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Hodge numbers and Hodge structures for 3-Calabi–Yau categories [(*)]

Roland Abuaf [(1)]

Abstract. — Let $\mathscr{A}$ be a smooth proper $\mathbb{C}$-linear triangulated category which is 3-Calabi–Yau endowed with a (non-trivial) rank function. Using the homological unit of $\mathscr{A}$ with respect to the given rank function, we define Hodge numbers for $\mathscr{A}$.

If the classes of unitary objects generate the rational numerical K-theory of $\mathscr{A}$, it is proved that these numbers are independent of the chosen rank function : they are intrinsic invariants of the triangulated category $\mathscr{A}$.

In the special case where $\mathscr{A}$ is a semi-orthogonal component of the derived category of a smooth complex projective variety and the homological unit of $\mathscr{A}$ is $\mathbb{C}\oplus\mathbb{C}[3]$, we define a Hodge structure on the Hochschild homology of $\mathscr{A}$. The dimensions of the Hodge spaces of this structure are the Hodge numbers aforementioned.

Finally, we give some numerical applications toward the Homological Mirror Symmetry conjecture for cubic sevenfolds and double quartic fivefolds.

Résumé. — Soit $\mathscr{A}$ une catégorie triangulée $\mathbb{C}$-linéaire, non singulière et propre, que l'on suppose être 3-Calabi–Yau et munie d'une fonction rang non-triviale. En nous basant sur la notion d'unité homologique pour $\mathscr{A}$ associée à la fonction rang, nous définissons des nombres de Hodge pour $\mathscr{A}$.

Si les classes d'objets unitaires engendrent la K-théorie numérique de $\mathscr{A}$, nous prouvons que ces nombres ne dépendent pas de la fonction rang choisie : ce sont alors des invariants intrinsèques de la catégorie $\mathscr{A}$.

Dans le cas particulier où $\mathscr{A}$ est une composante semi-orthogonale de la catégorie dérivée d'une variété projective non singulière définie sur $\mathbb{C}$ et que l'unité homologique de $\mathscr{A}$ est $\mathbb{C} \oplus \mathbb{C}[3]$, nous définissons une structure de Hodge sur l'homologie d'Hochschild de $\mathscr{A}$. Les dimensions des espaces de Hodge associés à cette structure sont les nombres de Hodge déjà mentionnés.

En conclusion, nous donnons quelques applications numériques de notre théorie en direction de la conjecture de Symétrie Miroir Homologique pour les hypersurfaces cubiques de dimension 5 et les recouvrements doubles quartiques de $\mathbb{P}^5$.

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Article proposé par Bertrand Toën.

1. Introduction

As part of his Homological Mirror Symmetry conjecture (see [20]) M. Kontsevich predicted that the Hochschild homology (or rather the cyclic homology) of a smooth proper triangulated category should be endowed with a Hodge structure. It has been suggested that this Hodge structure could be obtained via the degeneration of a certain spectral sequence, à la Deligne–Illusie ([14, 15, 16]). This approach is very promising. It seems however difficult, at the present time, to use it in practice to compute Hodge numbers of a given Calabi–Yau category outside of the realm of LG models of Fano threefolds and surfaces (see [10, 25] for instance).

In this paper, we use the more elementary theory of homological units (see [2]) in order to define and compute Hodge numbers for smooth proper $\mathbb{C}$-linear triangulated categories which are 3-Calabi–Yau. Our main definition and our main result are (see Definition 1.1 and Theorem 3.4):

Definition 1.1. — *Let $\mathscr{A}$ be a smooth proper triangulated category which is* 3*-Calabi–Yau and endowed with a non-trivial rank function. Let $\mathfrak{T}^{\bullet}_{\mathscr{A}}$ be a homological unit for $\mathscr{A}$ with respect to the rank function. We define the* Hodge numbers *of $\mathscr{A}$ as:*

(1) *for all $i \in [0, \dots, 3]$, $h^{i,0}(\mathscr{A}) = \mathfrak{T}^{3-i}_{\mathscr{A}}$,*
(2) $h^{3,1}(\mathscr{A}) = \dim \mathrm{HH}_{-2}(\mathscr{A}) - h^{2,0}(\mathscr{A})$,
(3) $h^{3,2}(\mathscr{A}) = h^{1,0}(\mathscr{A})$ *and* $h^{2,1}(\mathscr{A}) = \dim \mathrm{HH}_{-1}(\mathscr{A}) - h^{1,0}(\mathscr{A}) - h^{3,2}(\mathscr{A})$,
(4) $h^{3,3}(\mathscr{A}) = h^{0,0}(\mathscr{A})$ *and*
$$h^{1,1}(\mathscr{A}) = h^{2,2}(\mathscr{A}) = \frac{\dim \mathrm{HH}_0(\mathscr{A}) - h^{0,0}(\mathscr{A}) - h^{3,3}(\mathscr{A})}{2}.$$
(5) $h^{p,q}(\mathscr{A}) = h^{q,p}(\mathscr{A})$ *for all $p, q \in [0, \dots, 3]$.*

Theorem 1.2. — *Let $\mathscr{A}$ be a smooth proper triangulated* 3*-Calabi–Yau category. Let $\mathrm{rk}_1, \mathrm{rk}_2$ be a non-trivial numerical rank functions on $\mathscr{A}$ and let $\mathfrak{T}^{\bullet}_{\mathscr{A},1}, \mathfrak{T}^{\bullet}_{\mathscr{A},2}$ be homological units for $\mathscr{A}$ with respect to rk_1, rk_2. Let $\mathrm{cl} : \mathscr{A} \to \mathrm{K}_{\mathrm{num}}(\mathscr{A})$ be the class map and denote by $\mathscr{A}^{(i)}_{\mathrm{unitary}}$ the set of objects $\mathscr{F} \in \mathscr{A}$ such that $\mathrm{Hom}^{\bullet}_{\mathscr{A}}(\mathscr{F}, \mathscr{F}) \simeq \mathfrak{T}^{\bullet}_{\mathscr{A},i}$ as graded rings. Finally, for all $p, q \in [0, \dots, 3]$, we denote by $h^{p,q}_i(\mathscr{A})$ the Hodge numbers of $\mathscr{A}$ associated to $\mathfrak{T}^{\bullet}_{\mathscr{A},i}$ as in Definition 1.1. We have the following:*

(1) *If both $\mathrm{cl}(\mathscr{A}^{(1)}_{\mathrm{unitary}})$ and $\mathrm{cl}(\mathscr{A}^{(2)}_{\mathrm{unitary}})$ generate $\mathrm{K}_{\mathrm{num}}(\mathscr{A}) \otimes \mathbb{C}$, then:*
$$h^{p,q}_1(\mathscr{A}) = h^{p,q}_2(\mathscr{A}),$$
for all $p, q \in [0, \dots, 3]$.

(2) *If* $\mathfrak{T}^{\bullet}_{\mathscr{A},1} = \mathbb{C} \oplus \mathbb{C}[3]$, $\mathrm{cl}(\mathscr{A}^{(1)}_{\mathrm{unitary}})$ *generates* $\mathrm{K}_{\mathrm{num}}(\mathscr{A}) \otimes \mathbb{C}$ *and there exists a unitary object in* $\mathscr{A}$ *with respect to* $\mathfrak{T}^{\bullet}_{\mathscr{A},2}$, *then:*

$$h_1^{p,q}(\mathscr{A}) = h_2^{p,q}(\mathscr{A}),$$

for all $p, q \in [0, \ldots, 3]$.

The key hypothesis in the above Theorem, namely that the classes of unitary objects generate the rational numerical K-theory of $\mathscr{A}$ is satisfied for many examples of smooth proper 3-Calabi–Yau categories. Indeed, derived categories of projective varieties of dimension three, 3-Calabi–Yau categories obtained as semi-orthogonal components of hypersurfaces in weighted projective spaces and Calabi–Yau categories associated to quivers with potentials do satisfy this hypothesis. We shall discuss in details such examples in Section 2.2 of the paper.

In the last section of the paper, we give some applications of our definition of Hodge numbers to numerical Mirror Symmetry for some Greene–Plesser pairs, namely for the cubic sevenfold and the double quartic fivefold.

The plan of the paper is the following:

In Section 2.1, we recall the definitions of rank functions, homological units and their basic properties. We compute the units for many examples of smooth proper 3-Calabi–Yau category.

In Section 2.2, we study the invariance of the homological unit with respect to the chosen rank function.

In Section 3.1, we define the Hodge numbers for a smooth proper Calabi–Yau category endowed with a non-trivial rank function. We compute them for the examples introduced in Section 2.2.

In Section 3.2, we assume that our triangulated category is a semi-orthogonal component of the derived category of coherent sheaves on a smooth complex projective variety and that its homological unit (with respect to the rank function coming from the variety) is $\mathbb{C} \oplus \mathbb{C}[3]$. We then define a Hodge structure on the Hochschild homology of this category. The dimensions of the corresponding Hodge spaces are the Hodge numbers aforementioned.

In Section 3.3, we give some numerical applications toward the Homological Mirror Symmetry conjecture for cubic sevenfolds and double quartic fivefolds.

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Conventions

We work over the field of complex numbers. We only consider $\mathbb{C}$-linear triangulated categories which can be realized as derived categories of *DG*-modules over a smooth and proper *DG*-algebra (call them *smooth* and *proper*). In particular, for any such triangulated category $\mathscr{A}$ and any $\mathscr{F}, \mathscr{G} \in \mathscr{A}$, the graded vector space $\bigoplus_{k \in \mathbb{Z}} \operatorname{Ext}^k(\mathscr{F}, \mathscr{G})$ is finite dimensional over $\mathbb{C}$.

2. Homological units and invariance properties for 3-Calabi–Yau categories

In this section, we recall the definition of homological unit [2], compute the homological units for many examples of Calabi–Yau categories and prove some invariance properties for the homological units of 3-Calabi–Yau categories.

2.1. Calabi–Yau categories and homological units

We first recall some definitions related to such categories.

Definition 2.1. — *Let $\mathscr{A}$ be a triangulated category. We say that $\mathscr{A}$ is a* p-Calabi–Yau category *if the shift by $[p]$ is a Serre functor for $\mathscr{A}$. We say furthermore that $\mathscr{A}$ is a* geometric p-Calabi–Yau category *if it is p-Calabi–Yau and there exists a smooth projective variety X and a semi-orthogonal decomposition*

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathscr{A}, {}^{\perp}\mathscr{A} \rangle.$$

This definition already appeared many times in the literature. It has been studied in details when X is a cubic fourfold and $p = 2$ in [22] and more generally when X is a hypersurface in (or a double cover of) a rational homogeneous space for any p in [23]. The study of the $K3$ category appearing in a cubic fourfold in connection with rationality problems and the Hodge theory of cubic fourfolds has been carried out in many papers (with starting point [22]). The paper [11] provides a detailed study of recent works on the subject.

The notion of homological units has been introduced in [2] as a categorical substitute for the algebra $H^\bullet(\mathscr{O}_{\mathscr{X}})$. We recall its definition in the context of triangulated categories.

Definition 2.2. — *Let $\mathscr{A}$ be a smooth proper triangulated category. A* rank function *on $\mathscr{A}$ is a function* $\mathrm{rk} : \mathscr{A} \to \mathbb{Z}$ *which is additive with respect to exact triangles and such that* $\mathrm{rk}(\mathscr{F}[1]) = -\mathrm{rk}(\mathscr{F})$*, for any $\mathscr{F} \in \mathscr{A}$. We say that it is a* numerical rank function *if it descends to a map:*

$$\mathrm{rk} : \mathrm{K}_{\mathrm{num}}(\mathscr{A}) \longrightarrow \mathbb{Z},$$

where $\mathrm{K}_{\mathrm{num}}(\mathscr{A})$ is the quotient of the K*-theory of $\mathscr{A}$ by the kernel of the bilinear form:*

$$\begin{aligned}\chi : \mathrm{K}_0(\mathscr{A}) \times \mathrm{K}_0(\mathscr{A}) &\longrightarrow \mathbb{Z}\\ (\mathscr{E},\mathscr{F}) &\longmapsto \sum_{k\geqslant 0}(-1)^k \dim \mathrm{Ext}^k(\mathscr{E},\mathscr{F}).\end{aligned}$$

Let X be a smooth projective variety, the Grothendieck–Riemann–Roch Theorem shows that the rank of a numerically trivial object in $\mathrm{K}_0(X)$ is necessarily 0. Hence, the natural rank function on $\mathrm{D}^{\mathrm{b}}(X)$ is a numerical rank function. If $\mathscr{A}$ is a semi-orthogonal component of $\mathrm{D}^{\mathrm{b}}(X)$, then we have decompositions:

$$\mathrm{K}_0(X) = \mathrm{K}_0(\mathscr{A}) \oplus \mathrm{K}_0({}^{\perp}\mathscr{A})$$

and

$$\mathrm{K}_{\mathrm{num}}(X) = \mathrm{K}_{\mathrm{num}}(\mathscr{A}) \oplus \mathrm{K}_{\mathrm{num}}({}^{\perp}\mathscr{A}).$$

As a consequence, the natural rank function on $\mathrm{D}^{\mathrm{b}}(X)$ restricts to a numerical rank function on $\mathscr{A}$.

