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Jean Raimbault

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Analytic, Reidemeister and homological torsion for congruence three–manifolds

Jean Raimbault [1]

Pour les 60 ans de Jean-Pierre Otal

Abstract. — For a given Bianchi group Γ and certain natural coefficent modules $V_{\mathbb{Z}}$ and sequences Γ_n of congruence subgroups of Γ we give a conjecturally optimal upper bound for the size of the torsion subgroup of $H_1(\Gamma_n; V_{\mathbb{Z}})$. We also prove limit multiplicity results for the irreducible components of $L^2_{\mathrm{cusp}}(\Gamma_n \backslash \mathrm{SL}_2(\mathbb{C}))$.

Résumé. — Soit Γ un groupe de Bianchi. Pour certains $\mathbb{Z}\Gamma$-modules $V_{\mathbb{Z}}$, et suites Γ_n de sous-groupes de congruence de Γ nous démontrons une borne supérieure, conjecturée optimale, pour la taille du sous-groupe de torsion de l'homologie $H_1(\Gamma_n, V_{\mathbb{Z}})$. On démontre aussi des résultats de multiplicités limites pour les facteurs irréductibles des espaces $L^2_{\mathrm{cusp}}(\Gamma_n \backslash \mathrm{SL}_2(\mathbb{C}))$.

1. Introduction

1.1. Torsion in the homology of arithmetic groups and hyperbolic manifolds

Let Γ be a discrete group and $V_{\mathbb{Z}}$ a free, finitely generated $\mathbb{Z}$-module with a Γ-action. The cohomology $H^*(\Gamma; V_{\mathbb{Z}})$ is an important invariant of Γ since it is both accessible to computation (though not necessarily efficiently) and often contains nontrivial information. If Γ is the fundamental group of an aspherical manifold M then there is a local system $\mathcal{V}$ on M such that $H^*(\Gamma; V_{\mathbb{Z}}) = H^*(M; \mathcal{V})$. When M is endowed with a Riemannian metric this gives analytic tools for the study of the characteristic zero cohomology

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[1] Institut de Mathématiques de Toulouse, UMR5219, Université de Toulouse, CNRS, UPS IMT, F-31062 Toulouse Cedex 9, France —
jean.raimbault@math.univ-toulouse.fr

$H^*(\Gamma; V_{\mathbb{C}})$. Maybe the most famous instance of this is when Γ is a torsion-free congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ and $V_{\mathbb{Z}}$ is a space of homogeneous polynomials. In this case, by the Eichler–Shimura isomorphism the cohomology can be computed via classical modular forms which correspond to certain harmonic forms on the Riemann surface $\Gamma\backslash\mathbb{H}^2$ (where $\mathbb{H}^2$ is the hyperbolic plane). More generally, if Γ is an arithmetic lattice in a real Lie group then classes in $H^*(\Gamma; V_{\mathbb{C}})$ correspond to "automorphic forms" on G. This correspondance is interesting in both directions: the analytic side is easier to grasp to prove theoretical results (in particular asymptotic results, as we will see below) but on the other hand the combinatorial side makes the exact computation of the cohomology groups possible (this has been used for example to experimentally check special cases of Langlands functoriality, as in [17]).

The torsion part of the cohomology is somewhat less accessible from both the combinatorial and analytic viewpoint. On the other hand it is in certain cases of greater interest than the characteristic zero cohomology. In what follows we will be exclusively interested in arithmetic subgroups of the Lie group $\mathrm{SL}_2(\mathbb{C})$. In this case, if Γ is torsion-free, it acts freely and properly discontinuously on the hyperbolic space $\mathbb{H}^3$ and the associated manifold $M = \Gamma\backslash\mathbb{H}^3$ has finite Riemannian volume. It has been observed that very often we have $H^1(M; \mathbb{C}) = 0$. On the other hand the torsion part tends to be very large. For numerical illustrations of these points see [33]. There is also a form of functoriality for torsion classes which has been explored in [4], [7] and [32], which makes them of interest in number theory.

In this paper we will be interested in asymptotic statements about the size of the torsion subgroup of $H^1(\Gamma_n; V_{\mathbb{Z}})$, when Γ_n is a sequence of lattices with covolume tending to infinity. The characteristic zero counterpart of this is the "limit multiplicity problem" which was studied by many people and (at least in the case of congruence subgroups) received a definitive solution in [1] and [11]. We will be interested in the following conjecture (we also give a statement for nonarithmetic manifolds since we try to maintain an interest in the topological aspects of the problem). We will use the notion of a "arithmetic Γ-module", that is a lattice $V_{\mathbb{Z}} \subset V$ where $\rho : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(V)$ is a representation and $\rho(\Gamma)$ stabilises $V_{\mathbb{Z}}$.[(1)] This conjecture first appeared in print in [4], in the arithmetic setting, but in the topological case Thang Lê had independently formulated it (and dubbed it "topological volume conjecture", in analogy with the Volume Conjecture in quantum topology).

Conjecture 1.1. — *If $M = \Gamma\backslash\mathbb{H}^3$ is a closed or cusped hyperbolic 3–manifold then there exists a sequence of subgroups $\Gamma < \Gamma_1 < \cdots < \Gamma_n < \cdots$*

[(1)] These exist for nontrivial ρ if and only if Γ is arithmetic, on the other hand this includes the case of trivial coefficients which is the most important for topologists.

with $\bigcap_n \Gamma_n = \{1\}$ *and*

$$\lim_{n\to+\infty} \frac{\log|H^1(\Gamma_n;V_{\mathbb{Z}})_{\mathrm{tors}}|}{[\Gamma:\Gamma_n]} = \mathrm{vol}(M)c(V). \tag{1.1}$$

If Γ *is an arithmetic subgroup of* $\mathrm{SL}_2(\mathbb{C})$ *and* Γ_n *is a sequence of pairwise distinct congruence subgroups of* Γ *then* (1.1) *holds for* Γ_n*.*

The constant $c(V)$ equals $-t^{(2)}(V)$ where $t^{(2)}(V)$ is the L^2-torsion associated to V, see [4] or [26]. For the trivial representation it equals $-1/(6\pi)$, for the adjoint representation $-13/(6\pi)$. It is computed in full generality in [4].

The conjecture is completely open for trivial coefficients. There is a certain amount of computational evidence for a positive answer, see the tables in the paper by M. H. Şengün [33] and the graphs in Section 4 of the paper J. Brock and N. Dunfield [6]. For some coefficient systems–including the adjoint representation–the limit is proved to hold in [1] and [4] for a cocompact lattice Γ (see also [27, Section 6.1]). For trivial coefficients the upper bound on the upper limit in (1.1) was established by Thang Le [19] (the proof is purely topological and hence works for non-necessarily arithmetic lattices). See also [3] for some related results in the case of trivial coefficients.

In this paper we will consider the case where Γ is a Bianchi group, i.e. there is an imaginary quadratic field F such that $\Gamma = \mathrm{SL}_2(\mathcal{O}_F)$, and deal only with nontrivial coefficients as in [4]. It is well-known that these groups represent all commensurability classes of arithmetic nonuniform lattices in $\mathrm{SL}_2(\mathbb{C})$. We will be concerned in the upper limit in (1.1). We do not manage to deal with all sequences of congruence subgroups of such a Γ (see 1.4.1 below) and we do not address here the question of dealing with more general sequences of commensurable congruence groups. Also we do not prove that the torsion actually has an exponential growth, which is the most interesting part of the conjecture. This exponential growth (the fact that the limit inferior of the sequence $\log|H_1|/\mathrm{vol}$ is positive) is established for certain sequences by work of the author [27, Section 6.5] and independent work of J. Pfaff [25]. However, the method used in the present paper, which is different from those in these two references, gives a clear way to establishing the correct exponential growth rate. It is only because of certain number-theoretical complications that we were not able to get a complet proof. We will explain this in more detail later, for the moment let us state our main theorem.

Theorem 1.2. — *Let* Γ *be a Bianchi group,* Γ_n *a cusp-uniform sequence of torsion-free congruence subgroups and* $M_n = \Gamma_n\backslash\mathbb{H}^3$*. Let* ρ, V *be a real representation of* $\mathrm{SL}_2(\mathbb{C})$ *which is strongly acyclic and* $V_{\mathbb{Z}}$ *a lattice in* V

preserved by Γ. *Then we have*

$$\lim_{n\to\infty}\left(\frac{\log|H_1(\Gamma_n;V_{\mathbb{Z}})_{\mathrm{tors}}|}{\operatorname{vol}M_n}\right)=\lim_{n\to\infty}\left(\frac{\log|H^2(\Gamma_n;V_{\mathbb{Z}})_{\mathrm{tors}}|}{\operatorname{vol}M_n}\right)\leqslant -t^{(2)}(V).$$

Strong acyclicity of representations was introduced in [4], it means that the Hodge Laplace operators with coefficients in the local system induced by ρ have a uniform spectral gap for all hyperbolic manifolds; it was shown there to hold for all representations that are not fixed by the Cartan involution of $\mathrm{SL}_2(\mathbb{C})$, in particular its nontrivial complex representations. Cusp-uniformity means that the cross-sections of all cusps of all M_n form a relatively compact subset of the moduli space $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2$ of Euclidean tori. There are obvious sequences of congruence covers which are not cusp-uniform. The proofs of [26] actually apply not only to cusp-uniform sequences but to all BS-convergent sequences which satisfy a less restrictive condition on the geometry of their cusps, (1.5) below. However even this more relaxed hypothesis fails for some congruence sequences (see 1.4.1 below) and this raises the question of whether Question 1.1 actually has an affirmative answer in these cases. Examples of sequences to which our result does apply include the following congruence subgroups, which are all cusp-uniform (see 2.1.4 below):

$$\begin{aligned}\Gamma(\mathfrak{I})&=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(\mathcal{O}_F)\ :\ b,c\in\mathfrak{I},\ a,d\in 1+\mathfrak{I}\right\}\\ \Gamma_1(\mathfrak{I})&=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(\mathcal{O}_F)\ :\ c\in\mathfrak{I},\ a,d\in 1+\mathfrak{I}\right\}\\ \Gamma_0(\mathfrak{I})&=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(\mathcal{O}_F)\ :\ c\in\mathfrak{I}\right\}.\end{aligned}\tag{1.2}$$

Actually the $\Gamma_0(\mathfrak{I})$ contain torsion for all $\mathfrak{I}$ but we can apply Theorem 1.2 to the sequence $\Gamma_0(\mathfrak{I})\cap\Gamma'$ where $\Gamma'\subset\Gamma$ is a torsion-free congruence subgroup. As remarked below (see 1.4.2) our scheme of proof applies to orbifolds except at one point.

1.2. A few words about the proof

Let $\Gamma,\Gamma_n,\rho,V_{\mathbb{Z}}$ be as in the statement of Theorem 1.2. We refer to the introduction of [26] for more detailed information on how the scheme of proof of [4] can be adapted to the setting of nonuniform arithmetic lattices. We recall here that Theorems A and B in this paper, for a sequence M_n of finite volume hyperbolic 3–manifolds which is Benjamini–Schramm convergent to

$\mathbb{H}^3$[2] and cusp-uniform, under a certain additional technical assumption on the continuous part of the spectra of the M_n, we have the limit

$$\lim_{n\to\infty} \frac{\log \tau_{\mathrm{abs}}(M_n^{Y^n}; V)}{\operatorname{vol} M_n} = t^{(2)}(V) \tag{1.3}$$

where:

- $M_n^{Y^n}$ is a compact manifold with toric boundary, obtained from M_n by cutting off the cusps[3] along horospheres at a certain height Y^n;
- $\tau_{\mathrm{abs}}(M_n^{Y^n}; V)$ is the Reidemeister torsion associated to absolute boundary conditions on the Riemannian manifold $M_n^{Y^n}$ (see [26, 6.1.2]).

The first task we need to complete is to check that the hypotheses of these theorems are satisfied by cusp–uniform sequences of congruence manifolds. Regarding the BS-convergence we prove a quantitative result valid for any sequence of congruence subgroups (Theorem 3.1 below, we recall that $(M_n)_{\leqslant R}$ denotes, as is usual, the R-thin part of M_n).

Theorem 1.3. — *Let Γ_n be a sequence of congruence subgroups of a Bianchi group; then the manifolds $M_n = \Gamma_n \backslash \mathbb{H}^3$ are BS-convergent to $\mathbb{H}^3$ and we have in fact that there exists a $\delta > 0$ such that for all $R > 0$*

$$\operatorname{vol}(M_n)_{\leqslant R} \leqslant e^{CR} (\operatorname{vol} M_n)^{1-\delta}$$

where C depends on Γ.

We note that this result is much simpler to prove in the cusp-uniform case (see 3.1.1). For the estimates on intertwining operators as well our proof goes well for any sequence of congruence subgroups. It is when we study the trace formula along the sequence M_n that our arguments go awry, as some summands of the geometric side seem to diverge as $n \to \infty$.

Then we want to use (1.3) to study cohomological torsion. The Reidemeister torsion $\tau_{\mathrm{abs}}(M_n^{Y^n}; V)$ is related to the torsion in $H^2(M_n; V_{\mathbb{Z}})$, in fact it is defined as

$$\tau_{\mathrm{abs}}(M_n^{Y^n}; V) = \frac{R^1(M_n^{Y^n})}{R^2(M_n^{Y^n})} \cdot \frac{|H^1(M_n; V_{\mathbb{Z}})_{\mathrm{tors}}|}{|H^2(M_n; V_{\mathbb{Z}})_{\mathrm{tors}}|}$$

where $R^p(M_n^{Y^n})$ is the covolume of the lattice $H^p(M_n; V_{\mathbb{Z}})_{\mathrm{free}}$ in the space of harmonic forms satisfying absolute boundary conditions on the boundary of $M_n^{Y^n}$. The second thing to be done is to relate these to terms defined

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(2) See [1, Definition 1.1]; we recall that it means that for any $R > 0$ the volume of the R-thin part $(M_n)_{\leqslant R}$ is an $o(\operatorname{vol} M_n)$.

(3) One needs to specify how to proceed to choose at which height the cutting is performed, this question is adressed in the quoted paper.

on the manifolds M_n, using the description of $H^*(M_n; V_{\mathbb{C}})$ by non-cuspidal automorphic forms (namely, harmonic Eisenstein series). The latter define a "Reidemeister torsion" $\tau(M_n; V)$ (see (5.6) for the precise definition). We then get a limit

$$\lim_{n\to\infty} \frac{\log \tau(M_n; V)}{\operatorname{vol} M_n} = t^{(2)}(V).$$

It remains to show that the terms $|H^1(M_n; V_{\mathbb{Z}})_{\text{tors}}|$ and $R^2(M_n^{Y^n})$ disappear in the limit and that

$$\liminf_{n\to+\infty} \frac{\log R^1(M_n^{Y^n})}{\operatorname{vol}(M_n)} \geqslant 0.$$

The proofs of these claims use elementary manipulations with the long exact sequence for the Borel–Serre compactification $\overline{M}_n$ and its boundary and lemmas on the boundary cohomology.

This is also where our proof encounters an obstruction to proving the full conjecture, as for the term $R^1(M_n^{Y^n})$ we are not able to get that its limit inferior is positive. For this we would need statements on the integrality of Eisenstein classes which we were not able to establish. We can still isolate a number-theoretical statement which would ensure this as stated in the following proposition.

Proposition 1.4. — *Let F be a quadratic field. For χ a Hecke character with conductor $\mathfrak{I}_\chi$ we denote by $L(\chi, \cdot)$ the associated L-function and by $L^{\text{alg}}(\chi, \cdot)$ the normalisation which takes algebraic values at half-integers (see* [7, 6.7.2]*). Assume that there exists m such that*

$$\forall\, \chi, \forall\, s \in \frac{1}{2}\mathbb{Z} : |L^{\text{alg}}(\chi, s)|_{\overline{\mathbb{Q}}/\mathbb{Q}} \leqslant |\mathfrak{I}_\chi|^m. \tag{1.4}$$

Then we can change the lim sup *in Theorem 1.2 to a limit.*

We will not give the proof of this statement here. It is available in the old arXiv version of this paper [28] (we note that this version claims to prove the unconditional statement but in fact proves only Proposition 1.4 as at some point in the argument the bound (1.4) is assumed to hold without justification).

1.3. Limit multiplicities

Another problem about sequences of congruence groups is the question of limit multiplicities for unitary representations of $\mathrm{SL}_2(\mathbb{C})$. For such a representation π on a Hilbert space $\mathcal{H}_\pi$ and a lattice Γ in $\mathrm{SL}_2(\mathbb{C})$ one defines its multiplicity $m(\pi, \Gamma)$ to be the largest integer m such that there is a $\mathrm{SL}_2(\mathbb{C})$-equivariant embedding of $\mathcal{H}_\pi^m$ into $L^2(\Gamma\backslash \mathrm{SL}_2(\mathbb{C}))$. The question of limit multiplicities is then to determine the limit of the sequence $m(\pi,\Gamma_n)/\operatorname{vol}(\Gamma_n\backslash\mathbb{H}^3)$ as Γ_n ranges over the congruence subgroups of some arithmetic lattice. This question is of particular interest when π is a discrete series (when the limit is expected to be positive), and it has been considered in the uniform case by D. L. De George and N. R. Wallach in [14], by G. Savin [31] in the nonuniform case. In the case we consider there are no discrete series and thus we expect that the limit multiplicity of any representation will be 0.

A more precise question to ask is the following: the set $\widehat{G}$ of irreducible unitary representations (up to isomorphism) of $G = \mathrm{SL}_2(\mathbb{C})$ is endowed with a Borel measure ν^G (the Plancherel measure of Harish–Chandra), and for each lattice $\Gamma \subset G$ the multiplicities in $L^2(\Gamma\backslash G)$ define an atomic measure ν^Γ. For a congruence sequence Γ_n and a Borel set $A \subset \widehat{G}$, do we have $\nu^{\Gamma_n}(A) \sim \operatorname{vol}(\Gamma_n\backslash\mathbb{H}^3)\nu^G(A)$ as n tends to infinity? In the case where the Γ_n are congruence subgroups of a cocompact lattice this is shown to hold in [1, Section 6]. The non-compact case is much harder, but T. Finis, E. Lapid et W. Müller manage in [12] to deal with principal congruence subgroups in all groups SL_n (or GL_n) over a number field and this was generalised to all congruence subgroups in [10], [11]. Here we will, much more modestly, deal only with SL_2 over an imaginary quadratic field (note that the first prepublication of this results predates [10]).

Theorem 1.5. — *Let S be a regular Borel set in the unitary dual of $G = \mathrm{SL}_2(\mathbb{C})$, Γ a Bianchi group and Γ_n a cusp-uniform sequence of congruence subgroups. Then*

$$\lim_{n\to\infty} \frac{\sum_{\pi\in S} m(\pi,\Gamma_n)}{\operatorname{vol} M_n} = \nu^G(S)$$

The question of limit multiplicities is related to the growth of Betti numbers in sequences of congruence subgroups via Matsushima's formula and the Hodge-de Rham theorem. The latter has been studied in greater generality (for sequences of finite covers of finite CW-complexes) by W. Lück in [21] and M. Farber in [9]. Theorem 0.3 of the latter paper together with Theorem 1.12 of [1] imply that in a sequence of congruence subgroups of an arithmetic lattice the Betti numbers are sublinear in the volume in all degrees except possibly in the middle one where the growth is linear in the volume if the group has discrete series. Another proof of this for cocompact lattices, which actually yields explicit sublinear bounds in certain degrees, is also given in [1, Section 7]. For non-compact hyperbolic 3–manifolds we dealt with this problem in [26]; a corollary of [26, Proposition C] and of Theorem 1.3 is then:

Corollary 1.6. — *Let Γ_n be a sequence of torsion-free congruence subgroups of a Bianchi group Γ. Then we have:*

$$\frac{b_1(\Gamma_n)}{\operatorname{vol} M_n} \xrightarrow[n\to\infty]{} 0.$$

Note that one of the proofs given in [26] is actually a very short and easy argument if one admits [1, Theorem 1.8].

