ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

Frédéric Chapoton

A note on gamma triangles and local gamma vectors (with an appendix by Alin Bostan)

Tome XXIX, nº 4 (2020), p. 907-925.

<http://afst.centre-mersenne.org/item?id=AFST_2020_6_29_4_907_0>

Annales de la faculté des sciences de Toulouse Volume XXIX, n° 4, 2020
pp. 907-925

A note on gamma triangles and local gamma vectors (with an appendix by Alin Bostan) (*)

Frédéric Chapoton (1)

Abstract. — This article introduces Gamma-triangles, which are closely related to F-triangles and H-triangles that were used in the combinatorial study of cluster complexes, and in some sense are more fundamental. We prove that Gamma-triangles can be expressed as sums of local gamma-vectors, that were introduced by Athanasiadis as a refinement of the Stanley's local h-vector of simplicial subdivisions. We compute explicitly the Gamma-triangles for cluster complexes of finite type.

Résumé. — Cet article introduit les gamma-triangles, qui sont liés aux F-triangles et H-triangles utilisés dans l'étude combinatoire des complexes d'amas, et dont ils sont en quelque sorte une version plus fondamentale. On démontre que les gamma-triangles s'expriment comme des sommes de gamma-vecteurs locaux, introduits par Athanasiadis comme un raffinement des h-vecteurs locaux de subdivisions simpliciales, dûs à Stanley. On calcule ensuite explicitement les gamma-triangles des complexes d'amas de type fini.

When studying simplicial complexes, a basic invariant is the f-vector that counts faces according to their dimensions. It is now well-known that it is also interesting to consider the h-vector, obtained in a simple way from the f-vector. For example, when the simplicial complex comes from a complete toric fan, the h-vector records the dimensions of the homology groups of the associated toric variety. Even deeper stands the γ-vector introduced by Gal [11], which is not really well understood. It can be defined when the h-vector is symmetric, and is conjectured to be nonnegative under some precise hypotheses involving flagness of simplicial complexes.

Stanley has introduced in [19] a local variation on this theme, where the starting point is no longer any simplicial complex, but rather a subdivision of a standard simplex. This involves a local h-vector, which he proved to

(*) Reçu le 11 octobre 2018, accepté le 18 décembre 2018.

(1) Institut de Recherche Mathématique Avancée, Université de Strasbourg, 7 rue René Descartes, F-67084 STRASBOURG Cedex (France) — chapoton@math.unistra.fr

Article proposé par Bertrand Toën.

be nonnegative under some technical condition. The local analogue of the γ-vector was introduced and studied in depth in [10]. It is also conjectured there to be nonnegative under the appropriate hypothesis of flagness. We refer to the survey article [1] for much more details on this beautiful theory.

The author has introduced in [5], motivated by the study of the combinatorics of simplicial complexes attached to cluster algebras [8, 9, 17], a finer version of the f-vector, where faces are not only counted according to their dimension, but in a more refined way using the fact that the underlying set is split into negative and positive parts. This gives the F-triangle, a polynomial in two variables, that could be defined for any pure simplicial complex endowed with a prefered maximal simplex.

Later, an analogue of the h-vector in this context, called the H-triangle, has been introduced in [6]. It is related to the F-triangle by a simple birational change of variables, that extends the classical transformation from the f-vector to the h-vector.

The main aim of the present article is to introduce the analogue in this context of the γ-vector: we define a Γ-triangle starting from the H-triangle. To justify that it exists, this Γ-triangle is expressed as a sum of local γ-vectors. This implies that the Γ-triangle is a refinement of the γ-vector, that also contains the information of the local γ-vector. Conversely, the Γ-triangle is determined by the knowledge of all local γ-vectors of subdivisions of facets.

We then compute explicitly the Γ-triangle for all the cluster simplicial complexes of irreducible Coxeter groups, using the information on local γ-vectors for the related subdivisions from the article [2].

As a general reference on the relationships between the combinatorics and homology of simplicial complexes and commutative algebra, the reader may want to consult [20].

As a first side remark, it was observed by Athanasiadis (private communication) that a formula similar to the relation (3.2) between H-triangle and Γ-triangle appears in [21, Conjectures 1 and 8] (see also [3, Conjecture 10.2]), which presents a conjecture of Gessel about the distribution of descents and inverse descents in the symmetric groups (two-sided Eulerian polynomials).

As another side remark, let us note that the article [13] by Katz and Stapledon contains material which present some formal similarities with the present article, including versions of h-polynomial involving two variables (see their Lemma 5.8). Nevertheless, it seems that the two settings cannot be made identical by any appropriate change of notation.

It may also be interesting to see if the ideas presented here could have some impact on the study of the zero loci of general F-triangles made in [18].

Acknowledgements

Many thanks to Alin Bostan, who wrote the appendix and thereby allowed the author to complete this article.

1. Simplicial complexes and subdivisions

Let C be a finite simplicial complex, which means a collection of subsets of a fixed finite set, closed under taking subsets. The elements of C are called faces. The *dimension* of a face of C is the number of elements in that face minus 1. Faces of dimension 0 are called vertices. The *dimension* of C is the maximal dimension of the faces of C. The simplicial complex C is *pure* if all maximal faces have the same dimension. These faces of maximal dimension are then called *facets*.

Let us recall briefly the definition of *simplicial subdivisions*, see [19, Section 2] and [2, Section 2.2] for more context and details on this notion. Let I be a finite set and 2^I be the full simplex with vertex set I. A simplicial subdivision of 2^I is a simplicial complex C_+ together with a map σ from C_+ to 2^I such that for every $J \subseteq I$,

- $\sigma^{-1}(2^J)$ is a subcomplex $C_+(J)$ of C_+ which is a simplicial ball of dimension $|J| - 1$ and
- $\sigma^{-1}(J)$ consists of the interior faces of $C_+(J)$.

