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François Laudenbach

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The Poincaré-Lefschetz pairing viewed on Morse complexes (*)

François Laudenbach (1)

Abstract. — Given a compact manifold with a non-empty boundary and equipped with a generic Morse function (that is, no critical point on the boundary and the restriction to the boundary is a Morse function), we already knew how to construct two Morse complexes, one yielding the absolute homology and the other the relative homology. In this note, we construct a short exact sequence from both of them and the Morse complex of the boundary. Moreover, we define a pairing of the relative Morse complex with the absolute Morse complex which induces the intersection product in homology, in the form due to S. Lefschetz. This is a very first step in an ambitious approach towards A_∞-structures built from similar data.

Résumé. — Considérons une variété compacte à bord non vide munie d'une fonction de Morse générique ; en particulier sans point critique sur le bord mais dont la restriction au bord est une fonction de Morse. Dans un article antérieur, avec cette donnée et des gradients bien choisis, nous avons construit deux complexes de Morse, l'un calculant l'homologie absolue de la variété et l'autre son homologie relative au bord.

Dans la présente note, nous construisons une suite exacte courte à partir de ces deux complexes et du complexe de Morse du bord. En outre, nous definissons une forme bilinéaire du complexe absolu avec le complexe relatif qui induit la forme d'intersection en homologie, dans sa forme due à S. Lefschetz. Il s'agit là d'une étape élémentaire dans une démarche plus ambitieuse vers les structures multiplicatives d'ordre supérieur (structures A_∞) que l'on peut construire avec des données similaires.

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(1) Laboratoire de mathématiques Jean Leray, UMR 6629 du CNRS, Faculté des Sciences et Techniques, Université de Nantes, 2, rue de la Houssinière, F-44322 Nantes cedex 3, France — francois.laudenbach@univ-nantes.fr

Article proposé par Jean-Pierre Otal.

1. Introduction

We are given an n-dimensional compact manifold M with a non-empty boundary ∂M and a Morse function $f : M \to \mathbb{R}$ which is generic with respect to the boundary, meaning that f has no critical point on the boundary and that the restriction f_∂ of f to ∂M is a Morse function. It is well-known that the set of critical points of f_∂ is divided into two *types* $+$ and $-$:

$$\operatorname{crit} f_\partial = \operatorname{crit}^+ f_\partial \sqcup \operatorname{crit}^- f_\partial \,. \tag{1.1}$$

A point x belongs to $\operatorname{crit}^+ f_\partial$ (resp. $\operatorname{crit}^- f_\partial$) if it is a critical point of f_∂ and the differential $\mathrm{d}f(x)$ is positive (resp. negative) on a tangent vector at x pointing outwards.(1)

We have introduced in [3] the notion of quasi-gradients(2) *positively* (resp. *negatively*) *adapted to* f. Such vector fields, noted respectively noted X^+ and X^-, satisfy:

- X^+ vanishes only in $\operatorname{crit} f \cup \operatorname{crit}^+ f_\partial$ and $\langle \mathrm{d}f, X^+\rangle > 0$ elsewhere;
- X^- vanishes only in $\operatorname{crit} f \cup \operatorname{crit}^- f_\partial$ and $\langle \mathrm{d}f, X^-\rangle < 0$ elsewhere.

The zeroes of both of them are assumed hyperbolic, implying the existence of local stable and unstable manifolds. The quasi-gradient X^+ (resp. X^-) is required to be tangent to the boundary near $\operatorname{crit}^+ f_\partial$ (resp. $\operatorname{crit}^- f_\partial$). Globally, both X^+ and X^- are nowhere pointing outwards along ∂M. As a consequence, their flows are positively complete, and hence, global unstable manifolds exist. By taking inverse images of the local stable manifolds by the positive semi-flow, global stable manifolds are also defined (see Section 3).

