ANNALES
DE LA FACULTÉ
DES SCIENCES

DE
TOULOUSE

Mathématiques

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About relative polar varieties and Brasselet numbers

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Abstract. — In this work, we study the consequences of an empty polar variety on the topology of a function-germ with (possibly) nonisolated singularities defined on a singular variety.

Résumé. — Dans ce travail, nous étudions les conséquences d'une variété polaire vide sur le topologie d'une germe de fonction avec (possiblement) des singularités non isolées définies sur une variété singulière

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Introduction

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic function defined in a neighbourhood of the origin. In [15], Milnor described the topology of the germ f at the origin, proving that, if f has an isolated singularity at the origin, the set $f^{-1}(\delta) \cap B_\epsilon$, later called Milnor fiber, has the homotopy type of a bouquet of $\mu(f)$ spheres of dimension $n - 1$, where δ is a regular value of $f, 0 < |\delta| \ll \epsilon \ll 1$. The number $\mu(f)$ is the Milnor number of f, an invariant associated to the germ f which also counts the number of Morse points in a Morsefication of f in a neighbourhood of the origin.

The Milnor number was largely studied and generalized to many settings (for instance, for mention a few [4, 5, 6, 7, 12, 17]). If f is defined over a complex analytic space X and f has an isolated singularity at the origin, a generalization for the Milnor number is the Euler obstruction of the function f, introduced in [2], by Brasselet, Massey, Parameswaran and Seade. In [16],

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Seade, Tibăr and Verjovsky proved that, up to sign, this number is the number of Morse critical points of a stratified Morsefication of f appearing in the regular part of X in a neighbourhood of the origin.

In the case where f is defined over a complex analytic germ $(X, 0)$ equipped with a good stratification $\mathcal{V}$ relative to f and the function f has nonisolated singularities, Dutertre and Grulha provided a way to numerically describe the topology of the generalized Milnor fiber $X \cap f^{-1}(\delta) \cap B_\epsilon$. In [3], the authors defined the Brasselet number $B_{f,X}(0)$ of f at the origin and proved a Lê–Greuel type formula for this number: if $g : X \to \mathbb{C}$ is prepolar with respect to $\mathcal{V}$ at the origin and $0 < |\delta| \ll \epsilon \ll 1$, then $B_{f,X}(0) - B_{f,X^g}(0) = (-1)^{d-1} n_q$, where n_q is the number of Morse critical points of a partial Morsefication of $g|_{X \cap f^{-1}(\delta) \cap B_\epsilon}$ appearing in the regular part of X, and $X^g = X \cap \{g = 0\}$.

Computing these number of stratified Morse critical points is directly connected to relative polar varieties. Consider a linear form l in $\mathbb{C}^n$, a Whitney stratification of X and a function-germ $g : (U, 0) \to (\mathbb{C}, 0)$. If l is sufficiently generic, the polar variety (curve) $\Gamma_{g,l}$ defined by Lê and Teissier in [8] coincides with the relative polar curve defined by Massey in [11], and with the relative polar varieties defined by Massey in [12] (see [13] and [14]). Each of these polar varieties is useful to compute polar multiplicities ([9, 18]) and intersection numbers ([12]), but also to describe the critical loci of a pair of functions defined over X ([3, 11]), which is the approach we are interested the most.

In this work, we use polar varieties to compute a number of stratified Morse critical points of a specific type of deformation of a function-germ aiming to obtain informations about the Brasselet number of this germ.

1. Local Euler obstruction and Euler obstruction of a function

In this section, we will see the definition of the local Euler obstruction, a singular invariant defined by MacPherson and used as one of the main tools in his proof of the Deligne–Grothendieck conjecture about the existence and uniqueness of Chern classes for singular varities (see [10]).

Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be an equidimensional reduced complex analytic germ of dimension d in a open set $U \subset \mathbb{C}^n$. Consider a complex analytic Whitney stratification $\mathcal{V} = \{V_\lambda\}$ of U adapted to X such that $\{0\}$ is a stratum. We choose a small representative of $(X, 0)$, denoted by X, such that 0 belongs to the closure of all strata. We write $X = \bigcup_{i=0}^{q} V_i$, where $V_0 = \{0\}$ and $V_q = X_{\mathrm{reg}}$, where X_{reg} is the regular part of X. We suppose

that $V_0, V_1, \ldots, V_{q-1}$ are connected and that the analytic sets $\overline{V_0}, \overline{V_1}, \ldots, \overline{V_q}$ are reduced. We write $d_i = \dim(V_i)$, $i \in \{1, \ldots, q\}$. Note that $d_q = d$.