The simplicial subdivision (C_+, σ) is called *geometric* if there exists a geometric realisation of C_+ (where each face is realised as an Euclidean simplex) that subdivides geometrically a geometric realisation of 2^I.

Let us now describe a correspondence between simplicial subdivisions and some spherical complexes. This construction has already appeared (see (4-1) there) in [10, Section 4].

Given a simplicial subdivision C_+ of 2^I, one can define a new simplicial complex $\mathrm{Sphere}(C_+)$ as follows. Let $(C_+)_0$ be the underlying set of C_+. The underlying set of $\mathrm{Sphere}(C_+)$ is the disjoint union $(C_+)_0 \sqcup I$. The faces of $\mathrm{Sphere}(C_+)$ are pairs (F, J) (or rather their disjoint union) where F is a face of C_+ and $J \subseteq I$ such that $\sigma(F)$ does not intersect J.

For a geometric simplicial subdivision C_+, the simplicial complex $\mathrm{Sphere}(C_+)$ has a geometric realisation that subdivides a sphere, more precisely the boundary of a cross-polytope.

Conversely, given the spherical simplicial complex $\mathrm{Sphere}(C_+)$, one can recover C_+ from the knowledge of the distinguished subset of vertices I. Indeed, the faces of C_+ are the faces of $\mathrm{Sphere}(C_+)$ that do not contain any element of I.

An important class of examples of the situation just described are the cluster complexes, as appearing in the theory of cluster algebras [8, 9]. For every finite Weyl group W and for any choice of Coxeter element $c \in W$, there is an associated complete fan, called the cluster fan, whose rays are indexed by almost-positive roots (positive roots or negative simple roots) in the root system of W. The dual polytope of this fan is a generalized associahedra, whose edge graph is the flip graph of cluster variables. The simplicial complex associated to the cluster fan is naturally of the form $\mathrm{Sphere}(C_+)$ where C_+ is the subcomplex obtained by restriction to the set of positive roots. The set I is the set of simple roots, and the structure map σ of the subdivision C_+ is given by the support of sets of positive roots.

We will use one important property of the cluster fans, namely the property that the subcomplexes $C_+(J)$, for J a subset of the set I of simple roots, is isomorphic to the set C_+ for the cluster fan associated to the parabolic subgroup W_{I-J}.

We will also use freely the extension of this theory of cluster fans to all finite Coxeter groups as done under the name of cambrian fans by Reading and Speyer in [17]. All the properties that we need do extend to this more general setting.

1.1. f-vector, h-vector and γ-vector

Let us first recall the definition of the f-vector, h-vector and γ-vector attached to a simplicial complex.

Given a pure finite simplicial complex C of dimension $d-1$, its f-vector is the sequence of integers $(f_{-1}, f_0, \dots, f_{d-1})$ where f_i is the number of faces with $i+1$ vertices in C. The associated f-polynomial is defined as

$$f_C(x) = \sum_{0 \leqslant i \leqslant d} f_{i-1} x^i. \tag{1.1}$$

The h-polynomial of the simplicial complex C is then defined as

$$h_C(x) = (1-x)^d f_C\left(\frac{x}{1-x}\right) = \sum_{0 \leqslant i \leqslant d} f_{i-1} x^i (1-x)^{d-i}. \tag{1.2}$$

The reverse transformation is given by

$$f_C(x) = (1+x)^d h_C\left(\frac{x}{1+x}\right). \tag{1.3}$$

When the h-polynomial is written as

$$h_C(x) = \sum_{0\leqslant i\leqslant d} h_i x^i, \tag{1.4}$$

the sequence of integers $(h_0, h_1, \dots, h_d)$ is called the h-vector of C.

When C is an homology sphere, the h-vector satisfies the symmetry property $h_i = h_{d-i}$ for all $0 \leqslant i \leqslant d$. In this case, one can always write

$$h(x) = \sum_{0\leqslant i\leqslant d/2} \gamma_i x^i (1+x)^{d-2i}, \tag{1.5}$$

for some uniquely defined integer coefficients γ_i. These coefficients form the γ-vector attached to the simplicial complex C.

There is a famous conjecture of Gal about these coefficients [11]. Recall that a simplicial complex is said to be *flag* if all minimal non-faces have two elements.

Conjecture 1.1 ([11, Conjecture 2.1.7]). — *The γ-vector has nonnegative coordinates for every flag homology sphere.*

1.2. Local f-vector, h-vector and γ-vector

Let us now recall the definition of the local f-vector, local h-vector and local γ-vector attached to a simplicial subdivision.

Let I be a finite set of cardinality d and let C_+ be a simplicial subdivision of the simplex 2^I. The local h-polynomial $h^\ell_{C_+}(x)$ is the alternating sum of the h-polynomials of the restrictions of C_+ to the faces of 2^I. More precisely,

$$h^\ell_{C_+}(x) = \sum_{J\subseteq I} (-1)^{|I-J|} h_{C_+(J)}(x). \tag{1.6}$$

Conversely, by Möbius inversion on the boolean lattice of subsets,

$$h_{C_+}(x) = \sum_{J\subseteq I} h^\ell_{C_+(F)}(x). \tag{1.7}$$

When expanded as

$$h^\ell_{C_+}(x) = \sum_{i=0}^{n} h^\ell_i x^i, \tag{1.8}$$

the sequence $(h^\ell_0, \dots, h^\ell_d)$ is called the local h-vector of C_+.

The local h-vector is known to be symmetric ($h^\ell_i = h^\ell_{d-i}$ for all $0 \leqslant i \leqslant d$) for any simplicial subdivision and nonnegative for every geometric simplicial subdivision [19].