These invariant submanifolds are denoted by $W^s(x, X^\pm)$ and $W^u(x, X^\pm)$ respectively when x is a zero of the considered quasi-gradient. If $x \in \operatorname{crit} f$, the dimension of $W^s(x, X^+)$ (resp. $W^u(x, X^-)$) is equal to the Morse index of f at x. If $x \in \operatorname{crit}^- f_\partial$, the dimension of $W^u(x, X^-)$ is equal to the Morse index of f_∂ at x; but, if $x \in \operatorname{crit}^+ f_\partial$, we have

$$\dim W^s(x, X^+) = \operatorname{Ind}_x f_\partial + 1. \tag{1.2}$$

It makes sense to assume $X^\pm$ Morse–Smale (mutual transversality of stable and unstable manifolds); this property is open and dense. An orientation is chosen on each stable (resp. unstable) manifold arbitrarily when dealing with X^+ (resp. X^-). This makes the unstable (resp. stable) manifolds co-oriented and allows us to put a *sign* on the orbits in $W^s(x, X^+) \cap W^u(y, X^+)$ when the sum of the codimensions is equal to $n-1$; and similarly for X^-.

(1) Here, we choose to introduce notations which are more suggestive than in [3].

(2) In [3], these vector fields are named *pseudo-gradients* though they vanish at points of ∂M where $\mathrm{d}f$ does not vanish. So, we prefer to name them *quasi-gradients*.

Thus, two Morse complexes $C_*(f, X^+)$ and $C_*(f, X^-)$ are built whose homologies are respectively isomorphic to $H_*(M, \partial M; \mathbb{Z})$ and to $H_*(M; \mathbb{Z})$.(3) By abuse of notation, we first neglect to mention the choice of orientations; this will be corrected in 3.1 for further need. For brevity, they are also noted C_*^+ and C_*^-.

To be more precise, C_k^+ is freely generated by $\mathrm{crit}_k f \cup \mathrm{crit}_{k-1}^+ f_\partial$ (note the shift in the grading due to (1.2)) while C_k^- is freely generated by $\mathrm{crit}_k f \cup \mathrm{crit}_k^- f_\partial$. The differential $\partial^+ := \partial^{X^+}$ evaluated on a generator $x \in C_k^+$ is given by the algebraic counting of orbits of X^+ ending at x and starting from generators of C_{k-1}^+. And similarly for the complex C_*^-. The present note is aimed at proving two results which are stated below.

Theorem 1.1. — *Let X_∂ be a Morse–Smale descending pseudo-gradient of f_∂ on the boundary ∂M and let $C_*(f_\partial, X_\partial)$ be the associated Morse complex. Then for suitable adapted quasi-gradients X^- and X^+, there exist a quasi-isomorphic extension $\widehat{C}_*(f, X^-)$ of the complex $C_*(f, X^-)$ and a short exact sequence of complexes*

$$0 \longrightarrow C_*(f_\partial, X_\partial) \longrightarrow \widehat{C}_*(f, X^-) \longrightarrow C_*(f, X^+) \longrightarrow 0. \tag{1.3}$$

The second result is stated right below. I should add that Theorem 1.2 corrects something which was poorly said at the end of [3].

Theorem 1.2. — *Here, M is assumed* oriented. *For a generic choice of the adapted quasi-gradients X^+ and X^-, there is a pairing at the chain level*

$$C_k(f, X^+) \otimes C_{n-k}(f, X^-) \to \mathbb{Z}$$

which induces the intersection pairing in homology

$$\iota : H_*(M, \partial M; \mathbb{Z}) \otimes H_{n-*}(M; \mathbb{Z}) \to \mathbb{Z}$$

Intitially, this note was thought of as the beginning of an article on multiplicative structures, namely A_∞-algebra structures, on Morse complexes [1]. It appeared that the pairing $C_*^+ \otimes C_{n-*}^- \to \mathbb{Z}$ was not of the same type in nature as the multiplications of these A_∞-structures. Therefore, I decided to separate this piece from [1].

