ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE Mathématiques

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Tome XXXII, n° 1 (2023), p. 173–178.

https://doi.org/10.5802/afst.1733

Publication membre du centre Mersenne pour l'édition scientifique ouverte
http://www.centre-mersenne.org/
e-ISSN : 2258-7519

Annales de la faculté des sciences de Toulouse Volume XXXII, n° 1, 2023
pp. 173-178

Remark on an inequality for closed hypersurfaces in complete manifolds with nonnegative Ricci curvature (*)

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Abstract. — We give a simple proof of a recent result due to Agostiniani, Fogagnolo and Mazzieri [1].

Résumé. — Nous donnons une preuve simple d'un résultat récent dû à Agostiniani, Fogagnolo and Mazzieri [1].

The following result was proved by Agostiniani, Fogagnolo and Mazzieri [1].

Theorem. — *Let (M^n, g) $(n \geqslant 3)$ be a complete Riemannian manifold with nonnegative Ricci curvature and $\Omega \subset M$ a bounded open set with smooth boundary. Then*

$$\int_{\partial\Omega} \left|\frac{H}{n-1}\right|^{n-1} \mathrm{d}\sigma \geqslant \mathrm{AVR}(g)|\mathbb{S}^{n-1}|, \tag{1}$$

where H is the mean curvature of $\partial\Omega$ and $\mathrm{AVR}(g)$ is the asymptotic volume ratio of M. Moreover, if $\mathrm{AVR}(g) > 0$, equality holds iff $M\backslash\Omega$ is isometric to $\left([r_0, \infty) \times \partial\Omega, \mathrm{d}r^2 + (\frac{r}{r_0})^2 g_{\partial\Omega}\right)$ with

$$r_0 = \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}$$

In particular, $\partial\Omega$ is a connected totally umbilic submanifold with constant mean curvature.

(*) Reçu le 25 janvier 2021, accepté le 22 février 2021.

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Article proposé par Gilles Carron.

The proof in [1] is highly nontrivial. It is based on the study of the solution of the following problem

$$\begin{cases} \Delta u = 0, & \text{on } M\backslash\Omega \\ u = 1 & \text{on } \partial\Omega \\ u(x) \to 0 & \text{as } x \to \infty, \end{cases}$$

which exists when $\mathrm{AVR}(g) > 0$. The key step consists of showing that, with $\beta \geqslant (n-2)/(n-1)$

$$U_\beta(t) = t^{-\beta(\frac{n-1}{n-2})} \int_{u=t} |\nabla u|^{\beta+1} \mathrm{d}\sigma$$

is monotone in $t \in (0,1]$. The geometric inequality (1) then follows by analyzing the asymptotic behavior of $U_\beta(t)$ as $t \to 0$. It is a beautiful argument.

In this short note, we show that this theorem can be proved by standard comparison methods in Riemannian geometry.

To prove the inequality (1), we assume, without loss of generality, that Ω has no hole, i.e. $M\backslash\Omega$ has no bounded component. In the following we write $\Sigma = \partial\Omega$ and let ν be the outer unit normal along Σ. For each $p \in \Sigma$ let $\gamma_p(t) = \exp_p t\nu(p)$ be the normal geodesic with initial velocity $\nu(p)$. We define

$$\tau(p) = \sup\{L > 0 : \gamma_p \text{ is minimizing on } [0, L]\} \in (0, \infty].$$