1.4. Some remarks

1.4.1. Sequences that are not cusp-uniform

As noted there the results of [26] are valid under a slightly less restrictive condition than cusp-uniformity: it suffices that we have

$$\sum_{j=1}^{h_n} \left(\frac{\alpha_2(\Lambda_{n,j})}{\alpha_1(\Lambda_{n,j})}\right)^2 \leqslant (\operatorname{vol} M_n)^{1-\delta} \tag{1.5}$$

for some $\delta > 0$, where $\Lambda_{n,j}$ are the Euclidean lattices associated to the h_n cusps of M_n (and α_1, α_2 are respectively the first and the second minima of the Euclidean norm on a lattice). This is clearly implied by cusp-uniformity in view of Lemma 3.2, and implies the unipotent part of BS-convergence. It is not hard to see that there are examples of congruence sequence which satisfy this condition but are not cusp-uniform. However, there are congruence sequences which do not satisfy (1.5), for example those associated to the subgroups K_f^n which are the preimage in $K_f = \overline{\mathrm{SL}}_2(\mathcal{O}_F)$ of $\mathrm{SL}_2(\mathbb{Z}/n)$ under the map $K_f \to K_f/K_f(n) \cong \mathrm{SL}_2(\mathcal{O}_F/(n))$: in this case there are n cusps having $\alpha_1 \asymp 1$ and $\alpha_2 \asymp n$ and the index is about n^3. However I have no clue as to whether the limit multiplicities and approximation result should or not be valid for these sequences.

1.4.2. Orbifolds

Theorems 1.5 and 1.3 are valid for sequences of orbifolds as well (see [27]). We have not included the necessary additions here in order to keep this paper to a reasonable length and because they are quite straightforward. The approximation for analytic torsion carries to this setting as well, but the Cheeger–Müller equality for manifolds with boundary which is one of the main ingredients in the proof of Theorem 5.1 has not, to the best of my knowledge, been proven for orbifolds yet. As for the final steps of the proof of Theorem 1.2 they either remain identical (if the cuspidal subgroups are torsion-free) or are simplified by the presence of finite stabilizers for the cusps, which may kill the homology and the continuous spectrum. We will not adress this here, some details are given in [27].

1.4.3. Trivial coefficients

For the topologist or the group theorist the trivial local system is the most natural and interesting. The approximation of analytic L^2-torsion [26, Theorem A], [4, Theorem 4.5] extends to that setting if one assumes that the small eigenvalues on forms have a distribution which is uniformly similar to the spectral density of the Laplacian on L^2-forms on $\mathbb{H}^3$ (see Chapter 2 and Theorem 3.183 in [22] for a precise definition of the latter).

Let us describe more precisely what this means. Let M_n be a sequence of congruence covers of some arithmetic three–manifold, we know by Theorem 1.5 that for $p = 0, 1$ the number $m_p([0, \delta]; M_n) = \sum_{\lambda\in[0,\delta]} m_p(\lambda; M_n)$ of eigenvalues of the Laplace operator on p-forms on M_n in an interval $[0, \delta]$ behaves asymptotically as $m_p^{(2)}([0, \delta]) \operatorname{vol} M_n$ where $m_p^{(2)}$ is the pushforward of the Plancherel measure. We would need to know that we have in fact a uniform decay of $m_p([0, \delta]; M_n)/ \operatorname{vol} M_n$ as $\delta \to 0$, for example

$$\frac{m_p([0, \delta]; M_n)}{\operatorname{vol} M_n} \leqslant C\delta^c \tag{1.6}$$

for all $\delta > 0$ small enough and some absolute C, c. We will now describe a (very) idealized situation in which (1.6) would hold in a particularly nice form. Let $\alpha_p \in]0, \infty^+]$ be the pth Novikov–Shubin invariant of $\mathbb{H}^3$ (see [22, Chapters 2 and 5]) then there would be an absolute constant $C > 0$ such that for any $\delta > 0$ small enough and any congruence hyperbolic three–manifold M_n we have

$$\frac{\sum_{\lambda\in[0,\delta]} m_p(\lambda; M_n)}{\operatorname{vol} M_n} \leqslant C\delta^{\alpha_p}. \tag{1.7}$$

For functions we have $\alpha_0 = \infty^+$ (meaning there is a spectral gap on $\mathbb{H}^3$) and (1.6) is known to hold in this case, and in fact in a much more general situation, by L. Clozel's solution of the "Conjecture τ" [8]. For 1-forms $\alpha_1 = 1$ and one should probably not expect to prove (1.7) literally (or even for it to hold in this form). We ask the following question, which to the best of our knowledge is wide open (see [20] for some recent advances on the general question of lower bounds for the smallest positive eigenvalue).

Question 1.7. — *Does there exist $\lambda_0 > 0$ such that for any $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that for any congruence hyperbolic three–manifold M and*

$\delta \leqslant \lambda_0$ we have

$$\frac{\sum_{\lambda\in[0,\delta]} m_1(\lambda; M)}{\operatorname{vol} M} \leqslant C_\varepsilon \delta^{1+\varepsilon}\,?$$

(Or less precisely, does this hold for some exponent $c > 0$ in place of $1+\varepsilon$ on the right-hand side?)

A positive answer to this question is not enough to imply a positive answer to Conjecture 1.1 as one still has to analyze the "regulator" terms in the Reidemeister torsion: see [4, 9.1]. In some cases the latter problem is dealt with in work of Bergeron–Şengün–Venkatesh [3].

1.4.4. Non-arithmetic manifolds

If Γ is a non-arithmetic lattice in $\mathrm{SL}_2(\mathbb{C})$ then it is conjugated into $\mathrm{SL}_2(\mathcal{O}_E[a^{-1}])$ for some number field E and algebraic integer a. Thus we can define its congruence covers (whose level will be coprime to a), and it can be proved (see 3.1.1) that they are BS-convergent to $\mathbb{H}^3$. The statement of Conjecture 1.1 still makes sense for trivial coefficients, but it is not expected that it always holds in that setting. It is actually expected that for some sequences of non-arithmetic covers with $\operatorname{inj} M_n \to +\infty$ the order of the torsion part of $H_1(M_n;\mathbb{Z})$ does not satisfy (1.1). We refer to [4, 9.1] and [6] for more complete discussion around these questions.

1.4.5. Sequences of noncommensurable lattices

Let M_n be a sequence of finite-volume hyperbolic three–manifolds such that M_n BS-converges to $\mathbb{H}^3$ and their Cheeger constants are uniformly bounded from below. Do we have

$$\lim_{n\to+\infty} \frac{\log T(M_n)}{\operatorname{vol} M_n} = \frac{1}{6\pi}\,?$$

Here $T(M_n)$ is the Ray–Singer analytic torsion, regularized as in [24] if the M_n have cusps. For compact manifolds Conjecture 8.2 in [1] states that it does, and for covers soe does [6, Conjecture 1.13]. Examples of such Benjamini–Schramm convergent sequences are given by noncommensurable arithmetic lattices, for example the sequence of Bianchi groups $\mathrm{SL}_2(\mathcal{O}_F)$ as the discriminant of the field F goes to infinity. See [13] and [29].

In the case of uniform lattices with trace fields having bounded degree it is easy to find natural sequences of coefficient modules of bounded rank satisfying (1.1).

1.5. Outline

Section 2 introduces the background we use throughout the paper. In Section 3 we prove Theorem 1.3, and in Section 4 we estimate the norm of intertwining operators, thus completing the proof of Theorem 1.5 and of (1.3). Section 5 completes the proof of the asymptotic Cheeger–Müller equality between analytic and Reidemeister torsions of the manifolds M_n. The final section 6 analyses the individual behaviour of the terms in the Reidemeister torsion, finishing the proof of Theorem 1.2.

1.6. Acknowledgments

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2. Notation and preliminaries

2.1. Bianchi groups and congruence manifolds

For this section we fix an imaginary quadratic field F and let $\Gamma = \mathrm{SL}_2(\mathcal{O}_F)$ be the associated Bianchi group. We will denote by $\mathbb{A}_f$ the ring of finite adèles of F. At infinity we fix the maximal compact subgroup K_∞ of $\mathrm{SL}_2(\mathbb{C})$ to be $\mathrm{SU}(2)$; if K'_f is a compact-open subgroup of $\mathrm{SL}_2(\mathbb{A}_f)$ we will adopt the convention of denoting by K' the compact-open subgroup $K_\infty K'_f$ of $K_\infty \, \mathrm{SL}_2(\mathbb{A}_f)$.

2.1.1. Congruence subgroups

For any finite place v of F let K_v be the closure of Γ in $\mathrm{SL}_2(F_v)$; then $K_f = \prod_v K_v$ is the closure of Γ in $\mathrm{SL}_2(\mathbb{A}_f)$. A congruence subgroup of Γ is defined to be the intersection $\Gamma \cap K'_f$ where K'_f is a compact-open subgroup of K_f. Let $\Gamma(\mathfrak{I})$ be defined by (1.2) and $K_f(\mathfrak{I})$ its closure in K_f; then $\Gamma(\mathfrak{I}) = \mathrm{SL}_2(F) \cap K_f(\mathfrak{I})$ so that $\Gamma(\mathfrak{I})$ is indeed a congruence subgroup; likewise, Γ_0 and $\Gamma_1(\mathfrak{I})$ are "congruence-closed", i.e. they are equal to the intersection of their closure with $\mathrm{SL}_2(F)$.

For a compact-open subgroup $K'_f \subset K_f$ we will denote by $\Gamma_{K'} = \mathrm{SL}_2(F) \cap K'_f$ the associated congruence lattice; we define its level to be the largest $\mathfrak{I}$ such that $K_f(\mathfrak{I}) \subset K'_f$ (this is well–defined as $K_f(\mathfrak{I})K_f(\mathfrak{I}') = K_f(\mathfrak{J})$ where $\mathfrak{J} = \gcd(\mathfrak{I}, \mathfrak{I}')$). Then the following fact is well-known (see [27, Lemme 5.8]).

Lemma 2.1. — *For any compact-open $K'_f \subset K_f$ we have*

$$[K_f : K'_f] \geqslant \frac{1}{3}|\mathfrak{I}|^{\frac{1}{3}}$$

where $\mathfrak{I}$ is the level of K'_f.

2.1.2. Congruence manifolds

For a compact-open K'_f we denote by $M_{K'}$ the orbifold $\Gamma_{K'} \backslash \mathbb{H}^3$. The strong approximation theorem for SL_2 (which in this case is a rather direct consequence of the Chinese remainder theorem) states that the subgroup $\mathrm{SL}_2(F)$ is dense in $\mathrm{SL}_2(\mathbb{A}_f)$, and it follows that we have a homeomorphism

$$M_{K'} \cong \mathrm{SL}_2(F) \backslash \mathrm{SL}_2(\mathbb{A})/K'. \tag{2.1}$$

2.1.3. Unipotent subgroups

Let N be a unipotent subgroup of SL_2 defined over F, B its normalizer in SL_2 and α the morphism from B/N to the multiplicative group[(4)] given by the conjugacy action on N. For any place v of F we have the Iwasawa decomposition

$$\mathrm{SL}_2(F_v) = \mathrm{B}(F_v)K_v,$$

and this yields also that $\mathrm{SL}_2(\mathbb{A}) = \mathrm{B}(\mathbb{A})K$. We define a height function on $\mathrm{SL}_2(\mathbb{A})$ by

$$y(g) = \max\{|\alpha(b)|,\, b \in \mathrm{B}(\mathbb{A}) : \exists\, \gamma \in \mathrm{SL}_2(F), k \in K,\, \gamma g = bk\}, \tag{2.2}$$

this does not depend on the F-rational unipotent subgroup N. We fix the unipotent subgroup ${}_0\mathrm{N}$ to be the stabilizer of the point $(0,1)$ in affine 2-space, and we identify it with the additive group using the isomorphism ψ sending

(4) Which is isomorphic to the automorphism group of N since the latter is itself isomorphic to the additive group.

1 to the matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Then N is conjugated to ${}_0\mathrm{N}$ by some $g \in \mathrm{SL}_2(F)$ and we identify N with the additive group using the isomorphism given by the composition of conjugation by g with $\psi \circ \alpha(b)^{-1}$ where $g = bk$ (and we view $\alpha(b)$ as an automorphism of the additive group).

2.1.4. Cusps

The cusps of the manifold $M_{K'}$ are isometric to the quotient $\mathrm{B}(F)\backslash \mathrm{SL}_2(\mathbb{A})/K'$; in particular the number h of cusps is equal to the cardinality of the finite set

$$\mathcal{C}(K') = \mathrm{B}(F)\mathrm{B}(F_\infty)\backslash \mathrm{SL}_2(\mathbb{A})/K'.$$

We can describe accurately the cross-section of each cusp. Let N be the unipotent subgroup which is the commutator of the stabilizer in SL_2 of the point $(a : b) \in \mathbb{P}^1(F)$. We may suppose that a, b have no common divisor in $\mathcal{O}_F$ except for units, then the ideal (a, b) is equal to some ideal $\mathfrak{C}$ without principal factors and we write $(a) = \mathfrak{C}\mathfrak{A}$, $(b) = \mathfrak{C}\mathfrak{B}$, so that $(\mathfrak{A}, \mathfrak{B}) = 1$. Then we have [27, Proposition 5.1]:

$$K_f \cap \mathrm{N}(F) = 1 + \left\{ \begin{pmatrix} \frac{a}{b}c & -\frac{a^2}{b^2}c \\ c & -\frac{a}{b}c \end{pmatrix},\ c \in \mathfrak{B}^2 \right\} = 1 + \left\{ \begin{pmatrix} -\frac{b}{a}c & c \\ -\frac{b^2}{a^2}c & \frac{b}{a}c \end{pmatrix},\ c \in \mathfrak{A}^2 \right\}.$$

In particular, since the ideals of $\mathcal{O}_F$ are a uniform family of lattices it follows easily that the families $\Gamma(\mathfrak{I}), \Gamma_0(\mathfrak{I})$ or $\Gamma_1(\mathfrak{I})$ are cusp-uniform (see [27, Lemme 5.7]).

2.2. Analysis on $\mathrm{SL}_2(\mathbb{A})$ and Eisenstein series

We fix a Borel subgroup $\mathrm{B} \subset \mathrm{SL}_2$ defined over F and a maximal split torus T in B, and let N be the unipotent radical of B.

2.2.1. Haar measures

We fix the additive and multiplicative measures on each F_v and $F_v^\times$ as usual, and choose the Haar measure of total mass one on K_v. We take the Haar measure on $\mathrm{B}(F_v) = \mathrm{N}(F_v)\mathrm{T}(F_v) \cong F_v \rtimes F_v^\times$ to be given by $d(n_v a_v) = \frac{\mathrm{d}x_v}{|\cdot|_v} \otimes \mathrm{d}^\times x_v$. On a proper quotient we always take the pushforward measure, in particular the measure on $\mathrm{SL}_2(\mathbb{A})$ is the pushforward of $\mathrm{d}b\,\mathrm{d}k$ in the Iwasawa decomposition $\mathrm{SL}_2(\mathbb{A}) = \mathrm{B}(\mathbb{A})K$.

2.2.2. Spaces of functions

Let A^1 be the subgroup of $\mathrm{T}(\mathbb{A})$ such that every character from $\mathrm{T}(\mathbb{A})$ to $\mathbb{R}_+^\times$ factors through A^1; then $\mathrm{T}(F) \cong F^\times$ is contained in A^1 and we will denote by $\mathcal{H}$ the Hilbert space $L^2(F^\times \backslash (A^1 K))$. We have a natural isomorphism

$$\mathcal{H} \cong \mathbb{C}[C(F)] \otimes L^2(K)$$

where $C(F)$ is the class-group of F. Let χ be a Hecke character, then it induces characters of A^1 and $\mathrm{B}(\mathbb{A})$ that we will continue to denote by χ. For $s \in \mathbb{C}$ and $\phi \in \mathcal{H} \cap C^\infty(A^1 K)$ there is a unique extension ϕ_s of ϕ to $\mathrm{SL}_2(\mathbb{A})$ which satisfies:

$$\forall\, n \in N(\mathbb{A}), a_\infty \in A_\infty, a \in A^1, k \in K \;\; \phi_s(n a_\infty a k) = |\alpha(a_\infty)|_\infty^s \phi(ak). \quad (2.3)$$

We will denote by $\mathcal{H}_s$ the space of such extensions, and $\mathcal{H}_s(\chi)$ its subspace of functions having A^1-type χ on the left. The space $\mathcal{H}_s$ is acted upon by $\mathrm{SL}_2(\mathbb{A})$ by right translation; when $s \neq 0, 1$ the decomposition of $\mathcal{H}_s$ into $\mathrm{SL}_2(\mathbb{A})$-irreducible factors is given by the $\mathcal{H}_s(\chi)$.

If τ is a finite-dimensional complex continuous representation of K on a space V_τ we define $\mathcal{H}_s(\chi, \tau)$ to be the subspace of $\mathcal{H}_s(\chi)$ containing the functions which have K-type τ on the right (in other words, the projection to $\mathcal{H}_s(\chi)$ of the subspace of K-invariant vectors in $\mathcal{H}_s(\chi) \otimes V_\tau$).

2.2.3. Eisenstein series

For a function $f \in C^\infty(\mathrm{B}(F)\mathrm{N}(\mathbb{A}) \backslash \mathrm{SL}_2(\mathbb{A}))$ put

$$E(f)(g) = \sum_{\gamma \in \mathrm{SL}_2(F)/\mathrm{B}(F)} f(\gamma^{-1} g) \quad (2.4)$$

which is well-defined (i.e. the series converge) for f sufficiently decreasing at infinity (for example compactly supported).

The height function y defined by (2.2) is left $\mathrm{B}(F)\mathrm{N}(\mathbb{A})$-invariant, and it is well-known (cf. [27, Lemme 5.23]) that the series

$$E(y^s)(g) \sum_{\gamma \in \mathrm{SL}_2(F)/\mathrm{B}(F)} y(\gamma^{-1} g)^s$$

converges absolutely for all $g \in \mathrm{SL}_2(\mathbb{A})$ and $\mathrm{Re}(s) > 2$, uniformly on compact sets. For $\phi \in \mathcal{H}$ the function ϕ_s defined through (2.3) we denote $E(\phi_s) = E(\phi, s)$ which is convergent for $\mathrm{Re}(s) > 2$ according to the above. We have the following fundamental result, due to A. Selberg for $\mathrm{SL}_2/\mathbb{Q}$ and to R. Langlands in all generality (see [27, 5.4.1] for a simpler proof in this case, based on ideas of R. Godement [15]).

Proposition 2.2. — *The function $s \mapsto E(\phi, s)$ has a meromorphic continuation to $\mathbb{C}$, which is homolorphic everywhere if $\chi \neq 1$. If $\chi = 1$ there is only one pole of order one at $s = 1$.*

The main point of the theory of Eisenstein series is that they give the orthogonal complement to the discrete part of the regular representation on $L^2(\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A}))$. The space $L^2_{\mathrm{cusp}}(\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A}))$ of cusp forms is usually defined to be the closed subspace of all functions on $\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})$ whose constant term (defined by (2.5) below) vanishes.

Proposition 2.3. — *The map*

$$\int_{-\infty}^{+\infty} \mathcal{H}_{\frac{1}{2}+iu}(\chi) \frac{\mathrm{d}u}{2\pi} \ni \psi \mapsto E(\psi) \in L^2(\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A}))$$

is an isometry onto the orthogonal of the space $L^2_{\mathrm{cusp}}(\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})) \oplus \mathbb{C}$. Moreover, for any $\phi \in C_c^\infty(\mathrm{SL}_2(\mathbb{A})$ the associated operator on $L^2_{\mathrm{cusp}}(\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A}))$ is trace-class; in particular $L^2_{\mathrm{cusp}}(\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A}))$ decomposes as a Hilbert sum of irreducible, $\mathrm{SL}_2(\mathbb{A})$-invariant closed subspaces.