Definition 2.3. — *Let $\mathscr{A}$ be a smooth proper triangulated category endowed with a non-trivial rank function. A graded algebra $\mathfrak{T}^\bullet_{\mathscr{A}}$ is called a* homological unit *for $\mathscr{A}$, if $\mathfrak{T}^\bullet_{\mathscr{A}}$ is maximal, with respect to inclusion, for the following property. For any object $\mathscr{F} \in \mathscr{A}$, there exists a pair of morphisms $i_{\mathscr{F}} : \mathfrak{T}^\bullet_{\mathscr{A}} \to \mathrm{Hom}^\bullet_{\mathscr{A}}(\mathscr{F},\mathscr{F})$ and $t_{\mathscr{F}} : \mathrm{Hom}^\bullet_{\mathscr{A}}(\mathscr{F},\mathscr{F}) \to \mathfrak{T}^\bullet_{\mathscr{A}}$ such that:*

- *the morphism $i_{\mathscr{F}} : \mathfrak{T}^\bullet_{\mathscr{A}} \to \mathrm{Hom}^\bullet_{\mathscr{A}}(\mathscr{F},\mathscr{F})$ is a graded algebra morphism which is functorial in the following sense. Let $\mathscr{F},\mathscr{G} \in \mathscr{A}$,* *then, for any morphism $\psi : \mathscr{F} \to \mathscr{G}$, there is a commutative diagram:*

$$\begin{tikzcd} \mathscr{F} \arrow[r, "i_{\mathscr{F}}"] \arrow[d, "\psi"'] & \mathscr{F}[k] \arrow[d, "\psi[k]"] \\ \mathscr{G} \arrow[r, "i_{\mathscr{G}}"] & \mathscr{G}[k] \end{tikzcd}$$

- *the morphism* $t_{\mathscr{F}} : \mathrm{Hom}^{\bullet}_{\mathscr{A}}(\mathscr{F}, \mathscr{F}) \to \mathfrak{T}^{\bullet}_{\mathscr{A}}$ *is a graded vector spaces morphism such that for any* $\mathscr{F} \in \mathscr{A}$ *and any* $a \in \mathfrak{T}^{\bullet}_{\mathscr{A}}$, *we have* $t_{\mathscr{F}}(i_{\mathscr{F}}(a)) = \mathrm{rank}(\mathscr{F}).a$.

An object $\mathscr{F} \in \mathscr{A}$ *is said to be* unitary, *if* $\mathrm{Hom}^{\bullet}_{\mathscr{A}}(\mathscr{F}, \mathscr{F}) \simeq \mathfrak{T}^{\bullet}_{\mathscr{A}}$ *as graded rings, where* $\mathfrak{T}^{\bullet}_{\mathscr{A}}$ *is a homological unit for* $\mathscr{A}$.

Remark 2.4. —

(1) By "maximal with respect to inclusion", we (obviously) mean that for any algebra $\mathfrak{B}^{\bullet}$ satisfying the conditions in the above definition, a graded algebra monomorphism $r^{\bullet} : \mathfrak{T}^{\bullet}_{\mathscr{A}} \hookrightarrow \mathfrak{B}^{\bullet}$ which makes the following diagram commutative for any $\mathscr{F} \in \mathscr{A}$:

$$\begin{tikzcd} \mathfrak{T}^{\bullet}_{\mathscr{A}} \arrow[rr, hook, "r^{\bullet}"] \arrow[dr, "i^{\bullet}_{\mathscr{F}}"'] & & \mathfrak{B}^{\bullet} \arrow[dl, "i^{\bullet}_{\mathscr{F}}"] \\ & \mathrm{Hom}^{\bullet}_{\mathscr{A}}(\mathscr{F}, \mathscr{F}) & \end{tikzcd}$$

is necessarily an isomorphism

(2) Let $\mathscr{X}$ be a projective Deligne–Mumford stack which can be written as a global quotient $[X/\mathrm{G}]$ where X is a smooth projective variety and G is a reductive group acting linearly on X. Let $\mathscr{O}_X(1)$ be a G-equivariant line bundle. A minor modification of the arguments in Theorem 4 of [27] shows that there is an equivalence:

$$D^{\mathrm{perf}}(\mathscr{X}) \simeq D^{\mathrm{perf}}(\mathscr{C}),$$

where

$$\mathscr{C} = \mathrm{RHom}^{\mathrm{G}}_X \left(\bigoplus_{i=0}^{\dim X} \mathscr{O}_X(i), \bigoplus_{i=0}^{\dim X} \mathscr{O}_X(i) \right).$$

Let us consider the rank of an $\mathscr{O}_{\mathscr{X}}$-module as a rank function on $D^{\mathrm{perf}}(\mathscr{X})$. In such a case, we have $\mathfrak{T}^{\bullet}_{D^{\mathrm{perf}}(\mathscr{X})} = H^{\bullet}(\mathscr{O}_{\mathscr{X}})$. Furthermore, for any $\mathscr{F} \in \mathscr{D}^{\mathrm{perf}}(\mathscr{X})$, the morphism $i_{\mathscr{F}}$ is the tensor product (over $\mathscr{O}_{\mathscr{X}}$) with the identity map of $\mathscr{F}$ and the morphism $t_{\mathscr{F}}$ is the trace map $\mathrm{Hom}^{\bullet}_{\mathscr{X}}(\mathscr{F}, \mathscr{F}) \to H^{\bullet}(\mathscr{O}_{\mathscr{X}})$.

(3) In the above definition, the existence of the morphism $i_{\mathscr{F}}$for all $\mathscr{F} \in \mathscr{A}$ and its functorial properties is equivalent to the existence of a morphism of graded algebras:

$$\mathfrak{T}^{\bullet}_{\mathscr{A}} \longrightarrow \mathrm{HH}^{\bullet}(\mathscr{A}),$$

where $\mathrm{HH}^{\bullet}(\mathscr{A})$ is the Hochschild cohomology of $\mathscr{A}$. If the rank function on $\mathscr{A}$ is non-trivial, the splitting property of $t^{\bullet}$ implies that the map $\mathfrak{T}^{\bullet}_{A} \to \mathrm{HH}^{\bullet}(\mathscr{A})$ is injective.

(4) On the other hand, the definition and the (splitting) properties of the morphisms $t_{\mathscr{F}}$, for $\mathscr{F} \in \mathscr{A}$ with non-zero rank do not seem to be easily written using only the notion of graded morphisms between $\mathrm{HH}^{\bullet}(\mathscr{A})$ and $\mathfrak{T}^{\bullet}_{\mathscr{A}}$. It appears that there is no obvious way to write that $t_{\mathscr{F}}$ splits $i_{\mathscr{F}}$ whenever the rank of $\mathscr{F}$ is not zero only in terms of Hochschild cohomology.

(5) If $\mathscr{A}$ contains a unitary object whose rank is not zero, then the homological unit of $\mathscr{A}$ is necessarily unique (though the embedding of the homological unit in $\mathrm{HH}^{\bullet}(\mathscr{A})$ is certainly not unique). This follows from the maximality condition imposed in Definition 2.3.

(6) Let X and Y be smooth projective varieties of dimension less or equal to 4 such that $\mathrm{D^b}(X) \simeq \mathrm{D^b}(Y)$. It is proved in [2] that the algebras $H^{\bullet}(\mathscr{O}_X)$ and $H^{\bullet}(\mathscr{O}_Y)$ are isomorphic. This suggests that the homological unit of a DG category of geometric origin could be independent of the embedding into the derived category of a smooth projective Deligne–Mumford stack (at least if the dimensions of the varieties are small enough). In the next subsection, we will investigate in more details the invariance properties of homological units attached to geometric 3-Calabi–Yau categories.

Let $X \subset \mathbb{P}^5$ be a smooth cubic hypersurface. According to [22], we have a semi-orthogonal decomposition:

$$\mathrm{D^b}(X) = \langle \mathscr{A}_X, \mathscr{O}_X, \mathscr{O}_X(1), \mathscr{O}_X(2) \rangle,$$

where the Serre functor of $\mathscr{A}_X$ is the twist by [2]. This category has been studied in some details, most prominently in connection with the rationality problem for cubic fourfolds (see [4, 11, 22] for instance).

Proposition 2.5. — *We keep notations as above, the algebra $\mathbb{C} \oplus \mathbb{C}[2]$ satisfies all the properties of a homological unit for $\mathscr{A}_X$ with respect to the rank function coming from $\mathrm{D^b}(X)$, except that for any $\mathscr{E} \in \mathscr{A}_X$ which rank is not zero, the map $\mathbb{C} \oplus \mathbb{C}[2] \to \mathrm{Hom}^{\bullet}(\mathscr{E}, \mathscr{E})$ is perhaps only a graded vector space isomorphism (and not a graded ring isomorphism as expected).*

Proof. — The pure shift by [2] is a Serre functor for $\mathscr{A}_X$, as proved in [22]. The dimension of the Hochschild cohomology spaces of $\mathscr{A}_X$ can be found for instance in [4] :

$$\begin{aligned}
&\mathrm{HH}^1(\mathscr{A}_X) = \mathrm{HH}^3(\mathscr{A}_X) = 0\\
&\mathrm{HH}^0(\mathscr{A}_X) \simeq \mathrm{HH}^4(\mathscr{A}_X) \simeq \mathbb{C}\\
&\mathrm{HH}^2(\mathscr{A}_X) \simeq \mathbb{C}^{22}.
\end{aligned}$$

Let $l \subset X$ be a line in X and let $\mathscr{J}_l \subset \mathscr{O}_X$ be the ideal sheaf of l. The computations made in [24] show that $\mathscr{F}_l \in \mathscr{A}_X$ and that:

$$\mathrm{Hom}(\mathscr{F}_l, \mathscr{F}_l) = \mathbb{C}.$$

Since $\mathscr{F}_l \in \mathscr{A}_X$ and $\mathscr{A}_X$ is 2-Calabi–Yau category, we find $\mathrm{Ext}^2(\mathscr{F}_l\mathscr{F}_l) = \mathbb{C}$ and $\mathrm{Ext}^p(\mathscr{F}_l, \mathscr{F}_l) = 0$ for $p \geqslant 3$. Let $\mathfrak{T}^\bullet_{\mathscr{A}_X}$ be a homological unit for $\mathscr{A}_X$ with respect to the embedding in $\mathrm{D}^{\mathrm{b}}(X)$. As $\mathrm{rank}(\mathscr{F}_l) = 3$ and $\mathscr{F}_l \in \mathscr{A}_X$, we have embedding of graded algebras:

$$\mathfrak{T}^\bullet_{\mathscr{A}_X} \hookrightarrow \mathrm{Hom}^\bullet(\mathscr{F}_l, \mathscr{F}_l),$$

by definition of an homological unit for $\mathscr{A}_X$. We therefore find $\mathfrak{T}^0_{\mathscr{A}_X} \simeq \mathfrak{T}^2_{\mathscr{A}_X} \simeq \mathbb{C}$ and $\mathfrak{T}^p_{\mathscr{A}_X} = 0$ for $p \geqslant 3$ or $p < 0$. By item 2 of Remark 2.4 above, we have an embedding:

$$\mathfrak{T}^\bullet_{\mathscr{A}_X} \hookrightarrow \mathrm{HH}^\bullet(\mathscr{A}_X).$$

We have already observed that $\mathrm{HH}^1(\mathscr{A}_X) = 0$, which implies $\mathfrak{T}^1_{\mathscr{A}_X} = 0$. We conclude that if $\mathfrak{T}^\bullet(\mathscr{A}_X)$ is a homological unit for $\mathscr{A}_X$ with respect to the rank function coming from $\mathrm{D}^{\mathrm{b}}(X)$, then we necessarily have $\mathfrak{T}^\bullet_{\mathscr{A}_X} = \mathbb{C} \oplus \mathbb{C}[2]$.

We now prove that $\mathbb{C} \oplus \mathbb{C}[2]$ satisfies all the properties of a homological unit for $\mathscr{A}_X$ with respect to the rank function coming from $\mathrm{D}^{\mathrm{b}}(X)$, except perhaps the map $\mathbb{C} \oplus \mathbb{C}[2] \to \mathrm{Hom}^\bullet(\mathscr{E}, \mathscr{E})$ is not a ring morphism (and is only a graded vector space morphism). For any $\mathscr{E} \in \mathscr{A}_X$, any $a \in \mathbb{C}$ and any $f \in \mathrm{Hom}(\mathscr{E}, \mathscr{E})$, we put:

$$i^0_{\mathscr{E}}(a) = a.\mathrm{id}_{\mathscr{E}} \quad \text{and} \quad t^0_{\mathscr{E}}(f) = \mathrm{Trace}(f),$$

where Trace is the trace map inherited from $\mathrm{D}^{\mathrm{b}}(X)$. It is clear that $t^0_{\mathscr{E}}(i^0_{\mathscr{E}}(a)) = \mathrm{rank}(\mathscr{E}).a$, for any $a \in \mathbb{C}$. Let ω be a generator of $H^4(X, \omega_X)$. By the Hochschild–Kostant–Rosenberg isomorphism, we can see $\omega \in \mathrm{HH}_0(\mathrm{D}^{\mathrm{b}}(X))$. Let $\delta : \mathscr{A}_X \hookrightarrow \mathrm{D}^{\mathrm{b}}(X)$ be the admissible embedding of $\mathscr{A}_X$ in $\mathrm{D}^{\mathrm{b}}(X)$. It is clear that $\delta^!\omega \in \mathrm{HH}_0(\mathscr{A}_X)$ and that for any $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}(X)$:

$$\omega|_{\mathscr{F}} = \mathrm{id}_{\mathscr{F}} \otimes \omega : \mathscr{F} \longrightarrow \mathscr{F} \otimes \omega_X[4].$$

Furthermore, for any $\mathscr{E} \in \mathscr{A}_X$, we have:

$$\begin{aligned}
\mathrm{Hom}(\delta_*\mathscr{E}, \delta_* E \otimes \omega_X[4]) &= \mathrm{Hom}(\mathscr{E}, \delta^!(\delta_* E \otimes \omega_X[4])), \quad \text{by adjunction,}\\
&= \mathrm{Hom}(\mathscr{E}, \mathscr{E}[2]), \quad \text{because } \mathscr{A}_X \text{ is 2-Calabi–Yau.}
\end{aligned}$$

For any $\mathscr{E} \in \mathscr{A}_X$ and any $f \in \mathrm{Ext}^2(\mathscr{E},\mathscr{E})$, we then define $t^2_{\mathscr{E}}(f)$ as the trace of f seen as an element in $\mathrm{Hom}(\delta_*\mathscr{E}, \delta_*\mathscr{E} \otimes \omega_X[4])$.