Because of this symmetry property, one can define the local γ-vector in the same way as the γ-vector was defined from the h-vector. Namely, one can always write

$$h^\ell(x) = \sum_{0 \leqslant i \leqslant d/2} \gamma^\ell_i x^i (1+x)^{d-2i}, \tag{1.9}$$

for some uniquely defined integer coefficients γ^ℓ_i. These coefficients form the local γ-vector attached to the simplicial subdivision C_+.

The local γ-polynomial is multiplicative for the natural join operation on simplicial subdivisions, see [2, Lemma 2.2].

2. F-triangle, H-triangle

Let us now recall the definition of F-triangles and H-triangles, originally introduced in [5] in the context of cluster complexes. They were later related to a third polynomial, the M-triangle, that will not be considered here.

Let C be a pure finite spherical simplicial complex of dimension $d-1$, with a distinguished facet T. The F-triangle of the pair (C, T) is the generating polynomial

$$F_{C,T}(x, y) = \sum_{0 \leqslant i,j \leqslant d} F_{i,j} x^i y^j, \tag{2.1}$$

where $F_{i,j}$ is the number of faces of C of cardinality $i + j$ that are made of i elements not in T and j elements in T. When setting $y = x$, this reduces to the usual f-polynomial, that is $F_{C,T}(x, x) = f_C(x)$.

The H-triangle is then defined as

$$H_{C,T}(x, y) = (1-x)^d F_{C,T}\left(\frac{x}{1-x}, \frac{xy}{1-x}\right). \tag{2.2}$$

The reverse conversion formula is

$$F_{C,T}(x, y) = (1+x)^d H_{C,T}\left(\frac{x}{1+x}, \frac{y}{x}\right). \tag{2.3}$$

When setting $y = 1$ in the H-triangle, one gets back the usual h-polynomial of the simplicial complex C, that is $H_{C,T}(x, 1) = h_C(x)$. The conversion formulas also extend the usual ones between f-vectors and h-vectors.

In this article, we will only consider F-triangles and H-triangles in the case where $C = \mathrm{Sphere}(C_+)$ for some simplicial subdivision C_+ of 2^I, taking as distinguished facet T the unique facet of $\mathrm{Sphere}(C_+)$ with set of vertices I.

3. Γ-triangle

Let us now introduce the main novelty of the article, the Γ-triangle. It is closely related to the F-triangle and H-triangle, and can be seen as a condensed way to describe these polynomials, with half less coefficients.

Consider a simplicial sphere of the form $\mathrm{Sphere}(C_+)$. Assume for the moment that one can write its H-triangle in the following shape

$$H(x,y) = (1+x)^d \sum_{\substack{0\leqslant i\\ 0\leqslant j\leqslant d-2i}} \gamma_{i,j} \left(\frac{x}{(1+x)^2}\right)^i \left(\frac{1+xy}{1+x}\right)^j, \tag{3.1}$$

for some integer coefficients $\gamma_{i,j}$. If this is possible, there is a unique way to do so. The coefficients $\gamma_{i,j}$ are called the Γ-triangle of $\mathrm{Sphere}(C_+)$. Note that the coefficients fit inside a triangle, whence the name.

We will prove in the next section that this decomposition is always possible in this context, and give an expression for the coefficients $\gamma_{i,j}$ in terms of the local γ-vectors for sub-complexes of C_+.

The formula (3.1) can also be displayed as

$$H(x,y) = \sum_{i,j} \gamma_{i,j} x^i (1+xy)^j (1+x)^{d-2i-j}. \tag{3.2}$$

This reduces to the usual formula (1.5) for the γ-vector when $y = 1$. This implies that a necessary condition for (3.1) to exist is the symmetry of the h-vector. It also implies that the Γ-triangle is a refinement of the γ-vector, in the sense that

$$\gamma_i = \sum_j \gamma_{i,j}. \tag{3.3}$$

Using the relation (2.3) between F-triangle and H-triangle, one obtains that

$$F(x,y) = (1+2x)^d \sum_{i,j} \gamma_{i,j} \left(\frac{x(1+x)}{(1+2x)^2}\right)^i \left(\frac{1+x+y}{1+2x}\right)^j. \tag{3.4}$$

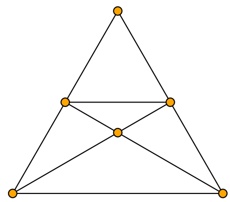

Figure 3.1. Positive part C_+ of a cluster complex of type $\mathbb{A}_3$.

As a simple concrete example, let us consider for $n \geqslant 4$ the regular n-polygon, whose F, H and Γ-triangles (with respect to any facet) are

$$\begin{pmatrix} 1 & & \\ 2 & 2 & \\ 1 & n-2 & n-3 \end{pmatrix}, \begin{pmatrix} & & 1 \\ & 2 & 0 \\ 1 & n-4 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \\ 0 & \\ 0 & n-4 \end{pmatrix}. \tag{3.5}$$

Here and after, the coefficients are displayed with the power of x (index i) increasing from left to right and the power of y (index j) increasing from bottom to top.

Here is another example, for the cluster complex of type $\mathbb{A}_3$ whose positive part C_+ is depicted in figure 3.1. Here, the F, H and Γ-triangles of $\mathrm{Sphere}(C_+)$ are

$$\begin{pmatrix} 1 & & & \\ 3 & 3 & & \\ 3 & 8 & 5 & \\ 1 & 6 & 10 & 5 \end{pmatrix}, \begin{pmatrix} & & & 1 \\ & & 3 & 0 \\ & 3 & 2 & 0 \\ 1 & 3 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \\ 0 & \\ 0 & 2 \\ 0 & 1 \end{pmatrix}. \tag{3.6}$$

3.1. Existence and local description of Γ-triangle

We will proceed here to a computation, ending with a formula that implies the existence of the Γ-triangle for $\mathrm{Sphere}(C_+)$, together with an expression for the $\gamma_{i,j}$ coefficients in terms of all the local γ-vectors for the subcomplexes $C_+(J)$.