Let $G(d, n)$ be the Grassmannian manifold, $x \in X_{\text{reg}}$ and consider the Gauss map $\phi : X_{\text{reg}} \to U \times G(d, n)$ given by $x \mapsto (x, T_x(X_{\text{reg}}))$.

DEFINITION 1.1. — *The closure of the image of the Gauss map ϕ in $U \times G(d, n)$, denoted by $\widetilde{X}$, is called* Nash modification *of X. It is a complex analytic space endowed with an analytic projection map $\nu : \widetilde{X} \to X$.*

Consider the extension of the tautological bundle $\mathcal{T}$ over $U \times G(d, n)$. Since $\widetilde{X} \subset U \times G(d, n)$, we consider $\widetilde{T}$ the restriction of $\mathcal{T}$ to $\widetilde{X}$, called the *Nash bundle*, and $\pi : \widetilde{T} \to \widetilde{X}$ the projection of this bundle.

In this context, denoting by φ the natural projection of $U \times G(d, n)$ at U, we have the following diagram:

$$
\begin{array}{ccc}
\widetilde{T} & \longrightarrow & \mathcal{T} \\
{\scriptstyle \pi}\downarrow & & \downarrow \\
\widetilde{X} & \longrightarrow & U \times G(d, n) \\
{\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \varphi} \\
X & \longrightarrow & U \subseteq \mathbb{C}^n
\end{array}
$$

Considering $\|z\| = \sqrt{z_1 \overline{z_1} + \cdots + z_n \overline{z_n}}$, the 1-differential form $w = d\|z\|^2$ over $\mathbb{C}^n$ defines a section in $T^*\mathbb{C}^n$ and its pullback $\varphi^* w$ is a 1- form over $U \times G(d, n)$. Denote by $\widetilde{w}$ the restriction of $\varphi^* w$ over $\widetilde{X}$, which is a section of the dual bundle $\widetilde{T}^*$.

Choose ϵ small enough for $\widetilde{w}$ be a non zero section over $\nu^{-1}(z), 0 < \|z\| \leqslant \epsilon$. Let $B_\epsilon \subset \mathbb{C}^n$ be the closed ball with center at the origin with radius ϵ and denote by:

(1) $\mathrm{Obs}(\widetilde{T}^*, \widetilde{w}) \in \mathbb{H}^{2d}(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon), \mathbb{Z})$ the obstruction for extending $\widetilde{w}$ from $\nu^{-1}(S_\epsilon)$ to $\nu^{-1}(B_\epsilon)$;

(2) $O_{\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon)}$ the fundamental class in $\mathbb{H}_{2d}(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon), \mathbb{Z})$.

We can obtain the integer number given by the value of the cohomology class $\mathrm{Obs}(\widetilde{T}^*, \widetilde{w})$ on the correspondent homology class $O_{\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon)}$.

DEFINITION 1.2. — *The* local Euler obstruction *of X at 0, $\mathrm{Eu}_X(0)$, is given by the evaluation*

$$\mathrm{Eu}_X(0) = \langle \mathrm{Obs}(\widetilde{T}^*, \widetilde{w}), O_{\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon)} \rangle.$$

In [1], Brasselet, Lê and Seade proved a formula to make the calculation of the Euler obstruction easier.

THEOREM 1.3 ([1, Thm. 3.1]). — *Let $(X, 0)$ and $\mathcal{V}$ be given as before, then for each generic linear form l, there exists ϵ_0 such that for any ϵ with $0 < \epsilon < \epsilon_0$ and $\delta \neq 0$ sufficiently small, the Euler obstruction of $(X, 0)$ is equal to*

$$\mathrm{Eu}_X(0) = \sum_{i=1}^{q} \chi(V_i \cap B_\epsilon \cap l^{-1}(\delta)).\,\mathrm{Eu}_X(V_i),$$

where χ is the Euler characteristic, $\mathrm{Eu}_X(V_i)$ is the Euler obstruction of X at a point of V_i, $i = 1, \ldots, q$, and $0 < |\delta| \ll \epsilon \ll 1$.

Let us give the definition of another invariant introduced by Brasselet, Massey, Parameswaran and Seade in [2]. Let $f : X \to \mathbb{C}$ be a holomorphic function with isolated singularity at the origin given by the restriction of a holomorphic function $F : U \to \mathbb{C}$ and denote by $\overline{\nabla} F(x)$ the conjugate of the gradient vector field of F in $x \in U$,

$$\overline{\nabla} F(x) := \left(\overline{\frac{\partial F}{\partial x_1}}, \ldots, \overline{\frac{\partial F}{\partial x_n}} \right).$$

Since f has an isolated singularity at the origin, for all $x \in X \setminus \{0\}$, the projection $\widehat{\zeta}_i(x)$ of $\overline{\nabla} F(x)$ over $T_x(V_i(x))$ is nonzero, where $V_i(x)$ is a stratum containing x. Using this projection, the authors constructed in [2] a stratified vector field over X, denoted by $\overline{\nabla} f(x)$. Let $\widetilde{\zeta}$ be the lifting of $\overline{\nabla} f(x)$ as a section of the Nash bundle $\widetilde{T}$ over $\widetilde{X}$, without singularity over $\nu^{-1}(X \cap S_\epsilon)$.