The category $\mathscr{A}_X$ is 2-Calabi–Yau, Theorem 4.5 and Proposition 4.6 of [21] ensure an isomorphism $\mathrm{HH}_0(\mathscr{A}_X) \simeq \mathrm{HH}^2(\mathscr{A}_X)$. We can therefore see $\delta^!\omega$ as an element in $\mathrm{HH}^2(\mathscr{A}_X)$ and for any $\mathscr{E} \in \mathscr{A}_X$ and any $a \in \mathbb{C}$, we define:

$$i^2_{\mathscr{E}}(a) = a.(\delta^!\omega)|_{\mathscr{E}},$$

where $(\delta^!\omega)|_{\mathscr{E}}$ is the restriction to $\mathrm{Ext}^2(\mathscr{E},\mathscr{E})$ of $\delta^!\omega$. Since for any $\mathscr{E} \in \mathscr{A}_X$, we have:

$$(\delta^!\omega)|_{\mathscr{E}} = \delta^!(\omega|_{\mathscr{E}}),$$

we deduce that, for all $a \in \mathbb{C}$:

$$t^2_{\mathscr{E}}(i^2_{\mathscr{E}}(a)) = t^2_{\mathscr{E}}(a.\delta^!(\omega|_{\mathscr{E}})) = \mathrm{Tr}_{\mathrm{D^b}(X)}(a.\mathrm{id}_{\mathscr{E}} \otimes \omega) = a.\mathrm{rk}(\mathscr{E}). \qquad \square$$

Remark 2.6. —

(1) Since $\mathscr{A}_X$ share many features with the derived category of a $K3$ surface [22], we naturally expect that for any $\mathscr{E} \in \mathscr{A}_X$, the map $\mathbb{C} \oplus \mathbb{C}[2] \to \mathrm{Hom}^\bullet(\mathscr{E},\mathscr{E})$ is a ring morphism (and not only a graded vector space morphism).
(2) If it were true, then the homological unit of $\mathscr{A}_X$ (with respect to the rank function inherited from X) would be $\mathbb{C} \oplus \mathbb{C}[2]$. This would be in sharp contrast with the fact that $H^\bullet(\mathscr{O}_X) = \mathbb{C}$. Nevertheless, it would match with the general principle that $\mathscr{A}_X$ should be seen as a "non-commutative" $K3$ surface.

In case $\mathscr{A}$ is an admissible odd-dimensional Calabi–Yau subcategory of the derived category of a smooth projective variety and contains a spherical object whose rank is non-zero, the homological unit of $\mathscr{A}$ is easily computed. Let us recall that an object $\mathscr{E} \in \mathscr{A}$ is $(2p+1)$-spherical if we have a ring isomorphism:

$$\mathrm{Hom}^\bullet(\mathscr{E},\mathscr{E}) \simeq \mathbb{C} \oplus \mathbb{C}[2p+1].$$

Proposition 2.7. — *Let X be a smooth projective variety and $\mathscr{A} \subset \mathrm{D^b}(X)$ be an admissible subcategory. Assume that $\mathscr{A}$ is a $(2p+1)$-Calabi–Yau category and that it contains a $(2p+1)$-spherical object whose rank (as an $\mathscr{O}_X$-module) is non-zero. Then, the homological unit of $\mathscr{A}$ (with respect to the rank function coming from $\mathrm{D^b}(X)$) is $\mathbb{C} \oplus \mathbb{C}[2p+1]$.*

Proof. — Let $\mathfrak{T}^\bullet_{\mathscr{A}}$ be a homological unit for $\mathscr{A}$ with respect to the rank function coming from $\mathrm{D^b}(X)$. Let $\mathscr{E}$ be a $(2p+1)$-spherical object in $\mathscr{A}$ which rank is not zero. By definition of homological unit, we must have $\mathfrak{T}^\bullet_{\mathscr{A}} \hookrightarrow \mathrm{Hom}^\bullet(\mathscr{E},\mathscr{E}) = \mathbb{C} \oplus \mathbb{C}[2p+1]$.

We now prove that $\mathbb{C}\oplus\mathbb{C}[2p+1]$ is a homological unit for $\mathscr{A}$. Let ω be a generator of $H^{\dim X}(X,\omega_X)$. By the Hochschild–Kostant–Rosenberg isomorphism, we can see $\omega\in \mathrm{HH}_0(\mathrm{D^b}(X))$. Let $\delta:\mathscr{A}_X\hookrightarrow \mathrm{D^b}(X)$ be the admissible embedding of $\mathscr{A}_X$ in $\mathrm{D^b}(X)$. It is again clear $\delta^!\omega\in\mathrm{HH}_0(\mathscr{A}_X)$. Since A is $(2p+1)$-Calabi–Yau and $\mathrm{HH}_0(\mathscr{A}_X)\simeq \mathrm{HH}^{2p+1}(\mathscr{A}_X)$, we can therefore see $\delta^!\omega$ as an element in $\mathrm{HH}^{2p+1}(\mathscr{A}_X)$. Hence, for any $\mathscr{E}\in\mathscr{A}_X$ and any $a\in\mathbb{C}$, we define:

$$i_{\mathscr{E}}^{2p+1}(a)=a.(\delta^!\omega)|_{\mathscr{E}},$$

where $(\delta^!\omega)|_{\mathscr{E}}$ is the restriction to $\mathrm{Ext}^{2p+1}(\mathscr{E},\mathscr{E})$ of $\delta^!\omega$. The first steps of the proof that $\mathbb{C}\oplus\mathbb{C}[2p+1]$ is a homological unit for $\mathscr{A}_X$ are identical to that of Proposition 2.5. We only focus on the last defining feature of a homological unit and we prove that the map $i_{\mathscr{E}}^{\bullet}:\mathbb{C}\oplus\mathbb{C}[2p+1]\to\mathrm{Hom}^\bullet(\mathscr{E},\mathscr{E})$ is a ring morphism for any $\mathscr{E}\in\mathscr{A}$. This is equivalent to the vanishing of $i_{\mathscr{E}}^{2p+1}(a)\circ i_{\mathscr{E}}^{2p+1}(a)$, for any $\mathscr{E}\in\mathscr{A}$ and any $a\in\mathbb{C}$. For any such a and $\mathscr{E}$, we have:

$$\begin{aligned} i_{\mathscr{E}}^{2p+1}(a)\circ i_{\mathscr{E}}^{2p+1}(a) &= a^2\delta^!(\omega)|_{\mathscr{E}}\circ\delta^!(\omega)|_{\mathscr{E}},\ \text{by definition,}\\ &= a^2(\delta^!\omega\circ\delta^!\omega)|_{\mathscr{E}},\ \text{by functoriality.}\end{aligned}$$

But the algebra $\mathrm{HH}^\bullet(\mathscr{A})$ is graded commutative (see [35] for instance). Since $\delta^!\omega\in\mathrm{HH}^{2p+1}(\mathscr{A})$, we deduce that $\delta^!\omega\circ\delta^!\omega=0\in\mathrm{HH}^{4p+2}(\mathscr{A})$. As a consequence, we have $i_{\mathscr{E}}^{2p+1}(a)\circ i_{\mathscr{E}}^{2p+1}(a)=0$, for any $\mathscr{E}\in\mathscr{A}$ and any $a\in\mathbb{C}$. This finally demonstrates that the ring $\mathbb{C}\oplus\mathbb{C}[2p+1]$ is a homological unit for $\mathscr{A}$. ☐

Remark 2.8. — We stated our result only in the odd-dimensional case in order to benefit from the graded-commutativity of the algebra $\mathrm{HH}^\bullet(\mathscr{A})$ and therefore get a quick proof that the graded vector space morphism $\mathbb{C}\oplus\mathbb{C}[2p+1]\hookrightarrow\mathrm{Hom}^\bullet(\mathscr{E},\mathscr{E})$ is indeed a ring morphism. We of course expect that Proposition 2.7 should be true also in the even dimensional case.

Example 2.9. —

(1) Let $X\subset\mathbb{P}^8$ be a generic cubic hypersurface. It was checked in [12] that X is a linear section of the E_6 invariant cubic $\mathscr{C}_{\mathrm{E}_6}\subset\mathbb{P}(V_{27})$, where V_{27} is the minuscule 27-dimensional representation of E_6. Denote by $L_X\subset V_{27}$ the 9-dimensional vector space such that $\mathscr{C}\cap\mathbb{P}(L_X)=X$. As explained in [23], we have a semi-orthogonal decomposition:

$$\mathrm{D^b}(X)=\langle\mathscr{A}_X,\mathscr{O}_X,\mathscr{O}_X(1),\dots,\mathscr{O}_X(5)\rangle,$$

where $\mathscr{A}_X$ is a 3-Calabi–Yau category. We will prove below that the homological unit of $\mathscr{A}_X$ (with respect to the rank function coming from X) is $\mathbb{C}\oplus\mathbb{C}[3]$ by exhibiting a 3-spherical vector bundle in $\mathscr{A}_X$.

Recall that a Jordan algebra structure can be put on V_{27}. Namely let :

$$V_{27} = \left\{ \begin{pmatrix} \lambda_1 & \sigma_1 & \overline{\sigma_2} \\ \overline{\sigma_1} & \lambda_2 & \sigma_3 \\ \sigma_2 & \overline{\sigma_3} & \lambda_3 \end{pmatrix},\ \sigma_1, \sigma_2, \sigma_3 \in \mathbb{O} \text{ and } \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\},$$

where $\mathbb{O}$ is the algebra of rational octonions and $\overline{\sigma}$ is the conjugate of σ with respect to the octonionic conjugation. For any $A, B \in V_{27}$, we put $A \star B = \frac{AB+BA}{2}$. This is a commutative (but non-associative) product on V_{27}, which endows V_{27} with a structure of Jordan algebra.

The determinant of any element of V_{27} is well defined and we get a determinant map, say $\mathrm{Det} \in S^3 V_{27}^*$. The vanishing locus of Det in $\mathbb{P}(V_{27})$ is the E_6 invariant cubic. The Hessian matrix of Det gives a $27 * 27$ symmetric matrix with linear entries (say M) which is part of a matrix factorization of Det. As a consequence, we get an exact sequence:

$$0 \longrightarrow V_{27} \otimes \mathscr{O}_{\mathbb{P}(V_{27})}(-1) \xrightarrow{M} V_{27} \otimes \mathscr{O}_{\mathbb{P}(V_{27})} \longrightarrow i_*(\mathscr{E}) \longrightarrow 0,$$

where $i_*(\mathscr{E})$ is the push-forward of a rank-9 coherent sheaf on $\mathscr{C}_{\mathrm{E}_6}$. The jumping locus of $\mathscr{E}$ is the singular locus of $\mathscr{C}_{\mathrm{E}_6}$, that is the Cayley plane $\mathbb{OP}^2$ (which is a 16-dimensional projective variety homogeneous under E_6).

The intersection $X = \mathscr{C} \cap \mathbb{P}(L_X)$ being transverse, we can restrict the above exact sequence to $\mathbb{P}(L_X)$ and we get:

$$0 \longrightarrow V_{27} \otimes \mathscr{O}_{\mathbb{P}(L_X)}(-1) \xrightarrow{M|_{L_X}} V_{27} \otimes \mathscr{O}_{\mathbb{P}(L_X)} \longrightarrow i_*(\mathscr{E}_X) \longrightarrow 0, \tag{2.1}$$

where $i_*(\mathscr{E}_X)$ is the push-forward of a rank-9 vector bundle on $\mathscr{C} \cap \mathbb{P}(L_X) = X$. It is shown in [12] that $\mathscr{E}_X$ is a 3-spherical vector bundle on X. Proposition 2.7 then implies that the homological unit of $\mathscr{A}_X$ with respect to the rank function coming from X is $\mathbb{C} \oplus \mathbb{C}[3]$.

(2) Let $X \subset \mathbb{P}(1,1,1,1,1,1,2)$ be a generic double quartic fivefold. We checked in [3] that X is a linear section of the double cover of $\mathbb{P}(\Delta_+)$ ramified along the Spin_{12} invariant quartic $\mathscr{Q} \subset \mathbb{P}(\Delta_+)$ (here Δ_+ is one of the 32-dimensional half-spin representations for Spin_{12}). Denote by $L_X \subset \Delta_+$ the 6 dimensional vector space such that X is the double cover of $\mathbb{P}(L_X)$ along $\mathscr{Q} \cap \mathbb{P}(L_X)$. As explained in [23], we have a semi-orthogonal decomposition:

$$\mathrm{D^b}(X) = \langle \mathscr{A}_X, \mathscr{O}_X, \mathscr{O}_X(1), \mathscr{O}_X(2), \mathscr{O}_X(3) \rangle,$$

where $\mathscr{A}_X$ is a 3-Calabi–Yau category. We will prove below that the homological unit of $\mathscr{A}_X$ (with respect to the rank function coming from X) is $\mathbb{C} \oplus \mathbb{C}[3]$ by exhibiting a 3-spherical vector bundle in $\mathscr{A}_X$.