It is well known that τ is a continuous function on Σ and the focus locus

$$C(\Sigma) = \{\exp_p \tau(p)\nu(p) : \tau(p) < \infty\}$$

is a closed set of measure zero in M. Moreover the map $\Phi(r,p) = \exp_p r\nu(p)$ is a diffeomorphism from

$$E = \{(r,p) \in \Sigma \times [0,\infty) : r < \tau(p)\}$$

onto $(M\backslash\Omega)\backslash C(\Sigma)$. And on E the pull back of the volume form takes the form $\mathrm{d}\mu = \mathcal{A}(r,p)\mathrm{d}r\mathrm{d}\sigma(p)$. We will also understand r as the distance function to Σ and it is smooth on $M\backslash\Omega$ away from $C(\Sigma)$. By the Bochner formula and nonnegative Ricci curvature condition

$$\begin{aligned} 0 &= \frac{1}{2}\Delta|\nabla r|^2 = |D^2 r|^2 + \langle \nabla r, \nabla \Delta r\rangle + \mathrm{Ric}(\nabla r, \nabla r) \\ &\geqslant \frac{(\Delta r)^2}{n-1} + \frac{\partial}{\partial r}\Delta r. \end{aligned}$$

In view of the initial condition $\Delta r|_{r=0} = H$, it is standard to deduce from the above inequality $\tau \leqslant \frac{n-1}{H^-}$ and

$$\frac{\mathcal{A}'}{\mathcal{A}} = \Delta r \leqslant \frac{(n-1)H}{n-1+Hr}$$

This shows that the function

$$\theta(r,p) = \frac{\mathcal{A}(r,p)}{\left(1+\frac{H(p)}{n-1}r\right)^{n-1}}$$

is non-increasing in r on $[0,\tau(p))$. As $\theta(0,p)=1$, we obtain

$$\mathcal{A}(r,p) \leqslant \left(1+\frac{H(p)}{n-1}r\right)^{n-1}. \tag{2}$$

The above analysis is by now standard in Riemannian geometry. For more details, cf. [6, 7] or the original work of Heintze and Karcher [3] where the analysis is done using Jacobi fields and the index form in a more general setting.

Therefore for any $R>0$

$$\begin{aligned}
&\mathrm{Vol}\{x\in M : d(x,\Omega)<R\} \\
&\qquad = |\Omega| + \int_\Sigma \int_0^{\min(R,\tau(p))} \mathcal{A}(r,p)\mathrm{d}r\mathrm{d}\sigma(p) \\
&\qquad \leqslant |\Omega| + \int_\Sigma \int_0^{\min(R,\tau(p))} \left(1+\frac{H(p)}{n-1}r\right)^{n-1} \mathrm{d}r\mathrm{d}\sigma(p) \\
&\qquad \leqslant |\Omega| + \int_\Sigma \int_0^{\min(R,\tau(p))} \left(1+\frac{H^+(p)}{n-1}r\right)^{n-1} \mathrm{d}r\mathrm{d}\sigma(p) \\
&\qquad \leqslant |\Omega| + \int_\Sigma \int_0^{R} \left(1+\frac{H^+(p)}{n-1}r\right)^{n-1} \mathrm{d}r\mathrm{d}\sigma(p) \\
&\qquad = |\Omega| + \frac{R^n}{n}\int_\Sigma \left(\frac{H^+(p)}{n-1}\right)^{n-1} \mathrm{d}\sigma(p) + O(R^{n-1}).
\end{aligned}$$

Dividing both sides by $|\mathbb{B}^n|R^n = |\mathbb{S}^{n-1}|R^n/n$ and letting $R\to\infty$ yields

$$\mathrm{AVR}(g) \leqslant \frac{1}{|\mathbb{S}^{n-1}|}\int_\Sigma \left(\frac{H^+}{n-1}\right)^{n-1}\mathrm{d}\sigma,$$

which implies (1).

We now analyze the equality case. Suppose

$$\mathrm{AVR}(g) = \frac{1}{|\mathbb{S}^{n-1}|}\int_\Sigma \left(\frac{H^+}{n-1}\right)^{n-1}\mathrm{d}\sigma > 0. \tag{3}$$

