z\right).$$

Lemma 4.2. — *There exists $c>0$ such that*

$$\forall\, z\in\mathbb{R}^d\quad \forall\, n\geqslant 1\qquad \sum_{i=1}^{n} g\big(n^{1/4}\langle z,a_{i,n}\rangle\big)\leqslant -\frac{c\|z\|^4}{1+\|z\|^2/\sqrt{n}}.$$

Proof. We define h by

$$\forall\, y\in\mathbb{R}\backslash\{0\}\qquad h(y)=\frac{1+y^2}{y^4}g(y).$$

It is a non-negative continuous function on $\mathbb{R}\backslash\{0\}$. Since $g(y)\sim -y^4/12$ in the neighbourhood of 0, the function h can be extended to a function continuous on $\mathbb{R}$ by putting $h(0)=-1/12$. Next we have

$$\forall\, y\in\mathbb{R}\backslash\{0\}\qquad h(y)=\frac{1+y^2}{y^2}\times\left(\frac{\ln\cosh y}{y^2}-\frac{1}{2}\right),$$

so that $h(y)$ goes to $-1/2$ when $|y|$ goes to $+\infty$. Therefore h is bounded by some constant $-c$ with $c>0$. Hence, for any $z\in\mathbb{R}$ and $n\geqslant 1$,

$$\sum_{i=1}^{n} g\big(n^{1/4}\langle z,a_{i,n}\rangle\big)\leqslant -nc\,\frac{1}{n}\sum_{i=1}^{n}\frac{\big(n^{1/4}\langle z,a_{i,n}\rangle\big)^4}{1+\big(n^{1/4}\langle z,a_{i,n}\rangle\big)^2}.$$

We easily check that $x\longmapsto x^2/(1+x)$ is convex on $[0,+\infty[$. As a consequence

$$\sum_{i=1}^{n} g\big(n^{1/4}\langle z,a_{i,n}\rangle\big)\leqslant -nc\,\frac{\left(\frac{1}{n}\sum_{i=1}^{n}\big(n^{1/4}\langle z,a_{i,n}\rangle\big)^2\right)^2}{1+\frac{1}{n}\sum_{i=1}^{n}\big(n^{1/4}\langle z,a_{i,n}\rangle\big)^2}=-\frac{c\|z\|^4}{1+\|z\|^2/\sqrt{n}},$$

since $\langle z,a_{1,n}\rangle^2+\cdots+\langle z,a_{n,n}\rangle^2=1$. $\square$

If $\|z\| \leqslant n^{1/4}$ then $1 + \|z\|^2/\sqrt{n} \leqslant 2$ and thus, by the previous lemma,

$$\left| \mathbb{1}_{\mathcal{B}_n} \mathbb{1}_{\|z\| \leqslant n^{1/4}} \exp\left(\sum_{i=1}^{n} g\big(n^{1/4}\langle z, a_{i,n}\rangle\big) \right) \right| \leqslant \exp\left(-\frac{c\|z\|^4}{2} \right).$$

Thus the dominated convergence theorem implies that

$$z \longmapsto \exp\left(-M_4\left(\Sigma^{-1/2} z\right)/12 \right)$$

is integrable on $\mathbb{R}^d$ and that

$$\mathbb{E}\left[\mathbb{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} \mathbb{1}_{|z| \leqslant n^{1/4}} f(z) \exp\left(\sum_{i=1}^{n} g\big(n^{1/4}\langle z, a_{i,n}\rangle\big) \right) dz \right] \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}^d} f(z) \exp\left(-\frac{1}{12} M_4\left(\Sigma^{-1/2} z\right) \right) dz.$$

If $\|z\| > n^{1/4}$ then $1 + \|z\|^2/\sqrt{n} \leqslant 2\|z\|^2/\sqrt{n}$ and thus, by the previous lemma,

$$\mathbb{E}\left[\mathbb{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} \mathbb{1}_{|z| > n^{1/4}} f(z) \exp\left(\sum_{i=1}^{n} g\big(n^{1/4}\langle z, a_{i,n}\rangle\big) \right) dz \right] \leqslant \|f\|_\infty \int_{\mathbb{R}^d} \exp\left(-\frac{c\sqrt{n}\|z\|^2}{2} \right) dz = \frac{\|f\|_\infty (2\pi)^{d/2}}{n^{d/4} c^{d/2}} \underset{n\to+\infty}{\longrightarrow} 0,$$

and thus

$$\frac{E_n}{C_n} = \mathbb{E}\left[\mathbb{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} f(z) \exp\left(\sum_{i=1}^{n} g\big(n^{1/4}\langle z, a_{i,n}\rangle\big) \right) dz \right] \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}^d} f(z) \exp\left(-\frac{1}{12} M_4\left(\Sigma^{-1/2} z\right) \right) dz.$$

If we take $f = 1$, we get

$$\frac{1}{C_n} = \frac{Z_n (2\pi)^{d/2}}{n^{d/4}} \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}^d} \exp\left(-\frac{1}{12} M_4\left(\Sigma^{-1/2} z\right) \right) dz.$$

Summarizing, we have proved that

$$\frac{W}{n^{1/4}} + \frac{1}{n^{1/4}} T_n^{-1/2} S_n \underset{n\to\infty}{\overset{\mathscr{L}}{\longrightarrow}} \frac{\exp\left(-\frac{1}{12} M_4\big(\Sigma^{-1/2} z\big) \right) dz}{\displaystyle\int_{\mathbb{R}^d} \exp\left(-\frac{1}{12} M_4\big(\Sigma^{-1/2} u\big) \right) du}.$$

Since $(W n^{-1/4})_{n \geqslant 1}$ converges in distribution to 0, Slutsky's lemma (theorem 3.9 of [1]) implies the convergence in distribution of theorem 4.1.

We remark that we needed the hypothesis that ρ has a finite fifth moment in order to use Taylor-Lagrange formula. This hypothesis may certainly be weakened by assuming instead that

$$\exists\,\varepsilon>0\qquad \int_{\mathbb{R}^d}\|z\|^{4+\varepsilon}\,d\rho(z)<+\infty.$$

4.2. Proof of the fluctuation result in theorem 1.1.

In section 3, we proved the law of large numbers in theorem 1.1. It implies that, under $\widetilde{\mu}_{n,\rho}$, T_n/n converges in probability to Σ. Moreover hypothesis $(H1)$ implies that $(0,O_d)\in \overset{o}{D}_\Lambda$ and thus ρ has finite moments of all orders. Theorem 4.1 and Slutsky lemma yield

$$\frac{S_n}{n^{3/4}}=\left(\frac{T_n}{n}\right)^{1/2}\times\frac{1}{n^{1/4}}\,T_n^{-1/2}\,S_n\ \underset{n\to\infty}{\overset{\mathscr{L}}{\longrightarrow}}\ \frac{\exp\left(-\frac{1}{12}M_4\big(\Sigma^{-1}z\big)\right)\,dz}{\displaystyle\int_{\mathbb{R}^d}\exp\left(-\frac{1}{12}M_4\big(\Sigma^{-1}u\big)\right)\,du}.$$

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