Let us start by the equality

$$F_{\mathrm{Sphere}(C_+)}(x,y) = \sum_{J \subseteq I} \sum_{f \in C_+(I-J)} x^{|f|} y^{|J|} \tag{3.7}$$

holding by definition of the F-triangle and by the description of the faces of $\mathrm{Sphere}(C_+)$ from the faces of C_+. Using f-vectors, one gets

$$F_{\mathrm{Sphere}(C_+)}(x,y) = \sum_{J \subseteq I} y^{|J|} f_{C_+(I-J)}(x). \tag{3.8}$$

Passing to the H-triangle by (2.2), one gets

$$H_{\mathrm{Sphere}(C_+)}(x,y) = (1-x)^{|I|}\sum_{J\subseteq I}\left(\frac{xy}{1-x}\right)^{|J|} f_{C_+(I-J)}\left(\frac{x}{1-x}\right). \tag{3.9}$$

Expressing f-vectors in terms of h-vectors by (1.2), this becomes

$$H_{\mathrm{Sphere}(C_+)}(x,y) = (1-x)^{|I|}\sum_{J\subseteq I}\left(\frac{xy}{1-x}\right)^{|J|}(1-x)^{-|I-J|}h_{C_+(I-J)}(x) \tag{3.10}$$

which is just

$$H_{\mathrm{Sphere}(C_+)}(x,y) = \sum_{J\subseteq I}(xy)^{|J|}h_{C_+(I-J)}(x). \tag{3.11}$$

Expressing h-vectors in terms of local h-vectors by (1.7), one gets

$$H_{\mathrm{Sphere}(C_+)}(x,y) = \sum_{J\subseteq I}(xy)^J\sum_{K\subseteq I-J}h^{\ell}_{C_+(K)}(x). \tag{3.12}$$

which can be rewritten

$$H_{\mathrm{Sphere}(C_+)}(x,y) = \sum_{K\subseteq I}(1+xy)^{|I-K|}h^{\ell}_{C_+(K)}(x). \tag{3.13}$$

Then passing from local h-vectors to local γ-vectors by (1.9), one gets

$$H_{\mathrm{Sphere}(C_+)}(x,y) = \sum_{K\subseteq I}(1+xy)^{|I-K|}\sum_i \gamma^{\ell}_{C_+(K),i}\left(\frac{x}{(1+x)^2}\right)^i(1+x)^{|K|}. \tag{3.14}$$

By comparing carefully with the desired expression (3.1), one finds at last the following expression for the coefficients $\gamma_{i,j}$.

Proposition 3.1. — *The Γ-triangle $\Gamma(x,y)$ of* Sphere(C_+) *can be expressed as*

$$\Gamma(x,y) = \sum_{i,j}\gamma_{i,j}x^iy^j = \sum_{K\subseteq I}\gamma^{\ell}_{C_+(K)}(x)y^{|I-K|}, \tag{3.15}$$

where the sum is over all subsets K of the set I.

This implies that Athanasiadis' version of the Gal's conjecture for the local γ-vectors ([1, Conjecture 3.6]) would imply the following version of Gal's conjecture for Γ-triangles.

Let Sphere(C_+) be a flag spherical simplicial complex of the type defined in Section 1.

Conjecture 3.2. — *The* Γ*-triangle for the pair* (Sphere$(C_+), I$) *has nonnegative coefficients.*

This holds true for all cluster complexes of finite type, see the next section and tables at the end of the article.

As an example of what happens without flagness, let us consider the case of the triangle made of 3 edges, and its F-triangle with respect to one edge. The H-triangle and Γ-triangle were already computed in (3.5) for $n = 3$, but the Γ-triangle has a negative coefficient.

4. Explicit values for finite cluster fans

In this section, we compute the Γ-triangle for cluster fans in types $\mathbb{A}, \mathbb{B}, \mathbb{D}$.

Using Proposition 3.1, the Γ-triangle of the cluster fan of a given Dynkin diagram Φ, seen as a polynomial $\Gamma(x, y)$, is therefore determined by

$$\Gamma_\Phi(x, y) = \sum_{J \subset I} \gamma^{\ell}_{I-J}(x) y^{|J|}, \tag{4.1}$$

where J is the set of vertices not in the subdiagram.

We will use this formula in the next sections, in the special cases of types $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{D}$, to obtain algebraic equations for the generating series of Γ-triangles. We also give explicit expressions for coefficients of these generating series.

Note that the coefficients $\gamma_{1,i}$ for cluster fans have an explicit description as the numbers of non-simple roots, in the root system of W, according to the size of their support. This follows directly from Remark 1 in [2, Section 5] and Proposition 3.1.

4.1. Type $\mathbb{A}$

Let us start by a computation in type $\mathbb{A}$.

Proposition 4.1. — *The coefficient of* $x^k y^\ell$ *in the* Γ*-triangle of the associahedra of type* $\mathbb{A}_n$ *is*

$$\frac{\ell+1}{n-k+1}\binom{n}{k}\binom{n-k-\ell-1}{k-1} \tag{4.2}$$

for $k \geqslant 0$, $\ell \geqslant 0$ *and* $\ell + 2k \leqslant n$.

The proof follows.