From the proof of the inequality and the fact that τ is continuous, we conclude that $\tau \equiv \infty$ on the open set $\Sigma^+ = \{p \in \Sigma : H(p) > 0\}$. We note from (2) that $\mathcal{A}(r,p) \leqslant 1$ for $p \in \Sigma \backslash \Sigma^+$. For any $R' < R$ we have

$$\begin{aligned}
&\mathrm{Vol}\{x \in M : d(x,\Omega) < R\} \\
&= |\Omega| + \int_{\Sigma^+}\int_0^R \mathcal{A}(r,p)\mathrm{d}r\mathrm{d}\sigma(p) + \int_{\Sigma\backslash\Sigma^+}\int_0^{\min(R,\tau(p))} \mathcal{A}(r,p)\mathrm{d}r\mathrm{d}\sigma(p) \\
&\leqslant |\Omega| + \int_{\Sigma^+}\int_0^R \theta(r,p)\left(1+\frac{H(p)}{n-1}r\right)^{n-1}\mathrm{d}r\mathrm{d}\sigma(p) + \int_{\Sigma\backslash\Sigma^+}\int_0^R \mathrm{d}r\mathrm{d}\sigma(p) \\
&\leqslant |\Omega| + \int_{\Sigma^+}\int_{R'}^R \theta(r,p)(1+\frac{H(p)}{n-1}r)^{n-1}\mathrm{d}r\mathrm{d}\sigma(p) \\
&\qquad + \int_{\Sigma^+}\int_0^{R'} \theta(r,p)\left(1+\frac{H(p)}{n-1}r\right)^{n-1}\mathrm{d}r\mathrm{d}\sigma(p) + O(R) \\
&\leqslant |\Omega| + \int_{\Sigma^+}\theta(R',p)\int_{R'}^R \left(1+\frac{H(p)}{n-1}r\right)^{n-1}\mathrm{d}r\mathrm{d}\sigma(p) \\
&\qquad + \int_{\Sigma^+}\int_0^{R'} \theta(r,p)\left(1+\frac{H(p)}{n-1}r\right)^{n-1}\mathrm{d}r\mathrm{d}\sigma(p) + O(R).
\end{aligned}$$

Dividing both sides by $|\mathbb{B}^n|R^n = |\mathbb{S}^{n-1}|R^n/n$ and letting $R \to \infty$ yields

$$\mathrm{AVR}(g) \leqslant \frac{1}{|\mathbb{S}^{n-1}|}\int_{\Sigma^+}\left(\frac{H(p)}{n-1}\right)^{n-1}\theta(R',p)\mathrm{d}\sigma(p).$$

Letting $R' \to \infty$ yields

$$\mathrm{AVR}(g) \leqslant \frac{1}{|\mathbb{S}^{n-1}|}\int_{\Sigma^+}\left(\frac{H(p)}{n-1}\right)^{n-1}\theta_\infty(p)\mathrm{d}\sigma(p),$$

where $\theta_\infty(p) = \lim_{r\to\infty}\theta(r,p) \leqslant 1$. As we have equality (3) we must have $\theta_\infty(p) = 1$ for a.e. $p \in \Sigma^+$. It follows that

$$\mathcal{A}(r,p) = \left(1+\frac{H(p)}{n-1}r\right)^{n-1} \quad \text{on } [0,\infty)$$

for a.e. $p \in \Sigma^+$. By continuity the above identity holds for all $p \in \Sigma^+$.

Inspecting the comparison argument, we must have on $\Phi([0,\infty)\times\Sigma^+)$

$$\mathrm{Ric}(\nabla r, \nabla r) = 0, \tag{4}$$

$$D^2 r = \frac{\Delta r}{n-1}g = \frac{H}{n-1+Hr}g. \tag{5}$$

As $\mathrm{Ric} \geqslant 0$, it follows from the 1st equation above that $\mathrm{Ric}(\nabla r, \cdot) = 0$. From the 2nd equation above Σ^+ is an umbilic hypersurface, i.e. the 2nd fundamental form $\Pi = \frac{H}{n-1} g_{\Sigma^+}$. Working with an orthonormal frame $\{e_0 = \nu, e_1, \ldots, e_{n-1}\}$ along Σ^+ we have by the Codazzi equation, with $1 \leqslant i, j, k \leqslant n-1$

$$R(e_k, e_j, e_i, \nu) = \Pi_{ij,k} - \Pi_{ik,j} = \frac{1}{n-1}(H_k \delta_{ij} - H_j \delta_{ik}).$$