2.2.4. Intertwining operators

Let f be a continuous function on $SL_2(F)\backslash \mathrm{SL}_2(\mathbb{A})$. We define its constant term to be

$$f_P(g) = \int_{\mathrm{N}(F)\backslash \mathrm{N}(\mathbb{A})} f(ng)\mathrm{d}n. \tag{2.5}$$

Let $\phi \in \mathcal{H}$, ϕ_s be defined by (2.3) and $f = E(\phi_s)$. We use the Bruhat decomposition $\mathrm{SL}_2(F)/\mathrm{B}(F) = \{\mathrm{B}(F)\} \cup \{\gamma w \mathrm{B}(F), \gamma \in N(F)\}$ and when $\mathrm{Re}(s) > 3/2$ we get:

$$\begin{aligned} f_P(g) &= \int_{\mathrm{N}(F)\backslash \mathrm{N}(\mathbb{A})} \sum_{\gamma \in \mathrm{SL}_2(F)/\mathrm{B}(F)} \phi_s(\gamma^{-1} ng)\mathrm{d}n \\ &= \int_{\mathrm{N}(F)\backslash \mathrm{N}(\mathbb{A})} \phi_s(ng)\mathrm{d}n + \sum_{\gamma \in \mathrm{N}(F)} \int_{\mathrm{N}(F)\backslash \mathrm{N}(\mathbb{A})} \phi_s(w\gamma ng)\mathrm{d}n \\ &= \phi_s(g) + \int_{\mathrm{N}(\mathbb{A})} \phi_s(wng)\mathrm{d}n. \end{aligned}$$

and we define the intertwining operator $\Psi(s)$ on $\mathcal{H}$ by

$$(\Psi(s)\phi)(ak) = \int_{N(\mathbb{A})} \phi_s(wnak)\mathrm{d}n\,. \tag{2.6}$$

We obtain (using the notation of (2.3)):

$$E(\phi_s)_P = \phi_s + (\Psi(s)\phi)_{1-s}. \tag{2.7}$$

One can check that $\Psi(s)$ induces an $\mathrm{SL}_2(\mathbb{A})$-equivariant endomorphism on $\mathcal{H}_s$, which sends the irreducible subspace $\mathcal{H}_s(\chi)$ to $\mathcal{H}_s(\chi^{-1})$. For $\mathrm{Re}(s) = \frac{1}{2}$ the map $\Psi(s)$ is an isometry for the inner product of $\mathcal{H}$.

2.2.5. Maass–Selberg

Finally we record the Maass–Selberg expansions [27, 5.4.4]; for $s \in \mathbb{C} - \mathbb{R} \cup (\frac{1}{2} + i\mathbb{R})$:

$$\begin{aligned}\langle T^Y E(s,\phi), T^Y E(s',\psi)\rangle_{L^2(G(F)\backslash G(\mathbb{A}))} \\ = \frac{1}{2(s+\overline{s}'-1)}\left(Y^{2(s+\overline{s'}-1)}\langle\phi,\psi\rangle_{\mathcal{H}} - Y^{-2(s+\overline{s}'-1)}\langle\Psi(s')^*\Psi(s)\phi,\psi\rangle_{\mathcal{H}}\right) \\ + \frac{1}{2(s-\overline{s}')}\left(Y^{2(s-\overline{s'})}\langle\phi,\Psi(s')\psi\rangle_{\mathcal{H}} - Y^{2(-s+\overline{s}')}\langle\Psi(s)\phi,\psi\rangle_{\mathcal{H}}\right).\end{aligned} \tag{2.8}$$

When $s \in \mathbb{R}$ this degenerates to

$$\begin{aligned}\langle T^Y E(\phi,s), T^Y E(\psi,s)\rangle^2_{L^2(G(F)\backslash G(\mathbb{A}))} \\ = \frac{Y^{4s-2}}{4s-2}\langle\phi,\psi\rangle_{\mathcal{H}} - \frac{Y^{-4s+2}}{4s-2}\langle\Psi(s)\phi,\Psi(s)\psi\rangle_{\mathcal{H}} \\ + \log Y\langle\Psi(s)\phi,\psi\rangle_{\mathcal{H}} + \left\langle \frac{\mathrm{d}\Psi(s+iu)}{\mathrm{d}u}\bigg|_{u=0}\phi,\psi\right\rangle_{\mathcal{H}}.\end{aligned} \tag{2.9}$$

2.3. Regularized traces

2.3.1. Differential forms and $L^2(\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A}))$

Let K' be a compact-open subgroup of K, and ρ a representation of $\mathrm{SL}_2(\mathbb{C})$ on a finite-dimensional real vector space V. We can associate to ρ a local system on $M = M_{K'}$ and we denote by $\Omega^p, L^2\Omega^p(M;V)$ the spaces of smooth and square-integrable p-forms on M with coefficients in V (see [26, 2.3]). It is well-known that there is an identification

$$\left(L^2(\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})) \otimes \wedge^p\mathfrak{p} \otimes V\right)^{K'} \to L^2\Omega^p(M;V_{\mathbb{C}}). \tag{2.10}$$

Let τ be the representation of K that has for it's finite part $\mathbb{C}[K/K']$ and whose infinite part is equal to the representation of K_∞ on $V_\mathbb{C} \otimes \wedge^p \mathfrak{p}^*$. We define the map $E(s, \cdot\,)$ to be such that the following diagram commutes:

$$\begin{array}{ccc} (\mathcal{H}_s \otimes V_\mathbb{C} \otimes \wedge^p \mathfrak{p}^*)^{K'} & \xrightarrow{E} & \left(L^2(\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})) \otimes V_\mathbb{C} \otimes \wedge^p \mathfrak{p}^*\right)^{K'} \\ \| & & \| \\ \mathbb{C}^h \otimes V_\mathbb{C} \otimes \wedge^p \mathfrak{p}^* & \xrightarrow{E(s,\cdot\,)} & L^2\Omega^p(\Gamma_{K'}\backslash \mathbb{H}^3; V_\mathbb{C} \end{array} \tag{2.11}$$

where h is the number of cusps of M and we identify $\mathrm{B}(F)A_\infty \mathrm{N}(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})/K'_f$ with $\mathbb{C}^h$. For $p = 0, 1$ we retrieve the maps defined in [26, 3.1.3 and 3.1.4] which associate to a section or 1-form on $\partial \overline{M}$ an element of $L^2\Omega^p(\Gamma_H\backslash \mathbb{H}^3; V_\mathbb{C})$.

2.3.2. Spectral trace

Let M be a congruence hyperbolic three–manifold and $\Delta^p[M]$ the Hodge Laplacian on p-forms on M with coefficients in $V_\mathbb{C}$. We recall the "spectral" definition of the regularized trace given in [26, 3.2.4]. Let $p = 0$ for now, and let ϕ be a function on $\mathbb{R}$ such that the associated automorphic kernel K^0_ϕ on $\mathbb{H}^3 \times \mathbb{H}^3$ has compact support (see [26, Section 3.2]). The regularized trace of $\phi(\Delta^0[M])$ is given by

$$\begin{aligned} \mathrm{Tr}_R\, \phi(\Delta^0[M]) &= \sum_{j \geqslant 0} m(\lambda_j; M)\phi(\lambda_j) + \frac{1}{4} \sum_{l=-2q}^{2q} d_l \phi\left(-l^2 + 4 + \lambda_V\right) \mathrm{tr}\, \Psi_l(0) \\ &- \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{l=-2q}^{2q} d_l \phi\left(-u^2 + 4 - l^2 + \lambda_V\right) \mathrm{tr} \left(\Psi_l(iu)^{-1} \frac{\mathrm{d}\Psi_l(iu)}{\mathrm{d}u}\right) \mathrm{d}u. \end{aligned} \tag{2.12}$$

where:

- The $\lambda_j, j \geqslant 0$ are the eigenvalues of $\Delta^0[M]$ in $L^2(M; V_\mathbb{C})$;
- For $\lambda \in [0, +\infty[$, $m(\lambda; M) = \dim\ker(\Delta^0[M] - \lambda\,\mathrm{Id})$;
- For $u \in \mathbb{R}$, $\Psi_l(iu)$ is the operator on $\mathbb{C}^h \otimes W_l$ corresponding to $\Psi(\frac{1}{2}(1+iu))$ under the identifications on the right-hand side of (2.11);
- W_l is a certain subspace of $V_\mathbb{C}$, and we have $V_\mathbb{C} = \bigoplus_l W_l$.

For more details see [26, Section 3.1]. We will skip the definition for 1-forms since it is basically the same (see loc. cit.).

2.4. Homology and cohomology

Here we consider a CW-complex X, $\Lambda = \pi_1(X)$ and L a free $\mathbb{Z}$-module of finite rank with a Λ-action. There are then defined chain an cochain complexes $C_*(X;L), d_*$ and $C^*(X;L), d^*$. If X is aspherical then $H_*(X;L) \cong H_*(\Lambda;L)$ and $H^*(X;L) \cong H^*(\Lambda;L)$.

2.4.1. Kronecker pairing

Let L^* be the dual $\mathrm{Hom}(L,\mathbb{Z})$, $Z_p(X;L) = \ker(d_p)$ et $Z^p(X;L^*) = \ker(d^p)$. There is a natural bilinear form on $Z_p(X;L) \times Z^p(X;L^*)$ which induces a nondegenerate bilinear form

$$(\,\cdot\,,\,\cdot\,)_X : H_1(X;L)_{\mathrm{free}} \times H^1(X;L^*)_{\mathrm{free}} \to \mathbb{Z}.$$

If Y is a sub-CW-complex of X and i its inclusion in X we have the following property

$$\forall\, \eta \in H_p(Y;L^*), \omega \in H^p(X;L): \quad (i_*\eta,\omega)_X = (\eta, i^*\omega)_Y. \tag{2.13}$$

In case there is a perfect duality between two Λ-modules L, L' which comes from a Λ-invariant bilinear form we get a Kronecker pairing on $H_p(X;L)_{\mathrm{free}} \times H^p(X;L')_{\mathrm{free}}$ satisfying (2.13); for Γ a lattice in $\mathrm{SL}_2(\mathbb{C})$ there exists such a self-duality for the Γ-modules $V = V_{n_1,n_2}$ (given by the form induced on V by the determinant pairing on V). This bilinear form, which we will denote by $\langle\,\cdot\,,\,\cdot\,\rangle_V$, is actually defined over $\mathbb{Z}[m^{-1}]$ for $m = n_1!n_2!$. This can be seen from the explicit formula in [2, 2.4]. If $V_\mathbb{Z} \subset V_\mathbb{Q}$ is a lattice we put

$$V'_\mathbb{Z} = \{v' \in V_\mathbb{Q} \;:\; \forall\, v \in V_\mathbb{Z},\, \langle v', v\rangle_V \in \mathbb{Z}\}$$

which is another lattice in $V_\mathbb{Q}$.

2.4.2. Poincaré duality

We suppose now that X is an n-dimensional compact manifold with boundary ∂X. Poincaré duality is an isomorphism of graded $\mathbb{Z}$-modules $H_*(X;V) \xrightarrow{\sim} H^{n-*}(X,\partial X;V)$ or $H_*(X,\partial X;V) \xrightarrow{\sim} H^{n-*}(X;V)$. It is compatible with the long exact sequences of the pair $X, \partial X$ in the following sense [5, Theorem V.9.3].

Proposition 2.4. — *The diagram:*

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
H_p(\partial\overline{M};V) & \longrightarrow & H^{n-1-p}(\partial\overline{M};V^*) \\
\downarrow & & \downarrow \\
H_p(M;V) & \longrightarrow & H^{n-p}(\overline{M},\partial\overline{M};V^*) \\
\downarrow & & \downarrow \\
H_p(\overline{M},\partial\overline{M};V) & \longrightarrow & H^{n-p}(M;V^*) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

is commutative, where vertical lines are the long exact sequences in homology and cohomology of $(X,\partial X)$ *and horizontal arrows are Poincaré duality morphisms.*

3. Asymptotic geometry of congruence manifolds and approximation of L^2-invariants

In this section we shall, assuming the results of the next section, prove Theorem 1.5 from the introduction and the approximation result for analytic torsion (Theorem 3.8 below).

3.1. Benjamini–Schramm convergence

The following result generalizes [1, Theorem 1.12] to the case of noncompact congruence subgroups of $\mathrm{SL}_2(\mathbb{C})$.

Theorem 3.1. — *There are* $\delta, c > 0$ *such that for any Bianchi group* $\Gamma = \Gamma(\mathcal{O}_F)$ *and sequence* Γ_n *of torsion-free congruence subgroups in* Γ*, for all* $R > 0$ *we have*

$$\mathrm{vol}\{x \in M_n \,:\, \mathrm{inj}_x(M_n) \leqslant R\} \leqslant e^{cR}[\Gamma:\Gamma_n]^{1-\delta}$$

In particular, the sequence of hyperbolic manifolds $M_n = \Gamma_n\backslash\mathbb{H}^3$ *is BS-convergent to* $\mathbb{H}^3$.

We record the following much weaker consequence (see Lemma 3.3 below) of this as a separate fact; note that this is actually the only part of Theorem 3.1 that we make full use of here, and a direct proof is much easier than that of the latter.

Lemma 3.2. — *Let Γ, Γ_n, M_n be as in the statement of the theorem above and let h_n be the number of cusps of M_n. Then*

$$h_n \ll [\Gamma : \Gamma_n]^{1-\delta}.$$

Recall that K_f is the closure in $\mathrm{SL}_2(\mathbb{A}_f)$ of $\Gamma = \mathrm{SL}_2(\mathcal{O}_F)$ and let K'_f be a closed finite-index subgroup with level $\mathfrak{I}$. We will show that for the subgroup $\Gamma' = \Gamma_{K'}$ of Γ and $M = \Gamma' \backslash \mathbb{H}^3$ we have $\mathrm{vol}\{x \in M \,:\, \mathrm{inj}_x(M) \leqslant R\} \leqslant C[K : K']^{1-\delta}$.

3.1.1. Remarks

(1) The mere BS-convergence (without the precise estimates) follows from [1, Theorem 1.11]: one can see that it implies that any invariant random subgroup which is a limit of a sequence of congruence covers has to be supported on unipotent subgroups, which is impossible if the limit is nontrivial (for example because of "Borel's density theorem" [1, Theorem 2.9]).

(2) As a corollary we get that there is an $\varepsilon > 0$ such that

$$\mathrm{vol}\, M_{\leqslant \varepsilon \log \mathrm{vol}\, M} \leqslant (\mathrm{vol}\, M)^{1-\delta}$$

for all (manifold) congruence covers M of a given Bianchi orbifold.

3.1.2. Benjamini–Schramm convergence of manifolds with cusps

We recall some notation: for a hyperbolic manifold M we let $N_R(M)$ be the number of closed geodesics of length less than R on M. If Λ is a lattice in $\mathbb{C}$ we define

$$\alpha_1(\Lambda) = \min\{|v| \,:\, v \in \Lambda, v \neq 0\}$$

and for any $v_1 \in \Lambda$ such that $|v_1| = \alpha_1(\Lambda)$,

$$\alpha_2(\Lambda) = \min\{|v| \,:\, v \in \Lambda, v \notin \mathbb{Z}v_1\}.$$

Then the ratio α_2/α_1 only depends on the conformal class of Λ, in particular if $\Gamma \not\ni -1$ is a lattice in $\mathrm{SL}_2(\mathbb{C})$ and N a unipotent subgroup such that $\Gamma \cap N$ is nontrivial (we will say that N is Γ-rational) then $\alpha_2/\alpha_1(\Gamma \cap N)$ is well-defined and depends only on the Γ-conjugacy class of N. We can then estimate the volume of the thin part as follows (in particular, to prove that a sequence of finite covers of a fixed orbifold is BS-convergent we need only give $o(\operatorname{vol} M_n)$-bounds for the right-hand side).

Lemma 3.3. — *Let $M = \Gamma\backslash\mathbb{H}^3$ be a finite–volume hyperbolic three–manifold and let $N_1, \dots, N_h$ be representatives for the Γ-conjugacy classes of unipotent subgroups. Put $\Lambda_j = \Gamma_n \cap N_j$, then there are constants C (depending on Γ) and $c > 0$ such that*

$$\operatorname{vol} M_{\leqslant R} \leqslant Ce^{cR}\left(RN_R(M) + \sum_{j=1}^{h} \frac{\alpha_2(\Lambda_j)}{\alpha_1(\Lambda_j)}\right).$$

Proof. — A point $x \in M$ lies in the R-thin part if and only if, for any lift $\tilde{x}$ of it to $\mathbb{H}^3$, there is $\gamma \in \Gamma \setminus \{\mathrm{Id}\}$ such that $d(\tilde{x}, \gamma\tilde{x}) \leqslant R$. It follows that if $\gamma_1, \dots, \gamma_N$ are elements in Γ representing the conjugacy classes of loxodromic elements with translation lengths at most R (so $N = N_R(M)$) the R-thin part of M is the union of the images of the R-thin parts in the manifolds $\langle\gamma_i\rangle\backslash\mathbb{H}^3$ and $\Lambda_j\backslash\mathbb{H}^3$.

The statement of the lemma then follows from the two following facts from elementary hyperbolic geometry:

(i) If $g \in \mathrm{SL}_2(\mathbb{C})$ is loxodromic the R-thin part of $\langle g\rangle\backslash\mathbb{H}^3$ has volume $\leqslant C\ell e^{cR}$, where C depends only on the minimal translation length ℓ of g;

(ii) If Λ is a lattice in a unipotent subgroup N of $\mathrm{SL}_2(\mathbb{C})$ then the R-thin part of $\Lambda\backslash\mathbb{H}^3$ is of volume $\leqslant e^{cR}\alpha_1(\Lambda)/\alpha_2(\Lambda)$.

The point (i) follows immediately from the fact that if L is the axis of g and x is a point at distance at most R from L, then $d(x, gx) \gg R$ with a constant independent of g, and the fact that the volume of a R-neighbourhood of the closed geodesic in $\langle g\rangle\backslash\mathbb{H}^3$ is of volume $\leqslant \ell e^{cR}$ (where e^{cR} is an upper bound for the volume of a radius R ball in $\mathbb{H}^3$).

For point (ii) we observe that we can parametrize $\Lambda\backslash\mathbb{H}^3$ as $T \times [0, +\infty[$, where T is the Euclidean torus $\Lambda\backslash\mathbb{C}$, which we suppose normalized so that $\alpha_1(\Lambda) = 1$ (we conformally identify N with $\mathbb{C}$) and the product metric $(\mathrm{d}x^2 + \mathrm{d}y^2)/y^2$. Then the R-thin part is contained in $T \times [e^{-cR}, +\infty[$ (where c is such that if $x, y \in \mathbb{H}^3$ belong to an horosphere H with Euclidean distance d_H, we have $d_{\mathbb{H}^3}(x, y) \geqslant \log(1 + d_H(x, y))$). The volume of the latter is easily seen to be $\leqslant \operatorname{vol}(T)e^{cR} \ll e^{cR}\alpha_2(\Lambda)$. ☐

3.1.3. Proof of Theorem 3.1, the loxodromic part

Here we recall how the bound on N_R follows from the results in [1, Section 5]. Let c be a closed geodesic in the orbifold $\Gamma\backslash\mathbb{H}^3$ and $\gamma \in \Gamma$ any element of the associated loxodromic conjugacy class in Γ. For any $g \in K_f$ we have that γ fixes the coset gK'_f if and only if $g\gamma g^{-1}$ belongs to K'_f, so that the number of lifts of c in $M_{K'}$ is equal to the number of fixed points of γ in K_f/K'_f. By Theorem 1.11 in [1] there are constants δ (depending on F) and C (depending on c) such that the latter is less than $C|K_f/K'_f|^{1-\delta}$. This shows that for a given R there is a C_R such that for all K' we have

$$N_R(M_{K'}) \leqslant C_R|K_f/K'_f|^{1-\delta} = \left(C_R \operatorname{vol}(\Gamma\backslash\mathbb{H}^3)\right) (\operatorname{vol} M_{K'})^{1-\delta}.$$

3.1.4. Proof of Theorem 3.1, the unipotent part

Now we have to bound the second term in Lemma 3.3: we want to show that

$$\sum_{j=1}^{h_{K'}} \frac{\alpha_2(\Lambda_{K',j})}{\alpha_1(\Lambda_{K',j})} \ll |K_f/K'_f|^{1-\delta} \tag{3.1}$$

where $\Lambda_{K',j} = {}_j\mathrm{N}(F) \cap K'_f$ where the ${}_j\mathrm{N}$ are representatives of the $\Gamma_{K'}$-conjugacy classes of unipotent subgroups in $\mathrm{SL}_2(F)$. We fix a unipotent subgroup N in SL_2/F, and let $N = \mathrm{N}(\mathbb{A}_f) \cap K_f$; clearly it suffices to prove that (3.1) holds if we sum only over the unipotent groups contained in the Γ-conjugacy class of $\mathrm{N}(F)$. These $\Gamma_{K'}$-conjugacy classes are in natural bijection with the set of double cosets $N\backslash K_f/K'_f$

For $p \in \mathbb{Z}$ a rational prime we will denote $F_p = \mathbb{Q}_p \otimes_{\mathbb{Q}} F$ and K_p the closure of Γ in $\mathrm{SL}_2(F_p)$. The latter is isomorphic to

- $\mathrm{SL}_2(\mathcal{O}_v)$ in case p is inert or ramified in F and v is the corresponding place of F;
- $\mathrm{SL}_2(\mathbb{Z}_p) \times \mathrm{SL}_2(\mathbb{Z}_p)$ when p is split.