2. A short exact sequence

We first describe the suitable adapted quasi-gradients X^+ and X^- in Theorem 1.1. Let X_∂ be a vector field on ∂M which is a Morse–Smale descending pseudo-gradient of f_∂ and gives rise to the usual Morse complex of

(3) For defining the differential of these complexes, only the *local* stable manifolds are needed.

the boundary $C_*(f_\partial, X_\partial)$; its differential is denoted by $\partial_{\partial M}$. By partition of unity, one easily constructs a quasi-gradient X of f which extends X_∂. This X is tangent to the boundary, and hence it is not an *adapted quasi-gradient*. But it satisfies $X \cdot f < 0$ everywhere except at the critical points of f and f_∂, where it vanishes with some non-degeneracy condition. The flow of X is complete, positively and negatively as well. Therefore, one can make X Morse–Smale.

When $x \in \operatorname{crit}_k^- f_\partial$, the unstable $W^u(x, X)$ coincides with $W^u(x, X_\partial) \cong \mathbb{R}^k$ and is contained in the boundary. The stable manifold $W^s(x, X)$ is diffeomorphic to $\mathbb{R}^{n-k}_{\geqslant 0}$ and is bounded by $W^s(x, X_\partial)$. In the same way, when $y \in \operatorname{crit}_k^+ f_\partial$, the unstable manifold $W^u(y, X)$ coincides with the unstable manifold $W^u(y, X_\partial) \cong \mathbb{R}^{n-1-k}$ and is contained in the boundary. Moreover, the unstable manifold $W^s(y, X)$ is diffeomorphic to $\mathbb{R}^{k+1}_{\geqslant 0}$ and is bounded by $W^s(y, X_\partial)$.

Remark 2.1. — Since X is tangent to the boundary there are no connecting orbits of X descending from $x \in \operatorname{crit}^- f_\partial$ to $y \in \operatorname{crit} f$. Similarly, there are no connecting orbits of X descending from $x \in \operatorname{crit} f$ to $y \in \operatorname{crit}^+ f_\partial$.

We now change X to $X^- = X + Y$, which will be negatively adapted to f, just by adding a small vector field Y which satisfies the following conditions:

(i) Y vanishes on a closed neighborhood U of $\operatorname{crit}^- f_\partial$ in M;
(ii) Y points inwards along $\partial M \smallsetminus U$ and satisfies $Y \cdot f \leqslant 0$ everywhere;
(iii) Y vanishes away from a neighborhood of ∂M.

Similarly, $-X$ can be perturbed to X^+, which will be positively adapted to f; just take $X^+ = -X + Z$ where Z is a small vector field vanishing on a neighborhood V of $\operatorname{crit}^+ f_\partial$ in M, pointing inwards along $\partial M \smallsetminus V$ and satisfying $Z \cdot f \geqslant 0$ everywhere. The perturbations Y and Z are small enough so that Remark 2.1 still applies. So, X^- and X^+ will be the desired quasi-gradients of Theorem 1.1.

Proposition 2.2. — *Assume* $\operatorname{crit} f_\partial^+$ *is empty. Then the Morse complex* $C_*(f_\partial, X_\partial)$ *embeds as a subcomplex of* $C_*(f, X^-)$. *Moreover, one has the following short exact sequence:*

$$0 \longrightarrow C_*(f_\partial, X_\partial) \xrightarrow{i} C_*(f, X^-) \longrightarrow C_*(f, X^+) \longrightarrow 0\,.$$

Proof. — The embedding i is induced by the inclusion

$$\operatorname{crit} f_\partial = \operatorname{crit}^- f_\partial \hookrightarrow \left(\operatorname{crit} f \cup \operatorname{crit}^- f_\partial\right).$$

We have to prove that i is a chain morphism. This will follow from equalities (2.1) and (2.2) below. Let $x \in \mathrm{crit}_k f_\partial = \mathrm{crit}_k^- f_\partial$. By Remark 2.1 applied to X^-, for every $y \in \mathrm{crit}_{k-1} f$ we have

$$\langle \partial^- x, y\rangle = 0. \tag{2.1}$$

If $y \in \mathrm{crit}_{k-1}^- f_\partial$, the intersection $W^u(x, X^-) \cap W^s_{loc}(y, X^-)$, which is transverse in M, can be pushed by an f-preserving isotopy to $W^u(x, X_\partial) \cap W^s_{loc}(y, X_\partial)$, which is a transverse intersection in ∂M – note that $W^s_{loc}(y, X_\partial)$ is the boundary of $W^s_{loc}(y, X^-)$. Then, the signed number of connecting orbits is the same for both quasi-gradients and we have

$$\langle \partial^- x, y\rangle = \langle \partial_{\partial M} x, y\rangle. \tag{2.2}$$