Taking trace over i and k yields

$$-\frac{n-2}{n-1} H_j = \mathrm{Ric}(e_j, \nu) = 0.$$

As a result H is locally constant on Σ^+. Therefore Σ^+ must be the union of several components of Σ. We know that Φ is a diffeomorphism form $[0,\infty) \times \Sigma^+$ onto its image and the pullback metric $\Phi^* g$ takes the following form

$$\mathrm{d}r^2 + h_r,$$

where h_r is a r-dependent family of metrics on Σ^+ and $h_0 = g_{\Sigma^+}$. In terms of local coordinates $\{x_1, \ldots, x_{n-1}\}$ on Σ^+ the equation (5) implies

$$\frac{1}{2} \frac{\partial}{\partial r} h_{ij} = \frac{H}{n-1+Hr} h_{ij}.$$

Therefore $h_r = (1 + \frac{H}{n-1} r)^2 g_{\Sigma^+}$. This proves that $\Phi([0,\infty) \times \Sigma^+)$ is isometric to $([r_0, \infty) \times \Sigma^+, \mathrm{d}r^2 + (\frac{r}{r_0})^2 g_{\Sigma^+})$, where $r_0 = \frac{n-1}{H}$.

Since M has nonnegative Ricci curvature and Euclidean volume growth, it has only one end by the Cheeger–Gromoll theorem. Therefore Σ^+ is connected and if Σ has other components besides Σ^+, they all bound bounded components of $M \backslash \Omega$.

If we have the stronger identity

$$\mathrm{AVR}(g) = \frac{1}{|\mathbb{S}^{n-1}|} \int_\Sigma \left| \frac{H}{n-1} \right|^{n-1} \mathrm{d}\sigma > 0,$$

inspecting the proof of the inequality (1) shows that we must have $H \geqslant 0$ on Σ. Then $\overline{\Omega}$ is compact Riemannian manifold with mean convex boundary. It is a classic fact that its boundary $\partial\Omega$ must be connected, see [4, 5] or [2] for an analytic argument. Therefore $\Sigma = \Sigma^+$ is connected and $M \backslash \Omega$ is isometric to $([r_0, \infty) \times \Sigma, \mathrm{d}r^2 + (\frac{r}{r_0})^2 g_\Sigma)$.

Acknowledgements

The author would like to thank Fengbo Hang for helpful discussions.

Bibliography

[1] V. Agostiniani, M. Fogagnolo & L. Mazzieri, "Sharp geometric inequalities for closed hypersurfaces in manifolds with nonnegative Ricci curvature", *Invent. Math.* **222** (2020), no. 3, p. 1033-1101.

[2] F. Hang & X. Wang, "Vanishing sectional curvature on the boundary and a conjecture of Schroeder and Strake", *Pac. J. Math.* **232** (2007), no. 2, p. 283-287.

[3] E. Heintze & H. Karcher, "A general comparison theorem with applications to volume estimates for submanifolds", *Ann. Sci. Éc. Norm. Supér.* **11** (1978), no. 4, p. 451-470.

[4] R. Ichida, "Riemannian manifolds with compact boundary", *Yokohama Math. J.* **29** (1981), no. 2, p. 169-177.

[5] A. Kasue, "Ricci curvature, geodesics and some geometric properties of Riemannian manifolds with boundary", *J. Math. Soc. Japan* **35** (1983), no. 1, p. 117-131.

[6] P. Petersen, *Riemannian Geometry*, 3rd ed., Graduate Texts in Mathematics, vol. 171, Springer, 2016.

[7] T. Sakai, *Riemannian geometry*, Translations of Mathematical Monographs, vol. 149, American Mathematical Society, 1996.