Let $\mathcal{O}(\widetilde{\zeta}) \in \mathbb{H}^{2n}(\nu^{-1}(X \cap B_\epsilon), \nu^{-1}(X \cap S_\epsilon))$ be the obstruction cocycle for extending $\widetilde{\zeta}$ as a non zero section of $\widetilde{T}$ inside $\nu^{-1}(X \cap B_\epsilon)$.

DEFINITION 1.4. — *The* local Euler obstruction of the function f, $\mathrm{Eu}_{f,X}(0)$ *is the evaluation of $\mathcal{O}(\widetilde{\zeta})$ on the fundamental class $[\nu^{-1}(X \cap B_\epsilon), \nu^{-1}(X \cap S_\epsilon)]$.*

The next theorem compares the Euler obstruction of a space X with the Euler obstruction of function defined over X.

THEOREM 1.5 ([2, Thm. 3.1]). — *Let $(X, 0)$ and $\mathcal{V}$ be given as before and let $f : (X, 0) \to (\mathbb{C}, 0)$ be a function with an isolated singularity at 0. For $0 < |\delta| \ll \epsilon \ll 1$, we have*

$$\mathrm{Eu}_{f,X}(0) = \mathrm{Eu}_X(0) - \sum_{i=1}^{q} \chi(V_i \cap B_\epsilon \cap f^{-1}(\delta)).\,\mathrm{Eu}_X(V_i).$$

Let us now see a definition we will need to define a generic point of a function-germ. Let $\mathcal{V} = \{V_\lambda\}$ be a stratification of a reduced complex analytic space X.

DEFINITION 1.6. — *Let p be a point in a stratum V_β of $\mathcal{V}$. A degenerate tangent plane of $\mathcal{V}$ at p is an element T of some Grassmanian manifold such that $T = \lim_{p_i \to p} T_{p_i} V_\alpha$, where $p_i \in V_\alpha$, $V_\alpha \neq V_\beta$.*

DEFINITION 1.7. — *Let $(X, 0) \subset (U, 0)$ be a germ of complex analytic space in $\mathbb{C}^n$ equipped with a Whitney stratification and let $f : (X, 0) \to (\mathbb{C}, 0)$ be an analytic function, given by the restriction of an analytic function $F : (U, 0) \to (\mathbb{C}, 0)$. Then 0 is said to be a* generic point *of f if the hyperplane $Ker(d_0 F)$ is transverse in $\mathbb{C}^n$ to all degenerate tangent planes of the Whitney stratification at 0.*

Now, let us see the definition of a Morsefication of a function.

DEFINITION 1.8. — *Let $\mathcal{W} = \{W_0, W_1, \ldots, W_q\}$, with $0 \in W_0$, a Whitney stratification of the complex analytic space X. A function $f : (X, 0) \to (\mathbb{C}, 0)$ is said to be* Morse stratified *if $\dim W_0 \geqslant 1, f|_{W_0} : W_0 \to \mathbb{C}$ has a Morse point at 0, and 0 is a generic point of f with respect to W_i, for all $i \neq 0$.*

A *stratified Morsefication* of a germ of analytic function $f : (X, 0) \to (\mathbb{C}, 0)$ is a deformation $\widetilde{f}$ of f such that $\widetilde{f}$ is Morse stratified.

In [16], Seade, Tibăr and Verjovsky proved that the Euler obstruction of a function f is also related to the number of Morse critical points of a stratified Morsefication of f.

PROPOSITION 1.9 ([16, Prop. 2.3]). — *Let $f : (X, 0) \to (\mathbb{C}, 0)$ be a germ of analytic function with isolated singularity at the origin. Then,*

$$\mathrm{Eu}_{f,X}(0) = (-1)^d n_{\mathrm{reg}},$$

where n_{reg} is the number of Morse points in X_{reg} in a stratified Morsefication of f.

2. Brasselet number

In this section, we introduce the definitions and results needed in the development of this work. The main reference for this section is [11].