We will denote by $K_p(p^k)$ the compact-open subgroup of matrices congruent to 1 modulo p^k, and by $\mathfrak{g}_p$ the Lie algebra of K_p.

The crucial case is when we have $K'_p = K_p$ for all but one rational prime p. We identify $\mathrm{N}(F_p)$ with F_p (see 2.1.3 above) and for a finite-index subgroup L of $\mathrm{N}(F_p)$ we put

$$\alpha_1(L) = \min\{|v|_p^{-1} \ : \ v \in L, v \neq 0\}$$

and for any $v_1 \in L$ such that $|v_1|_p = \alpha_1(\Lambda)$,

$$\alpha_2(L) = \min\{|v|_p^{-1} \ : \ v \in L, v \notin \mathbb{Z}v_1\}.$$

If $g \in K_p$ and $\Lambda = g\mathrm{N}(F)g^{-1} \cap K'_p$, $L = gN_pg^{-1} \cap K'_p$ (where $N_p = K_p \cap \mathrm{N}(F_p)$) then we have

$$\alpha_i(\Lambda) \asymp \alpha_i(L),\ i = 1, 2,$$

with absolute constants, so that we must bound the sum

$$S_p = \sum_{g \in N_p \backslash K_p / K'_p} \frac{\alpha_2(g^{-1}N_pg \cap K'_p)}{\alpha_1(g^{-1}N_pg \cap K'_p)}. \tag{3.2}$$

We rewrite S_p as follows: we fix $k \geqslant 1$ such that $K_p(p^k) \subset K'_p$. Then the quantities $\alpha_i(g^{-1}N_pg \cap K'_p)$ are constant on a K'_p-orbit in $K_p/N_pK_p(p^k)$; on the other hand the cardinality of the K'_p-orbit of $gN_pK_p(p^k)$ in $K_p/N_pK_p(p^k)$ is equal to $\frac{|K'_p/K_p(p^k)|}{|(g^{-1}N_pg \cap K'_p)K_p(p^k)/K_p(p^k)|}$ so that

$$S_p = \sum_{g \in K_p/N_pK_p(p^k)} \frac{|(gN_pg^{-1} \cap K'_p)K_p(p^k)/K_p(p^k)|}{[K'_p : K_p(p^k)]} \times \frac{\alpha_2(gN_pg^{-1} \cap K'_p)}{\alpha_1(gN_pg^{-1} \cap K'_p)}.$$

From the equality

$$\alpha_2(gN_pg^{-1} \cap K'_p)\alpha_1(g^{-1}N_pg \cap K'_p) = |gN_pg^{-1}/(gN_pg^{-1} \cap K'_p)|$$

it follows that $\alpha_1\alpha_2 = \frac{|N_p/(N_p \cap K_p(p^k))|}{|g^{-1}N_pg \cap K'_p|}$ and then that:

$$\begin{aligned} S_p &= \frac{[N_p : N_p \cap K_p(p^k)]}{[K'_p : K_p(p^k)]} \sum_{g \in K_p/N_pK_p(p^k)} \frac{1}{\alpha_1(gN_pg^{-1} \cap K'_p)^2} \\ &= [K_p : K'_p] \cdot \frac{[N_p : N_p \cap K_p(p^k)]}{[K_p : K_p(p^k)]} \sum_{g \in K_p/N_pK_p(p^k)} \frac{1}{\alpha_1(gN_pg^{-1} \cap K'_p)^2}. \end{aligned}$$

On the other hand B_p normalizes N_p and we can mod out on the right to get:

$$S_p = [K_p : K'_p] \cdot \frac{|B_p : (B_p \cap K_p(p^k))]}{[K_p : K_p(p^k)]} \sum_{g \in K_p/B_pK_p(p^k)} \frac{1}{\alpha_1(gN_pg^{-1} \cap K'_p)^2} \tag{3.3}$$

which is the sum that we will now estimate.

For $l = 0, \dots, k-1$ we define

$$X_l = \{g \in K_p/B_pK_p(p^k) \ : \ gN_pg^{-1} \cap K'_p \subset K_p(p^l),\ \not\subset K_p(p^{l+1})\} \tag{3.4}$$

and put $d_l = |X_l|$. Then for $g \in X_l$ we have $\alpha_1(gN_pg^{-1} \cap K'_p) = p^l$ and so:

$$S_p \leqslant \frac{[K_p : K'_p]}{2p^{2k}} \sum_{l=0}^{k-1} d_l p^{-2l}$$

We may suppose that $K_p(p^{k-1}) \not\subset K'_p$, and we will prove in 3.1.5 below the following estimate for d_l when $l \leqslant k/3$:

$$d_l \leqslant p^{\frac{17k}{9}}. \tag{3.5}$$

In general we have trivially that $d_l \leqslant |K_p/B_pK_p(p^k)| \ll p^{2k}$. It follows that

$$\begin{aligned} S_p &\ll \frac{[K_p : K'_p]}{p^{2k}} \sum_{l=0}^{\lfloor k/3 \rfloor} p^{\frac{17k}{9}} + [K_p : K'_p] \sum_{l=\lfloor k/3 \rfloor + 1}^{k-1} p^{-2l} \\ &\leqslant k\frac{[K_p : K'_p]}{p^{k/9}} + 2\frac{[K_p : K'_p]}{p^{2k/3}}. \end{aligned}$$

On the other hand we can estimate trivially $[K_p : K'_p] \leqslant 2p^{6k}$ and $k \ll p^{\varepsilon k}$ for any $\varepsilon > 0$, uniformly in k and p, so we finally get

$$\sum_{j=1}^{h_{K'}} \frac{\alpha_2(\Lambda_{K',j})}{\alpha_1(\Lambda_{K',j})} \ll [K_p : K'_p]^{1-\frac{1}{55}}. \tag{3.6}$$

Now we return to the general case; let m be an integer such that $K_f(m) \subset K'_f$, as above we have that

$$S = \frac{[K_f : K'_f]}{[K_f : N_fK_f(m)]} \sum_{g \in K_f/N_fK_f(m)} \frac{1}{\alpha_1(gN_fg^{-1} \cap K'_f)^2}.$$

Let $N_f = \mathrm{N}(\mathbb{A}) \cap K_f$. For any prime p dividing m, $gN_pg^{-1} \cap K'_p$ is the pro-p summand of $gN_fg^{-1} \cap K'_f$, so that we have

$$gN_fg^{-1} \cap K'_f = \prod_{p|m} gN_pg^{-1} \cap K'_p$$

and it follows that

$$\alpha_1(gN_fg^{-1} \cap K'_f) = \prod_p \alpha_1(gN_pg^{-1} \cap K'_p).$$

So we get, writing $m = \prod_p p^{k_p}$:

$$S = \frac{[K_f : K'_f]}{\prod_p [K_p : N_pK_p(p_p^k)]} \prod_p \left(\sum_{g \in K_p/N_pK_{(p_p^k)}} \frac{1}{\alpha_1(gN_pg^{-1} \cap K'_p)^2} \right).$$

We can rewrite this as

$$S = \frac{[K_f : K'_f]}{\prod_p [K_p : K'_p]} \prod_p S_p \ll \frac{[K_f : K'_f]}{\prod_p [K_p : K'_p]^{\frac{1}{55}}}$$

where the second inequality follows from (3.6). It follows from [1, Lemma 5.11] that there are constants $c, C \geqslant 1$ such that for any compact-open subgroup $K'_f \subset K_f$, if K'_p is its projection to K_p then we have

$$[K_f : K'_f] \leqslant C \left(\prod_p [K_p : K'_p] \right)^c$$

where the product runs over all rational primes such that $K'_p \neq K_p$, so that we get

$$S \ll [K_f : K'_f]^{1 - \frac{1}{55c}}$$

which finishes the proof of (3.1) (we get $\delta = \frac{1}{55c}$).

3.1.5. Proof of (3.5)

This proof is reminescent of that of [1, Proposition 5.13], albeit much more cumbersome due to the fact that we cannot identify the precise elements of N_p which are conjugated into K'_p. Under the hypothesis that $K'_p \not\supset K_p(p^{k-1})$, for any $l = 1, \ldots, k-1$ we have that $K'_p K_p(p^{l+1})/K_p(p^{l+1})$ cannot contain a generating set for the 6-dimensional $\mathbb{F}_p$-Lie algebra $\mathfrak{g}_p / p\mathfrak{g}_p = K_p(p^l)/K_p(p^{l+1})$. For a subset $Y \subset K_p$ define

$$q_Y(j) = \max_{h \in K_p/K_p(p^j)} |(hK_p(p^j) \cap Y) B_p K_p(p^{j+1}) / B_p K_p(p^{j+1})|.$$

Then we have :

$$|X_l| \leqslant \prod_{j=0}^{k-1} q_{X_l}(j). \tag{3.7}$$

We will prove the following lemma at the end of the section (recall that $\mathfrak{g}_p$ is the $\mathbb{Z}_p$-Lie algebra associated to K_p).

Lemma 3.4. — *If p is unramified and $p \neq 2, 3$ then a proper subgroup of $K_p/K_p(p)$ (resp. a proper Lie subalgebra of $\mathfrak{g}_p/p\mathfrak{g}_p$) cannot contain more than $p+1$ pairwise noncommuting unipotent (resp. nilpotent) elements.*

This implies that $q_{X_l}(0) \leqslant p+1$ for all $l \leqslant k/3$ (since there are only finitely many ramified primes we get $q_{X_l}(0) \leqslant Cp$ for a $C > 0$ depending on F, we will work with $C = 1$ to simplify notation). Now we deal with $j \geqslant 1$: we will prove that when $j < (k-2l)/3$ we must have $q_{X_l}(j) \leqslant p$, which in view of (3.7) implies immediately (3.5) for $l \leqslant k/3$. Suppose that there is an $h \in K_p/K_p(p^j)$ such that

$$|(hK_p(p^j) \cap Y)B_pK_p(p^{j+1})/B_pK_p(p^{j+1})| > p;$$

conjugating K'_p by h we may suppose that $h = 1$. This means that there exists pairwise distinct $c_i \in \mathcal{O}_p - p\mathcal{O}_p$, $i = 0, \ldots, p$, such that for each i there is $t_i \in \mathcal{O}_p - p\mathcal{O}_p$ satisfying

$$\left(1 + \begin{pmatrix} * & * \\ p^j c_i & * \end{pmatrix}\right) \begin{pmatrix} 1 & p^l t_i \\ & 1 \end{pmatrix} \left(1 - \begin{pmatrix} * & * \\ p^j c_i & * \end{pmatrix}\right) \in K'_p.$$

Computing the right-hand side yields that

$$g_i = 1 + \begin{pmatrix} p^{l+j} t_i c_i & p^l t_i \\ p^{l+2j} t_i c_i^2 & -p^{l+j} t_i c_i \end{pmatrix} \in K'_p. \tag{3.8}$$

Now the worst that can happen is that we are (up to conjugation) in at most one of the following situations:

(a) All t_i are in $\mathbb{Z}_p$;
(b) All $t_i c_i$ are in $\mathbb{Z}_p$;
(c) All $t_i c_i^2$ are in $\mathbb{Z}_p$.

In case (a) we get that $K'_p K_p(p^{l+2j+1})$ contains the subgroup $1 + p^{l+2j} V$ where

$$V = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \,:\, x, z \in \mathcal{O}_p/p\mathcal{O}_p,\, y \in \mathbb{F}_p \right\}.$$

We may suppose that all $t_m = 1$; now let i, i' such that $a = c_i + c_{i'} \notin \mathbb{F}_p$, modulo p^{2l+j+1} we have

$$g_i g_{i'}^{-1} = 1 + p^{2l+j} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$$

which is not in $1 + p^{2l+j} V$ so that we see that K'_p contains $K_p(p^{2(l+j)})$. In case (c) we can do exactly the same reasoning to get that $K'_p \supset K_p(p^{2l+3j})$. It remains to deal with case (b), which is again similar: we have that $K'_p K_p(p^{l+2j+1})$ contains $1 + p^{l+2j} V'$,

$$V' = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \,:\, y, z \in \mathcal{O}_p/P\mathcal{O}_p,\, x \in \mathbb{F}_p \right\}.$$

and if we suppose that all $t_m c_m = 1$ and $t_i + t_{i'} \notin \mathbb{F}_p$ we get that

$$g_i g_{i'}^{-1} = 1 + p^{2l} \begin{pmatrix} p^j (t_i + t_{i'}) & t_i - t_{i'} \\ 0 & p^j (t_i + t_{i'}) \end{pmatrix}$$

modulo p^{2l+j}, and multiplying by some other g_ms to kill the top-right coefficients we get that $1 + p^{2l+j} u \in K'_p K_p(p^{2l+j+1})$ for some $u \notin V'$, which shows that K'_p contains $K_p(p^{2(l+j)})$ also in this case.

In conclusion, we have seen that if $q_{X_l}(j) > p$ then K'_p contains $K_p(p^{2l+3j})$ which implies that $j \geqslant (k-2l)/3$, which finishes the proof of (3.5).

3.1.6. Proof of Lemma 3.4

It follows from the following classification as the image X of any of the proper subgroups listed here contains less than $p+1$ unipotent, pairwise noncommuting elements.

Lemma 3.5. — *Let H be a subgroup of $K_p/K_p(p)$ such that H contains two unipotent elements which do not commute ; then*

- *If p is inert then either $H = \mathrm{SL}_2(\mathbb{F}_{p^2})$ or H is conjugated to $\mathrm{SL}_2(\mathbb{F}_p)$;*
- *If p is split then either $H = \mathrm{SL}_2(\mathbb{F}_p) \times \mathrm{SL}_2(\mathbb{F}_p)$ or $H = \phi(\mathrm{SL}_2(\mathbb{F}_p))$ where $\phi = (\phi_1, \phi_2)$ for some endomorphisms ϕ_1, ϕ_2 of $\mathrm{SL}_2(\mathbb{F}_p)$.*

There is a similar statement for proper Lie subalgebras of $\mathfrak{g}_p$.

In case $p \neq 3$ is inert this follows immediately from Dickson's Theorem [16, Theorem 8.4 in Chapter 2] (this is where we use $p \neq 3$). In the remaining case where p is split the lemma is actually much simpler, since the projection of H on one of the factors $\mathrm{SL}_2(\mathbb{F}_p)$ must contain two noncommuting unipotent elements and we can then apply Dickson's theorem (which in this case is almost trivial).

The result for Lie algebras is easier in the inert case: if a subalgebra contains two noncommuting nilpotent elements then their Lie bracket is an element in the Cartan subalgebra contained in the intersection of their normalizers. If this bracket is not $\mathbb{F}_p$-rational then we get the whole algebra since its adjoint action on each of the nilpotent $\mathbb{F}_{p^2}$-subalgebras is irreducible (here we use $p \neq 2$), if it is then they generate a subalgebra conjugated to $\mathfrak{sl}_2(\mathbb{F}_p)$. The ramified case is dealt with as for the case of groups.

3.2. Limit multiplicities

Let ρ, V be a finite-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$ and for an hyperbolic orbifold M let $\Delta^p[M]$ be the Hodge Laplacian on $L^2\Omega^p(M;V)$, and $m_V^p(\lambda; M) = \dim\ker(\Delta^p[M] - \lambda)$. There are also L^2-spectral measures ν^p, which are Borel measures on $[0, +\infty[$ obtained by pushing forward the Plancherel measure. We will prove in 3.2.3 below that Theorem 1.5 follows from the following less precise result.

Theorem 3.6. — *For any regular Borel set $S \subset [0,+\infty[$ and $p = 0,\dots,3$ we have*

$$\lim_{n\to+\infty} \frac{\sum_{\lambda\in S} m_V^p(\lambda; M_n)}{\operatorname{vol} M_n} = \nu^p(S)$$

where M_n is as in Theorem 1.5.

3.2.1. Regularized trace

The first step towards Theorem 3.6 is to prove the convergence of regularized traces; the following result is an immediate consequence of Theorem 4.5 in [26] and Theorem 3.1 above.

Proposition 3.7. — *Let Γ_n be a sequence of cusp-uniform congruence subgroups of a given Bianchi group. Then we have the limit*

$$\lim_{n\to+\infty} \frac{\operatorname{Tr}_R \phi(\Delta^p[M_n])}{\operatorname{vol} M_n} = \operatorname{Tr}^{(2)} \phi(\Delta^p[\mathbb{H}^3])$$

for any $\phi \in \mathcal{A}(\mathbb{R})$.

Recall from loc. cit. that $\mathcal{A}(\mathbb{R})$ is a C^∞-dense subset of the Schwartz functions on $\mathbb{R}$ whose Fourier transforms yield point-pair invariants of rapid decay on $\mathbb{H}^3 \times \mathbb{H}^3$.

3.2.2. Proof of Theorem 3.6

Because $\mathcal{A}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ it suffices (by approximating the characteristic function of regular sets) to prove that for any $\phi \in \mathcal{A}(\mathbb{R})$ we have that $\operatorname{Tr}\phi(\Delta^p_{\mathrm{cusp}}[M_n])/\operatorname{vol} M_n$ converges to $\operatorname{Tr}^{(2)} \phi(\Delta^p[\mathbb{H}^3])$. Let $\Delta^p_{\mathrm{cusp}}[M_n]$ be the restriction of $\Delta^p[M_n]$ to the subspace of cusp forms. By standard arguments one deduces Theorem 3.6 from the fact that for all ϕ in a L^1-dense subset of $C_c^\infty(\mathbb{R})$ we have

$$\frac{\operatorname{Tr}\phi(\Delta^p_{\mathrm{cusp}}[M_n])}{\operatorname{vol} M_n} = \operatorname{Tr}^{(2)} \phi(\Delta^p[\mathbb{H}^3]). \tag{3.9}$$

According to Proposition 3.7 this would follow if we can prove

$$\operatorname{Tr}_R \phi(\Delta^p[M_n]) - \operatorname{Tr}\phi(\Delta^p_{\mathrm{cusp}}[M_n]) = o(\operatorname{vol} M_n).$$

We will show this when $p = 0$. From (2.12) we get:

$$\begin{aligned}
&\operatorname{Tr}_R \phi(\Delta^0[M_n]) - \operatorname{Tr}\phi(\Delta^0_{\mathrm{cusp}}[M_n])\\
&= 0/1 + \frac{1}{4}\sum_{l=-2q}^{2q} d_l \phi\left(\left(1-\frac{|l|}{2}\right)^2 + \lambda_V\right)\operatorname{tr}\Psi_l(0)\\
&\quad - \frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{l=-2q}^{2q} d_l\phi\left(\left(1-\frac{|l|}{2}\right)^2 + u^2 + \lambda_V\right)\operatorname{tr}\left(\Psi_l(iu)^{-1}\frac{\mathrm{d}\Psi_l(iu)}{\mathrm{d}u}\right)\mathrm{d}u
\end{aligned}$$

(where the summand $0/1$ comes from the subspace of harmonic sections, equals 0 when V is acyclic). As $\Psi(iu)$ is unitary for $u \in \mathbb{R}$ we obtain

$$\frac{1}{4}\sum_{l=-2q}^{2q} d_l\phi\left(\left(1-\frac{|l|}{2}\right)^2 + \lambda_V\right)\operatorname{tr}\Psi_l(0) \leqslant C\sum_{l=-2q}^{2q} d_l \leqslant Ch_n.$$

Putting

$$\xi(u) = \max_{l=-2q,\dots,2q}\phi\left(\left(1-\frac{|l|}{2}\right)^2 + u^2 + \lambda_V\right)$$

and applying Proposition 4.1 we get for any $\varepsilon > 0$ the bound

$$|\operatorname{Tr}_R K^{\Gamma_n}_{\phi,0} - \operatorname{Tr}(K^{\Gamma_n}_{\phi,0})_{\mathrm{disc}}| \leqslant Ch_n + \int_{-\infty}^{+\infty}\xi(u)C_\varepsilon(u)\mathrm{d}u\, h_n[\Gamma:\Gamma_n]^\varepsilon$$

where the integral on the right-hand side converges absolutely since $C_\varepsilon(u)$ is polynomially bounded. By Lemma 3.2 we get that for ε small enough it is in fact $o(\operatorname{vol} M_n)$, which finishes the proof of (3.9) and of the theorem.