Let $g_{\mathbb{A}}$ be the following generating series

$$\sum_{m\geqslant 0}\sum_{k\geqslant 0}\frac{1}{k+m+1}\binom{2k+m}{k}\binom{k+m-1}{k-1}x^k t^{2k+m}. \tag{4.3}$$

This is the generating series for the known local γ-vectors for the positive part of the cluster complex of type $\mathbb{A}$, see [2, Proposition 3.1, Equation (8)]. In this generating series and later on in similar ones, the exponent of the variable x is the index in the local γ-vector, and the exponent of the variable t is the rank n of the Dynkin diagram $\mathbb{A}_n$. The summation variables (m,k) are chosen in such a way that the summation range get simplified.

Let $G_{\mathbb{A}}$ be the following generating series

$$\sum_{m\geqslant 0}\sum_{k\geqslant 0}\sum_{\ell\geqslant 0}\frac{\ell+1}{\ell+k+m+1}\binom{\ell+2k+m}{k}\binom{k+m-1}{k-1}x^k y^\ell t^{2k+m+\ell}. \tag{4.4}$$

This is the generating series for the expected Γ-triangles for the cluster complex of type $\mathbb{A}$. Here the exponents of x and y are the indices in the Γ-triangles.

Proposition 4.2. — *We have the following relation:*

$$G_{\mathbb{A}} = g_{\mathbb{A}} + yt\, g_{\mathbb{A}} G_{\mathbb{A}}. \tag{4.5}$$

Proof. — Let us compute the coefficient of $(xt^2)^k(yt)^\ell t^m$ in $g_{\mathbb{A}}G_{\mathbb{A}}$. This is given by the finite sum

$$\sum_{\substack{k_1+k_2=k\\ m_1+m_2=m}}\frac{1}{k_1+m_1+1}\binom{2k_1+m_1}{k_1}\binom{k_1+m_1-1}{k_1-1}\\ \frac{\ell+1}{\ell+k_2+m_2+1}\binom{\ell+2k_2+m_2}{k_2}\binom{k_2+m_2-1}{k_2-1}$$

with $k_1\geqslant 0$, $k_2\geqslant 0$, $m_1\geqslant 0$ and $m_2\geqslant 0$. This expression can be rewritten as

$$\sum_{\substack{k_1+k_2=k\\ m_1+m_2=m}}\frac{k_1}{(2k_1+m_1+1)(k_1+m_1)}\binom{2k_1+m_1+1}{k_1}\binom{k_1+m_1}{m_1}\\ \frac{(\ell+1)k_2}{(2k_2+m_2+\ell+1)(k_2+m_2)}\binom{2k_2+m_2+\ell+1}{k_2}\binom{k_2+m_2}{m_2}$$

By a summation formula of Carlitz [4, Theorem 6 (5.14)], this is equal to

$$\frac{(\ell+2)k}{(2k+m+\ell+2)(k+m)}\binom{2k+m+\ell+2}{k}\binom{k+m}{m} \tag{4.6}$$

which is exactly the coefficient of $(xt^2)^k(yt)^{\ell+1}t^m$ in $G_{\mathbb{A}} - g_{\mathbb{A}}$. □

But the equation (4.5) is exactly the relation given by applying (4.1) in type $\mathbb{A}$, where Dynkin diagrams are line-shaped graphs. Namely, either J is empty and the subdiagram on $I - J$ is the full diagram, or there exists a leftmost vertex that is in J. The first case correspond to the term $g_{\mathbb{A}}$ in the right hand side of (4.5). In the second case, one can use the multiplicativity of local γ-vectors to separate the leftmost connected component of $I - J$. This gives the second term in the right hand side of (4.5).

4.2. Type $\mathbb{B}$

Let us proceed to the similar computation in type $\mathbb{B}$.

PROPOSITION 4.3. — *The coefficient of $x^k y^\ell$ in the Γ-triangle of the associahedra of type $\mathbb{B}_n$ is*

$$\binom{n}{k}\binom{n-k-\ell-1}{k-1}, \tag{4.7}$$

for $k \geqslant 0$, $\ell \geqslant 0$ and $\ell + 2k \leqslant n$.

The proof, similar to the case of type $\mathbb{A}$, follows.

Let $g_{\mathbb{B}}$ be the following generating series

$$\sum_{m\geqslant 0}\sum_{k\geqslant 0}\binom{2k+m}{k}\binom{k+m-1}{k-1}x^k t^{2k+m}. \tag{4.8}$$

This is the generating series for the known local γ-vectors for type $\mathbb{B}$, see [2, Proposition 3.2, Equation (12)].

Let $G_{\mathbb{B}}$ be the following generating series

$$\sum_{m\geqslant 0}\sum_{k\geqslant 0}\sum_{\ell\geqslant 0}\binom{2k+\ell+m}{k}\binom{k+m-1}{k-1}x^k y^\ell t^{2k+\ell+m}. \tag{4.9}$$

This is the generating series for the expected Γ-triangles for type $\mathbb{B}$.

PROPOSITION 4.4. — *We have the following relations:*

$$G_{\mathbb{B}} = g_{\mathbb{B}} + yt\, g_{\mathbb{A}} G_{\mathbb{B}} \quad \textit{and} \quad G_{\mathbb{B}} = g_{\mathbb{B}} + yt\, g_{\mathbb{B}} G_{\mathbb{A}}. \tag{4.10}$$

Proof. — The proof of these two equations is very similar. Let us give some details only for the first one. One computes the coefficient of

$(xt^2)^k(yt)^\ell t^m$ in $g_{\mathbb{A}}G_{\mathbb{B}}$. This is given by

$$\sum_{\substack{k_1+k_2=k\\ m_1+m_2=m}} \frac{1}{k_1+m_1+1}\binom{2k_1+m_1}{k_1}\binom{k_1+m_1-1}{k_1-1} \binom{\ell+2k_2+m_2}{k_2}\binom{k_2+m_2-1}{k_2-1},$$

which can be rewritten

$$\sum_{\substack{k_1+k_2=k\\ m_1+m_2=m}} \frac{k_1}{(2k_1+m_1+1)(k_1+m_1)}\binom{2k_1+m_1+1}{k_1}\binom{k_1+m_1}{m_1} \binom{2k_2+m_2+\ell}{k_2}\binom{k_2+m_2-1}{m_2}.$$