Let X be a reduced complex analytic space (not necessarily equidimensional) of dimension d in an open set $U \subseteq \mathbb{C}^n$ and let $f : (X, 0) \to (\mathbb{C}, 0)$ be an analytic map. We denote $V(f) = f^{-1}(0)$, despite the fact it will always be considered as a germ of set at the origin.

DEFINITION 2.1. — *A good stratification of X relative to f is a stratification $\mathcal{V}$ of X which is adapted to $V(f)$ such that $\{V_\lambda \in \mathcal{V}, V_\lambda \nsubseteq V(f)\}$ is a Whitney stratification of $X \setminus V(f)$ and such that for any pair (V_λ, V_γ) such that $V_\lambda \nsubseteq V(f)$ and $V_\gamma \subseteq V(f)$, the (a_f)-Thom condition is satisfied. That is, if $p \in V_\gamma$ and $p_i \in V_\lambda$ are such that $p_i \to p$ and $T_{p_i} V(f|_{V_\lambda} - f|_{V_\lambda}(p_i))$ converges to some $\mathcal{T}$, then $T_p V_\gamma \subseteq \mathcal{T}$.*

If $f : X \to \mathbb{C}$ has a stratified isolated critical point and $\mathcal{V}$ is a Whitney stratification of X, then

$$\{V_\lambda \setminus X^f, V_\lambda \cap X^f \setminus \{0\}, \{0\}, V_\lambda \in \mathcal{V}\} \tag{2.1}$$

is a good stratification of X relative to f, called the good stratification induced by f.

Let $\mathcal{V}$ be a good stratification of X relative to f.

DEFINITION 2.2. — *The critical locus of f relative to $\mathcal{V}$, $\Sigma_\mathcal{V} f$, is given by the union*

$$\Sigma_\mathcal{V} f = \bigcup_{V_\lambda \in \mathcal{V}} \Sigma(f|_{V_\lambda}).$$

DEFINITION 2.3. — *If $\mathcal{V} = \{V_\lambda\}$ is a stratification of X, the relative polar variety of f and g with respect to $\mathcal{V}$, denoted by $\Gamma_{f,g}(\mathcal{V})$, is the union $\bigcup_\lambda \Gamma_{f,g}(V_\lambda)$, where $\Gamma_{f,g}(V_\lambda)$ denotes the closure in X of the critical locus of $(f,g)|_{V_\lambda \setminus X^f}$, where $X^f = X \cap \{f = 0\}$.*

DEFINITION 2.4. — *If $\mathcal{V} = \{V_\lambda\}$ is a stratification of X, the symmetric relative polar variety of f and g with respect to $\mathcal{V}$, $\widetilde{\Gamma}_{f,g}(\mathcal{V})$, is the union $\bigcup_\lambda \widetilde{\Gamma}_{f,g}(V_\lambda)$, where $\Gamma_{f,g}(V_\lambda)$ denotes the closure in X of the critical locus of $(f,g)|_{V_\lambda \setminus (X^f \cup X^g)}$, $X^f = X \cap \{f = 0\}$ and $X^g = X \cap \{g = 0\}$.*

DEFINITION 2.5. — *Let $\mathcal{V}$ be a good stratification of X relative to a function $f : (X, 0) \to (\mathbb{C}, 0)$. A function $g : (X, 0) \to (\mathbb{C}, 0)$ is prepolar with respect to $\mathcal{V}$ at the origin if the origin is a stratified isolated critical point of g, that is, 0 is an isolated point of $\Sigma_\mathcal{V} g$.*

DEFINITION 2.6. — *A function $g : (X, 0) \to (\mathbb{C}, 0)$ is tractable at the origin with respect to a good stratification $\mathcal{V}$ of X relative to $f : (X, 0) \to (\mathbb{C}, 0)$, if $\dim_0 \widetilde{\Gamma}_{f,g}(\mathcal{V}) \leqslant 1$ and, for all strata $V_\alpha \subseteq X^f$, $g|_{V_\alpha}$ has no critical point in a neighbourhood of the origin except perhaps at the origin itself.*

Another usefulconcept for this work is the notion of constructible functions. Consider a Whitney stratification $\mathcal{W} = \{W_1, \ldots, W_q\}$ of X such that each stratum W_i is connected.

DEFINITION 2.7. — *A constructible function with respect to the stratification $\mathcal{W}$ of X is a function $\beta : X \to \mathbb{Z}$ which is constant on each stratum W_i, that is, there exist integers $t_1, \ldots, t_q$, such that $\beta = \sum_{i=1}^{q} t_i.1_{W_i}$, where 1_{W_i} is the characteristic function of W_i.*

DEFINITION 2.8. — *The Euler characteristic $\chi(X, \beta)$ of a constructible function $\beta : X \to \mathbb{Z}$ with respect to the stratification $\mathcal{W}$ of X, given by $\beta = \sum_{i=1}^{q} t_i.1_{W_i}$, is defined by $\chi(X, \beta) = \sum_{i=1}^{q} t_i.\chi(W_i)$.*

We present now the definition of the Brasselet number and the main theorems of [3], used as inspiration for this work.