For the exactness of the sequence, observe that the complex $C_*(f, X^+)$ is generated by the critical points of f. Both vector fields X^+ and X^- are approximations of the Morse–Smale vector field X (up to sign). Therefore, for every $x \in \mathrm{crit}_k f$ and $y \in \mathrm{crit}_{k-1} f$, the signed number of connecting orbits is the same when counted with X^- or X^+:

$$\langle \partial^+ x, y\rangle = \langle \partial^- x, y\rangle.$$

The quotient kills $\mathrm{crit}^- f_\partial$, which generates the image of $C_*(f_\partial, X_\partial)$, and also the connecting orbits from $\mathrm{crit}\, f$ to $\mathrm{crit}^- f_\partial$. The exactness follows. $\square$

Proof of Theorem 1.1. — It was shown in [3, Lemma 2.4],(4) that there is a C^0-small deformation, supported in a neighborhood U of $\mathrm{crit}^- f_\partial$, of the generic Morse function f to a new generic Morse function f' with the following property: each $x \in \mathrm{crit}_k^- f_\partial$ becomes a critical point of *positive* type and index k. The degree of x as generator of $C_*(f')$ is $k+1$. This is obtained at the cost of a new critical point $x' \in \mathrm{int}\, M$ for f', of index k and close to x. The two critical points x and x' of f' are indeed linked by a unique gradient line; since x belongs to the boundary, this pair is not cancellable but its fusion cancels x' only and changes the type of x from $+$ to $-$.

Arguing this way with the function $-f$ leads to the following. There exists a C^0-small deformation of f, supported in a neighborhood V of $\mathrm{crit}^+ f_\partial$, to some generic function $\widehat{f}$ having the following property: $\widehat{f}_\partial = f_\partial$ and each $x \in \mathrm{crit}_k^+ f_\partial$ becomes a critical point of negative type and index k, that is, $x \in \mathrm{crit}_k^- \widehat{f}_\partial$. This is made at the cost of a critical point $\widehat{x} \in \mathrm{int}\, M$ for $\widehat{f}$ of index $k+1$ and close to x and satisfying

$$\widehat{f}(\widehat{x}) > \widehat{f}(x)\,.$$

(4) After that [3] appeared, I was informed that a similar lemma exists in [5] in a setting where only the Morse inequalities are discussed.

The extension which is mentioned in Theorem 1.1 consists of adding to C_*^- a pair of new generators $\{x, \widehat{x}\}$ for each $x \in \operatorname{crit}^+ f_\partial$. More precisely,

$$\widehat{C}_*(f, X^-) := C_*(\widehat{f}, \widehat{X}^-)$$

for some quasi-gradient $\widehat{X}^-$ negatively adapted to $\widehat{f}$. According to [3], the new complex is quasi-isomophic to the old one $C_*(f, X^-)$. Since the restriction $f_\partial = \widehat{f}|_{\partial M}$ has no critical point of positive type, Proposition 2.2 applies and there is an exact sequence

$$0 \longrightarrow C_*(f_\partial, X_\partial) \longrightarrow C_*(\widehat{f}, \widehat{X}^-) \ \longrightarrow C_*(\widehat{f}, \widehat{X}^+) \longrightarrow 0,$$

where $\widehat{X}^+$ denotes a suitable vector field positively adapted to $\widehat{f}$. In order to identify the quotient in this exact sequence, it is necessary to specify this vector field $\widehat{X}^+$.