Let us fix a system of coordinates on Δ_+^*. Using the theory of exceptional quaternionic representations, we constructed in [3] a 12×12 matrix, say M, with quadratic entries in the variables of Δ such that $M \times M = P_{\mathscr{Q}}.\mathrm{I}_{12}$, where $P_{\mathscr{Q}}$ is an equation for $\mathscr{Q}$ in the chosen variables of Δ_+. As a consequence, we have an exact sequence:

$$0 \longrightarrow \mathbb{C}^{12} \otimes \mathscr{O}_{\mathbb{P}(\Delta_+)}(-2) \xrightarrow{M} \mathbb{C}^{12} \otimes \mathscr{O}_{\mathbb{P}(\Delta_+)} \longrightarrow i_*(\mathscr{F}) \longrightarrow 0,$$

where $i_*(\mathscr{F})$ is the push-forward of a rank-6 coherent sheaf on $\mathscr{Q}$. The jumping locus of $\mathscr{F}$ is the singular locus of $\mathscr{Q}$, that is the closure in $\mathbb{P}(\Delta_+)$ of a 24-dimensional quasi-projective variety homogeneous under Spin_{12}.

The intersection $X = \mathscr{Q} \cap \mathbb{P}(L_X)$ being transverse, we can restrict the above exact sequence to $\mathbb{P}(L_X)$ and we get:

$$0 \longrightarrow \mathbb{C}^{12} \otimes \mathscr{O}_{\mathbb{P}(L_X)}(-2) \xrightarrow{M} \mathbb{C}^{12} \otimes \mathscr{O}_{\mathbb{P}(L_X)} \longrightarrow i_*(\mathscr{F}_X) \longrightarrow 0, \qquad (2.2)$$

where $i_*(\mathscr{F}_X)$ is the push-forward of a rank-6 vector bundle on $\mathscr{Q} \cap \mathbb{P}(L_X) = X$. We showed in [3] that $\mathscr{F}_X$ is a 3-spherical vector bundle on X. Furthermore, if we consider the semi-orthogonal decomposition:

$$\mathrm{D^b}(X) = \langle \mathscr{A}_X, \mathscr{O}_X, \mathscr{O}_X(1), \ldots, \mathscr{O}_X(3) \rangle$$

described above, the exact sequence (2.2) shows that $\mathscr{F}_X(-1)$ and $\mathscr{F}_X(-2)$ are in $\mathscr{A}_X$. As they are 3-spherical vector bundles, Proposition 2.7 allows to conclude that the homological unit of $\mathscr{A}_X$ with respect to the rank function coming from X is $\mathbb{C} \oplus \mathbb{C}[3]$.

Remark 2.10. — Let X be a generic cubic fourfold an $\mathscr{A}_X$ the $K3$-category associated to X. It is shown in [11] that $\mathscr{A}_X$ does not contain any 2-spherical object. Proposition 2.5 nevertheless strongly suggests that the homological unit of $\mathscr{A}_X$ with respect to the rank function coming from $\mathrm{D^b}(X)$ is $\mathbb{C} \oplus \mathbb{C}[2]$. This highlights the fact that the homological unit is useful in order to capture the Hodge-theoretic properties of a Calabi–Yau category even in the absence of spherical objects.

There is an analogous statement to that of 2.7 holds for Calabi–Yau categories coming from quivers with potentials, namely we have:

Proposition 2.11. — *Let (Q, W) be a quiver without loops and W be a reduced potential for* Q. *Let $\mathscr{A}_{(\mathrm{Q},W)}$ be the* 3*-Calabi–Yau algebra associated to (Q, W). Let $D_{fd}(\mathscr{A}_{(\mathrm{Q},W)})$ be the derived category of finite dimensional DG $\mathscr{A}_{(\mathrm{Q},W)}$-modules. For $\mathscr{E} \in D_{fd}(\mathscr{A}_{(\mathrm{Q},W)})$, we let* $\mathrm{rk}(\mathscr{E})$ *be the rank of $\mathscr{E}$ as a representation of* Q. *Then, the homological unit of $D_{fd}(\mathscr{A}_{(\mathrm{Q},W)})$ with respect to* rk *is* $\mathbb{C} \oplus \mathbb{C}[3]$.

We refer to [7, 19] for the construction of $\mathscr{A}_{(Q,W)}$ and to [7, 18] for proofs that the category $D_{fd}(\mathscr{A}_{(Q,W)})$ is 3-Calabi–Yau.

Proof. — We denote by $\mathfrak{T}^{\bullet}_{D_{fd}(\mathscr{A}_{(Q,W)})}$ a homological unit of $\mathscr{A}_{(Q,W)}$ with respect to the rank. Let i be a vertex of Q and let S_i be the simple module corresponding to i. Its rank as a Q-module is 1. Since there are no loops at i, we know that $\mathrm{Hom}^{\bullet}(S_i, S_i) = \mathbb{C} \oplus \mathbb{C}[3]$ (see [19, Lemma 2.15], for instance). As $\mathrm{rank}(S_i) = 1 \neq 0$, we have an injection of graded algebras:

$$\mathfrak{T}^{\bullet}_{D_{fd}(\mathscr{A}_{(Q,W)})} \hookrightarrow \mathrm{Hom}^{\bullet}(S_i, S_i) = \mathbb{C} \oplus \mathbb{C}[3].$$

We now prove that $\mathbb{C}\oplus\mathbb{C}[3]$ is indeed the homological unit of $D_{fd}(\mathscr{A}_{(Q,W)})$ with respect to the rank function coming from representations of Q. By [29], for any $\mathscr{E} \in D_{fd}(A_{(Q,W)})$, we have a linear map (called the *bulk-boundary map* in [29]):

$$\Theta_{\mathscr{E}} : \mathrm{Hom}^{\bullet}(\mathscr{E}, \mathscr{E}) \longrightarrow \mathrm{HH}_{\bullet}(D_{fd}(\mathscr{A}_{(Q,W)})).$$

The vector $\Theta_{\mathscr{E}}(\mathrm{id}_{\mathscr{E}}) \in \mathrm{HH}_0(D_{fd}(\mathscr{A}_{(Q,W)}))$ is called the *Chern character* of $\mathscr{E}$. The set $\{S_i\}_{i\in Q}$ is a minimal system of generators of $D_{fd}(\mathscr{A}_{(Q,W)})$, from which we deduce that the set $\{\Theta_{S_i}(\mathrm{id}_{S_i})\}_{i\in Q}$ is a free family in $\mathrm{HH}_0(\mathscr{D}_{fd}(\mathscr{A}_{(Q,W)}))$.

We fill $\{\Theta_{S_i}(\mathrm{id}_{S_i})\}_{i\in Q}$ into a basis of $\mathrm{HH}_0(\mathscr{D}_{fd}(\mathscr{A}_{(Q,W)}))$ and we let $\Theta_{S_i}(\mathrm{id}_{S_i})^*$ be the corresponding vectors of the dual basis in $(\mathrm{HH}_0(\mathscr{D}_{fd}(\mathscr{A}_{(Q,W)})))^*$. For any $\mathscr{E} \in D_{fd}(\mathscr{A}_{(Q,W)})$, we let:

$$t^0_{\mathscr{E}} := \left(\bigoplus_{i\in Q} \Theta_{S_i}(\mathrm{id}_{S_i})^*\right) \circ \Theta_{\mathscr{E}} : \mathrm{Hom}(\mathscr{E}, \mathscr{E}) \longrightarrow \mathbb{C}.$$

By definition of the rank function in$D_{fd}(\mathscr{A}_{(Q,W)})$ we have $t^0_{\mathscr{E}}(\mathrm{id}_{\mathscr{E}}) = \mathrm{rank}(\mathscr{E})$, for any $\mathscr{E} \in D_{fd}(\mathscr{A}_{(Q,W)})$. For all $\mathscr{E} \in D_{fd}(\mathscr{A}_{(Q,W)})$, we then we define:

$$\begin{aligned} i^0_{\mathscr{E}}(a) : \mathbb{C} &\longrightarrow \mathrm{Hom}(\mathscr{E}, \mathscr{E}) \\ a &\longmapsto a.\mathrm{id}_{\mathscr{E}} \end{aligned}$$

It is obvious that $i_{\mathscr{E}}$ is functorial in $\mathscr{E}$ and that $t^0_{\mathscr{E}}(i^0_{\mathscr{E}})(a) = a.\mathrm{rank}(\mathscr{E})$, for any $\mathscr{E} \in D_{fd}(\mathscr{A}_{(Q,W)})$.

For any $i \in Q$, we denote by ω_i a generator of $\mathrm{Ext}^3(S_i, S_i)$ such that:

$$\mathscr{S}_{S_i}(\mathrm{id}_{S_i}, w_i) = 1,$$

where $\mathscr{S}_{S_i}$ is the perfect pairing between $\mathrm{Hom}(S_i, S_i)$ and $\mathrm{Ext}^3(S_i, S_i)$ provided by Serre duality. We let $\omega = \bigoplus_{i\in Q} \omega_i \in \mathrm{Hom}^{\bullet}(\bigoplus_{i\in Q} S_i, \bigoplus_{i\in Q} S_i)$. Since S_i is the simple module corresponding to $i \in Q$, the morphism ω

induces (functorially) a morphism $\omega_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}[3]$, for any finite dimensional representation $\mathscr{F}$ of Q. Hence, for any $\mathscr{E} \in D_{fd}(\mathscr{A}_{(Q,W)})$, we have (functorially) a morphism:

$$\omega_{\mathscr{E}} : \mathscr{E} \longrightarrow \mathscr{E}[3].$$

We then define:

$$\begin{aligned} i^3_{\mathscr{E}} : \mathbb{C} &\longrightarrow \operatorname{Ext}^3(\mathscr{E},\mathscr{E}) \\ a &\longrightarrow a.\omega_{\mathscr{E}}, \end{aligned}$$

and

$$\begin{aligned} t^3_{\mathscr{E}} : \operatorname{Ext}^3(\mathscr{E},\mathscr{E}) &\longrightarrow \mathbb{C} \\ f &\longrightarrow \mathscr{S}_{\mathscr{E}}(\mathrm{id}_{\mathscr{E}}, f), \end{aligned}$$

where $\mathscr{S}_{\mathscr{E}}$ is the perfect pairing between $\operatorname{Hom}(\mathscr{E},\mathscr{E})$ and $\operatorname{Ext}^3(\mathscr{E},\mathscr{E})$ provided by Serre duality.

By construction, we have $t^3_{S_i} \circ i^3_{S_i}(a) = a$, for any $a \in \mathbb{C}$ and any $i \in Q$. As a consequence, the definition of the rank function on $\mathscr{D}_{fd}(\mathscr{A}_{(Q,W)})$ implies that $t^3_{\mathscr{E}} \circ i^3_{\mathscr{E}}(a) = a.\mathrm{rank}(E)$, for any $a \in \mathbb{C}$ and any $\mathscr{E} \in \mathscr{D}_{fd}(\mathscr{A}_{(Q,W)})$.

We are left to prove that, for any $\mathscr{E} \in \mathscr{D}_{fd}(\mathscr{A}_{(Q,W)})$, the graded vector space morphism $i^{\bullet}_{\mathscr{E}} : \mathbb{C} \oplus \mathbb{C}[3] \to \operatorname{Hom}^{\bullet}(\mathscr{E},\mathscr{E})$ is a ring morphism. We only have to check that $i^p_{\mathscr{E}}(a) \circ i^3_{\mathscr{E}}(a) = 0$, for any $a \in \mathbb{C}$ and $\mathscr{E} \in \mathscr{D}_{fd}(\mathscr{A}_{(Q,W)})$. This is obvious as $\operatorname{Ext}^k(\bigoplus_{i\in Q} S_i, \bigoplus_{i \in Q} S_i) = 0$, for any $k \geqslant 4$. $\square$

2.2. Invariance of the homological unit

In this section, we will be interested in the following question:

Question 2.12. — *Let $\mathscr{A}$ be a smooth proper triangulated category and let rk_1, rk_2 be two non-trivial rank functions on $\mathscr{A}$. We denote by $\mathfrak{T}^{\bullet}_{\mathscr{A},1}$, $\mathfrak{T}^{\bullet}_{\mathscr{A},2}$ homological units of $\mathscr{A}$ with respect to rk_1 and rk_2. Is there always a ring isomorphism $\mathfrak{T}^{\bullet}_{\mathscr{A},1} \simeq \mathfrak{T}^{\bullet}_{\mathscr{A},2}$?*

In the geometric setting, a special case of the above question is the:

Question 2.13. — *Let X, Y be smooth projective varieties (over $\mathbb{C}$). Let $\mathscr{A}_X$ and $\mathscr{A}_Y$ be full admissible subcategories of $\mathrm{D^b}(X)$ and $\mathrm{D^b}(Y)$. Let $\Phi : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ be a Fourier–Mukai functor such that Φ induces an equivalence between $\mathscr{A}_X$ and $\mathscr{A}_Y$. Let $\mathfrak{T}^{\bullet}_{\mathscr{A}_X}$ and $\mathfrak{T}^{\bullet}_{\mathscr{A}_Y}$ be homological units of $\mathscr{A}_X$ and $\mathscr{A}_Y$ with respect to the rank function coming from $\mathrm{D^b}(X)$ and $\mathrm{D^b}(Y)$. Assume that $\mathscr{A}_X$ and $\mathscr{A}_Y$ both contain an object whose rank is not zero. Is there always a ring isomorphism $\mathfrak{T}^{\bullet}_{\mathscr{A}_X} \simeq \mathfrak{T}^{\bullet}_{\mathscr{A}_Y}$?*

When $\mathscr{A}_X = \mathrm{D^b}(X)$ and $\mathscr{A}_Y = \mathrm{D^b}(Y)$, we proved in [2] that there is a ring isomorphism $\mathfrak{T}^\bullet_{\mathscr{A}_X} \simeq \mathfrak{T}^\bullet_{\mathscr{A}_Y}$ provided that *one* of the following conditions holds:

- *The kernel giving the equivalence is generically a (possibly shifted) vector bundle on* $X \times Y$,
- *the Chern classes of the kernel giving the equivalence vanish in degree less than* $2\dim X - 1$,
- *the Hodge algebra* $\bigoplus_{p=0}^{\dim X} H^p(X, \Omega_X^p) \cap H^\bullet(X, \mathbb{Q})$ *and* $\bigoplus_{p=0}^{\dim X} H^p(Y, \Omega_Y^p) \cap H^\bullet(Y, \mathbb{Q})$ *are generated in degree* 1,
- *the varieties* X *and* Y *have dimension less or equal to* 4.