Let $f : (X, 0) \to (\mathbb{C}, 0)$ be a complex analytic function germ and let $\mathcal{V}$ be a good stratification of X relative to f. We denote by $V_1, \ldots, V_q$ the strata of $\mathcal{V}$ that are not contained in $\{f = 0\}$ and we assume that $V_1, \ldots, V_{q-1}$ are connected and that $V_q = X_{\text{reg}} \setminus \{f = 0\}$. Note that V_q may not be connected.

DEFINITION 2.9. — *Suppose that X is equidimensional. Let $\mathcal{V}$ be a good stratification of X relative to f. The* Brasselet number *of f at the origin, $B_{f,X}(0)$, is defined by*

$$B_{f,X}(0) = \sum_{i=1}^{q} \chi(V_i \cap f^{-1}(\delta) \cap B_\epsilon) \, \mathrm{Eu}_X(V_i),$$

where $0 < |\delta| \ll \epsilon \ll 1$.

Remark. — If V_q^i is a connected component of V_q, $\mathrm{Eu}_X(V_q^i) = 1$.

Notice that if f has a stratified isolated singularity at the origin, then $B_{f,X}(0) = \mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0)$ (see Theorem 1.5).

In [3], Dutertre and Grulha proved interesting formulas describing the topological relation between the Brasselet number and a number of certain critical points of a special type of deformation of functions. Let us now remind some of these results. Fist, we need the definition of a special type of Morsefication, introduced by the authors in [3].

DEFINITION 2.10. — *A partial Morsefication of $g : f^{-1}(\delta) \cap X \cap B_\epsilon \to \mathbb{C}$ is a function $\widetilde{g} : f^{-1}(\delta) \cap X \cap B_\epsilon \to \mathbb{C}$ (not necessarily holomorphic) which is a local Morsefication of all isolated critical points of g in $f^{-1}(\delta) \cap X \cap \{g \neq 0\} \cap B_\epsilon$ and which coincides with g outside a small neighbourhood of these critical points.*

Let $g : (X, 0) \to (\mathbb{C}, 0)$ be a complex analytic function which is tractable at the origin with respect to $\mathcal{V}$ relative to f. Then $\widetilde{\Gamma}_{f,g}$ is a complex analytic curve and for $0 < |\delta| \ll 1$ the critical points of $g|_{f^{-1}(\delta) \cap X}$ in B_ϵ lying outside

$\{g = 0\}$ are isolated. Let $\widetilde{g}$ be a partial Morsefication of $g : f^{-1}(\delta) \cap X \cap B_\epsilon \to \mathbb{C}$ and, for each $i \in \{1, \ldots, q\}$, let n_q be the number of stratified Morse critical points of $\widetilde{g}$ appearing on $X_{\text{reg}} \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B_\epsilon$.

THEOREM 2.11 ([3, Thm. 4.4]). — *Suppose that X is equidimensional and that g is prepolar with respect to $\mathcal{V}$ at the origin. For $0 < |\delta| \ll \epsilon \ll 1$, we have*

$$B_{f,X}(0) - B_{f,X^g}(0) = (-1)^{d-1} n_q,$$

where n_q is the number of stratified Morse critical points on the top stratum $V_q \cap f^{-1}(\delta) \cap B_\epsilon$ appearing in a Morsefication of $g : X \cap f^{-1}(\delta) \cap B_\epsilon \to \mathbb{C}$.

In the case where X is equipped with a Whitney stratification $\mathcal{V} = \{V_0, V_1, \ldots, V_q\}$ with $V_0 = \{0\}$, and $f, g : X \to \mathbb{C}$ have an isolated stratified singularity at the origin with respect to this stratification. In [3], the authors also related the topology of the generalized Milnor fibres of f and g with the number of Morse points of a partial Morsefication of $g : X \cap f^{-1}(\delta) \cap B_\epsilon \to \mathbb{C}$ and $f : X \cap g^{-1}(\delta) \cap B_\epsilon \to \mathbb{C}$.