3.2.3. Laplacian eigenvalues and representations

The fact that we can deduce limit multiplicities for representations (Theorem 1.5) from limit multiplicities for Laplacian eigenvalues (Theorem 3.6) is a consequence of the fact, which is specific to real-rank-one groups, that a unitary representation of $\mathrm{SL}_2(\mathbb{C})$ is determined by its Casimir eigenvalue and its SU(2)-types. More precisely, the unitary representations of $\mathrm{SL}_2(\mathbb{C})$ are parametrized by $(\mathbb{Z}\times i\mathbb{R})/\sim \cup\,]0,2[$ where $(l, ia) \sim (-l, -ia)$. On the other hand the SU(2)-types are the restrictions to SU(2) of the holomorphic representations $V_n = V_{n,0}$ of $\mathrm{SL}_2(\mathbb{C})$, and by Frobenius reciprocity the representations containing the SU(2)-type V_n are the $\pi_{\pm l,ia}$ for $0 \leqslant l \leqslant n$ and $n - l = 0 \pmod 2$, so we can deduce Theorem 1.5 by an easy induction on l using the limit multiplicities for $m^0_{V_l}$ or $m^1_{V_l}$.

3.3. Approximation for analytic torsion

Theorem 3.8. — *Let Γ_n be a cusp-uniform sequence of torsion-free congruence subgroups of Γ and $M_n = \Gamma_n\backslash\mathbb{H}^3$. Then we have*

$$\lim_{n\to\infty} \frac{\log T_R(M_n;V)}{\operatorname{vol} M_n} = t^{(2)}(V).$$

Proof. — According to Theorem A in [26] we have to check two conditions:

- The sequence M_n is BS-convergent to $\mathbb{H}^3$;
- There is an $\varepsilon > 0$ such that there exists a $C > 0$ so that for all $u \in [-\varepsilon, \varepsilon]$ we have

$$\operatorname{tr}\left(\Psi_l(iu)^{-1}\frac{\mathrm{d}\Psi_l(iu)}{\mathrm{d}u}\right),\ \operatorname{tr}\left(\Phi_l(iu)^{-1}\frac{d\Phi_l(iu)}{\mathrm{d}u}\right) = o(\operatorname{vol} M_n). \tag{3.10}$$

The BS-convergence is the content of Theorem 3.1 above. To prove (3.10) note that we have for all $u \in \mathbb{R}$ the bound

$$\operatorname{tr}\left(\Psi_l(iu)^{-1}\frac{\mathrm{d}\Psi_l(iu)}{\mathrm{d}u}\right) \ll \left|\frac{\mathrm{d}\Psi_l(iu)}{\mathrm{d}u}\right| h_n d \tag{3.11}$$

where $d = \dim V$ and h_n is the number of cusps of M_n, since $\Psi(s)$ operates on a vector space of dimension $h_n d$ and it is unitary for $\operatorname{Re}(s) = 1/2$. Now we have $h_n \leqslant (\operatorname{vol} M_n)^{1-\delta}$ for some $\delta > 0$, according to Lemma 3.2, and on the other hand according to Proposition 4.1[5] for all $\varepsilon > 0$ there exists $C_\varepsilon(u)$ such that

$$\left|\frac{\mathrm{d}\Psi_l(iu)}{\mathrm{d}u}\right| \leqslant C_\varepsilon(u)|\mathfrak{I}_n|^\varepsilon \ll C_\varepsilon(u)(\operatorname{vol} M_n)^{3\varepsilon}$$

(where the second majoration follows from Lemma 2.1), and taking $\varepsilon = \delta/6$ shows that the right-hand side of (3.11) is indeed $o(\operatorname{vol} M_n)$, uniformly for u in a compact set. □

4. Estimates on the logarithmic derivatives of intertwining operators

This section is devoted to the proof of the following result (see Section 2.2 for notations).

[5] The operators $\Psi_l(s)$ are intertwined with the K'_f-invariant, χ_l, ρ-isotypic matrix block of $\Psi(s)$ according to (2.11).

Proposition 4.1. — *Let τ_∞ be a finite-dimensional representation of K_∞, $\mathfrak{I}$ an ideal of $\mathcal{O}_F$ and χ a Hecke character such that $\mathfrak{f}_\chi|\mathfrak{I}$. Let τ_f be the representation of K_f on $\mathbb{C}[K_f/K_f(\mathfrak{I})]$, $\tau = \tau_\infty \otimes \tau_f$ and $\phi \in \mathcal{H}(\chi,\tau)$. Let $\varepsilon > 0$. Then there exists a polynomially bounded function C_ε on $\mathbb{R}$ depending only on F, ε and τ_∞ such that*

$$\left\| \frac{\mathrm{d}}{\mathrm{d}u} \Psi\left(\frac{1}{2} + iu\right)\phi \right\|_{\mathcal{H}} \leqslant C_\varepsilon(u)|\mathfrak{f}_\chi|^\varepsilon \|\phi\|_{\mathcal{H}}. \tag{4.1}$$

for all $u \in \mathbb{R}$.

We will suppose that $\phi \in C^\infty(K)$ is equal to a product $\bigotimes_v \phi_v$; then it is enough to show that

$$\left\| \frac{\mathrm{d}}{\mathrm{d}u} \Psi\left(\frac{1}{2} + iu\right)\phi \right\|_{L^2(K)} \ll |\mathfrak{f}_\chi|^\varepsilon$$

because $\|\cdot\|_{\mathcal{H}} = \sqrt{h_F}\|\cdot\|_{L^2(K)}$ for A^1-equivariant functions. Since $\mathcal{H}_s(\chi)$ is an irreducible unitary representation of $\mathrm{SL}_2(\mathbb{A})$ when $\mathrm{Re}(s) = 1/2$ and $\Psi(s)^{-1}\frac{\mathrm{d}}{\mathrm{d}u}\Psi(s)$ is a $\mathrm{SL}_2(\mathbb{A})$-equivariant endomorphism of $\mathcal{H}_s(\chi)$, it is a scalar operator, say $c\,\mathrm{Id}$. Now $\Psi(s)^{-1}$ is unitary and thus for any two $\phi, \phi' \in \mathcal{H}_s(\chi)$ we have

$$\frac{\left|\frac{\mathrm{d}}{\mathrm{d}u}\Psi(s)\phi\right|_{\mathcal{H}}}{|\phi|_{\mathcal{H}}} = \frac{\left|\frac{\mathrm{d}}{\mathrm{d}u}\Psi(s)\phi'\right|_{\mathcal{H}}}{|\phi'|_{\mathcal{H}}}$$

so that it suffices to consider a single function $\phi \in \mathcal{H}_s(\chi)$; we will take ϕ_v to be the spherical vector at unramified places and specify ϕ_v for each ramified place; the infinite place does not matter very much for our purposes here.

4.1. Computation of the intertwining integrals

Let v be a finite place; we will make repeated use of the matrix decomposition:

$$w\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} x^{-1} & -1 \\ & x \end{pmatrix}\begin{pmatrix} 1 & \\ x^{-1} & 1 \end{pmatrix} \in \mathrm{B}(F_v)K_v \text{ when } x \in F_v - \mathcal{O}_v. \tag{4.2}$$

4.1.1. Ramified places

We suppose that $v \in S_\chi$, that is χ is non-trivial on $\mathcal{O}_v^\times$; we suppose that χ_v is trivial on $1+\pi_v^{m_v}\mathcal{O}_v$. From (4.2) and (2.3) it follows that:

$$\begin{aligned}
&\Psi_v(s)\phi_v \\
&= \int_{|x|_v>1} |x|^{-2s}\chi^{-1}(x)\phi_v\left(\begin{pmatrix}1 & \\ x & 1\end{pmatrix}k\right)\mathrm{d}x + \int_{\mathcal{O}_v}\phi_v\left(\begin{pmatrix}1 & \\ x & 1\end{pmatrix}w^{-1}k\right)\mathrm{d}x \\
&=: I_v(k)+J_v(k).
\end{aligned}$$

We now compute the L^2-norms of I_v and J_v for the function ϕ_v defined as follows:

$$\phi_v(k) = \begin{cases}\chi(m) & \text{if } k = mn \in B_v; \\ 0 & \text{otherwise.}\end{cases}$$

One can then compute that for $k = w\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ we have:

$$\begin{aligned}
J_v(k) &= \frac{1}{q_v^{m_v}}\sum_{x\in\mathcal{O}_v/\pi_v^{m_v}\mathcal{O}_v}\phi_v\left(\begin{pmatrix}a & b\\ ax+c & bx+d\end{pmatrix}\right) = \frac{1}{q_v^{m_v}}\sum_{x,ax+c=0}\chi(a) \\
&= \begin{cases}\frac{1}{q_v^{m_v}}\chi(a) & \text{if } a\in\mathcal{O}_v^\times; \\ 0 & \text{otherwise}\end{cases}
\end{aligned}$$

and it follows immediately that

$$|J_v|^2_{L^2(K_v)} = \frac{1}{q_v^{2m_v-1}(q_v+1)} = q_v^{-m_v}|\phi_v|^2_{L^2(K_v)}. \tag{4.3}$$

Since $\int_{\mathcal{O}_v^\times}\chi(x)\mathrm{d}x = 0$ and our function ϕ_v is $K_v(\pi_v^{m_v})$-left-invariant we have

$$\int_{|x|_v\geqslant m_v}|x|^{-2s}\chi^{-1}(x)\phi_v\left(\begin{pmatrix}1 & \\ x & 1\end{pmatrix}k\right)\mathrm{d}x = 0$$

and it follows that

$$\begin{aligned}
I_v(k) &= \sum_{l=1}^{m_v-1} q_v^{-2s}\int_{|x_v|=l}\chi(x)^{-1}\phi_v\left(\begin{pmatrix}1 & \\ x^{-1} & 1\end{pmatrix}k\right)\mathrm{d}x \\
&= \sum_{l=1}^{m_v-1} q_v^{l-2s}\chi(\pi_v)^{-l}\int_{\mathcal{O}_v^\times}\chi(x)^{-1}\phi_v\left(\begin{pmatrix}1 & \\ \pi_v^l x & 1\end{pmatrix}k\right)\mathrm{d}x.
\end{aligned}$$

For $k = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$, $c\in\mathcal{O}_v^\times$ the right-hand side equals 0. The summands are also zero if $c\in\pi_v^m\mathcal{O}_v^\times$ for some $m=1,\dots,m_v-1$: indeed, the sum restricts to the summand $l = m$ and we get

$$\begin{aligned} I_v(k) &= q_v^{l-2s}\chi(\pi_v)^{-m}\int_{\mathcal{O}_v^\times}\chi(x)^{-1}\phi_v\left(\begin{pmatrix} a & b \\ \pi_v^m ax + c & \pi_v^m bx + d\end{pmatrix}\right)\mathrm{d}x \\ &= q_v^{l-2s}\chi(\pi_v)^{-m}\chi(a)\int_{(-\pi_v^{-m}ca^{-1})(1+\pi_v^{m_v-m}\mathcal{O}_v)}\chi(x)\mathrm{d}x \\ &= q_v^{l-2s}\chi(\pi_v)^{-2m}\chi(c)\int_{1+\pi_v^{m_v-m}\mathcal{O}_v}\chi(x)\mathrm{d}x \end{aligned}$$

and since χ is nontrivial on $1+\pi_v^{m_v-m}\mathcal{O}_v$ the integral on the right-hand side vanishes.

4.1.2. Unramified places

If χ is not ramified at v then we choose ϕ_v to be the function in $\mathcal{H}_s$ which is identically equal to 1 on K_v. We recall the following well-knwown computation:

$$\begin{aligned} \Psi_v(s)\phi_v(k) &= \int_{F_v-\mathcal{O}_v}\phi_s\left(w\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}k\right)\mathrm{d}x + 1 \\ &= \int_{F_v-\mathcal{O}_v}|x|_v^{-2s}\chi(x)^{-1}\phi\left(\begin{pmatrix} 1 & \\ x^{-1} & 1\end{pmatrix}k\right)\mathrm{d}x + 1 \\ &= 1 + \sum_{l\geqslant 1}(1-q_v^{-1})q_v^l\chi(\pi_v)^k q_v^{-2sl} = \frac{1-\chi(\pi_v)q_v^{-2s}}{1-\chi(\pi_v)q_v^{-2s+1}}. \end{aligned}$$

4.1.3. Infinite place

We finally compute

$$I_\infty(k) = \Psi_\infty(s)\phi_\infty(k).$$

For $z\in\mathbb{C}$ we have

$$w\begin{pmatrix} 1 & z \\ & 1\end{pmatrix} = \begin{pmatrix} u^{-1} & u^{-1}\bar{z} \\ & u\end{pmatrix}k_z,\ u = \sqrt{|z|_\infty + 1},\ k_z = \begin{pmatrix} -u^{-1}\bar{z} & u^{-1} \\ u^{-1} & u^{-1}z\end{pmatrix}$$

and it follows that

$$I_\infty(k) = \int_{\mathbb{C}}(|z|_\infty + 1)^{-2s}\phi_\infty(k_z k)\mathrm{d}z\mathrm{d}\bar{z}$$

As $|z| \to +\infty$ we have $\left|k_z - \begin{pmatrix} \bar{z}/|z| & \\ & z/|z| \end{pmatrix}\right| \ll |z|_\infty^{\frac{-1}{2}}$ (for any norm $|\cdot|$ on K_∞). For $\mathrm{Re}(s) > 1/2$ we get:

$$I_\infty = \phi_\infty(k) \int_{\mathbb{C}} \chi_\infty(z)(|z|_\infty + 1)^{-2s} \mathrm{d}z \mathrm{d}\bar{z} + O(1) \tag{4.4}$$

where the $O(1)$ depends on τ_∞ but not on s. If $\chi_\infty \neq 1$ then the integral is zero so that I_∞ is bounded independantly of s for $\mathrm{Re}(s) \geqslant 1/2$. If $\chi_\infty = 1$ we have

$$\int_{\mathbb{C}} (|z|_\infty + 1)^{-2s} \mathrm{d}z \mathrm{d}\bar{z} = \pi \frac{\Gamma(2s-1)}{\Gamma(2s)}$$

and it follows that I_∞ has a meromorphic continuation to $\mathrm{Re}\, s > 0$ such that $I_\infty(k) - \pi \frac{\Gamma(2s-1)}{\Gamma(2s)} \phi_\infty(k)$ is bounded independantly of s for $\mathrm{Re}(s) \geqslant 1/2$.

4.1.4. Final expression

For $\mathrm{Re}(s) \geqslant 1/2$ and our specific ϕ we get the formula

$$\Psi(s)\phi(k) = I_\infty \times \frac{L(\chi, 2s-1)}{L(\chi, 2s)} \prod_{v \in S_\chi} J_v(k). \tag{4.5}$$

4.2. Proof of Proposition 4.1

We will write $s = \sigma + iu$ for this whole subsection and suppose (unless otherwise stated) that $\sigma = 1/2$. Taking the derivative of the product (4.5) yields

$$\frac{\mathrm{d}}{\mathrm{d}u} \Psi(s)\phi(k) = \left(\frac{\frac{\mathrm{d}}{\mathrm{d}u} L(\chi, 2s)}{L(\chi, 2s)} + \frac{\frac{\mathrm{d}}{\mathrm{d}u} \left(L(\chi, 2s-1) I_\infty(k) \right)}{L(\chi, 2s-1) I_\infty(k)} \right) \Psi(s)\phi(k)$$

so that we get, using the functional equation for $L(\chi, \cdot)$:

$$\begin{aligned} \left| \frac{\mathrm{d}}{\mathrm{d}u} \Psi(s)\phi \right|_{L^2(K)} &\leqslant 2 \left| \frac{\frac{\mathrm{d}}{\mathrm{d}u} L(\chi, 2s)}{L(\chi, 2s)} \right| + \frac{\left| \frac{\mathrm{d}}{\mathrm{d}u} (\gamma(s) I_\infty) \right|_{L^2(K_\infty)}}{|\gamma(s) I_\infty|_{L^2(K_\infty)}} \\ &=: L_1 + L_2. \end{aligned}$$

We will suppose at first that χ is non-trivial so that the L-function $L(\chi, \cdot)$ is holomorphic on $\mathrm{Re}(s) > 0$. To bound both L_1 and L_2 we use the following well-known lemma.

Lemma 4.2. — *Suppose that χ is non-trivial; then we have*

$$\left|\frac{\frac{\mathrm{d}}{\mathrm{d}u}L(\chi,2s)}{L(\chi,2s)}\right| \ll (\log|\mathfrak{f}_\chi|)^2 \tag{4.6}$$

with a constant depending only on s and F, growing polynomially in $\mathrm{Im}(s)$*.*

Proof. — The Euler product for $L(\chi,2s)$ yields for $\mathrm{Re}(s)>1/2$ the absolutely converging series expansion

$$\frac{\frac{\mathrm{d}}{\mathrm{d}u}L(\chi,2s)}{L(\chi,2s)} = \sum_{v\notin S_\chi}\frac{2i(\log q_v)q_v^{-2s}\chi(\pi_v)}{1-\chi(\pi_v)q_v^{-2s}}.$$

We get

$$\frac{\frac{\mathrm{d}}{\mathrm{d}u}L(\chi,2s)}{L(\chi,2s)} = \sum_{v\notin S_\chi} 2i(\log q_v)q_v^{-2s}\chi(\pi_v)\sum_{k\geqslant 0}\chi(\pi_v)^k q_v^{-2ks}.$$

The series $\sum_{v\notin S_\chi}(\log q_v)q_v^{-2s}\chi(\pi_v)\sum_{k\geqslant 1}\chi(\pi_v)^k q_v^{-2ks}$ converges absolutely for $\mathrm{Re}(s)>1/4$ and its sum is bounded by a constant depending only on F, so that we are left with estimating $\sum_{v\notin S_\chi}(\log q_v)q_v^{-2s}\chi(\pi_v)$.

Let $\chi_1,\dots,\chi_{h_F}$ be all the Hecke characters on $F^\times\backslash\mathbb{A}^1$ such that $\ker(\chi_j)\supset M$. If χ is any Hecke character there exists some j such that $\chi(\pi_v)=\chi_j(\pi_v)$ for all places $v\notin S_\chi$. We then have that

$$\sum_{v\notin S_\chi} q_v^{-2s}\log(q_v)\chi(\pi_v) = \sum_{v\notin S_{\chi_j}} q_v^{-2s}\log(q_v)\chi_j(\pi_v) - \sum_{v\in S_\chi - S_{\chi_j}} q_v^{-2s}\log(q_v)\chi_j(\pi_v).$$

As χ_j is non-trivial the function $H: s\mapsto \sum_{v\notin S_{\chi_j}} q_v^{-2s}\log(q_v)\chi_j(\pi_v)$ has an holomorphic extension to an open subset of $\mathbb{C}$ containing the half-plane $\mathrm{Re}(s)\geqslant 1/2$, and by standard arguments[6] there is a polynomial bound (depending only on F as $\chi_1,\dots,\chi_{h_F}$ are fixed) in u for $H(1/2+iu)$. On the other hand, putting $q=\max_{v\in S_\chi} q_v$ we get that for $\mathrm{Re}(s)=1/2$ we have

$$\left|\sum_{v\in S-S_\chi}(\log q_v)q_v^{-2s}\chi(\pi_v)\right| \leqslant \log q\sum_{v\in S-S_\chi}\frac{1}{q_v} \ll (\log q)^2$$

and as $q\leqslant|\mathfrak{f}_\chi|$ we are left with

$$\left|\frac{\frac{\mathrm{d}}{\mathrm{d}u}L(\chi,2s)}{L(\chi,2s)}\right| \leqslant C(u)(\log|\mathfrak{f}_\chi|)^2$$

[6] It suffices to prove that there is a polynomial bound for the logarithmic derivative of a given Hecke L-function.

where $C(u)$ has polynomial growth in u, which finishes the proof of the lemma. □

Since $\frac{\mathrm{d}}{\mathrm{d}u} I_\infty \gamma(s)$ is also bounded by $\log|\mathfrak{f}_\chi|$ we thus get

$$|L_1|, |L_2| \leqslant \left| \frac{\frac{\mathrm{d}}{\mathrm{d}u} L(\chi, 2s)}{L(\chi, 2s)} \right| \leqslant C(u)(\log|\mathfrak{f}_\chi|)^2 \tag{4.7}$$

where $C(u)$ is growing polynomially.