In case $\mathscr{A}_X = \mathrm{D^b}(X)$ and $\mathscr{A}_Y = \mathrm{D^b}(Y)$, we expect that there is always a ring isomorphism $\mathfrak{T}^\bullet_{\mathscr{A}_X} \simeq \mathfrak{T}^\bullet_{\mathscr{A}_Y}$. Below, we prove such invariance result for the homological unit of any triangulated category provided that this category contains enough unitary objects with respect to the homological unit under study. We will provide many examples where this result can be applied.

Theorem 2.14. —

(1) *Let* $\mathscr{A}$ *be a smooth proper triangulated category and let* rk_1, rk_2 *be two non-trivial rank functions on* $\mathscr{A}$. *We denote by* $\mathfrak{T}^\bullet_{\mathscr{A},1}$, $\mathfrak{T}^\bullet_{\mathscr{A},2}$ *the homological units associated to* rk_1 *and* rk_2. *Let*

$$\mathrm{cl} : \mathscr{A} \longrightarrow \mathrm{K}_0(\mathscr{A})$$

be the class map and let $\mathscr{A}^{(1)}_{\mathrm{unitary}}$ *the subset of* $\mathscr{A}$ *consisting of unitary objects with respect to* $\mathfrak{T}^\bullet_{\mathscr{A},1}$. *Assume that* $\mathrm{cl}(\mathscr{A}^{(1)}_{\mathrm{unitary}})$ *generates* $\mathrm{K}_0(\mathscr{A}) \otimes \mathbb{C}$. *Then, there is a injection of graded rings:*

$$\mathfrak{T}^\bullet_{\mathscr{A},2} \hookrightarrow \mathfrak{T}^\bullet_{\mathscr{A},1}.$$

(2) *The same conclusion as above holds if we only assume that* $\mathrm{cl}(\mathscr{A}^{(1)}_{\mathrm{unitary}})$ *generates* $\mathrm{K}_{\mathrm{num}}(\mathscr{A}) \otimes \mathbb{C}$ *provided that* rk_1 *and* rk_2 *are numerical rank functions on* $\mathscr{A}$.

Proof. — By definition of rank functions, both rk_1 and rk_2 can be lifted to rank functions:

$$\mathrm{rk}_i \otimes \mathbb{C} \ : \mathrm{K}_0(\mathscr{A}) \otimes \mathbb{C} \longrightarrow \mathbb{C}.$$

The function rk_2 is non trivial, so the same holds for $\mathrm{rk}_2 \otimes \mathbb{C}$. Furthermore, we know by hypothesis that $\mathrm{cl}(\mathscr{A}^{(1)}_{\mathrm{unitary}})$ generates $\mathrm{K}_0(\mathscr{A}) \otimes \mathbb{C}$. Hence, there exists $\mathscr{F} \in \mathscr{A}^{(1)}_{\mathrm{unitary}}$ such that $\mathrm{rk}_2 \otimes \mathbb{C}(\mathscr{F}) \neq 0$, that is $\mathrm{rk}_2(\mathscr{F}) \neq 0$. By definition of homological units, we have a graded ring embedding:

$$\mathfrak{T}^\bullet_{\mathscr{A},2} \hookrightarrow \mathrm{Hom}^\bullet(\mathscr{F}, \mathscr{F}).$$

Since $\mathscr{F} \in \mathscr{A}^{(1)}_{\mathrm{unitary}}$, this turns into a graded ring embedding:

$$\mathfrak{T}^{\bullet}_{\mathscr{A},2} \hookrightarrow \mathfrak{T}^{\bullet}_{\mathscr{A},1}.$$

The proof in the numerical case is exactly the same. $\square$

As a consequence of the above result, we get an effective criterion to determine whether the homological units related to different rank functions on a given triangulated category are isomorphic:

Corollary 2.15. —

(1) *Let $\mathscr{A}$ be a smooth proper triangulated category and let* rk_1, rk_2 *be two non-trivial rank functions on $\mathscr{A}$. We denote by $\mathfrak{T}^{\bullet}_{\mathscr{A},1}$, $\mathfrak{T}^{\bullet}_{\mathscr{A},2}$ the homological units associated to* rk_1 *and* rk_2. *Assume that both* $\mathrm{cl}(\mathscr{A}^{(1)}_{\mathrm{unitary}})$ *and* $\mathrm{cl}(\mathscr{A}^{(2)}_{\mathrm{unitary}})$ *generate* $\mathrm{K}_0(\mathscr{A}) \otimes \mathbb{C}$. *Then, there is an isomorphism of graded rings:*

$$\mathfrak{T}^{\bullet}_{\mathscr{A},2} \simeq \mathfrak{T}^{\bullet}_{\mathscr{A},1}.$$

(2) *The same conclusion as above holds if we only assume that* $\mathrm{cl}(\mathscr{A}^{(1)}_{\mathrm{unitary}})$ *and* $(\mathscr{A}^{(2)}_{\mathrm{unitary}})$ *generate* $\mathrm{K}_{\mathrm{num}}(\mathscr{A}) \otimes \mathbb{C}$ *provided that* rk_1 *and* rk_2 *are numerical rank functions on $\mathscr{A}$.*

Theorem 2.14 happens to be equally useful when the given category is $(2p+1)$-Calabi–Yau with $p \geqslant 0$ and the homological unit related to a rank function is $\mathbb{C} \oplus \mathbb{C}[2p+1]$. If the classes of unitary objects related to this homological unit generate $\mathrm{K}_{0/num}(\mathscr{A}) \otimes \mathbb{C}$, then the homological unit associated to any other rank function on $\mathscr{A}$ is necessarily $\mathbb{C} \oplus \mathbb{C}[2p+1]$.

Corollary 2.16. —

(1) *Let $\mathscr{A}$ be a triangulated category which is a $(2p+1)$-Calabi–Yau category with $p \geqslant 0$. Let* rk_1 *be a non-trivial rank function on $\mathscr{A}$ such that $\mathbb{C} \oplus \mathbb{C}[2p+1]$ is a homological unit for $\mathscr{A}$ with respect to* rk_1. *Assume that* $\mathrm{cl}(\mathscr{A}^{(1)}_{\mathrm{unitary}})$ *generates* $\mathrm{K}_0(\mathscr{A}) \otimes \mathbb{C}$. *Let* rk_2 *be another non-trivial rank function on $\mathscr{A}$ and let $\mathfrak{T}^{\bullet}_{\mathscr{A},2}$ be a homological unit on $\mathscr{A}$ with respect to* rk_2. *Assume that there exists $\mathscr{E} \in \mathscr{A}$ with* $\mathrm{rk}_2(\mathscr{E}) \neq 0$ *and which is unitary with respect to $\mathfrak{T}^{\bullet}_{\mathscr{A},2}$. Then we have a graded ring isomorphism:*

$$\mathfrak{T}^{\bullet}_{\mathscr{A},2} \simeq \mathbb{C} \oplus \mathbb{C}[2p+1].$$

(2) *The same conclusion as above holds if we only assume that* $\mathrm{cl}(\mathscr{A}^{(1)}_{\mathrm{unitary}})$ *generates* $\mathrm{K}_{\mathrm{num}}(\mathscr{A}) \otimes \mathbb{C}$ *provided that* rk_1 *and* rk_2 *are numerical rank functions on $\mathscr{A}$.*

Proof. — By Theorem 2.14, we have a graded injection:

$$\mathfrak{T}^{\bullet}_{\mathscr{A},2} \hookrightarrow \mathbb{C} \oplus \mathbb{C}[2p+1].$$

Let $\mathscr{E} \in \mathscr{A}$ be a unitary object with respect to $\mathfrak{T}^{\bullet}_{\mathscr{A},2}$ such that $\mathrm{rk}_2(\mathscr{E}) \neq 0$. Since $\mathscr{A}$ is a $(2p+1)$-Calabi–Yau category, we have a graded injection (see for instance the proof of Proposition 2.7):

$$\mathbb{C} \oplus \mathbb{C}[2p+1] \hookrightarrow \mathrm{Hom}^{\bullet}(\mathscr{E}, \mathscr{E}) \simeq \mathfrak{T}^{\bullet}_{\mathscr{A},2}.$$

We conclude that there is a graded ring isomorphism : $\mathfrak{T}^{\bullet}_{\mathscr{A},2} \simeq \mathbb{C} \oplus \mathbb{C}[2p+1]$. □

Example 2.17. — There are numerous situations where Corollaries 2.15 and 2.16 apply:

(1) Let X be a smooth projective threefold over $\mathbb{C}$. All line bundles on X are unitary objects. Because numerical and homological equivalence coincide on X, the cohomological Chern character gives an injection:

$$\mathrm{Ch}_X(.) \otimes \mathbb{C} : \mathrm{K}_{\mathrm{num}}(X) \otimes \mathbb{C} \hookrightarrow \bigoplus_{p \geqslant 0} H^p(X, \Omega_X^p) \cap H^{\bullet}(X, \mathbb{C}).$$

Furthermore, the Lefschetz $1-1$ Theorem and the Hard Lefschetz Theorem easily imply that the Chern characters of line bundles generate $\bigoplus_{p\geqslant 0} H^p(X, \Omega_X^p) \cap H^{\bullet}(X, \mathbb{C})$. We deduce that $\mathrm{cl}(\mathrm{D^b}(X)_{\mathrm{unitary}})$ generates $K_{\mathrm{num}}(X) \otimes \mathbb{C}$. In particular, if X is a strict Calabi–Yau variety (that is $K_X \simeq \mathscr{O}_X$ and $H^{\bullet}(\mathscr{O}_X) = \mathbb{C} \oplus \mathbb{C}[3]$), then Corollary 2.16 shows that the homological unit associated to any other non-trivial numerical rank function on $\mathrm{D^b}(X)$ having a unitary object whose rank is non zero is necessarily $\mathbb{C} \oplus \mathbb{C}[3]$.

(2) Let $X \subset \mathbb{P}^8$ be a generic cubic hypersurface. Consider the semi-orthogonal decomposition:

$$\mathrm{D^b}(X) = \langle \mathscr{A}_X, \mathscr{O}_X, \mathscr{O}_X(1), \ldots, \mathscr{O}_X(5) \rangle,$$

where $\mathscr{A}_X$ is a 3-Calabi–Yau category. As explained in Example 2.9, there exists a rank 9 vector bundle $\mathscr{E}_X$ on X which is 3-spherical and such that $\mathscr{E}_X(-1)$ and $\mathscr{E}_X(-2)$ are in $\mathscr{A}_X$. From this, we deduced that the homological unit of $\mathscr{A}_X$ with respect to the rank function coming from $\mathrm{D^b}(X)$ is $\mathbb{C} \oplus \mathbb{C}[3]$ (see Proposition 2.7).

Let us note that $\mathrm{K}_{\mathrm{num}}(\mathscr{A}_X) \otimes \mathbb{C} = \mathbb{C}^2$. Indeed, the Grothendieck–Riemann–Roch Theorem implies that the Chern character:

$$\mathrm{ch} : \mathrm{K}_0(\mathscr{A}_X) \otimes \mathbb{C} \longrightarrow \mathrm{HH}_0(\mathscr{A}_X),$$

descends to an injective map:

$$\mathrm{ch} : \mathrm{K}_{\mathrm{num}}(\mathscr{A}_X) \otimes \mathbb{C} \longrightarrow \left(\mathrm{HH}_0(\mathscr{A}_X)\right)_{\mathrm{num}},$$

where $(\mathrm{HH}_0(\mathscr{A}_X))_{\mathrm{num}}$ is the quotient of $\mathrm{HH}_0(\mathscr{A}_X)$ by the numerically trivial classes. Griffiths computations of the Hodge numbers of a smooth hypersurface [9], the Hochschild–Kostant–Rosenberg isomorphism for Hochschild homology and Corollary 9.2 of [21] show that $(\mathrm{HH}_0(\mathscr{A}_X))_{\mathrm{num}} \otimes \mathbb{C} = \mathbb{C}^2$. Since $\mathrm{ch}(\mathscr{E}_X(-1))$ and $\mathrm{ch}(\mathscr{E}_X(-2))$ are linearly independent in $(\mathrm{HH}_0(\mathscr{A}_X))_{\mathrm{num}}$, we deduce that the classes of $\mathscr{E}_X(-1)$ and $\mathscr{E}_X(-2)$ generate $\mathrm{K}_{\mathrm{num}}(\mathscr{A}_X) \otimes \mathbb{C}$. In particular, $\mathrm{cl}((\mathscr{A}_X)_{\mathrm{unitary}})$ generate $\mathrm{K}_{\mathrm{num}}(\mathscr{A}_X) \otimes \mathbb{C}$. Corollary 2.16 then shows that the homological unit associated to any other non-trivial rank function on $\mathscr{A}_X$ having a unitary object whose rank is non zero is necessarily $\mathbb{C} \oplus \mathbb{C}[3]$.