COROLLARY 2.12. — *Suppose that X is equidimensional and that g (resp. f) is prepolar with respect to the good stratification induced by f (resp. g) at the origin. Then*

$$B_{f,X}(0) - B_{g,X}(0) = (-1)^{d-1} (n_q - m_q),$$

where n_q (resp. m_q) is the number of stratified Morse critical points on the top stratum $V_q \cap f^{-1}(\delta) \cap B_\epsilon$ (resp. $V_q \cap g^{-1}(\delta) \cap B_\epsilon$) appearing in a Morsefication of $g : X \cap f^{-1}(\delta) \cap B_\epsilon \to \mathbb{C}$ (resp. $f : X \cap g^{-1}(\delta) \cap B_\epsilon \to \mathbb{C}$).

3. Brasselet numbers and empty polar varieties

In this final section, we relate the Brasselet numbers of two function-germs in the case where the relative polar variety associated to these germs is empty.

Let $(X, 0)$ be a complex analytic space in $U \subset \mathbb{C}^{n+1}$. Let $f, g : (U, 0) \to (\mathbb{C}, 0)$ be germs of holomorphic functions and let $\mathcal{V}$ be a good stratification of X relative to f. Suppose that $\Sigma_{\mathcal{V}} g \cap \{f = 0\} = \{0\}$. We aim to obtain information about the Brasselet number of f and the Brasselet number of g in the case where the relative polar variety $\Gamma_{f,g}(\mathcal{V})$ is empty. We begin with a description of two relevant subsets of $\Gamma_{f,g}(\mathcal{V})$.

PROPOSITION 3.1. — *The stratified critical set $\Sigma_{\mathcal{V}} g$ of g and the symmetric relative polar variety $\widetilde{\Gamma}_{f,g}(\mathcal{V})$ are subsets of $\Gamma_{f,g}(\mathcal{V})$.*

Proof. — If $x \in \Sigma_{\mathcal{V}} g$, then $d_x \tilde{g}|_{V_\alpha} = 0$, for a stratum $V_\alpha \in \mathcal{V}$ containing x and an analytic extension $\tilde{g}$ of g in a neighbourhood of x. If $V_\alpha \subset \{f = 0\}$, then $x = 0$, since $\Sigma_{\mathcal{V}} g \cap \{f = 0\} = \{0\}$. If $V_\alpha \subset X \setminus \{f = 0\}$, $\mathrm{rk}(d_x \tilde{f}, d_x \tilde{g}) \leqslant 1$, where $\tilde{f}$ is an analytic extension of f in a neighbourhood of x. Hence, $x \in \Sigma(f, g)|_{V_\alpha} = \Sigma(f, g)|_{V_\alpha \setminus \{f = 0\}}$, that is, $x \in \Gamma_{f,g}(\mathcal{V})$.

Furthermore, $\widetilde{\Gamma}_{f,g}(\mathcal{V})$ is given by the components of $\Gamma_{f,g}(\mathcal{V})$ not contained in $\{g = 0\}$, that is, $\widetilde{\Gamma}_{f,g}(\mathcal{V}) = \overline{\Gamma_{f,g}(\mathcal{V}) \setminus \{\} = \prime} \subset \Gamma_{f,g}(\mathcal{V})$. $\qquad\square$

Using this proposition, we obtain the following useful information about the behaviour of g with respect to $\mathcal{V}$.

COROLLARY 3.2. — *If $\Gamma_{f,g}(\mathcal{V})$ is empty, then g is prepolar at the origin with respect to the good stratification $\mathcal{V}$ of X relative to f.*

Proof. — By Proposition 3.1, if $\Gamma_{f,g}(\mathcal{V})$ is empty, then $\Sigma_{\mathcal{V}} g$ is empty, that is, g has no stratified critical point with respect to $\mathcal{V}$. $\qquad\square$

Let n_i be the number of stratified Morse critical points of a Morsefication of $g : X \cap f^{-1}(\delta) \cap B_\epsilon \to \mathbb{C}$ in $V_i \cap f^{-1}(\delta) \cap B_\epsilon$, for each $i \in \{1, \ldots, q\}$. The next proposition uses the relative polar variety $\Gamma_{f,g}(\mathcal{V})$ for counting the number n_i.