It remains to deal with the case where $\chi = 1$, which is the same except that we have to group the terms I_∞ and $L(\chi, 2s-1) = \zeta_F(2s-1)$ to cancel their poles at $s = 1/2$. The details will be left to the reader (see also [27, 5.5.3]).

4.3. Estimates off the critical line

The estimates above are still valid for any real number s (except if χ is trivial and $s = 1$) but different exponents for $|\mathfrak{I}|$ and $\mathfrak{f}_\chi$ are obtained. We will not require sharp estimates for those, and be content with stating the following rough result.

Proposition 4.3. — *For any $\sigma \in \mathbb{R}$, $\chi \neq 1$ and $\phi \in \mathcal{H}(\chi, \tau)$ we have*

$$\|\Psi(\sigma)\phi\|_{\mathcal{H}}, \left\| \frac{\mathrm{d}\Psi(\sigma + iu)}{\mathrm{d}u} \phi \right\|_{\mathcal{H}} \leqslant C|\mathfrak{I}|^c \|\phi\|_{\mathcal{H}}$$

where C depends on F, σ and τ_∞ and c only on σ.

5. Reidemeister torsion and asymptotic Cheeger–Müller equality

The aim of this section is to define a Reidemeister torsion τ for congruence manifolds with cusps and then prove the following result. We fix a Bianchi group Γ and a strongly acyclic Γ-module $V_\mathbb{Z}$.

Theorem 5.1. — *Let Γ_n be a cusp-uniform sequence of pairwise distinct torsion-free congruence subgroups of Γ. Let $M_n = \Gamma_n \backslash \mathbb{H}^3$ and let $\tau(M_n; V_\mathbb{Z})$ be defined by (5.6) below, then we have*

$$\lim_{n \to \infty} \frac{\log \tau(M_n; V_\mathbb{Z}) - \log T_R(M_n; V)}{\operatorname{vol} M_n} = 0.$$

We now explain how this result follows from [26] and the results in Sections 5.4 and 5.5 below. According to Theorem 3.1 and the proof of Theorem 3.8 we can apply Theorem B in [26] to the sequence M_n, so that we get

$$\lim_{n\to\infty} \frac{\log \tau_{\mathrm{abs}}(M_n^{Y^n}; V) - \log T_R(M_n; V)}{\mathrm{vol}\, M_n} = 0$$

for the sequence Y^n described there. According to Proposition 5.7 below we can replace Y^n by any Υ^n such that $\max \Upsilon_j^n \leqslant |\mathfrak{I}_n|^c$ for some constant c, and the result now follows from Proposition 5.4 below.

Before giving the definition of $\tau(M; V_{\mathbb{Z}})$ and the proof of Proposition 5.4 we will recall from scratch how to describe analytically the cohomology of the boundary $\partial\overline{M}$ of the Borel–Serre compactification $\overline{M}$ of an hyperbolic manifold with cusps, and how to construct a section of the pull-back map $H^*(M) \to H^*(\partial\overline{M})$ using Eisenstein series as in [18] (see also [2, Section 3]).

5.1. Boundary cohomology

In this subsection and the next we fix a congruence manifold $M = M_{K'}$ (we note that everything in the next three sections applies to all finite-volume hyperbolic three–manifolds). We recall notation from [26]: $V = V(n_1, n_2)$ is the $\mathrm{SL}_2(\mathbb{C})$-module $\mathrm{Sym}^{n_1}\mathbb{C}^2 \otimes \mathrm{Sym}^{n_1}\overline{\mathbb{C}}^2$, for a pair k, l we denote by $V_{k,l}$ the $\mathrm{T}(\mathbb{C})$-eigenspace of V associated to the character $z \mapsto z^l/\bar{z}^k$ and $e_{k,l}$ a norm 1 vector generating it. Recall that we have the decomposition

$$V(n_1, n_2) = \bigoplus_{\substack{0\leqslant i\leqslant n_1\\ 0\leqslant j\leqslant n_2}} V_{2i-n_1, 2j-n_2} = \bigoplus_{\substack{0\leqslant i\leqslant n_1\\ 0\leqslant j\leqslant n_2}} \mathbb{C}e_{2i-n_1,2j-n_2}$$

The set of cusps of M is in bijection with $\mathcal{C}(K') = C(F) \times (K'_f \backslash K_f / N_f)$. In degree 1 we have an isomorphism

$$H^1(\partial M; V_{\mathbb{C}}) \cong \mathbb{C}[\mathcal{C}(K')] \otimes (V_{-n_1,n_2} \oplus V_{n_1,-n_2}) \tag{5.1}$$

defined as follows: to a $2h$-tuple of vectors $v_1, \dots, v_h \in V_{-n_1,n_2}$ and $\bar{v}_1, \dots, \bar{v}_h \in V_{n_1,-n_2}$ we associate the de Rham cohomology class $[\omega]$ of the 1-form ω given by

$$\omega = \sum_{j=1}^{h} \mathrm{d}\bar{z}_j \otimes (g_j\rho(n_{z_j})v_j) + \mathrm{d}z_j \otimes (g_j\rho(n_{z_j})\bar{v}_j), \quad n_z = \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix}.$$

Let us check that ω is indeed a closed form. We have $v_j = w_j \otimes u_j$ where $w_j = g_j\lambda_j e_0$ and $u_j = g_j\bar{e}_{n_2}$, so that $\rho(g_j n_z g_j^{-1}).v_j = w_j \otimes (\sum_{l=0}^{n_2} Q_l(\bar{z}) g_j \bar{e}_{n_1-l})$ where Q_l is a polynomial depending only on n_2. It follows that $z \mapsto \rho(n_z)v_j$

is anti-holomorphic. We can see in the same way that $z \mapsto \rho(n_z)\bar{v}_j$ is holomorphic and all this yields that

$$d(\mathrm{d}\bar{z}_j \otimes (g_j \rho(n_{z_j}) v_j)) = 0 = d(\mathrm{d}z_j \otimes (g_j \rho(n_{z_j}) \bar{v}_j)).$$

A similar computation shows that ω cannot be exact, hence the right-hand side of (5.1) injects into the de Rham cohomology. By the equality of dimensions of both sides (which follows for example from the combinatorial description in Section 6.1 below) we get that this inclusion is an isomorphism.

We will denote by $H^{1,0}(\partial\overline{M}; V_{\mathbb{C}}), H^{0,1}(\partial\overline{M}; V_{\mathbb{C}})$ the subspaces of H^1 corresponding respectively to $\mathbb{C}[\mathcal{C}(K')] \otimes V_{\mp n_1, \pm n_2}$.

As for degrees 0 and 2 we have isomorphisms

$$H^0(\partial\overline{M}; V_{\mathbb{C}}) \cong \mathbb{C}[\mathcal{C}(K')] \otimes V_{n_1,n_2} \cong H^2(\partial\overline{M}; V_{\mathbb{C}}). \tag{5.2}$$

Indeed, the space V_{n_1,n_2} is the space of fixed vectors of ${}_0N$ in V, and to $v_1, \dots, v_h \in V_{n_1-n_2}$ we associate the holomorphic section $\sum_{j=1}^h g_j v_j$ or the holomorphic 2-form $\sum_{j=1}^h (dz_j \wedge d\bar{z}_j) \otimes (g_j v_j))$.

5.2. Eisenstein cohomology

In our strongly acyclic case it is very easy to determine the dimensions of the cohomology spaces $H^p(M; V_{\mathbb{C}})$. The L^2-cohomology of M with coefficients in $V_{\mathbb{C}}$ vanishes and the map $i_p^* : H^p(M; V_{\mathbb{C}}) \to H^p(\partial\overline{M}; V_{\mathbb{C}})$ is thus an embedding for $p = 1, 2$ (cf. [23, Theorem 2.1]). One can then show using the long exact sequence of the pair $\overline{M}, \partial\overline{M}$ and Kronecker duality that

$$\dim H^1(M; V_{\mathbb{C}}) = 1/2 \dim H^1(\partial\overline{M}; V_{\mathbb{C}}) \tag{5.3}$$

(cf. [34, Lemme 11]); see also [23, Corollaries 3.6 and 3.7]. As $H^0(M; V_{\mathbb{C}}) = 0$ the long exact sequence also yields that

$$\dim H^2(M; V_{\mathbb{C}}) = \dim H^2(\partial\overline{M}; V_{\mathbb{C}}) - \dim H^0(M; V_{\mathbb{C}}) = \dim H^2(\partial\overline{M}; V_{\mathbb{C}}).$$

In what follows we will give an explicit description of the restriction maps i_p^* following [18]. For a closed p-form $f \in \Omega^p(M; V_{\mathbb{C}})$ we denote by $[f]$ its de Rham cohomology class. Given a harmonic form $\omega \in H^1(\partial\overline{M}; V_{\mathbb{C}})$ and $s \in \mathbb{C}$ we can form the Eisenstein series $E(s, \omega) \in \Omega^1(M; V_{\mathbb{C}})$. The following result is well-known, see for instance the proof of [18, Theorem 2].

Lemma 5.2. — *Let $s_V^1 = n_2 - n_1$ and $\omega \in H^{1,0}(\partial\overline{M}; V_{\mathbb{C}})$ (resp. $\overline{\omega} \in H^{0,1}(\partial\overline{M}; V_{\mathbb{C}})$). The Eisenstein series $E(s_V^1, \omega)$ (resp. $E(-s_V^1, \overline{\omega})$) is then a closed 1-form. Moreover the classes $[E(s_V^1, \omega)]$ span $H^1(M; V_{\mathbb{C}})$.*

Proof. — We need only check that if P is a Γ-rational parabolic subgroup the constant term of $E(s_V^1, \omega)$ at P is a closed form on $\Gamma_P\backslash\mathbb{H}^3$. It is equal (in the $\mathrm{SL}_2(\mathbb{C})$-equivariant model for E_ρ, see [26, (2.5)]) to $\omega + \Phi^+(s_V^1)\omega$ and as $\omega, \Phi^+(s_V^1)\omega$ are closed forms on $\partial\overline{M}$ we have $d(\omega + \Phi^+(s_V^1)\omega) = 0$. Moreover, we have $E(-s_V^1, \overline{\omega}) = E(s_V^1, \Phi^-(-s_V^1)\overline{\omega})$ and thus the second statement follows.

The constant term of $E(s_V^1, \omega)$ is also not an exact form since its restriction to $\partial\overline{M}$ is not, and it follows that the map $H^{1,0}(\partial\overline{M}; V_\mathbb{C}) \ni \omega \mapsto [E(s_V^1), \omega]$ is injective. As we have the equality of dimensions

$$\dim H^{1,0}(\partial\overline{M}; V_\mathbb{C}) = 1/2 \dim H^1(\partial\overline{M}; V_\mathbb{C}) = \dim H^1(M; V_\mathbb{C})$$

it is in fact an isomorphism. □

We can thus define a mapping $E^1 : H^1(\partial\overline{M}; V_\mathbb{C}) \to H^1(M; V_\mathbb{C})$ by $E^1(\omega + \overline{\omega}) = [E(s_V^1, \omega) + E(-s_V^1, \overline{\omega})]$ for $\omega \in H^{1,0}(\partial\overline{M}; V_\mathbb{C}), \overline{\omega} \in H^{0,1}(\partial\overline{M}; V_\mathbb{C})$. From the formula for the constant term of Eisenstein series we get

$$i_1^* E^1(\omega) = \omega + \Phi^+(s_V^1)\omega, \quad \omega \in H^{1,0}(\partial\overline{M}; V_\mathbb{C})$$

and it follows that

$$\begin{aligned} \operatorname{im} i_1^* &= \{\omega + \Phi^+(s_V^1)\omega,\, \omega \in H^{1,0}(\partial\overline{M}; V_\mathbb{C})\} \\ &= \{\Phi^-(-s_V^1)\overline{\omega} + \overline{\omega},\, \overline{\omega} \in H^{0,1}(\partial\overline{M}; V_\mathbb{C})\}. \end{aligned}$$

In degree 2 the long exact sequence shows that i_2^* is onto (since $H^3(\overline{M}, \partial\overline{M}; V_\mathbb{C}) \cong H^0(M; V_\mathbb{C}) = 0$). We have a result akin to Lemma 5.2 for this case, whose proof is very similar.

Lemma 5.3. — *Let $s_V^0 = n_1 + n_2 + 1$ and $v \in V_N := \bigoplus_{j=1}^h V_\mathbb{C}^{N_j} \cong H^0(\partial\overline{M}; V_\mathbb{C})$. The 2-form $*\mathrm{d}E(s_V^0, v)$ is closed, and the classes $[*\mathrm{d}E(s_V^0, v)]$ for $v \in V_N$ span $H^2(M; V_\mathbb{C})$.*

Proof. — Computing the Casimir eigenvalue (cf. [26, (2.4)]) one sees that $E(s_V^0, v)$ is harmonic, so that $d * \mathrm{d}E(s_V^0, v) = 0$. The constant term of $*\mathrm{d}E(s_V^0, v)$ is a nonzero harmonic 2-form so that $[*\mathrm{d}E(s_V^0, v)]$ is nonzero, and by equality of dimensions we get that these classes span $H^2(M; V_\mathbb{C})$. □

We denote by E^2 the map $H^0(\partial\overline{M}; V_\mathbb{C}) \to H^2(M; V_\mathbb{C})$ defined by $v \mapsto [*\mathrm{d}E(s_V^0, v)]$.

5.3. Inner products on cohomology and Reidemeister torsion

From now on we will suppose that $V = V_{n_1,n_2}$ with $n_1 > n_2$, so that $s_v^1 \geqslant 1$. It follows from the Maass–Selberg relations (2.9) that for $\omega \in$

$H^{1,0}(\partial\overline{M};V_{\mathbb{C}})$ we have the limit

$$\lim_{Y\to\infty} Y^{-2s_V^1+1}\|T^Y E(s_V^1,\omega)\|^2_{L^2(M)} = (s_V^1)^{-1}\|\omega\|^2_{L^2(\partial\overline{M})}$$

and we define an inner product on $H^1_{\mathrm{Eis}}(M;V_{\mathbb{C}})$ by

$$\begin{aligned}\langle i_1^*[E^1(\omega)], i_1^*[E^1(\omega')]\rangle_{H^1_{\mathrm{Eis}}(M)} \\ &= \langle \omega,\omega'\rangle^2_{L^2\Omega^1(\partial\overline{M})} \\ &= \lim_{Y\to\infty} s_V^1\cdot Y^{-2s_V^1}\langle T^Y E(s_V^1,\omega), T^Y E(s_V^1,\omega')\rangle^2_{L^2\Omega^1(M)}. \end{aligned} \tag{5.4}$$

Similarly, we can put

$$\begin{aligned}\langle i_2^*[E^2(v)], i_2^*[E^2(v')]\rangle_{H^2_{\mathrm{Eis}}(M)} \\ &= \langle v,v'\rangle_{L^2(\partial\overline{M})} \\ &= \lim_{Y\to\infty} (s_V^0)^{\frac12} Y^{-2s_V^0}\langle T^Y(*\mathrm{d}E(s_V^0,v)), T^Y(*\mathrm{d}E(s_V^0,v'))\rangle_{L^2(M)}. \end{aligned} \tag{5.5}$$

Now for $p=1,2$ the integral cohomology $H^p(M;V_{\mathbb{Z}})_{\mathrm{free}}$ is a lattice in the hermitian vector space $H^p(M;V_{\mathbb{C}})$. We finally define the Reidemeister torsion of M with coefficients in V by the formula

$$\tau(M;V_{\mathbb{Z}}) = \frac{|H^1(M;V_{\mathbb{Z}})_{\mathrm{tors}}|}{\mathrm{vol}\,H^1(M;V_{\mathbb{Z}})_{\mathrm{free}}}\times\frac{\mathrm{vol}\,H^2(M;V_{\mathbb{Z}})_{\mathrm{free}}}{|H^2(M;V_{\mathbb{Z}})_{\mathrm{tors}}|}. \tag{5.6}$$

5.4. Asymptotic equality of Reidemeister torsions

We prove now that the Reidemeister torsion we just defined is asymptotically equal to the absolute Reidemeister torsion of the truncated manifolds (for a certain choice of truncations).

Proposition 5.4. — *Let Γ_n, V be as in the statement of Theorem 5.1. There exists a sequence Υ^n such that*

$$\frac{\log\tau_{\mathrm{abs}}(M_n^{\Upsilon^n};V_{\mathbb{Z}}) - \log\tau(M_n;V_{\mathbb{Z}})}{\mathrm{vol}\,M_n}\xrightarrow[n\to\infty]{} 0. \tag{5.7}$$

and $\max_j \Upsilon_j^n \leqslant |\mathfrak{I}|^c$ *for some* $c>0$.

The first step is the following result, whose proof is essentially contained in [7, 6.8.3].

Lemma 5.5. — *There are $C,c>0$ depending only on F such that the following holds. Let $\Gamma'\subset\Gamma$ be a congruence subgroup, $M=\Gamma'\backslash\mathbb{H}^3$, h its number of cusps, $\alpha_j=\alpha_1(\Lambda_{n,j})$ where $\Lambda_{n,j}, j=1,\dots,h$ are the euclidean lattices corresponding to the cusps of M'. Then for all $Y\in[1,+\infty)^h$ such* *that for all j, $Y_j \geqslant C\alpha_j$, $\omega \in H^{1,0}(\partial\overline{M}; V_{\mathbb{C}})$, $f = E(s_V^1, \omega)$ and f_Y the projection of $f|_{M^Y}$ on the subspace $H^1_{\mathrm{abs}}(M^Y; V_{\mathbb{C}})$ and we have*

$$\|f - f_Y\|_{L^2(M^Y)} \leqslant C\|f\|_{L^2(M^Y)} e^{-c\min_j(Y_j/2\alpha_j)} \operatorname{vol}(M^Y - M^{Y/2}).$$

Proof. — Let $h : [1, +\infty[\to [0,1]$ be a smooth function such that $h(1) = 1, h(2) = 0$ and define f'_Y on M^Y by $f'_Y = f - h(Y/y)(f - f_P)$ (where $y = \max_j y_j$). It follows from (6.16) of [26] that

$$\begin{aligned} \|f - f'_Y\|_{L^2(M^Y)} &\leqslant \|f - f_P\|_{L^2(M^Y - M^{Y/2})} \\ &\ll \|f\|_{L^2(M^Y)} e^{-c\min_j(Y_j/2\alpha_j)}. \end{aligned} \quad (5.8)$$

Now we check that f'_Y satisfies absolute boundary conditions: close enough to the boundary we have $f'_Y = f_P$, and since $\mathrm{d}y \wedge *f_P = 0$ and $\mathrm{d}f_P = 0$ we conclude that $f'_Y \in \Omega^1_{\mathrm{abs}}(M^Y; V_{\mathbb{C}})$. Thus, we have

$$\begin{aligned} \Delta^1_{\mathrm{abs}}[M^Y] f'_Y = \Delta^1[M^Y] f'_Y &= -\Delta^1[M^Y](h(Y/y)(f - f_P)) \\ &= (f_P - f)\Delta^1[M^Y] h(Y/y) \end{aligned} \quad (5.9)$$

and the L^2-norm of the right-hand side is bounded by $C\|f\|_{L^2(M^Y)} \times e^{-c\min_j(Y_j/2\alpha_j)}$.