(3) Let $X \subset \mathbb{P}(1,1,1,1,1,1,2)$ be a generic double quartic fivefold. Consider the semi-orthogonal decomposition:

$$\mathrm{D^b}(X) = \langle \mathscr{A}_X, \mathscr{O}_X, \mathscr{O}_X(1), \mathscr{O}_X(2), \mathscr{O}_X(3) \rangle,$$

where $\mathscr{A}_X$ is a 3-Calabi–Yau category. As explained in Example 2.9, there exists a rank 6 vector bundle $\mathscr{F}_X$ on X which is 3-spherical and such that $\mathscr{F}_X(-1)$ and $\mathscr{F}_X(-2)$ are in $\mathscr{A}_X$. As a consequence, the homological unit of $\mathscr{A}_X$ with respect to the rank function coming from $\mathrm{D^b}(X)$ is $\mathbb{C} \oplus \mathbb{C}[3]$ (see Proposition 2.7).

The same computation as in the previous example (with [34] instead of [9] for the Hodge numbers of a smooth hypersurface in a weighted projective space) show that $\mathrm{K}_{\mathrm{num}}(\mathscr{A}_X) \otimes \mathbb{C} = \mathbb{C}^2$. Hence, the classes of $\mathscr{F}_X(-1)$ and $\mathscr{F}_X(-2)$ generate $\mathrm{K}_{\mathrm{num}}(\mathscr{A}_X) \otimes \mathbb{C}$. In particular, $\mathrm{cl}((\mathscr{A}_X)_{\mathrm{unitary}})$ generate $\mathrm{K}_{\mathrm{num}}(\mathscr{A}_X) \otimes \mathbb{C}$. Corollary 2.16 then guarantees that the homological unit associated to any other non-trivial rank function on $\mathscr{A}_X$ having a unitary object whose rank is non zero is necessarily $\mathbb{C} \oplus \mathbb{C}[3]$.

(4) Let (Q, W) be a quiver without loops and W be a reduced potential for Q. Let $\mathscr{A}_{(\mathrm{Q},W)}$ be the 3-Calabi–Yau algebra associated to (Q, W). Let $D_{fd}(\mathscr{A}_{(\mathrm{Q},W)})$ be the derived category of finite dimensional DG $\mathscr{A}_{(\mathrm{Q},W)}$-modules. For $\mathscr{E} \in D_{fd}(\mathscr{A}_{(\mathrm{Q},W)})$, we let $\mathrm{rk}(\mathscr{E})$ be the rank of $\mathscr{E}$ as a representation of Q. As proved in Proposition 2.11, the homological unit of $D_{fd}(\mathscr{A}_{(\mathrm{Q},W)})$ with respect to rk is $\mathbb{C} \oplus \mathbb{C}[3]$.

Let i be a vertex of Q and let S_i be the simple module corresponding to i. We recalled in the proof of Proposition 2.11 that the $\{S_i\}_{i \in Q}$ are 3-spherical objects which generate $D_{fd}(\mathscr{A}_{(\mathrm{Q},W)})$. Hence, $\mathrm{cl}\left((D_{fd}(\mathscr{A}_{(\mathrm{Q},W)})_{\mathrm{unitary}}\right)$ generate $\mathrm{K}_0(\mathscr{A}_{(\mathrm{Q},W)}) \otimes \mathbb{C}$. Corollary 2.16 then shows that the homological unit associated to any other non-trivial rank function on $D_{fd}(\mathscr{A}_{(\mathrm{Q},W)})$ having a unitary object whose rank is non zero is necessarily $\mathbb{C} \oplus \mathbb{C}[3]$.

3. A Hodge structure on the Hochschild homology of some 3-Calabi–Yau categories

3.1. Hodge numbers for 3-Calabi–Yau categories

In this section, using the homological unit, we define Hodge numbers for 3-Calabi–Yau categories and we provide examples of computations of such numbers. In the following, we shall use the following terminology for a smooth proper triangulated category endowed with a non trivial rank function and a homological unit associated to this rank function:

Definition 3.1. — *A* TC/HU *triple is a triple* $(\mathscr{A}, \mathrm{rk}, \mathfrak{T}^{\bullet}_{\mathscr{A}})$ *where* $\mathscr{A}$ *is a smooth proper triangulated category,* rk *is a non-trivial rank function on* $\mathscr{A}$ *and* $\mathfrak{T}^{\bullet}_{\mathscr{A}}$ *is a homological unit for* $\mathscr{A}$ *with respect to* rk.

Definition 3.2. — *Let* $(\mathscr{A}, \mathrm{rk}, \mathfrak{T}^{\bullet}_{\mathscr{A}})$ *be a TC/HU triple, with* A *a* 3-*Calabi–Yau category. We define the* Hodge numbers *of* $\mathscr{A}$ *as:*

(1) *for all* $i \in [0, \dots, 3]$, $h^{i,0}(\mathscr{A}) = \mathfrak{T}^{3-i}_{\mathscr{A}}$,
(2) $h^{3,1}(\mathscr{A}) = \dim \mathrm{HH}_{-2}(\mathscr{A}) - h^{2,0}(\mathscr{A})$,
(3) $h^{3,2}(\mathscr{A}) = h^{1,0}(\mathscr{A})$ *and* $h^{2,1}(\mathscr{A}) = \dim \mathrm{HH}_{-1}(\mathscr{A}) - h^{1,0}(\mathscr{A}) - h^{3,2}(\mathscr{A})$,
(4) $h^{3,3}(\mathscr{A}) = h^{0,0}(\mathscr{A})$ *and*
$$h^{1,1}(\mathscr{A}) = h^{2,2}(\mathscr{A}) = \frac{\dim \mathrm{HH}_0(\mathscr{A}) - h^{0,0}(\mathscr{A}) - h^{3,3}(\mathscr{A})}{2}.$$
(5) $h^{p,q}(\mathscr{A}) = h^{q,p}(\mathscr{A})$ *for any* $p, q \in [0, \dots, 3]$.

In case $\mathscr{A} = \mathrm{D^b}(X)$, where X is a smooth complex projective variety of dimension 3 with $K_X \simeq \mathscr{O}_X$, one immediately checks (with the Hochschild–Kostant–Rosenberg isomorphism) that the Hodge numbers defined above match with the "classical" Hodge numbers. In the general case, the equality $h^{3,1}(\mathscr{A}) = h^{2,0}(\mathscr{A})$, which is certainly expected for 3-Calabi–Yau categories, is not obvious at all.[1] As a matter of fact, it isn't even clear that all these Hodge numbers are positive (and integral as far as $h^{1,1}$ and $h^{2,2}$ are concerned). We provide an easy criterion to check that all these numbers (except possibly $h^{2,1}$) are non-negative integers.

Proposition 3.3. — *Let* $(\mathscr{A}, \mathrm{rk}, \mathfrak{T}^{\bullet}_{\mathscr{A}})$ *be a TC/HU triple, where* $\mathscr{A}$ *is a* 3-*Calabi–Yau category which is the derived category of perfect DG-modules over a DG-algebra* $\mathscr{C}$. *Assume that* $\mathscr{A}$ *is connected (that is* $\mathrm{HH}_{-3}(\mathscr{A}) = \mathbb{C}$*) and that there exists a unitary object with respect to* $\mathfrak{T}^{\bullet}_{\mathscr{A}}$ *in* $\mathscr{A}$. *Then all* $h^{p,q}(\mathscr{A})$ *(except possibly* $h^{2,1}(\mathscr{A}) = h^{1,2}(\mathscr{A})$*) are non-negative integers.*

[1] Note however that $\mathrm{HH}_{-2}(\mathscr{A}) = 0$ in many cases of interest, so that we automatically get $h^{3,1}(\mathscr{A}) = h^{2,0}(\mathscr{A}) = 0$.

Proof. — We first notice that for all $i \in [0, \ldots, 3]$, the numbers $h^{i,0}(\mathscr{A})$ are non-negative integers by definition. We also remark that $h^{3,0}(\mathscr{A}) = h^{0,0}(\mathscr{A}) \neq 0$ and that $h^{1,0}(\mathscr{A}) = h^{2,0}(\mathscr{A})$. Indeed, we know by hypothesis that there exists an unitary object in $\mathscr{A}$ and Serre duality applies to its endomorphism algebra.

The definition of homological unit implies that there is a graded ring embedding:

$$\mathfrak{T}^{\bullet}_{\mathscr{A}} \hookrightarrow \mathrm{HH}^{\bullet}(\mathscr{A}).$$

Since $\mathscr{A}$ is a 3-Calabi–Yau category, there is an isomorphism of graded vector spaces $\mathrm{HH}^{\bullet}(\mathscr{A}) \simeq \mathrm{HH}_{\bullet-3}(\mathscr{A})$. We deduce that there is an embedding of graded vector spaces:

$$\mathfrak{T}^{\bullet}_{\mathscr{A}} \hookrightarrow \mathrm{HH}_{\bullet-3}(\mathscr{A}).$$

As a consequence, the number $h^{3,1}(\mathscr{A})$ is a non-negative integer. As noticed earlier,we have $\mathfrak{T}^{3}_{\mathscr{A}} \simeq \left(\mathfrak{T}^{0}_{\mathscr{A}}\right)^{*} \neq 0$. Moreover, we have $\mathrm{HH}_{-3}(\mathscr{A}) = \mathbb{C}$ by hypothesis (connectivity of $\mathscr{A}$). We find that $\mathbb{C} \simeq \mathrm{HH}_{-3}(\mathscr{A}) \simeq \mathfrak{T}^{3}_{\mathscr{A}}(\mathscr{A}) \simeq \mathfrak{T}^{0}_{\mathscr{A}}$. In particular $h^{3,0}(\mathscr{A}) = h^{0,0}(\mathscr{A}) = 1$.

As mentioned in the statement of Proposition 3.3, we do not know if $h^{2,1}(\mathscr{A})$ is always a non-negative integer. We move on to prove that $h^{1,1}(\mathscr{A}) = h^{2,2}(\mathscr{A})$ are non-negative integers. Since $\mathscr{A}$ is the derived category of perfect DG-modules over the DG-algebra $\mathscr{C}$, we have an identification:

$$\mathrm{HH}^{\bullet}(\mathscr{A}) = \mathrm{Hom}^{\bullet}_{D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C})}(\Delta_{\mathscr{C}}, \Delta_{\mathscr{C}}),$$

where $\Delta_{\mathscr{C}}$ is the diagonal bimodule over $\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C}$. Let us check that $D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C})$ is a 6-Calabi–Yau categories. For any $(\mathscr{F}_1, \mathscr{F}_2, \mathscr{G}_1, \mathscr{G}_2) \in (D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}}))^2 \times (D^{\mathrm{perf}}(\mathscr{C}))^2$, we have functorial isomorphisms:

$$\begin{aligned}&\mathrm{Hom}^{\bullet}_{D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C})}(\mathscr{F}_1 \otimes \mathscr{F}_2, \mathscr{G}_1 \otimes \mathscr{G}_2[6]) \\ &\qquad \simeq \mathrm{Hom}^{\bullet}_{D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}})}(\mathscr{F}_1, \mathscr{G}_1[3]) \otimes^{\bullet} \mathrm{Hom}^{\bullet}_{D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}})}(\mathscr{F}_2, \mathscr{G}_2[3]),\end{aligned}$$

where $\otimes^{\bullet}$ is the graded tensor product (see [17, Sections 4 and 6]). Since $D^{\mathrm{perf}}(\mathscr{C})$ and $D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}})$ are 3-Calabi–Yau categories, we have a functorial isomorphism:

$$\begin{aligned}&\mathrm{Hom}^{\bullet}_{D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}})}(\mathscr{F}_1, \mathscr{G}_1[3]) \otimes^{\bullet} \mathrm{Hom}^{\bullet}_{D^{\mathrm{perf}}}(\mathscr{C}^{\mathrm{op}})(\mathscr{F}_2, \mathscr{G}_2[3]) \\ &\qquad \simeq \mathrm{Hom}^{\bullet}_{D^{\mathrm{perf}}}(\mathscr{C}^{\mathrm{op}})(\mathscr{G}_1, \mathscr{F}_1)^{*} \otimes^{\bullet} \mathrm{Hom}^{\bullet}_{D^{\mathrm{perf}}}(\mathscr{C}^{\mathrm{op}})(\mathscr{G}_2, \mathscr{F}_2)^{*}.\end{aligned}$$

Hence, any $(\mathscr{F}_1, \mathscr{F}_2, \mathscr{G}_1, \mathscr{G}_2) \in (D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}}))^2 \times (D^{\mathrm{perf}}(\mathscr{C}))^2$, we have functorial isomorphisms:

$$\begin{aligned}&\mathrm{Hom}^{\bullet}_{D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C})}(\mathscr{F}_1 \otimes \mathscr{F}_2, \mathscr{G}_1 \otimes \mathscr{G}_2[6]) \\ &\qquad\qquad \simeq \mathrm{Hom}^{\bullet}_{D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C})}(\mathscr{G}_1 \otimes \mathscr{G}_2, \mathscr{F}_1 \otimes \mathscr{F}_2)^{*},\end{aligned}$$

which proves that $D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C})$ is 6-Calabi–Yau. Serre duality then provides a graded-commutative perfect pairing (given by composition of morphisms followed by a trace map):

$$S^{\bullet}_{(\mathscr{C}^{\mathrm{op}}\otimes\mathscr{C})} : \mathrm{Hom}^{\bullet}_{D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}}\otimes\mathscr{C})}(\Delta_{\mathscr{C}}, \Delta_{\mathscr{C}}) \times \mathrm{Hom}^{6-\bullet}_{D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}}\otimes\mathscr{C})}(\Delta_{\mathscr{C}}, \Delta_{\mathscr{C}}) \longrightarrow \mathbb{C}.$$