PROPOSITION 3.3. — *If $\Gamma_{f,g}(\mathcal{V})$ is empty, then $n_i = 0$, for all $i \in \{1, \ldots, q\}$.*

Proof. — Let V_i be a stratum of $\mathcal{V}$ and x be a critical point of $g|_{V_i \cap f^{-1}(\delta) \cap B_\epsilon}$. If $\tilde{f}$ and $\tilde{g}$ are analytic extensions of f and g in a neighbourhood of x, respectively, then $x \in V_i \cap f^{-1}(\delta) \cap B_\epsilon$ and $\mathrm{rk}(d_x \tilde{f}, d_x \tilde{g}) \leqslant 1$, that is, $x \in (V_i \cap f^{-1}(\delta) \cap B_\epsilon) \cap (\Sigma_{\mathcal{V}} f \cup \Sigma_{\mathcal{V}} g \cup \widetilde{\Gamma}_{f,g}(\mathcal{V}))$. By Proposition 1.3 of [11], $\Sigma_{\mathcal{V}} f \subset \{f = 0\}$. Therefore, by Proposition 3.1,

$$\Sigma g|_{V_i \cap f^{-1}(\delta) \cap B_\epsilon} = V_i \cap f^{-1}(\delta) \cap B_\epsilon \cap (\Sigma g|_{V_i} \cup \widetilde{\Gamma}_{f,g}(V_i))$$
$$\subset V_i \cap f^{-1}(\delta) \cap B_\epsilon \cap \Gamma_{f,g}(V_i).$$

Since $\Gamma_{f,g}(\mathcal{V})$ is empty, $\Sigma g|_{V_i \cap f^{-1}(\delta) \cap B_\epsilon}$ is empty which implies $n_i = 0$, for all $i \in \{1, \ldots, q\}$. $\qquad\square$

In [3], Dutertre and Grulha proved a Lê–Greuel type formula for the Brasselet number, making possible to count the number of stratified Morse critical points using Brasselet numbers. We apply their result to compute Brasselet numbers in the setting we already know the number of Morse critical points. First, let us show a more general result.

PROPOSITION 3.4. — *Let $\beta : X \to \mathbb{Z}$ be a constructible function with respect to the good stratification $\mathcal{V}$ of X relative to f. If $\Gamma_{f,g}(\mathcal{V})$ is empty, then*

$$\chi(X \cap f^{-1}(\delta) \cap B_\epsilon, \beta) = \chi(X \cap g^{-1}(0) \cap f^{-1}(\delta) \cap B_\epsilon, \beta).$$

Proof. — By Corollary 3.2, since $\Gamma_{f,g}(\mathcal{V})$ is empty, g is prepolar at the origin with respect to $\mathcal{V}$ and, by Proposition 1.12 of [11], g is tractable at the origin with respect to $\mathcal{V}$. Then, by Theorem 4.2 of [3], we obtain

$$\chi(X \cap f^{-1}(\delta) \cap B_\epsilon, \beta)$$

$$= \chi(X \cap g^{-1}(0) \cap f^{-1}(\delta) \cap B_\epsilon, \beta) + \sum_{i=1}^{q} (-1)^{d-1} n_i \eta(V_i, \beta),$$

where n_i is the number of stratified Morse critical points of a Morsefication of $g|_{V_i \cap f^{-1}(\delta) \cap B_\epsilon}$ appearing in $V_i \cap f^{-1}(\delta) \cap B_\epsilon$. By Proposition 3.3, $n_i = 0$, for all $i \in \{1, \ldots, q\}$, and the equality holds. $\qquad\square$

If the constructible function β is the local Euler obstruction, we obtain a relation between Brasselet numbers.

COROLLARY 3.5. — *If X is equidimensional and $\Gamma_{f,g}(\mathcal{V})$ is empty, then* $B_{f,X}(0) = B_{f,X^g}(0)$.

Proof. — By Theorem 4.4 of [3], $B_{f,X}(0) = B_{f,X^g}(0) + (-1)^{d-1} n_q$, where n_q is the number of stratified Morse critical points of a Morsefication of $g|_{X_{\mathrm{reg}} \cap f^{-1}(\delta) \cap B_\epsilon}$ appearing in $X_{\mathrm{reg}} \cap f^{-1}(\delta) \cap B_\epsilon$. Since $\Gamma_{f,g}(\mathcal{V})$ is empty, n_q vanishes and the equality holds. $\qquad\square$

When f is a generic linear form on $\mathbb{C}^n$, $B_{f,X}(0) = \mathrm{Eu}_X(0)$ and $B_{f,X^g}(0) = \mathrm{Eu}_{X^g}(0)$. Therefore, the last corollary leads the following consequence.

COROLLARY 3.6. — *If X is equidimensional and $\Gamma_{f,g}(\mathcal{V})$ is empty, then* $\mathrm{Eu}_X(0) = \mathrm{Eu}_{X^g}(0)$.