In its support V, the modification from f to $\widehat{f}$ is modelled similarly to the birth of a pair of critical points in usual Morse Theory. The model produces also a descending quasi-gradient $\widehat{X}$ of $\widehat{f}$ from the quasi-gradient X of f, which coincides with X out of V and on ∂M. Then, $-\widehat{X}$ (which is tangent to the boundary) is changed to $\widehat{X}^+$ by adding a vector field Z which is small with respect to $\widehat{X}$ and satisfies the conditions (i)–(iii) up to sign.

Claim. — *The bijection $j : \operatorname{crit}^+ f_\partial \cup \operatorname{crit} f \to \operatorname{crit} \widehat{f}$ which maps $x \in \operatorname{crit}^+ f_\partial$ to $\widehat{x} \in \operatorname{crit} \widehat{f}$ and which is the identity on $\operatorname{crit} f \subset \operatorname{crit} \widehat{f}$ induces a chain isomorphism $C_*(\widehat{f}, \widehat{X}^+) \cong C_*(f, X^+)$.*

Proof of the claim. — Say $x \in \operatorname{crit}_k^+ f_\partial$. On the one hand, each X-orbit descending from x to $y \in \operatorname{crit}_k f$ gives rise to an $\widehat{X}$-orbit from $\widehat{x}$ to y and hence, an $\widehat{X}^+$-orbit from y to $\widehat{x}$. Similarly, each X-orbit on ∂M descending from x to $y \in \operatorname{crit}_{k-1}^+ f_\partial$ gives rise to an $\widehat{X}^+$-orbit from $\widehat{y}$ to $\widehat{x}$. And conversely. Making j an identification, this proves the following:

$$\partial^{X^+} x = \partial \widehat{X}^+ \, \widehat{x}\,.$$

On the other hand, we have to consider $y \in \operatorname{crit}_{k+2} f$ and compute its two differentials with respect to X^+ and $\widehat{X}^+$ and evaluate them at x (recall that x has degree $k+1$ in C_*^+). When x and y have consecutive critical values, as a consequence of Remark 2.1, there are no X^+-connecting orbits from x to $\operatorname{crit}_{k+2} f$.

But, if their critical values are not consecutive, one could have a broken X-orbit from y to x made of an orbit from y to $z \in \operatorname{crit}_{k+1}^- f_\partial$ and an orbit from z to x on ∂M. By using the deformation formula $X^+ = -X + Z$, such a broken orbit gives rise to an X^+-orbit from x to y, and hence to an $\widehat{X}^+$-orbit from $\widehat{x}$ to y. Then, such connecting orbits may exist. Conversely, by looking at the fusion of the pair $(x,\widehat{x})$ we get that every $\widehat{X}^+$-orbit from $\widehat{x}$ to y is produced by an X^+-orbit from x to y. Then, via j the following equality holds true:

$$\langle \partial^{X^+} y, x\rangle = \langle \partial^{\widehat{X}^+} y, \widehat{x}\rangle\,.$$

This finishes the proof of the claim and Theorem 1.1 follows. □

3. Global stable manifolds and application to intersection pairing

We now discuss the question of *global stable manifolds* for adapted quasi-gradients. We only consider X^- in the definition below; there is a similar definition for X^+. If $x\in\partial M$ is a critical point of negative type, so far we have only considered its local stable manifold $W^s_{loc}(x,X^-)$. If x is of index k, it is a small half-disc D^{n-k}_- whose planar boundary lies in a level set of f and spherical boundary lies in ∂M. Since the flow of X^-, noted X^-_t at time t, is positively complete, the following definition makes sense:

Definition 3.1. — *For $x\in \operatorname{crit} f\cup \operatorname{crit}^-(f_\partial)$, the global stable manifold of x with respect to X^- is defined as the union*

$$W^s(x,X^-) = \bigcup_{t>0} \left(X^-_t\right)^{-1}\left(W^s_{loc}(x,X^-)\right).$$

Under mild assumptions, it is a (non-proper) submanifold with boundary and its closure is a stratified set. The following assumption (Morse-Model-Transversality) is made in what follows.