By applying [4, Theorem 6 (5.15)] (with the correct right-hand side that involves $+cn$), one gets that this is equal to

$$\binom{2k+m+\ell+1}{k}\binom{k+m-1}{m}, \tag{4.11}$$

which is readily seen to be the coefficient of $(xt^2)^k(yt)^{\ell+1}t^m$ in $G_{\mathbb{B}}-g_{\mathbb{B}}$. $\square$

By the same proof as in type $\mathbb{A}$, the equations (4.10) are exactly the relations given by (4.1) between the local γ-vectors and the Γ-triangle in type $\mathbb{B}$.

4.3. Type $\mathbb{D}$

There is an amusing and unexpected relation between the Γ-triangles of cluster fans of type $\mathbb{B}$ and $\mathbb{D}$.

Proposition 4.5. — *For every $n \geqslant 3$, the Γ-triangle of type $\mathbb{D}_n$ is obtained from the Γ-triangle of type $\mathbb{B}_{n-1}$ by adding a bottom line, which is the local γ-vector of type $\mathbb{D}_n$.*

Proof. — This follows from the statements below.

Let $g_{\mathbb{D}}$ be the following generating series

$$\sum_{m\geqslant 0}\sum_{k\geqslant 1} \frac{2k+m-2}{k}\binom{2k-2}{k-1}\binom{2k+m-2}{2k-2}x^k t^{2k+m}. \tag{4.12}$$

This is the generating series for the known local γ-vectors for type $\mathbb{D}$ see [2, Proposition 3.3].

Let $G_{\mathbb{D}}$ be the generating series for the expected Γ-triangles for type $\mathbb{D}$ for $n \geqslant 2$. The proposition above is equivalent to

$$G_{\mathbb{D}} = yt(G_{\mathbb{B}} - 1) + g_{\mathbb{D}}, \tag{4.13}$$

which is therefore what we want to prove. □

Proposition 4.6. — *We have the following relation:*

$$G_{\mathbb{D}} = g_{\mathbb{D}} + 2yt(g_{\mathbb{A}} - 1) + (yt)^2 g_{\mathbb{A}} + ytg_{\mathbb{A}}G_{\mathbb{D}}. \tag{4.14}$$

Proof. — This is the consequence of the general relation (4.1) from γ-vector to Γ-triangle. If the subset J is empty, we get the first term. If the subset J is made of one of the two vertices in the fork of the $\mathbb{D}$ Dynkin diagram, we get the next term. If J is made of both, then we get the third term. Otherwise, J has at least one element on the tail part of the Dynkin diagram, and one can cut at the farthest one from the forking point. □

To deduce (4.13) from (4.14), it is enough to prove the following.

Lemma 4.7. —

$$g_{\mathbb{B}} - 1 = 2(g_{\mathbb{A}} - 1) + g_{\mathbb{A}}g_{\mathbb{D}}. \tag{4.15}$$

Proof. — This follows from the algebraicity of these 3 series, and more precisely from the equations that they satisfy, see the appendix for a detailed proof. □

Indeed, by taking the sum of (4.14) and yt times (4.15), one gets

$$(G_{\mathbb{D}} - g_{\mathbb{D}})(1 - ytg_{\mathbb{A}}) = yt(g_{\mathbb{B}} - 1 + ytg_{\mathbb{A}}). \tag{4.16}$$

But one can deduce from (4.10) that

$$(G_{\mathbb{B}} - 1)(1 - ytg_{\mathbb{A}}) = g_{\mathbb{B}} - 1 + ytg_{\mathbb{A}}. \tag{4.17}$$

Comparing these two equations implies (4.13).

For example, here are the Γ-triangles of type $\mathbb{B}_5$ and type $\mathbb{D}_6$:

$$\begin{pmatrix} 1 & & \\ 0 & & \\ 0 & 5 & \\ 0 & 5 & \\ 0 & 5 & 10 \\ 0 & 5 & 20 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ 0 & & & \\ 0 & 5 & & \\ 0 & 5 & & \\ 0 & 5 & 10 & \\ 0 & 5 & 20 & \\ 0 & 4 & 24 & 8 \end{pmatrix}.$$

5. Quadrangulations and dissections

Another large class of pure flag spherical simplicial complexes of the shape Sphere(C_+) is obtained from Stokes complexes associated to quadrangulations [7] and more general dissection complexes as introduced by Garver and McConville in [12] and further studied and extended in [15, 16]. We will only briefly mention some interesting examples, with no proof.

Among these simplicial complexes, one can find families indexed by n that should be the simplest possible, in the sense that the dual polytopes are not products of simpler cases and have as few vertices as possible for a given dimension n.

For quadrangulations and Stokes complexes, there is a family having vertices enumerated by the Lucas numbers (OEIS A32), see [7, Section 4.1]. The Γ-triangles for this family satisfy the linear recursion

$$u_{n+1} = (y^2 + 2x)u_n - x^2 u_{n-1} + xyu_{n-1}, \tag{5.1}$$

with initial values $0, 1$.

For general dissections, there is a family having their vertices counted by the Pell numbers (OEIS A129). This seems to be closely related to the objects considered in [14]. The Γ-triangles for this family satisfy the linear recursion

$$u_{n+1} = yu_n + xu_{n-1}, \tag{5.2}$$

with initial values $0, 1$. These are therefore some kind of Fibonacci polynomials. Strangely, the discriminant of this recursion is the Γ-triangle for $\mathbb{I}_2(6)$.