Specializing to the case $\bullet = 3$, we find that $\mathrm{Hom}^{3}_{D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}}\otimes\mathscr{C})}(\Delta_{\mathscr{C}}, \Delta_{\mathscr{C}}) = \mathrm{HH}^3(\mathscr{A})$ is a symplectic vector space, in particular its dimension is even. We deduce that $\dim \mathrm{HH}_0(\mathscr{A})$ is even. As $\mathbb{C} \simeq \mathfrak{T}^3_{\mathscr{A}}(\mathscr{A})$ embeds in $\mathrm{HH}_0(\mathscr{A})$, the dimension of $\mathrm{HH}_0(\mathscr{A})$ must necessarily be an even integer strictly positive. Since $h^{0,0}(\mathscr{A}) = h^{3,3}(\mathscr{A}) = 1$, we have proved that the numbers $h^{1,1}(\mathscr{A})$ and $h^{2,2}(\mathscr{A})$ defined by:

$$h^{1,1}(\mathscr{A}) = h^{2,2}(\mathscr{A}) = \frac{\dim \mathrm{HH}_0(\mathscr{A}) - h^{0,0}(\mathscr{A}) - h^{3,3}(\mathscr{A})}{2}$$

are integral and non-negative. □

Using the results of the previous sections, one can prove the invariance (with respect to the rank function) of the afore-mentioned Hodge numbers in many situations. Namely we have the:

Theorem 3.4. —

(1) *Let $(\mathscr{A}, \mathrm{rk}_1, \mathfrak{T}^{\bullet}_{\mathscr{A},1})$ and $(\mathscr{A}, \mathrm{rk}_2, \mathfrak{T}^{\bullet}_{\mathscr{A},2})$ be two TC/HU triples based on the same smooth proper 3-Calabi–Yau category A. Let $\mathrm{cl} : \mathscr{A} \to \mathrm{K}_0(\mathscr{A})$ be the class map and denote by $\mathscr{A}^{(i)}_{\mathrm{unitary}}$ the set of objects in $\mathscr{A}$ which are unitary with respect to $\mathfrak{T}^{\bullet}_{\mathscr{A},i}$. Finally, for all $p, q \in [0, \ldots, 3]$, we denote by $h^{p,q}_i(\mathscr{A})$ the Hodge numbers of $\mathscr{A}$ associated to $\mathfrak{T}^{\bullet}_{\mathscr{A},i}$ as in Definition 3.2.*

  (a) *If both $\mathrm{cl}(\mathscr{A}^{(1)}_{\mathrm{unitary}})$ and $\mathrm{cl}(\mathscr{A}^{(2)}_{\mathrm{unitary}})$ generate $\mathrm{K}_0(\mathscr{A}) \otimes \mathbb{C}$, then we have:*

$$h^{p,q}_1(\mathscr{A}) = h^{p,q}_2(\mathscr{A}),$$

  for all $p, q \in [0, \ldots, 3]$.

  (b) *If $\mathfrak{T}^{\bullet}_{\mathscr{A},1} = \mathbb{C} \oplus \mathbb{C}[3]$, $\mathrm{cl}(\mathscr{A}^{(1)}_{\mathrm{unitary}})$ generates $\mathrm{K}_0(\mathscr{A}) \otimes \mathbb{C}$ and there exists a unitary object in $\mathscr{A}$ with respect to $\mathfrak{T}^{\bullet}_{\mathscr{A},2}$ whose* rank_2 *is not zero, then we have:*

$$h^{p,q}_1(\mathscr{A}) = h^{p,q}_2(\mathscr{A}),$$

  for all $p, q \in [0, \ldots, 3]$.

(2) *The same conclusions as above hold if we only assume that $\mathrm{cl}(\mathscr{A}^{(1)}_{\mathrm{unitary}})$ and $\mathrm{cl}(\mathscr{A}^{(2)}_{\mathrm{unitary}})$ generate $\mathrm{K}_{\mathrm{num}}(\mathscr{A}) \otimes \mathbb{C}$ (resp. $\mathrm{cl}(\mathscr{A}^{(1)}_{\mathrm{unitary}})$ generates $\mathrm{K}_{\mathrm{num}}(\mathscr{A}) \otimes \mathbb{C}$) provided that rk_1 and rk_2 are numerical rank functions on $\mathscr{A}$.*

Proof. — In light of the definition of Hodge numbers given in 1.1, the statements above are easy consequences of Corollaries 2.15 and 2.16. □

Example 3.5. —

(1) Let $X \subset \mathbb{P}^8$ be a generic cubic hypersurface. Consider the semi-orthogonal decomposition:

$$\mathrm{D^b}(X) = \langle \mathscr{A}_X, \mathscr{O}_X, \mathscr{O}_X(1), \dots, \mathscr{O}_X(5) \rangle,$$

where $\mathscr{A}_X$ is a 3-Calabi–Yau category. As explained in Example 2.17, the homological unit of $\mathscr{A}_X$ with respect to the rank function coming from $\mathrm{D^b}(X)$ is $\mathbb{C} \oplus \mathbb{C}[3]$. The Hochschild homology numbers for X are (see [13], Section 3):

- $\mathrm{hh}_0(X) = 8$,
- $\mathrm{hh}_1(X) = \mathrm{hh}_{-1}(X) = 84$,
- $\mathrm{hh}_2(X) = \mathrm{hh}_{-2}(X) = 0$,
- $\mathrm{hh}_3(X) = \mathrm{hh}_{-3}(X) = 1$,

The direct sum decomposition $\mathrm{HH}_\bullet(X) = \mathrm{HH}_\bullet(\mathscr{A}_X) \oplus \mathbb{C}^6$ finally implies that the Hodge diamond of $\mathscr{A}_X$ with respect to the rank function coming from $\mathrm{D^b}(X)$ is:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & 84 & & 84 & & 1 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

As mentioned in Example 2.17, we know that $\mathrm{cl}((\mathscr{A}_X)_{\mathrm{unitary}})$ generate $\mathrm{K_{num}}(\mathscr{A}_X) \otimes \mathbb{C}$. As a consequence, Theorem 3.4 guarantees that the Hodge numbers of $\mathscr{A}_X$ defined for any other non-trivial rank function on $\mathscr{A}_X$ having a unitary object whose rank is not zero are equal are equal to the numbers appearing in the above diamond.

(2) Let $X \subset \mathbb{P}(1,1,1,1,1,1,2)$ be a generic double quartic fivefold. Consider the semi-orthogonal decomposition:

$$\mathrm{D^b}(X) = \langle \mathscr{A}_X, \mathscr{O}_X, \mathscr{O}_X(1), \mathscr{O}_X(2), \mathscr{O}_X(3) \rangle,$$

where $\mathscr{A}_X$ is a 3-Calabi–Yau category. As explained in 2.17, the homological unit of $\mathscr{A}_X$ with respect to the rank function coming from $\mathrm{D^b}(X)$ is $\mathbb{C} \oplus \mathbb{C}[3]$. The Hochschild homology numbers for X are (see [13, Section 3]):

- $\mathrm{hh}_0(X) = 6$,
- $\mathrm{hh}_1(X) = \mathrm{hh}_{-1}(X) = 90$,
- $\mathrm{hh}_2(X) = \mathrm{hh}_{-2}(X) = 0$,

- $\mathrm{hh}_3(X) = \mathrm{hh}_{-3}(X) = 1$,

The direct sum decomposition $\mathrm{HH}_\bullet(X) = \mathrm{HH}_\bullet(\mathscr{A}_X) \oplus \mathbb{C}^6$ finally implies that the Hodge diamond of $\mathscr{A}_X$ with respect to the rank function coming from $\mathrm{D^b}(X)$ is:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & 90 & & 90 & & 1 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

As mentioned in Example 2.17, we know that $\mathrm{cl}((\mathscr{A}_X)_{\mathrm{unitary}})$ generate $\mathrm{K_{num}}(\mathscr{A}_X) \otimes \mathbb{C}$. As a consequence, Theorem 3.4 guarantees that the Hodge numbers of $\mathscr{A}_X$ defined for any other non-trivial rank function on $\mathscr{A}_X$ having a unitary object whose rank is not zero are equal to the numbers appearing in the above diamond.

The cubic sevenfold and the double quartic fivefolds are examples of Fano manifolds of Calabi–Yau type that were introduced in [13]. As far as complete intersections in weighted projective spaces are concerned, there is a another example of Fano manifolds of Calabi–Yau type exhibited in [13] : the (transverse) complete intersection of a smooth cubic and a smooth quadric in $\mathbb{P}^7$. Let X such a complete intersection. It is known (see [23]) that there is a semi-orthogonal decomposition:

$$\mathrm{D^b}(X) = \langle \mathscr{A}_X, \mathscr{O}_X, \mathscr{O}_X(1), \mathscr{O}_X(2), \mathscr{S}(2) \rangle,$$

where $\mathscr{A}_X$ is 3-Calabi–Yau and $\mathscr{S}$ is the restriction of one of the Spinor bundles from $\mathbb{Q}^6$ where $\mathbb{Q}^6$ denotes the smooth hyperquadric above-mentioned. It is easily computed that $\mathrm{K}_0(\mathscr{A}_X) = \mathbb{Z}^2$.

Question 3.6. — *Can we find a spherical bundle $\mathscr{E}$ on X such that $\mathscr{E}(-1) \in \mathscr{A}_X$ and $\mathscr{E}(-2) \in \mathscr{A}_X$?*

A positive answer to this question would show that the homological unit of $\mathscr{A}_X$ with respect to the rank function coming from $\mathrm{D^b}(X)$ is $\mathbb{C} \oplus \mathbb{C}[3]$. Hence, the Hodge diamond of $\mathscr{A}_X$ (with respect to the rank function on $\mathscr{A}_X$

coming from $\mathrm{D}^{\mathrm{b}}(X)$) would be:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & 83 & & 83 & & 1 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Theorem 3.4 would guarantee that Hodge numbers of $\mathscr{A}_X$ with respect to any other rank function on $\mathscr{A}_X$ having a unitary object whose rank is not zero are equal are equal to the above numbers.

Remark 3.7. — It would also certainly be interesting to compute the Hodge numbers of the corresponding categories in the case of quiver with potentials. We already know that their homological units are $\mathbb{C} \oplus \mathbb{C}[3]$ and that $\mathrm{cl}(\mathscr{A}_{\mathrm{unitary}})$ generates $\mathrm{K}_0(\mathscr{A}) \otimes \mathbb{C}$ (see Proposition 2.11). As a consequence of Theorem 3.4, the Hodge numbers of the 3-Calabi–Yau categories coming from quiver with potentials are independent of the rank function.

3.2. A Hodge structure

In this section we assume that $\mathscr{A}$ is a geometric 3-Calabi–Yau : there exists is a smooth projective variety over $\mathbb{C}$, say X, and a semi-orthogonal decomposition:

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathscr{A}, {}^{\perp}\mathscr{A} \rangle$$

such that $\mathscr{A}$ is 3-Calabi–Yau. By the Hochschild–Kostant–Rosenberg isomorphism, we have an isomorphism:

$$\tau_{\mathrm{HH}_\bullet} : \ \mathrm{HH}_\bullet(\mathrm{D}^{\mathrm{b}}(X)) \simeq \bigoplus_{p-q=\bullet} H^q(X, \Omega_X^p),$$

Furthermore, by Hodge symmetry, the complex conjugation induces an isomorphism:

$$\mathrm{HS}_X : H^p(X, \Omega_X^q) \simeq H^q(X, \Omega_X^p).$$

Composing complex conjugation with the inverse of the map $\tau_{\mathrm{HH}_\bullet}$, we find an involution:

$$\mathrm{c}_X : \mathrm{HH}_\bullet(\mathrm{D}^{\mathrm{b}}(X)) \simeq \mathrm{HH}_{-\bullet}(\mathrm{D}^{\mathrm{b}}(X))$$

From now, we make the following hypothesis:

Hypothesis 3.8. — *The map* c_X *stabilizes the Hochschild homology of* ${}^{\perp}\mathscr{A}$ *in the decomposition:*

$$\mathrm{HH}_\bullet(\mathrm{D}^{\mathrm{b}}(X)) = \mathrm{HH}_\bullet(\mathscr{A}) \oplus \mathrm{HH}_\bullet({}^{\perp}\mathscr{A}).$$

This hypothesis is satisfied in many situations which we shall describe. Let $Y_1, \dots, Y_k$ be smooth projective varieties and $\mathscr{F}_1, \dots, \mathscr{F}_k$ be objects in $\mathrm{D^b}(Y_1 \times X), \dots, \mathrm{D^b}(Y_k \times X)$. We denote by p_k and q_k the natural projections in the diagram:

$$\begin{tikzcd} & Y_k \times X \arrow[dl, "q_k"'] \arrow[dr, "p_k"] & \\ Y_k & & X \end{tikzcd}$$

Assume that the Fourier–Mukai functors:

$$\Phi_k(?) = (p_k)_*(q_k^*(?) \otimes \mathscr{F}_k) : \mathrm{D^b}(Y_k) \longrightarrow \mathrm{D^b}(X)$$

are fully faithful and that there is a semi-orthogonal decomposition:

$$\mathrm{D^b}(X) = \langle \mathscr{A}, \Phi_1(\mathrm{D^b}(Y_1)), \dots, \Phi_k(\mathrm{D^b}(Y_k)) \rangle.$$

Then Hypothesis 3.8 holds in that case. In particular, since we have an equality:

$$\mathrm{HH}_\bullet(\mathscr{A}) = \mathrm{HH}_\bullet(X)/\mathrm{HH}_\bullet({}^\perp\mathscr{A}),$$

the map c_X descends to an involution:

$$\mathrm{c}_{\mathscr{A}} : \mathrm{HH}_\bullet(\mathscr{A}) \simeq \mathrm{HH}_{-\bullet}(\mathscr{A}).$$

Since $\mathscr{A}$ is a semi-orthogonal component of $\mathrm{D^b}(X)$ with X smooth projective, we can write $\mathscr{A}$ as the derived category of *DG*-modules over some *DG*-algebra, say $\mathscr{C}$. In particular, we have an identification:

$$\mathrm{HH}^\bullet(\mathscr{A}) = \mathrm{Hom}^\bullet_{D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C})}(\Delta_{\mathscr{C}}, \Delta_{\mathscr{C}}),$$

where $\Delta_{\mathscr{C}}$ is the diagonal bimodule over $\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C}$. Assume that $\mathscr{A}$ is a 3-Calabi–Yau category, the category $D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C})$ is then a 6-Calabi–Yau category. Serre duality then provides a graded-commutative perfect pairing (given by composition of morphisms followed by a trace map):

$$S^\bullet_{(\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C})} : \mathrm{Hom}^\bullet_{D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C})}(\Delta_{\mathscr{C}}, \Delta_{\mathscr{C}}) \times \mathrm{Hom}^{6-\bullet}_{D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C})}(\Delta_{\mathscr{C}}, \Delta_{\mathscr{C}}) \longrightarrow \mathbb{C}.$$

Specializing to the case $\bullet = 3$, we find that $\mathrm{Hom}^3_{D^{\mathrm{perf}}(\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C})}(\Delta_{\mathscr{C}}, \Delta_{\mathscr{C}}) = \mathrm{HH}^3(\mathscr{A})$ is a symplectic vector space with symplectic form $S^3_{(\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C})}$. As $\mathscr{A}$ is 3-Calabi–Yau, we have an isomorphism $\mathrm{HH}_0(\mathscr{A}) \simeq \mathrm{HH}^3(\mathscr{A})$. Hence we can lift $S^3_{(\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C})}$ to a symplectic form on $\mathrm{HH}_0(\mathscr{A})$, which we denote by $\omega_{\mathrm{HH}_0(\mathscr{A})}$.