According to the proof of Proposition 8.2 in [26], up to making C larger we may suppose that for $Y_j \geqslant C\alpha_j$ the Laplace operator $\Delta^1_{\mathrm{abs}}[M^Y]$ has no eigenvalue in the open interval $]0, \lambda_1[$ (for some $\lambda_1 > 0$ depending only on V) as soon as $Y_j \geqslant C\alpha_j$, and we then get from (5.8) and (5.9) that

$$\begin{aligned} \|f - f_Y\|_{L^2(M^Y)} &\leqslant \|f - f'_Y\|_{L^2(M^Y)} + \|f'_Y - f_Y\|_{L^2(M^Y)} \\ &\leqslant \|f\|_{L^2(M^Y)} \left(\int_{M^Y - M^{Y/2}} \left(\sum_{j=1}^{h} e^{-c\min_j(y_j(x)/\alpha_j)} \right)^2 \mathrm{d}x \right)^{\frac{1}{2}} \\ &\quad + \frac{2}{\lambda_1} \|(f_P - f)\Delta^1[M^Y] h(Y/y)\|_{L^2(M^Y)} \\ &\ll \|f\|_{L^2(M^Y)} \left(\int_{M^Y - M^{Y/2}} e^{-c\min_j y_j(x)/\alpha_j} \mathrm{d}x \right)^{\frac{1}{2}} \\ &\leqslant \|f\|_{L^2(M^Y)} \operatorname{vol}(M^Y - M^{Y/2}) e^{-c\min_j(Y_j/\alpha_j)} \end{aligned}$$

where the last line is a consequence of Cauchy–Schwarz inequality. □

Proof of Proposition 5.4. — Let $\Upsilon \in [1, +\infty)^{h_n}$; we have

$$\frac{\tau(M_n; V_{\mathbb{Z}})}{\tau_{\mathrm{abs}}(M_n^{\Upsilon}; V_{\mathbb{Z}})} = \frac{\operatorname{vol} H^2(M_n; V_{\mathbb{Z}})_{\mathrm{free}}}{\operatorname{vol} H^2(M_n^{\Upsilon}; V_{\mathbb{Z}})_{\mathrm{free}}} \frac{\operatorname{vol} H^1(M_n^{\Upsilon}; V_{\mathbb{Z}})_{\mathrm{free}}}{\operatorname{vol} H^1(M_n; V_{\mathbb{Z}})_{\mathrm{free}}}$$

and we will thus show that for $p = 1, 2$ we have

$$\log \mathrm{vol}\, H^p(M_n; V_{\mathbb{Z}})_{\mathrm{free}} - \log \mathrm{vol}\, H^p(M_n^{\Upsilon^n}; V_{\mathbb{Z}})_{\mathrm{free}} = o(\mathrm{vol}\, M_n)$$

for a well-chosen sequence Υ^n.

We will deal only with $p = 1$, the case $p = 2$ being similar. Let r_n be the restriction map $H^1(M_n; V_{\mathbb{C}}) \to H^1(M_n^{\Upsilon}; V_{\mathbb{C}})$. As the inclusion $M_n^{\Upsilon} \subset M_n$ is an homotopy equivalence, it induces an isomorphism between the cohomology groups and we get that

$$\mathrm{vol}\, H^1(M_n^{\Upsilon}; V_{\mathbb{Z}})_{\mathrm{free}} = |\det(r_n)|\, \mathrm{vol}\, H^1(M_n; V_{\mathbb{Z}})_{\mathrm{free}}$$

where the determinant is taken with respect to unitary bases on each space (the left-hand space being endowed with the inner product defined by (5.4) and the right-hand on with the L^2 inner product coming from harmonic forms). We will show below that $\log|\det(r_n)| = o(\mathrm{vol}\, M_n)$, in fact that $|r_n|, |r_n|^{-1} \leqslant 1 + \varepsilon_n$ for some sequence ε_n such that $b_1(M_n; V_{\mathbb{C}}) \log \varepsilon_n = o(\mathrm{vol}\, M_n))$.

We take back the notation f_{Υ} from Lemma 5.5, if f is a closed form on M_n we have $r_n[f] = [f_{\Upsilon}]$. To bound $\|f_{\Upsilon}\|_{L^2(M^Y)}$ above we write

$$\begin{aligned} \|f_{\Upsilon}\|_{L^2(M^{\Upsilon})} &\leqslant \|f\|_{L^2(M^{\Upsilon})} + \|f - f_{\Upsilon}\|_{L^2(M^{\Upsilon})} \\ &\leqslant \left(1 + C\sum_{j=1}^{h_n} \frac{\alpha_2(\Lambda_{n,j})}{\alpha_1(\Lambda_{n,j})}\right) \|f\|_{L^2(M^{\Upsilon})}. \end{aligned} \tag{5.10}$$

where the second inequality follows from Lemma 5.5 and the rough bound $\mathrm{vol}(M^{\Upsilon} - M^{\Upsilon/2}) \leqslant C\sum_{j=1}^{h_n} \frac{\alpha_2(\Lambda_{n,j})}{\alpha_1(\Lambda_{n,j})}$. Now we will bound the right-hand side using the following lemma.

Lemma 5.6. — *Let $Y = \max_j \Upsilon_j$, $a = s_V^1$ and $\mathfrak{I}_n$ be the level of Γ_n. There are $b, C > 0$ (depending on F and V) such that*

$$\begin{aligned} (C^{-1}Y^a - C|\mathfrak{I}_n|^b)\|[f]\|_{H^1(M_n)} &\leqslant \|f\|_{L^2(M_n^{\Upsilon})} \\ &\leqslant C(Y^a + |\mathfrak{I}_n|^b)\|[f]\|_{H^1(M_n)}. \end{aligned} \tag{5.11}$$

Proof. — Let M_n^Y be the truncated manifold at height $(Y, \dots, Y) \in [1, +\infty)^{h_n}$ so that $M_n^{\Upsilon} \subset M_n^Y$ and $\|f\|_{L^2(M_n^{Y^n})} \leqslant \|f\|_{L^2(M_n^Y)} \leqslant \|T^Y f\|_{L^2(M_n)}$.

The Maass–Selberg relations (2.9) yield that

$$\begin{aligned}\|T^Y f\|^2_{L^2(M_n)} \leqslant \frac{Y^{2s^1_V-1}}{2s^1_V-1}\|\omega\|^2_{L^2\partial\overline{M}} + \frac{Y^{-2s^1_V+1}}{2s^1_V-1}\|\Phi^+(s^1_V)\omega\|^2_{L^2(\partial\overline{M})} \\ + \log Y\|\Phi^+(s^1_V)\omega\|_{L^1(\partial\overline{M})}\|\omega\|_{L^2(\partial\overline{M})} \\ + \left\| \left.\frac{\mathrm{d}\Phi^+(s^1_V+iu)}{\mathrm{d}u}\right|_{u=0}\omega\right\|_{L^2(\partial\overline{M})}\|\omega\|_{L^2(\partial\overline{M})}.\end{aligned}$$

From Proposition 4.3 it now follows that

$$\frac{\|T^Y f\|^2_{L^2(M_n)}}{|\omega|^2_{L^2(\partial\overline{M})}} \ll Y^{2s^1_V} + |\mathfrak{I}_n|^c(1+\log Y) \ll Y^{2a} + |\mathfrak{I}_n|^{2c}$$

which deals with the upper bound; the lower bound is proved in a similar manner. □

We have

$$\sum_{j=1}^{h_n}\frac{\alpha_2(\Lambda_{n,j})}{\alpha_1(\Lambda_{n,j})} \leqslant |\mathfrak{I}_n|h_n \leqslant h_F|K_f/N_{\mathfrak{I}_n}K_f(\mathfrak{I}_n)|\cdot|\mathfrak{I}_n| \leqslant 2h_F|\mathfrak{I}_n|^3$$

and it now follows from (5.10) and Lemma 5.6 that for some $e>0$ we have

$$\|f_\Upsilon\|_{L^2(M^\Upsilon)} \ll |\mathfrak{I}_n|^e Y^e \tag{5.12}$$

(we keep the notation $Y = \max_j \Upsilon_j$).

The lower bound for $\|f_\Upsilon\|_{L^2(M^\Upsilon)}$ is more subtle. We have

$$\begin{aligned}\|f_\Upsilon\|_{L^2(M^\Upsilon)} &\geqslant \|f\|_{L^2(M^\Upsilon)} - \|f-f_\Upsilon\|_{L^2(M^\Upsilon)} \\ &\geqslant (1-\mathrm{vol}(M^\Upsilon - M^{\Upsilon/2})e^{-cY/\max\alpha_n^j})\,\|f\|_{L^2(M^\Upsilon)}\end{aligned}$$

where the second minoration follows from Lemma 5.5. We have $\max\alpha_n^j \ll |\mathfrak{I}|^{\frac12}$ and also $\mathrm{vol}(M^\Upsilon - M^{\Upsilon/2}) \ll \sum_j \frac{\alpha_2}{\alpha_1}$ which is bounded by $|\mathfrak{I}_n|^3$, and it follows from Lemma 5.6 that

$$\begin{aligned}&\|f_\Upsilon\|_{L^2(M^\Upsilon)} \\ &\geqslant \left(1 - C|\mathfrak{I}_n|^2\exp\left(-c\frac{Y}{|\mathfrak{I}_n|^{\frac12}}\right)\right)(C^{-1}Y^a - C|\mathfrak{I}_n|^b)\|[f]\|_{H^1(M_n)}.\end{aligned}\tag{5.13}$$

For A large enough and $\Upsilon^n_j = |\mathfrak{I}_n|^{A-1}$ we get from (5.12) and (5.13) that

$$1/2\|[f]\|_{H^1(M_n)} \leqslant \|f_Y\|_{L^2(M_n^{\Upsilon^n})} \leqslant C|\mathfrak{I}_n|^{Ae}\|[f]\|_{H^1(M_n)}.$$

Thus $|r_n|^{-1}\leqslant 2$ and $|r_n| \leqslant C|\mathfrak{I}_n|^{Ae}$ and as $\dim H^1(M_n;V_\mathbb{C}) = h_n$ it follows that

$$|\log\det(r_n)| \ll h_n\log|\mathfrak{I}_n|,$$

and as $h_n \ll (\operatorname{vol} M_n)^{1-\delta}$ (Lemma 3.2) the right-hand side is an $o(\operatorname{vol} M_n)$, as we wanted to show. $\square$

5.5. Comparing absolute torsions

The following result is necessary to be able to use together Proposition 5.4 below and Theorem B in [26], and its proof completes that of Theorem 5.1.

Proposition 5.7. — *Let Y^n be the sequence from Theorem B of* [26]*. For any sequence $\Upsilon^n \in [1,+\infty)^{h_n}$ such that there is $c>0$ for which $Y_j^n \leqslant \Upsilon_j^n \leqslant |\mathfrak{I}|^c$ we have*

$$|\log \tau_{\mathrm{abs}}(M_n^{\Upsilon^n}) - \log \tau_{\mathrm{abs}}(M_n^{Y^n})| \ll \dim H^*(M_n; V_{\mathbb{C}}) \log |\mathfrak{I}_n|.$$

Proof. — It suffices to prove the result for $\Upsilon_j^n = |\mathfrak{I}_n|^c$. We will use a smooth family of Riemannian metrics g_u, $u \in [1,+\infty)$ on $\overline{M}$ such that:

(i) $(\overline{M}, g_u)$ is isometric to M^u through a diffeomorphism ϕ_u.
(ii) For $u/2 \leqslant v \leqslant u$, $\phi_u \circ \phi_v^{-1}|_{M^{v/2}}$ is the inclusion map $M^{v/2} \subset M^u$.
(iii) Let V be the line field perpendicular to horospheres (defined on $M - M^1$). We have

$$\frac{\mathrm{d}g_u}{\mathrm{d}u} \ll \frac{1}{u} g_u|_{V^\perp} + g_u|_V. \tag{5.14}$$

Let us prove that such a family exists. We identify a collar neighbourhood N of the boundary in $\overline{M}$ with $\bigcup_j T_j \times [0,1]$ where the T_j are the boundary components of ∂M^1. The metrics g_u defined as follows do the job, as can be checked by an easy computation: on $\overline{M} - N \cong M^1$, g_u is the hyperbolic metric, an in the cusps we put

$$|(v_1,v_2)|_{g_u}^2 = \frac{1}{(u\,h(ut+1-u))^2}\left(|v_1|^2 + u^2 h'(ut+1-u)|v_2|^2\right), \qquad v_2 \in V_{(x,t)},\ v_1 \in V_{(x,t)}^\perp$$

for $x \in T_j, t \in [0,1]$ where h is a bump function which takes the value 1 for $t \leqslant 0$ and 1 for $t \geqslant 1$ and we identify $V_{(x,t)}^\perp = T_x T_j$ and $V_{(x,t)}$ with the orthogonal complement of the latter in $T_x M^1$.

Let $*_u$ be the Hodge star for g_u, put $\overset{\circ}{*} = \mathrm{d}*/\mathrm{d}u$ and $\alpha_u = *_u^{-1}\overset{\circ}{*} \in \operatorname{End}_{\mathbb{C}} \ker \Delta_{\mathrm{abs}}[M, g_u]$. Then we have [30, Theorem 7.6]

$$\frac{\mathrm{d}}{\mathrm{d}u} \log \tau_{abs}(M; g_u) = \operatorname{tr}(\alpha_u).$$

Let λ_u be the largest eigenvalue of α_u, so that

$$|\log \tau_{\text{abs}}(M^{Y^n}) - \log \tau_{\text{abs}}(M^{\Upsilon^n})| \leqslant \sum_{j=1}^{h_n} \int_{Y_j}^{|\mathfrak{I}_n|^c} |\lambda_u| \mathrm{d}u.$$

Thus the result would follow if we proved that $|\lambda_u| \ll \frac{1}{u}$ for $u \geqslant Y_j$. First we compute the eigenvalues: if f is an eigenform of α_u with norm 1 and eigenvalue λ we have

$$\lambda = \frac{\mathrm{d}\|v\|_{g_u}}{\mathrm{d}u}.$$

Indeed, $\langle *^{-1} \overset{\circ}{*} f, f\rangle = \lambda$, so that we get $\lambda = \int_{\overline{M}} \overset{\circ}{*} f \wedge f = \frac{\mathrm{d}}{\mathrm{d}u} \int_{\overline{M}} *f \wedge f$.

Now let f be a harmonic 1-form for the metric g_u which is an eigenform for α_u; we want to see that $\mathrm{d}|f|_{g_u}/\mathrm{d}u \ll u^{-1}$. On $M - M^1$ write $f = f_1 + f_2$ according to the decompostion $TM = V \oplus V^\perp$ (in coordinates f_1 is the composant on $\mathrm{d}y$), then according to (5.14) we have the pointwise inequality

$$\left| \frac{\mathrm{d}|f|_{g_u}}{\mathrm{d}u} \right| \ll |f_1|_{g_u} + u^{-1}|f_2|_{g_u}$$

so that we need to show that $|f_1|_{g_u} \ll u^{-1}$ on $M^u - M^{\frac{u}{2}}$. The fact that f is co-closed implies that f_1 has a vanishing constant term and it follows that

$$|f_1| = |f_1 - (f_1)_P| \leqslant |f - f_P| \ll e^{-y_j/\alpha_1(\Lambda_j)}$$

where the estimate is a consequence of [7, Lemma 6.2.1]. The right-hand side is $\ll u^{-1}$: indeed, the sequence Y_j^n was defined in [26] as

$$Y_j^n = \alpha_1(\Lambda_{n,j}) \times \left(\frac{\operatorname{vol} M_n}{\sum_{j=1}^{h_n} (\alpha_2(\Lambda_{n,j})/\alpha_1(\Lambda_{n,j}))^2} \right)^{\frac{1}{10}}$$

and it follows from Lemma 3.2 and the cusp–uniformity of the M_n that $Y_j \gg \alpha_1(\lambda_j)|\mathfrak{I}_n|^\delta$ for some $\delta > 0$. Thus, as we consider only $|\mathfrak{I}_n|^c \geqslant u \geqslant Y_j^n/2$ we get $\frac{u}{\alpha_1(\Lambda_j)} \gg u^\eta$ for some $\eta > 0$ (depending on Υ_n) and clearly $e^{-u^\eta} \ll u^{-1}$. □

6. Torsion in (co)homology

We can now finish the proof of Theorem 1.2, whose statement we recall below.

Theorem 6.1. — *Let Γ be a Bianchi group, Γ_n a cusp-uniform sequence of torsion-free congruence subgroups and $M_n = \Gamma_n\backslash\mathbb{H}^3$. Let V be a real* *representation of* $\mathrm{SL}_2(\mathbb{C})$ *and* $V_\mathbb{Z}$ *a lattice in* V *preserved by* Γ. *If* V *is strongly acyclic then we have*

$$\limsup_{n\to\infty} \frac{\log |H_1(\Gamma_n; V_\mathbb{Z})_{\mathrm{tors}}|}{\mathrm{vol}\, M_n} \leqslant -t^{(2)}(V). \tag{6.1}$$

and

$$\limsup_{n\to\infty} \frac{\log |H^2(\Gamma_n; V_\mathbb{Z})_{\mathrm{tors}}|}{\mathrm{vol}\, M_n} \leqslant -t^{(2)}(V). \tag{6.2}$$

Let us describe how the results in this section articulate to yield this. Recall that in (5.6) we have defined a Reidemeister torsion for the congruence manifolds $M_n = \Gamma_n \backslash \mathbb{H}^3$ the logarithm of which is given by

$$\begin{aligned}\log \tau(M_n; V_\mathbb{Z}) = \log |H^1(M_n; V_\mathbb{Z})_{\mathrm{tors}}| - \log \mathrm{vol}\, H^1(M_n; V_\mathbb{Z})_{\mathrm{free}} \\ + \log \mathrm{vol}\, H^2(M_n; V_\mathbb{Z})_{\mathrm{free}} - \log |H^2(M_n; V_\mathbb{Z})_{\mathrm{tors}}|. \end{aligned} \tag{6.3}$$

It follows from Theorems 3.8 and 5.1 that

$$\lim_{n\to\infty} \frac{\log \tau(M_n; V)}{\mathrm{vol}\, M_n} = t^{(2)}(V)$$

and by Lemmas 6.5, 6.8 and 6.7 below all terms in (6.3) except $-\log |H^2(M_n; V_\mathbb{Z})_{\mathrm{tors}}|$ have a negative limit superior as $n \to \infty$. This proves (6.2); we will deduce (6.1) from it in 6.4 at the end of the section.

6.1. Integral homology of the boundary

We have previously described the cohomology of the boundary with coefficients in $V_\mathbb{C}$ using differential forms; to analyze the terms (6.3) we will need a precise description of the integral homology and cohomology through cell complexes.