6. Tables of Γ-triangles

Types of rank 2 and 3 :

$$\begin{pmatrix} 1 & \\ 0 & \\ 0 & h-2 \end{pmatrix}, \begin{pmatrix} 1 & \\ 0 & \\ 0 & \frac{6(h-2)}{h+2} \\ 0 & \frac{3(h-2)^2}{2(h+2)} \end{pmatrix}.$$

For rank 2, the parameter $h \geqslant 2$ is the Coxeter number, corresponding to $\mathbb{I}_2(h)$. For rank 3, the parameter is also the Coxeter number h, with possible values $2, 4, 6, 10$ corresponding to $\mathbb{A}_1^3, \mathbb{A}_3, \mathbb{B}_3$ and $\mathbb{H}_3$.

Types $\mathbb{A}_4$, $\mathbb{B}_4$, $\mathbb{D}_4$, $\mathbb{F}_4$ and $\mathbb{H}_4$:

$$\begin{pmatrix} 1 & & \\ 0 & & \\ 0 & 3 & \\ 0 & 2 & \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & & \\ 0 & & \\ 0 & 4 & \\ 0 & 4 & \\ 0 & 4 & 6 \end{pmatrix}, \begin{pmatrix} 1 & & \\ 0 & & \\ 0 & 3 & \\ 0 & 3 & \\ 0 & 2 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & & \\ 0 & & \\ 0 & 4 & \\ 0 & 6 & \\ 0 & 10 & 9 \end{pmatrix}, \begin{pmatrix} 1 & & \\ 0 & & \\ 0 & 5 & \\ 0 & 9 & \\ 0 & 42 & 40 \end{pmatrix}.$$

Types $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$:

$$\begin{pmatrix} 1 & & & \\ 0 & & & \\ 0 & 5 & & \\ 0 & 5 & & \\ 0 & 6 & 11 & \\ 0 & 7 & 23 & \\ 0 & 7 & 35 & 13 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ 0 & & & \\ 0 & 6 & & \\ 0 & 6 & & \\ 0 & 7 & 16 & \\ 0 & 9 & 36 & \\ 0 & 12 & 69 & 28 \\ 0 & 16 & 124 & 112 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ 0 & & & & \\ 0 & 7 & & & \\ 0 & 7 & & & \\ 0 & 8 & 22 & & \\ 0 & 10 & 48 & & \\ 0 & 14 & 94 & 46 & \\ 0 & 22 & 192 & 194 & \\ 0 & 44 & 484 & 784 & 120 \end{pmatrix}.$$

Appendix

Recall that $g_{\mathbb{A}}, g_{\mathbb{B}}, g_{\mathbb{D}}$ are power series in $\mathbb{Q}[x][\![t]\!]$ defined by

$$g_{\mathbb{A}} := 1 + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\lfloor n/2 \rfloor} \frac{\binom{n}{i}\binom{n-i-1}{i-1}}{n-i+1} x^i \right) t^n = 1 + xt^2 + xt^3 + \left(2\,x^2 + x\right) t^4 + \cdots,$$

$$g_{\mathbb{B}} := 1 + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n}{i}\binom{n-i-1}{i-1} x^i \right) t^n = 1 + 2\,xt^2 + 3\,xt^3 + \left(6\,x^2 + 4\,x\right) t^4 + \cdots$$

and

$$g_{\mathbb{D}} := \sum_{n=0}^{\infty} \left(\sum_{i=1}^{\lfloor n/2 \rfloor} \frac{\binom{2\,i-2}{i-1}\binom{n-2}{2\,i-2}(n-2)}{i} x^i \right) t^n = xt^3 + \left(2\,x^2 + 2\,x\right) t^4 + \cdots.$$

We prove here that these power series are algebraic, and that they satisfy the following identity:

$$g_{\mathbb{B}} - 1 = 2(g_{\mathbb{A}} - 1) + g_{\mathbb{A}} g_{\mathbb{D}}.$$

To do this, we introduce an auxiliary algebraic power series in $\mathbb{Q}[x][\![t]\!]$,

$$g = \sqrt{(1-t)^2 - 4\,xt^2} = 1 - t - 2\,xt^2 - 2\,xt^3 - \left(2\,x^2 + 2\,x\right)t^4 + \cdots$$

We claim that the following relations hold in $\mathbb{Q}[\![x,t]\!]$:

$$g_{\mathbb{A}} = \frac{1+t-g}{2t\,(tx+1)}, \tag{A.1}$$

$$g_{\mathbb{B}} = \frac{2\,tx + g - t + 1}{2\,g\,\,(tx+1)}, \tag{A.2}$$

$$g_{\mathbb{D}} = \frac{(g-1)\,(g-1+t)}{2\,g}. \tag{A.3}$$

Assuming these identities, it is immediate to check our claims, namely that $g_{\mathbb{A}}, g_{\mathbb{B}}, g_{\mathbb{D}}$ are algebraic, and related by

$$g_{\mathbb{B}} - 1 - 2(g_{\mathbb{A}} - 1) - g_{\mathbb{A}} g_{\mathbb{D}} = \frac{(1+g)\left(g^2 - (1-t)^2 + 4\,xt^2\right)}{4\,t\,(tx+1)\,g} = 0.$$

It is therefore enough to prove identities (A.1), (A.2) and (A.3).