For every $x\in \operatorname{crit} f\cup\operatorname{crit}^- f_\partial$ and $y\in\operatorname{crit}^- f_\partial$, the neighborhood U_y of y in ∂M where X^- is tangent to the boundary is mapped by the flow transversely to $W^s_{loc}(x,X^-)$. (MMT)

Notice that if X^- is Morse–Smale, the transversality condition is satisfied along a small neighborhood U of the local unstable manifold $W^u_{loc}(y,X^-)$. Then, after some small perturbation of X^- on $U_y\smallsetminus U$ destroying the tangency of X^- to ∂M, condition (MMT) is fulfilled for the pair (y,x). Thus, condition (MMT) is generic among the negatively adapted vector fields.

Proposition 3.2. — *If the negatively quasi-gradient X^- is Morse–Smale and fulfils condition* (MMT) *then the following holds:*

1. *The global stable manifold $W^s(x,X^-)$ is a submanifold with boundary (non-closed in general); its boundary lies in ∂M.*
2. *If z lies in the frontier of $W^s(x,X^-)$ in M, then it belongs to the stable manifold of some critical point y in $\operatorname{crit} f\cup\operatorname{crit}^- f_\partial$ such that $\dim W^s(y,X^-) < \dim W^s(x,X^-)$.*

This statement also holds for stable manifolds of critical points in $\operatorname{crit} f \cup \operatorname{crit}^+ f_\partial$ with respect to positively adapted vector fields.

Proof.

(1). — According to the Implicit Function Theorem, the conclusion is clear near any point where X^- is transverse to the boundary. Near a point z of U_y, it follows from (MMT).

(2). — This fact is well known in the case of closed manifolds. It is an easy consequence of the Morse–Smale assumption. The proof is alike if the boundary is non-empty. □

Remark 3.3. — Due to the transversality assumptions, a small perturbation of X^- (resp. X^+) moves each of stable and unstable manifolds by a small isotopy, and hence, preserves the complex C_*^- (resp. C_*^+) up to a canonical isomorphism.

As a consequence, without changing the above-mentioned complexes, we are allowed to assume that the X^--unstable manifolds of $\operatorname{crit} f \cup \operatorname{crit}^- f_\partial$ intersect the global X^+-stable manifolds of $\operatorname{crit} f \cup \operatorname{crit}^+ f_\partial$ transversely.

3.1. Where an abusive notation is corrected

If the orientation of some of the unstable manifolds is changed then the differential of the considered Morse complex (absolute or relative) is changed by a non-trivial isomorphism. So, to understand the role of the orientability of M in what follows, it will be better to replace $C_*(f, X^-)$ with $C_*(f, X^-, \varepsilon_f^-)$ where ε_f^- denotes the chosen *orientation map* which associates an orientation of $W^u(x, X^-)$ with each $x \in \operatorname{crit} f \cup \operatorname{crit}^- f_\partial$. Note that ε_f^- orients the unstable manifolds regardless of the quasi-gradient since they all have isotopic germs at the critical points. And similarly for $C_*(f, X^+)$. Actually, we will only apply this change of notation at the places where it will be crucial.

3.2. The Poincaré-Lefschetz isomorphism

At the homology level, this isomorphism is a isomorphism

$$P : H_*(M, \partial M; \mathbb{Z}) \to H^{n-*}(M; \mathbb{Z}),$$

We wish to describe it by means of our Morse complexes in order to deduce a Morse theoretical description of the homological intersection. There are several steps to achieve.

Step 1. — First, we recall that there is a *natural* isomorphism at the homology level

$$I_*(f, X^-) : H_*\left(C_*(f, X^-)\right) \to H_*(M; \mathbb{Z}).$$

Indeed, we have described in [3] a canonical process for removing the critical points of f_∂ of negative type. Once this is done, the unstable manifolds of X^- emerging from $\mathrm{crit}(f)$ yield a cell decomposition of M (see [2][5]) whose homology is canonically isomorphic to the singular homology of M (see [4, p. 90]).