When both f and g have isolated singularity at the origin, Dutertre and Grulha proved several formulas about the Brasselet numbers of f and g. Using these formulas and supposing that $\Gamma_{f,g}(\mathcal{V})$ is empty, we obtain further information about these numbers. If this is the case, then g is prepolar at the origin with respect to the good stratification $\mathcal{V}$ of X induced by f, given as a refinement of a Whitney stratification $\mathcal{W} = \{W_i\}$ of X. Hence, by Lemma 6.1 of [3], we conclude that f is prepolar at the origin with respect to the good stratification $\overline{\mathcal{V}}$ of X induced by g, also given by a refinement of $\mathcal{W}$. By Proposition 1.12 of [11], we obtain that $\Gamma_{f,g}(\mathcal{V}) = \widetilde{\Gamma}_{f,g}(\mathcal{V})$ and $\Gamma_{g,f}(\overline{\mathcal{V}}) = \widetilde{\Gamma}_{g,f}(\overline{\mathcal{V}})$. On the other hand,

$$\widetilde{\Gamma}_{f,g}(\mathcal{V}) = \bigcup_{V_i \in \mathcal{V}} \overline{\Sigma(f,g)|_{V_i \setminus (\{f=0\} \cup \{g=0\})}} = \bigcup_{W_i \in \mathcal{W}} \overline{\Sigma(f,g)|_{W_i \setminus (\{f=0\} \cup \{g=0\})}}$$

$$= \bigcup_{\overline{V_i} \in \overline{\mathcal{V}}} \overline{\Sigma(f,g)|_{\overline{V_i} \setminus (\{f=0\} \cup \{g=0\})}} = \widetilde{\Gamma}_{g,f}(\overline{\mathcal{V}}).$$

Hence, the four mentioned polar varieties are equal. We can then compute the following.

PROPOSITION 3.7. — *Let $\beta : X \to \mathbb{Z}$ be a constructible function with respect to the Whitney stratification $\mathcal{W}$ and $\mathcal{V}$ is a good stratification of X induced by f. If $\Gamma_{f,g}(\mathcal{V})$ is empty, then $\chi(X \cap f^{-1}(\delta) \cap B_\epsilon, \beta) = \chi(X \cap g^{-1}(\delta) \cap B_\epsilon, \beta)$.*

Proof. — Since $\Gamma_{f,g}(\mathcal{V})$ is empty, by Corollary 3.2, g is prepolar at the origin with respect to $\mathcal{V}$. By Lemma 6.1 of [3], f is prepolar at the origin with respect to the good stratification $\overline{\mathcal{V}}$ induced by g. By Theorem 6.4 of [3],

$$\chi(X \cap f^{-1}(\delta) \cap B_\epsilon, \beta)$$

$$= \chi(X \cap g^{-1}(\delta) \cap B_\epsilon, \beta) + \sum_{i=1}^{q} (-1)^{d_i - 1}(n_i - m_i)\eta(W_i, \beta),$$

where d_i denotes the dimension of $W_i \in \mathcal{W}, n_i$ is the number of stratified Morse critical points of a Morsefication of $g|_{V_i \cap f^{-1}(\delta) \cap B_\epsilon}$ appearing in $V_i \cap f^{-1}(\delta) \cap B_\epsilon$ and m_i is the number of stratified Morse critical points of a Morsefication of $f|_{\overline{V_i} \cap g^{-1}(\delta) \cap B_\epsilon}$ appearing in $\overline{V_i} \cap g^{-1}(\delta) \cap B_\epsilon$. By Proposition 3.3, since $\Gamma_{f,g}(\mathcal{V})$ is empty, $m_i = n_i = 0$, for all $i \in \{1, \ldots, q\}$, and the equality is proved. □

In the case that β is the local Euler obstruction, we obtain a relation between the Brasselet numbers.

COROLLARY 3.8. — *If X is equidimensional and $\Gamma_{f,g}(\mathcal{V})$ is empty, then $B_{f,X}(0) = B_{g,X}(0)$.*

Proof. — Since $\Gamma_{f,g}(\mathcal{V})$ is empty, g (resp. f) is prepolar at the origin with respect to the good stratification of f (resp. g). By Corollary 6.5 of [3],

$$B_{f,X}(0) = B_{g,X}(0) + (-1)^{d-1}(n_q - m_q),$$

where n_q (resp. m_q) is the number of Morse critical points of a Morsefication of $g|_{X_{\mathrm{reg}} \cap f^{-1}(\delta) \cap B_\epsilon}$ (resp. $f|_{X_{\mathrm{reg}} \cap g^{-1}(\delta) \cap B_\epsilon}$) appearing in $X_{\mathrm{reg}} \cap f^{-1}(\delta) \cap B_\epsilon$ (resp. $X_{\mathrm{reg}} \cap g^{-1}(\delta) \cap B_\epsilon$). Using again that $\Gamma_{f,g}(\mathcal{V})$ is empty, we have that $n_q = m_q = 0$ and the equality holds. □

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