First, let us remark that g is equal to

$$g = (1-t)\sqrt{1 - 4x\left(\frac{t}{1-t}\right)^2}.$$

Setting $u = xt^2/(1-t)^2$ in the binomial formula

$$\sqrt{1-4u} = 1 - 2\sum_{i=2}^{\infty} \frac{\binom{2\,i-2}{i-1}}{i} u^i$$

yields the expansion

$$g = 1 - t - 2\sum_{i\geqslant 1} \frac{\binom{2\,i-2}{i-1}}{i} x^i \frac{t^{2i}}{(1-t)^{2i-1}}.$$

Combining this with the classical expansion

$$\frac{1}{(1-t)^{2\,i-1}} = \sum_{n=0}^{\infty} \binom{n+2\,i-2}{2\,i-2} t^n$$

proves that

$$g = 1 - t - 2 \sum_{i \geqslant 1} \left(\sum_{n \geqslant 0} \frac{\binom{2\,i-2}{i-1}\binom{n+2i-2}{2\,i-2}}{i} t^{n+2i} \right) x^i,$$

in other words:

$$g = 1 - t - 2 \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\lfloor n/2 \rfloor} \frac{\binom{2\,i-2}{i-1}\binom{n-2}{2\,i-2}}{i} x^i \right) t^n. \tag{A.4}$$

Now the proof of identity (A.1) amounts to a direct verification, based on the binomial identity

$$\frac{\binom{n-2}{i-1}\binom{n-i-2}{i-2}+\binom{n-1}{i}\binom{n-i-2}{i-1}}{n-i} = \frac{\binom{2\,i-2}{i-1}\binom{n-2}{2\,i-2}}{i} \quad \text{for } n \geqslant 2 \text{ and } 1 \leqslant i \leqslant \lfloor n/2 \rfloor.$$

Identity (A.3) follows from (A.4) and the definition of $g_{\mathbb{D}}$, by using the following observation, where $\theta_t = t\,\frac{\partial}{\partial t}$ stands for the Euler derivation:

$$g_{\mathbb{D}} = (2 - \theta_t) \left(\frac{g - 1 + t}{2} \right).$$

Indeed, proving (A.3) amounts to checking that g satisfies the differential equation

$$tg\frac{\partial g}{\partial t} - g^2 - t + 1 = 0,$$

which is obvious from the definition of g.

Finally, identity (A.2) follows from (A.1) using the following observation:

$$g_{\mathbb{B}}(x,t) = \frac{\partial\,(t\,g_{\mathbb{A}}(x/t,t))}{\partial t}(xt,t).$$

Bibliography

[1] C. A. Athanasiadis, "A survey of subdivisions and local h-vectors", in *The mathematical legacy of Richard P. Stanley*, American Mathematical Society, 2016, p. 39-51.

[2] C. A. Athanasiadis & C. Savvidou, "The local h-vector of the cluster subdivision of a simplex", *Sémin. Lothar. Comb.* **66** (2011), article ID B66c (21 pages).

[3] P. Brändén, "Actions on permutations and unimodality of descent polynomials", *Eur. J. Comb.* **29** (2008), no. 2, p. 514-531.

[4] L. Carlitz, "Some expansions and convolution formulas related to MacMahon's master theorem", *SIAM J. Math. Anal.* **8** (1977), no. 2, p. 320-336.

[5] F. Chapoton, "Enumerative properties of generalized associahedra", *Sémin. Lothar. Comb.* **51** (2004), article ID B51b (16 pages).

[6] ———, "Sur le nombre de réflexions pleines dans les groupes de Coxeter finis", *Bull. Belg. Math. Soc. Simon Stevin* **13** (2006), no. 4, p. 585-596.

[7] ———, "Stokes posets and serpent nests", *Discrete Math. Theor. Comput. Sci.* **18** (2016), no. 3, article ID 18 (30 pages).

[8] S. Fomin & A. Zelevinsky, "Cluster algebras. II. Finite type classification", *Invent. Math.* **154** (2003), no. 1, p. 63-121.

[9] ———, "Y-systems and generalized associahedra", *Ann. Math.* **158** (2003), no. 3, p. 977-1018.

[10] ———, "Flag subdivisions and γ-vectors", *Pac. J. Math.* **259** (2012), no. 2, p. 257-278.

[11] S. R. Gal, "Real root conjecture fails for five- and higher-dimensional spheres", *Discrete Comput. Geom.* **34** (2005), no. 2, p. 269-284.

[12] A. Garver & T. McConville, "Enumerative properties of Grid-Associahedra", `https://arxiv.org/abs/1705.04901`, 2017.

[13] E. Katz & A. Stapledon, "Local h-polynomials, invariants of subdivisions, and mixed Ehrhart theory", *Adv. Math.* **286** (2016), p. 181-239.

[14] S. Law, "Combinatorial realization of the Hopf algebra of sashes", in *26th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2014)*, Discrete Mathematics and Theoretical Computer Science, The Association, 2014, p. 621-632.

[15] T. Manneville & V. Pilaud, "Geometric Realizations of the Accordion Complex of a Dissection", *Discrete Comput. Geom.* **61** (2019), no. 3, p. 507-540.

[16] Y. Palu, V. Pilaud & P.-G. Plamondon, "Non-kissing complexes and tau-tilting for gentle algebras", `https://arxiv.org/abs/1707.07574`, 2017.

[17] N. Reading & D. E. Speyer, "Cambrian fans", *J. Eur. Math. Soc.* **11** (2009), no. 2, p. 407-447.

[18] K. Saito, "The zero loci of F-triangles", *Acta Math. Vietnam.* **42** (2017), no. 2, p. 209-236.

[19] R. P. Stanley, "Subdivisions and local h-vectors", *J. Am. Math. Soc.* **5** (1992), no. 4, p. 805-851.

[20] ———, *Combinatorics and commutative algebra*, second ed., Progress in Mathematics, vol. 41, Birkhäuser, 1996, x+164 pages.

[21] M. Visontai, "Some remarks on the joint distribution of descents and inverse descents", *Electron. J. Comb.* **20** (2013), no. 1, article ID 52 (12 pages).