We now explain the naturality of this isomorphism. Let (g, Y^-) be another pair of generic Morse function and negatively adapted quasi-gradient. The choice of a generic path γ from (g, Y^-) to (f, X^-) gives rise to some *simple homotopy equivalence*

$$\gamma_* : C_*(g, Y^-) \to C_*(f, X^-) \tag{3.1}$$

well defined up to the orientations.[6] At each time that γ crosses a stratum corresponding to a codimension-one defect of genericity of the pair (*function, negatively adapted quasi-gradient*) this yields an *elementary* modification of the Morse complex, indeed a *quasi-isomorphism* [3]. One checks at each occurrence that this quasi-isomorphism is compatible to the isomorphism with $H_*(M; \mathbb{Z})$. Finally, γ_* is the composition of all these quasi-isomorphisms. It induces an isomorphism $[\gamma_*]$ in homology making the next diagram commute:

$$\begin{array}{ccc} H_*\left(C_*(g, Y^-)\right) & \xrightarrow{[\gamma_*]} & H_*\left(C_*(f, X^-)\right) \\ & \searrow^{I_*(g,Y^-)} & \downarrow{\scriptstyle I_*(f,X^-)} \\ & & H_*(M; \mathbb{Z}) \end{array} \tag{3.2}$$

By taking the transpose of all morphisms of chain complexes we get a similar diagram in cohomology made of isomorphisms:

$$\begin{array}{ccc} H^*\left(C^*(g, Y^-)\right) & \xleftarrow{[\gamma^*]} & H^*\left(C^*(f, X^-)\right) \\ & \searrow^{I^*(g,Y^-)} & \downarrow{\scriptstyle I^*(f,X^-)} \\ & & H^*(M; \mathbb{Z}) \end{array}$$

Note that a change of orientations of some unstable manifolds has the same effect on $[\gamma_*]$ and on $I_*(-,-)$. So, the commutativity of the above diagrams is not affected.

(5) In this reference, a stronger assumption is made on the vector field which implies this cell decomposition to be a CW-complex. Without this assumption, the cell decompositon has only the homotopy type of a CW-complex. This is sufficient for our discussion.

(6) The creation/cancellation times of pair of critical points along γ do not allow us to carry orientations along the path.

Step 2. — We can do the same for the Morse complex $C_*(f, X^+)$ which calculates the relative homology. Here, we will use the stable manifolds of X^+ that we introduced in the beginning of Section 3. More precisely, there is a canonical process, similar to the one above-mentioned for the complex $C_*(f, X^-)$, which removes the positive type critical points of f_∂. After removing them, the stable manifolds of X^+ associated with crit f give rise to a filtration of M starting from ∂M:

$$\partial M \subset M_{[1]} \subset \cdots \subset M_{[k]} \subset \cdots \subset M_{[n]} = M\,.$$

Here, $M_{[k]}$ is the union of ∂M and the closure of the stable manifolds of X^+ converging to $\mathrm{crit}_k\, f$. The cellular homology associated with this filtration gives a canonical isomorphism

$$I_*(f, X^+) : H_*\left(C_*(f, X^+)\right) \to H_*(M, \partial M; \mathbb{Z}).$$

Moreover, this isomorphism is *natural* with respect to change of function and quasi-gradient in the same sense as it is detailed in Step 1 above.

Step 3. — Here comes the important point for orientations. Let ε_f^+ be a choice of orientations of the stable manifolds of X^+. Since M is oriented, the unstable manifolds of X^+ are not only co-oriented but they are also oriented.[7] The latter orientations are denoted by $\varepsilon_f^\perp$.

We recall that X^+ is a negatively adapted quasi-gradient of $-f$; we denote it by $Y^- := X^+$ when it is considered as a descending quasi-gradient of $-f$. So, we have a chain complex $C_*(-f, Y^-, \varepsilon_{-f}^-)$ where ε_{-f}^- is determined by ε_f^+ by the rule

$$\varepsilon_{-f}^- = \varepsilon_f^\perp. \tag{3.3}$$

By applying the functor $Hom(-, \mathbb{Z})$ we have its dual, a co-chain complex, $C^*(-f, Y^-, \varepsilon_f^\perp)$. By construction of C_*^+, we have

$$\eta_* : C_*(f, X^+, \varepsilon_f^+) \xrightarrow{=} C^{n-*}(-f, Y^-, \varepsilon_f^\perp)\,. \tag{3.4}$$