6.1.1. Cell complexes

Let T be a 2–torus, U a finite-rank free $\mathbb{Z}$-module with a representation $\rho : \pi_1(T) \to \mathrm{SL}(U)$. We fix a cell structure on T with one 2–cell e^2, two 1–cells e_1^1, e_2^1 and one 0–cell e^0 and denote by u_1, u_2 the associated basis for $\pi_1(T)$ (i.e. u_i is the homotopy class of the loop e_i^1). Then we have an isomorphism of $\mathbb{Z}$-complexes $C_*(\widetilde{T}; U) \cong C_*(\widetilde{T}) \otimes U$ which yields an isomorphism of graded modules

$$C_*(T; U) = C_*(\widetilde{T}; U) \underset{\mathbb{Z}[\pi_1(T)]}{\otimes} \mathbb{Z} \cong C_*(T) \otimes U.$$

In this model the differentials for $C_*(T;U)$ are given by

$$\begin{aligned} d_2(e^2\otimes v) &= e_1^1\otimes(v-\rho(u_2)v)+e_2^1\otimes(\rho(u_1)v-v),\\ d_1(e_i^1\otimes v) &= e^0\otimes(v-\rho(u_i)v). \end{aligned} \tag{6.4}$$

6.1.2. Growth of torsion

Lemma 6.2. — *Let Λ be a lattice in a unipotent F-rational subgroup N, then for any sequence of pairwise distinct finite-index subgroups Λ_n in Λ we have*

$$\log|H_0(\Lambda_n;V_{\mathbb{Z}})_{\mathrm{tors}}|,\log|H_1(\Lambda_n;V_{\mathbb{Z}})_{\mathrm{tors}}|=o([\Lambda:\Lambda_n]).$$

Proof. — We prove the result only for $\Lambda=1+\mathcal{O}_F X_\infty$ (where $X_\infty=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$), the general case can be reduced to that particular one. In the proof of Lemma 6.4 below we will show that if $1+aX_\infty\in\Lambda'$ then $(\Lambda'-1)V_{\mathbb{Z}}\supset Na\overline{V}_{\mathbb{Z}}$, where $\overline{V}_{\mathbb{Z}}=\ker\rho(X_\infty)$, where N is a positive integer which does not depend on Γ. In particular, putting $d=\dim V$ we get:

$$|(V_{\mathbb{Z}}/(\Lambda'-1)V_{\mathbb{Z}})_{\mathrm{tors}}|\leqslant(N[\Lambda:\Lambda'])^d$$

and the result about H_0 follows at once.

Write now $\Lambda'=\mathbb{Z}u_1\oplus\mathbb{Z}u_2$. From (6.4) we know that $H_1(\Lambda';V_{\mathbb{Z}})$ embeds in $(V_{\mathbb{Z}}\oplus V_{\mathbb{Z}})/\operatorname{im}(\rho(u_1)-1)\oplus(\rho(u_2)-1)$. The $\mathbb{Z}$-torsion of the latter itself embeds into

$$V_{\mathbb{Z}}/(\operatorname{im}(\rho(u_1)-1)\oplus V_{\mathbb{Z}}(\rho(u_2)-1)).$$

Now this last module has a torsion the order of which is bounded by $(N|u_1|^2\times N|u_2|^2)^d\ll[\Lambda:\Lambda']^{4d}$, and this finishes the proof for H_1. □

6.1.3. Free part of the homology

Suppose now that T has an Euclidean structure, so that its homology groups with coefficients in $V_{\mathbb{C}}$ are endowed with the L^2 inner product and have a Hodge decomposition $H^1=H^{1,0}\oplus H^{0,1}$, which in the case of a boundary component of an hyperbolic manifold corresponds to the decomposition in Section 5.1. We use the rational structure on the F-vector space $V_{\mathbb{Q}}=(\operatorname{Sym}^{n_1}F^2)\otimes(\operatorname{Sym}^{n_2}\overline{F}^2)$ given by restricting the scalars from F to $\mathbb{Q}$, which induces a rational structure on $H^*(\Gamma;V_{\mathbb{Q}})$; recall that a $\mathbb{C}$-subspace $W\subset V_{\mathbb{C}}$ is called rational when $\dim_{\mathbb{Q}}(W\cap V_{\mathbb{Q}})=\dim_{\mathbb{C}}W$.

Lemma 6.3. — *The subspaces* $H^{1,0}(T;V_{\mathbb{C}})$ *and* $H^{0,1}(T;V_{\mathbb{C}})$ *are rational; moreover there exists* $C>0$ *depending only on* T,V *such that for any finite cover* T' *of* T *of degree* D*, we have*

$$[H^1(T';V_{\mathbb{Z}})_{\mathrm{free}} : H^{1,0}(T';V_{\mathbb{Z}})\oplus H^{0,1}(T';V_{\mathbb{Z}})] \leqslant CD^2$$

(where we use the notation $H^{1,0}(T';V_{\mathbb{Z}})$ *for the subgroup* $H^1(T';V_{\mathbb{Z}})\cap H^{1,0}(T';V_{\mathbb{C}})$ *and similarly for* $H^{0,1}(T';V_{\mathbb{Z}})$*).*

Proof. — We first prove that $H^{1,0}(T;V_{\mathbb{C}})$ and $H^{0,1}(T;V_{\mathbb{C}})$ are rational. This is a special case of results in [18], we will give a short completely explicit proof in our setting. Let z be a complex coordinate for T, we can assume that all periods of $\mathrm{d}z, \mathrm{d}\bar{z}$ are in F. Recall that $e_{k,l}\in V_{\mathbb{Z}}$ were defined in Section 5.1 and put

$$\omega_1 = \mathrm{d}\bar{z}\otimes(\rho(n_z)e_{-n_1,n_2}), \quad \omega_2 = \mathrm{d}z\otimes(\rho(n_z)e_{n_1,-n_2}),$$

then ω_1,ω_2 are generators (over $\mathbb{C}$) for $H^{1,0}(T;V_{\mathbb{C}})$ and $H^{0,1}(T;V_{\mathbb{C}})$ respectively according to Section 5.1. It is clear that the cohomology classes they define are rational, as for any chain $c\in C_1(T;V_{\mathbb{Z}})$ we have $(\omega_i,c)\in F$.

Now let $H=\pi_1(T)/\pi_1(T')$ be the group of deck transformations of the covering $T'\to T$. Let $\pi_* : H_1(T';V_{\mathbb{Z}})\to H_1(T;V_{\mathbb{Z}})$ be the map induced by the finite covering $T'\to T$, let $c\in Z_1(T;V_{\mathbb{Z}})$ any cycle and $\tilde{c}$ any lift of c to T' (which is a chain, but not necessarily a cycle). Then $\sum_{g\in H} g_*\tilde{c}$ is a cycle and $\pi_*(\sum_{g\in H} g_*\tilde{c}) = Dc$. So we see that the image of the map π_* contains $DH_1(T;V_{\mathbb{Z}})$. On the other hand all harmonic forms on T' with coefficients in $V_{\mathbb{C}}$ are H-invariant, so we can consider them as harmonic forms on T. If $\omega\in H^1(T',V_{\mathbb{Z}})$ then it follows from the above that $D\omega\in H^1(T;V_{\mathbb{Z}})$, so that $[H^1(T';V_{\mathbb{Z}}) : H^1(T;V_{\mathbb{Z}})]\leqslant D^2$. Since $H^{1,0}(T;V_{\mathbb{C}})$ and $H^{0,1}(T;V_{\mathbb{C}})$ are rational $H^{1,0}(T;V_{\mathbb{Z}})\oplus H^{0,1}(T;V_{\mathbb{Z}})$ is a finite-index subgroup in $H^1(T;V_{\mathbb{Z}})$ and letting C be the index of $H^{1,0}(T;V_{\mathbb{Z}})\oplus H^{0,1}(T;V_{\mathbb{Z}})$ in $H^1(T;V_{\mathbb{Z}})$ it follows that $[H^1(T';V_{\mathbb{Z}}) : H^{1,0}(T';V_{\mathbb{Z}})\oplus H^{0,1}(T';V_{\mathbb{Z}})]\leqslant CD^2$. □

6.2. Subexponential growth of torsion

We prove here that in degrees other than 2 the torsion in cohomology has subexponential growth.

6.2.1. Homology in degree 0

Lemma 6.4. — *Let* Γ_n *be a sequence of congruence subgroups in* $\Gamma(\mathcal{O}_F)$*,* $M_n=\Gamma_n\backslash\mathbb{H}^3$*. We have that* $\log|H_0(M_n;V_{\mathbb{Z}})| = o(\operatorname{vol} M_n)$*.*

Proof. — We prove the result for principal congruence subgroups and then deduce the general case. To do the former we will show that $N\mathfrak{I}V_{\mathbb{Z}} \subset (\Gamma(\mathfrak{I})-1)V_{\mathbb{Z}}$ for all $\mathfrak{I}$ and some integer N depending only on n_1, n_2, so that $|H_0(M_{\mathfrak{I}}; V_{\mathbb{Z}})| \leqslant (N|\mathfrak{I}|)^{\dim V}$ from which it follows at once that $\log|H_0(M_{\mathfrak{I}}; V_{\mathbb{Z}})| = O(\log|\mathfrak{I}|)$ is an $o(\mathrm{vol}\, M_{\mathfrak{I}})$. Let $X_\infty = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $X_0 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. For $a \in \mathfrak{I}$ we have that $\eta_a = 1 + aX_\infty \in \Gamma(\mathfrak{I})$. We begin by studying the case where $n_2 = 0, n_1 = n$; put $e_1 = (1,0)$ and $e_2 = (0,1)$, then the family $e_1^n, e_1^{n-1}e_2, \ldots, e_2^n$ is an $\mathcal{O}_F$-basis of the free module $V_{\mathbb{Z}}$. Let N be the product of all binomial coefficients $\binom{n}{k}$, we will see that $Nae_1^{n-k}e_2^k \in (\Gamma(\mathfrak{I})-1)V_{\mathbb{Z}}$ for all $k < n$. Indeed, we have $ae_2^n = \eta_a.(e_1e_2^{n-1}) - e_1e_2^{n-1}$, and on the other hand $\eta_a.(e_1^{k+1}e_2^{n-k-1}) - e_1^{k+1}e_2^{n-k-1}$ is a linear combination of the $e_1^l e_2^{n-l}$ for $l \geqslant k$ so we can prove this by induction on k. We also have that $ae_1^n = (1+aX_0)e_1^{n-1}e_2 - e_1^{n-1}e_2 \in (\Gamma(\mathfrak{I})-1)V_{\mathbb{Z}}$, which finishes the proof in this case. The same arguments work in general.

If $H \subset G_{\mathfrak{I}}$ is a proper subgroup there is an epimorphism $H_0(M_H; V_{\mathbb{Z}}) \to H_0(M_{\mathfrak{I}}; V_{\mathbb{Z}})$. Letting $\mathfrak{I}_n$ be the level of Γ_n we get that $\log|H_0(M_n; V_{\mathbb{Z}})| = O(\log|\mathfrak{I}_n|)$, and it follows from this and Lemma 2.1 that we have $\log|H_0(M_n; V_{\mathbb{Z}})| = o(\mathrm{vol}\, M_n)$. □

6.2.2. Cohomology in degree 1

We will use the following elementary lemma in what follows.

Lemma 6.5. — *Let $A \in \mathrm{Hom}(\mathbb{Z}^m, \mathbb{Z}^n)$ and $B \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}^m)$ such that for all $\varphi \in \mathrm{Hom}(\mathbb{Z}^m, \mathbb{Z})$ and $v \in \mathbb{Z}^n$ we have $(\varphi, Bv) = (\varphi \circ A, v)$. Then $\mathbb{Z}^m/B\mathbb{Z}^n$ and $\mathbb{Z}^n/A\mathbb{Z}^m$ have the same torsion subgroup.*

Proof. — In appropriate bases of $\mathbb{Z}^m, \mathbb{Z}^n$ the matrices of A and B are transpose of each other. □

Lemma 6.6. — *We have*

$$\frac{\log|H^1(M_n; V_{\mathbb{Z}})_{\mathrm{tors}}|}{\mathrm{vol}\, M_n} \xrightarrow[n\to\infty]{} 0. \tag{6.5}$$

Proof. — Recall that there is a $\Gamma(\mathcal{O}_F)$-invariant pairing on $V_{\mathbb{Q}}$ and let $V'_{\mathbb{Z}}$ is the lattice in V which is dual to $V_{\mathbb{Z}}$ through this pairing. For notational ease we will use H_*, H^* to denote (co)homology with coefficients in $V_{\mathbb{Z}}$ and H'_*, H'^* for $V'_{\mathbb{Z}}$-coefficients. The existence of the Kronecker pairing and the property (2.13), together with Lemma 6.5 imply that

$$[H_2(M)_{\mathrm{free}} : (\mathrm{im}\, i_*^2)_{\mathrm{free}}] = [H'^2(\partial\overline{M})_{\mathrm{free}} : (\mathrm{im}\, i_2^*)_{\mathrm{free}}]$$

and it further follows that

$$[H_2(M)_{\mathrm{free}} : (\mathrm{im}\, i_*^2)_{\mathrm{free}}] = [H'_0(\partial\overline{M})_{\mathrm{free}} : \mathrm{im}(\delta^1)_{\mathrm{free}}] \leqslant |H'_0(M)|$$

where the equality follows from Poincaré duality and the majoration from the segment $H_1'(\overline{M},\partial\overline{M}) \xrightarrow{\delta^1} H_0'(\partial\overline{M}) \to H_0'(M)$ in the homology long exact sequence of the pair $(\overline{M},\partial\overline{M})$. Applying once more Poincaré duality we get

$$[H^1(\overline{M},\partial\overline{M}) : \operatorname{im}\delta_0] = [H_2(M)_{\text{free}} : (\operatorname{im} i_*^2)_{\text{free}}] \leqslant |H_0'(M)_{\text{tors}}|. \tag{6.6}$$

On the other hand the cohomology long exact sequence for $\overline{M},\partial\overline{M}$ contains

$$H^0(\partial\overline{M}) \xrightarrow{\delta_0} H^1(\overline{M},\partial\overline{M}) \to H^1(M) \to H^1(\partial\overline{M})$$

which in turn yields

$$\begin{aligned}\log|H^1(M)_{\text{tors}}| &\leqslant \log[H^1(\overline{M},\partial\overline{M}) : \operatorname{im}\delta_0] + \log|H^1(\partial\overline{M})_{\text{tors}}|\\ &\leqslant \log|H_0'(M)_{\text{tors}}| + \log|H_1(\partial\overline{M})_{\text{tors}}|\end{aligned}$$

where the inequality on the second line follows from (6.6). The right-hand side is an $o(\operatorname{vol} M_n)$, as follows from Lemmas 6.4 and 6.2, which finishes the proof. □

6.3. Growth of regulators

6.3.1. Degree 1

Lemma 6.7. — *We have*

$$\liminf_{n\to\infty} \frac{\log\operatorname{vol} H^1(M_n;V_{\mathbb{Z}})}{\operatorname{vol} M_n} \geqslant 0.$$

Proof. — The embedding $H^1(M_n;V_{\mathbb{C}}) \to H^1(\partial\overline{M}_n;V_{\mathbb{C}})$ is isometric by definition of the inner product on $H^1(M_n;V_{\mathbb{C}})$ and its image is the subspace

$$\{\omega + \Phi^+(s_V^1)\omega, \omega \in H^{1,0}(\partial\overline{M}_n;V_{\mathbb{C}})\}.$$

Let π be the orthogonal projection of $H^1(\partial\overline{M}_n;V_{\mathbb{C}})$ onto $H^{1,0}(\partial\overline{M}_n;V_{\mathbb{C}})$. Then Lemma 6.3 implies that the image $\pi(H^1(\partial\overline{M}_n;V_{\mathbb{Z}}))$ contains $H^{1,0}(\partial\overline{M}_n;V_{\mathbb{Z}})$ with an index which is $\ll (\operatorname{vol} M_n)^{4h_n}$. As

$$\operatorname{vol}\left(i_1^* H^1(M_n;V_{\mathbb{Z}})\right) \geqslant \operatorname{vol}\left(\pi(i_1^* H^1(M_n;V_{\mathbb{Z}}))\right)$$

we get that

$$\operatorname{vol} H^1(M_n;V_{\mathbb{Z}}) \gg (\operatorname{vol} M_n)^{-4h_n} \operatorname{vol} H^{1,0}(\partial\overline{M}_n;V_{\mathbb{Z}}). \tag{6.7}$$

Let $M_0 = \Gamma\backslash\mathbb{H}^3$ (recall that Γ is the Bianchi group containing $\Gamma_n = \pi_1(M_n)$) and let T be a component of $\partial\overline{M}_0$; to simplify we assume that T is a torus, otherwise we would need to use its minimal manifold cover. Let $T_1,\dots,T_r$ the components of the preimage of T in M_n, and D_i the degree of the cover $T_i \to T$ (so in particular $D_i \leqslant [\Gamma : \Gamma_n] \ll \operatorname{vol}(M_n)$). Let $\omega_i \in H^{(1,0)}(T_i;V_{\mathbb{Z}})$

be a generator. Then, as we saw in the proof of Lemma 6.3 we have that $D_i\omega \in H^{(1,0)}(T;V_{\mathbb{Z}})$. It follows that

$$\operatorname{vol} H^{(1,0)}(T_i;V_{\mathbb{Z}}) = \|\omega_i\|_{L^2(T_i;V_{\mathbb{C}})} \geqslant D_i^{-1} \operatorname{vol} H^{(1,0)}(T;V_{\mathbb{Z}}).$$

Now as $H^{(1,0)}(\partial\overline{M}_n;V_{\mathbb{Z}}) = \bigoplus_{T,i} H^{(1,0)}(T_i;V_{\mathbb{Z}})$ we get that

$$\operatorname{vol} H^{(1,0)}(\partial\overline{M}_n;V_{\mathbb{Z}}) \geqslant \prod_{T,i} D_i^{-1} \operatorname{vol} H^{(1,0)}(T;V_{\mathbb{Z}}) \geqslant (C \operatorname{vol}(M_n))^{-h_n}.$$

Using (6.7) we finally obtain that

$$\frac{\log \operatorname{vol} H^1(M_n;V_{\mathbb{Z}})}{\operatorname{vol} M_n} \gg \frac{-h_n \log \operatorname{vol}(M_n)}{\operatorname{vol} M_n}$$

and by Lemma 3.2 the right-hand side goes to zero as $n \to +\infty$. □

6.3.2. Degree 2

Lemma 6.8. — *We have*

$$\frac{\log \operatorname{vol} H^2(M_n;V_{\mathbb{Z}})_{\mathrm{free}}}{\operatorname{vol} M_n} \xrightarrow[n\to\infty]{} 0.$$

Proof. — The map $i_2^* : H^2(M_n;V_{\mathbb{C}}) \to H^2(\partial\overline{M}_n;V_{\mathbb{C}})$ is an isometry according to the definition (5.5) of the inner product on $H^2(M_n;V_{\mathbb{C}})$. Moreover, using the long exact sequence we get

$$[H^2(\partial\overline{M}_n;V_{\mathbb{Z}}) : \operatorname{im} i_2^*] = |H^3(\overline{M}_n,\partial\overline{M}_n;V_{\mathbb{Z}})| = |H_0(M_n;V_{\mathbb{Z}})|$$

and it follows that

$$|\log \operatorname{vol} H^2(M_n;V_{\mathbb{Z}})_{\mathrm{free}}| \leqslant \log|H_0(M_n;V_{\mathbb{Z}})| + |\log \operatorname{vol} H^2(\partial\overline{M};V_{\mathbb{Z}})|.$$

For a torus T we have $\operatorname{vol} H^2(T,V_{\mathbb{Z}}) \ll \operatorname{vol}(T)$ so we get the bound

$$|\log \operatorname{vol} H^2(M_n;V_{\mathbb{Z}})_{\mathrm{free}}| \leqslant \log|H_0(M_n;V_{\mathbb{Z}})| + O(h_n \log(\operatorname{vol} M_n)).$$

Now the right-hand side is an $o(\operatorname{vol} M_n)$, as follows from Lemmas 6.4 and 3.2 so that the proof is complete. □

6.4. Homology from cohomology

We can finally deduce (6.1) from (6.2): from the sequence

$$H_1(\partial\overline{M}) \to H_1(M) \to H_1(\overline{M},\partial\overline{M}) \to H_0(\partial\overline{M}),$$

Lemmas 6.2 and 6.4, and Poincaré duality it is clear that it suffices to show that the index of the sublattice $i_*H_1(\partial\overline{M}_n)_{\mathrm{free}}$ in $H_1(M_n)_{\mathrm{free}}$ is an $o(\operatorname{vol} M_n)$.

We will not detail how to prove this, as it follows from the proof of Lemma 6.7 (where it was shown that the torsion subgroup of $H^1(\partial\overline{M}_n)/i^*H^1(M_n)$ is of order $o(\operatorname{vol} M_n)$) and Kronecker duality as in the proof of Lemma 6.5. We could also have applied the universal coefficients theorem as in [25, Lemma 3.1] to deduce it from (6.2) applied to the dual lattice of $V_{\mathbb{Z}}$ in $V_{\mathbb{Q}}$.

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