This equality means same generators and same differential; only the grading is reversed. It induces the equality $H_*(C_*(f, X^+, \varepsilon_f^+)) = H^{n-*}(C^{n-*}(-f, Y^-, \varepsilon_f^\perp))$ and by combining it with the isomorphims $I_*(f, X^+)$ and $I^{n-*}(-f, Y^-)$ we get a description at the Morse complex level of the Poincaré-Lefschetz isomorphism:

$$P : H_*(M, \partial M; \mathbb{Z}) \longrightarrow H^{n-*}(M; \mathbb{Z})\,.$$

[7] Here, some convention has to be used, for instance: co-or(-)$\wedge$ or(-)= or(M).

3.3. Application to the intersection pairing

We are interested in describing a pairing at the chain level

$$\sigma : C_k(f, X^+) \otimes C_{n-k}(f, X^-) \to \mathbb{Z}$$

which induces the intersection pairing in homology. This is achieved in the following way.

At the homology level the Poincaré-Lefschetz isomorphism P carries the intersection product

$$\iota : H_*(M, \partial M; \mathbb{Z}) \otimes H_{n-*}(M, \mathbb{Z}) \to \mathbb{Z}$$

to the evaluation map $ev : H^{n-*}(M;\mathbb{Z}) \otimes H_{n-*}(M;\mathbb{Z}) \to \mathbb{Z}$.

After what was done in the previous subsection, we only have to understand this evaluation map in the setting of Morse homology. First, there is a canonical evaluation map

$$ev = \langle -, - \rangle : C^{n-*}(-f, Y^-) \otimes C_{n-*}(-f, Y^-) \to \mathbb{Z}$$

which on the basis elements is the Kronecker product. A more sophisticated way to say the same thing is to count the transverse intersection $W^s(x, Y^-) \cap W^u(y, Y^-)$ for every pair of critical points of the same degree, that is both in $\mathrm{crit}_k f \cup \mathrm{crit}^+_{k-1} f_\partial$ for some integer k. Here, it is essential Y^- to be Morse–Smale for avoiding undesirable orbits connecting points of the same degree.

We choose a generic path[8] Γ from (f, X^-) to $(-f, Y^-)$ which yields a quasi-isomorphism $\Gamma_* : C_{n-*}(f, X^-, \varepsilon^-_f) \to C_{n-*}(-f, Y^-, \varepsilon^{\perp}_f)$. Thanks to (3.4), the desired evalution map is given by

$$\sigma = ev \circ (\eta_* \otimes \Gamma_*)\,. \tag{3.5}$$

If necessary, by Remark 3.3 we may approximate X^- in order to make mutually transverse $W^s(x, X^+)$ and $W^u(y, X^-)$ for every $x \in \mathrm{crit}_k f \cup \mathrm{crit}^+_{k-1} f_\partial$ and $y \in \mathrm{crit}_{n-k} f \cup \mathrm{crit}^-_{n-k} f_\partial$.

Claim. — *For every pair of cycles $\alpha \in C_k(f, X^+)$ and $\beta \in C_{n-k}(f, X^-)$, the geometric formula for $\sigma(\alpha, \beta)$ is given by counting the signed intersection number of the respective stable and unstable manifolds entering in the linear combinations forming α and β.*

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[8] First, choose a generic path (f_t) in the space of functions; then, complete with a path of quasi-gradients. For this second step, use the convexity of the set of quasi-gradients adapted to f_t for a given t. If the function f_t has a codimension-one singularity, the involved critical point needs to be a zero of X_t of corank one and the previous argument still works.

Indeed, by (3.2) the cycles β and $\Gamma_*(\beta)$ are homologous in M. Therefore, they have the same algebraic intersection with the cycle α. Notice that the frontier of the involved invariant manifolds does not appear in this counting since it is made of invariant manifolds of less dimension.

Corollary 3.4. — *The pairing σ induces the homological intersection.*

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