Definition 3.9. — *Let X be smooth projective variety and let $(\mathscr{A}, \mathrm{rk}, \mathfrak{T}^\bullet_{\mathscr{A}})$ be a TC/HU triple where $\mathscr{A}$ is a semi-orthogonal component of $\mathrm{D^b}(X)$ which is 3-Calabi–Yau and connected (that is $\mathrm{HH}_{-3}(\mathscr{A}) = \mathbb{C}$). Assume that $\mathfrak{T}^\bullet_{\mathscr{A}} = \mathbb{C} \oplus \mathbb{C}[3]$ and that the Hypothesis 3.8 is satisfied. We define the* Hodge spaces *of $\mathscr{A}$ as:*

(1) $H^{3,0}(\mathscr{A}) = \mathrm{HH}_{-3}(\mathscr{A}) = \mathbb{C}$, $H^{0,0}(\mathscr{A}) = \mathbb{C} \subset \mathrm{HH}_0(\mathscr{A})$, $H^{1,0}(\mathscr{A}) = H^{2,0}(\mathscr{A}) = 0$.
(2) $H^{3,1}(\mathscr{A}) = \mathrm{HH}_{-2}(\mathscr{A})$, $H^{3,2}(\mathscr{A}) = 0$ *and* $H^{2,1}(\mathscr{A}) = \mathrm{HH}_{-1}(\mathscr{A})$.
(3) *We choose* V_1 *a maximal isotropic subspace of* $\mathrm{HH}_0(\mathscr{A})$ *(for* $\omega_{\mathrm{HH}_0(\mathscr{A})}$*) containing* $H^{0,0}(\mathscr{A})$ *and* V_2 *a maximal isotropic subspace in* $\mathrm{HH}_0(\mathscr{A})$ *which is complementary to* V_1*. We let* $H^{1,1}(\mathscr{A})$ *be a complementary subspace of* $H^{0,0}(\mathscr{A})$ *in* V_1*. We let* $H^{3,3}(\mathscr{A})$ *be a line in* V_2 *and* $H^{2,2}(\mathscr{A})$ *be a complementary subspace of* $H^{3,3}(\mathscr{A})$ *in* V_2*.*
(4) $H^{p,q}(\mathscr{A}) = \mathrm{c}_{\mathscr{A}}(H^{q,p}(\mathscr{A}))$ *for any* $(p,q) \in [0,\dots,3]$ *such that* $p < q$*.*

Remark 3.10. —

(1) By definition of homological units, we have a graded embedding $\mathbb{C} \oplus \mathbb{C}[3] \hookrightarrow \mathrm{HH}^{\bullet}(\mathscr{A})$. The category $\mathscr{A}$ is 3-Calabi–Yau, so that there is a graded embedding:

$$\mathbb{C} \oplus \mathbb{C}[3] \longhookrightarrow \mathrm{HH}_{\bullet-3}(\mathscr{A}).$$

This accounts for the definition of $H^{0,0}(\mathscr{A})$ and its embedding in $\mathrm{HH}_0(\mathscr{A})$.
(2) Since $H^{0,0}(\mathscr{A})$ is a line in the symplectic vector space $(\mathrm{HH}_0(\mathscr{A}), \omega_{\mathrm{HH}_0(\mathscr{A})})$, it is automatically an isotropic subspace of $\mathrm{HH}_0(\mathscr{A})$. The definition of V_1 and $H^{1,1}(\mathscr{A})$ is accordingly meaningful.
(3) The definition of $H^{3,3}(\mathscr{A})$ and $H^{2,2}(\mathscr{A})$ looks rather arbitrary, but it seems difficult to do better in the absence of the a reasonable Lefschetz operator on $\mathrm{HH}_0(\mathscr{A})$. Note that for $\mathscr{A}$ the 3-Calabi–Yau category inside the derived category of the cubic sevenfold or the double quartic fivefold, this potential Leschetz operator is probably to be defined as 0. Indeed, we know $h^{1,1}(\mathscr{A}) = 0$ in both cases.
(4) It doesn't seem impossible to extend the definition of these Hodge spaces beyond the case where the homological unit with respect to the chosen rank function is $\mathbb{C} \oplus \mathbb{C}[3]$. One would naturally define $H^{i,0}(\mathscr{A}) = \mathfrak{T}_{\mathscr{A},3-i}$, for $i \in [0,\dots,3]$. The definition of $H^{1,1}(\mathscr{A})$, $H^{2,2}(\mathscr{A})$, $H^{3,3}(\mathscr{A})$ is carried out exactly as in Definition 3.9. On the other hand, the definition of $H^{3,1}(\mathscr{A})$, $H^{3,2}(\mathscr{A})$ and $H^{2,1}(\mathscr{A})$ looks less obvious, but could probably be found out in many special cases.

Proposition 3.11. — *Let* X *be smooth projective variety and let* $(\mathscr{A}, \mathrm{rk}, \mathfrak{T}^{\bullet}_{\mathscr{A}})$ *be a TC/HU triple where* $\mathscr{A}$ *is a semi-orthogonal component of* $\mathrm{D^b}(X)$ *which is* 3*-Calabi–Yau and connected (that is* $\mathrm{HH}_{-3}(\mathscr{A}) = \mathbb{C}$*). Assume that* $\mathfrak{T}^{\bullet}_{\mathscr{A}} = \mathbb{C} \oplus \mathbb{C}[3]$ *and that the Hypothesis 3.8 is satisfied. Consider the Hodge spaces of* $\mathscr{A}$ *as in Definition 3.9. Then we have a graded* *decomposition:*

$$\mathrm{HH}_\bullet(\mathscr{A}) = \bigoplus_{p-q=\bullet} H^{p,q}(\mathscr{A})$$

and the direct sum $\bigoplus_{p,q\geqslant 0} H^{p,q}(\mathscr{A})$ *is a Hodge structure on* $\mathrm{HH}_\bullet(\mathscr{A})$. *If* $\mathrm{cl}(\mathscr{A}_{\mathrm{unitary}})$ *generates* $\mathrm{K}_0(\mathscr{A})\otimes\mathbb{C}$ *(resp.* $\mathrm{K}_{\mathrm{num}}(\mathscr{A})\otimes\mathbb{C}$ *provided that* rk *is a numerical rank function), then the dimensions of the Hodge spaces defined for any other non-trivial rank function (resp. non-trivial numerical rank function) having a unitary object on* $\mathscr{A}$ *are equal to the* $h^{p,q}(\mathscr{A}) = \dim H^{p,q}(\mathscr{A})$.

Proof. — The graded decomposition and the fact that $\bigoplus_{p,q\geqslant 0} H^{p,q}(\mathscr{A})$ is a Hodge structure on $\mathrm{HH}_\bullet(\mathscr{A})$ follow immediately from Definition 3.9 and the existence of the involution $c_{\mathscr{A}} : \mathrm{HH}_\bullet(\mathscr{A}) \simeq \mathrm{HH}_{-\bullet}(\mathscr{A})$ which sends $H^{p,q}(\mathscr{A})$ on $H^{q,p}(\mathscr{A})$ (by definition). The second part of the proposition follows from Theorem 3.4. □

3.3. Some observations toward Homological Mirror Symmetry for the cubic sevenfold and the double quartic fivefold

This final section (mainly observational) relates the computations of Hodge numbers for the Calabi–Yau categories associated to the cubic sevenfold and the double quartic fivefold to some mirror symmetry phenomena expected for quotients of products of elliptic curves by finite groups. We also discuss the possible existence of generators of these categories in connection with the spherical vector bundles they contain. This would be a first step toward a natural generalization of the proof of Homological Mirror Symmetry for elliptic curves discovered in [30, 36].

3.3.1. Cubic sevenfold

Let $T = E\times E\times E$ be the triple product of an elliptic curve E, given by the equation $\{Z_1^3+Z_2^3+Z_3^3=0\}\subset\mathbb{P}^2$ and let $\mathbb{Z}_3\times\mathbb{Z}_3$ acts on T as follows:

$$(1,0).(z_1,z_2,z_3,z_4,z_5,z_6,z_7,z_8,z_9) = (\alpha.z_1,z_2,z_3,\alpha^2.z_4,z_5,z_6,z_7,z_8,z_9)$$
$$(0,1).(z_1,z_2,z_3,z_4,z_5,z_6,z_7,z_8,z_9) = (\alpha.z_1,z_2,z_3,z_4,z_5,z_6,\alpha^2.z_7,z_8,z_9),$$

where α is a cubic root of unity. The quotient $T/\mathbb{Z}_3\times\mathbb{Z}_3$ has a crepant resolution which is a Calabi–Yau threefold. We denote it by Z_1. The Hodge diamond of Z_1 is:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 84 & & 0 & \\ 1 & & 0 & & 0 & & 1 \\ & 0 & & 84 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

As explained in [5, 6, 31], the mirror of Z_1 ought to be a Landau–Ginzburg model related to a smooth cubic sevenfold. Let $X_1 \subset \mathbb{P}^8$ be a smooth cubic sevenfold. We have a semi-orthogonal decomposition:

$$\mathrm{D}^{\mathrm{b}}(X_1) = \langle \mathscr{A}_1, \mathscr{O}_{X_1}, \dots, \mathscr{O}_{X_1}(5) \rangle,$$

where $\mathscr{A}_1$ is a 3-Calabi–Yau category. It follows from [26] that $\mathscr{A}_1$ is the homotopy category of the DG-category of graded matrix factorizations of the equation of X_1. Therefore, the category $\mathscr{A}_1$ can be interpreted as a Landau–Ginzburg model for the cubic sevenfold X_1. We found out in Example 3.5 that the Hodge diamond of $\mathscr{A}_1$ is:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & 84 & & 84 & & 1 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

One observes that both diamonds are obtained from each other by a $\pi/2$-rotation. This is certainly a favorable presage as far as mirror symmetry is concerned.

We have shown in Example 2.9 that there exists a 3-spherical bundle $\mathscr{E}_{X_1}$ on X_1 such that $\mathscr{E}_{X_1}(-1)$ and $\mathscr{E}_{X_1}(-2)$ are in $\mathscr{A}_1$ and that the Chern characters of these two bundles generate $\mathrm{HH}_0(\mathscr{A}_1)$. The following questions naturally come to mind:

Question 3.12. —

(1) *Do the objects $\mathscr{E}_{X_1}(-1)$ and $\mathscr{E}_{X_1}(-2)$ split-generate the category $\mathscr{A}_1$?*
(2) *For $n \geqslant 0$, denote by $T^n_{\mathscr{E}_{X_1}(-2)}$ the n-th composition with itself of the spherical twist along $\mathscr{E}_{X_1}(-2)$. Can we reconstruct a smooth elliptic curve, say E, and an action of $\mathbb{Z}_3 \times \mathbb{Z}_3$ on $E \times E \times E$ from the ring $\bigoplus_{n \geqslant 0} \mathrm{Hom}(\mathscr{E}_{X_1}(-1), T^n_{\mathscr{E}_{X_1}(-2)}(\mathscr{E}_{X_1}(-1)))$?*

(3) *Is there a smooth cubic sevenfold* X_1 *such that the Fukaya category of* Z_1 *is equivalent to the category of* $\mathscr{A}_\infty$*-modules over the* $\mathscr{A}_\infty$*-algebra* $\mathrm{RHom}(\mathscr{E}_{X_1}(-1) \oplus \mathscr{E}_{X_1}(-2), \mathscr{E}_{X_1}(-1) \oplus \mathscr{E}_{X_1}(-2))$*?*

The first equation is quite natural and has a positive answer in the context of Fukaya categories (see [1, Theorem 1.1]) or [32, Lemma 9.2]). Unfortunately, the analogues of such results are not known in algebraic geometry. We state as a question:

Question 3.13. — *Let* X *be a smooth projective variety and* $\mathscr{A}$ *be a semi-orthogonal component of* $\mathrm{D^b}(X)$ *which is p-Calabi–Yau. Let* $\mathscr{E}_1, \dots, \mathscr{E}_k$ *be p-spherical objects in* $\mathscr{A}$ *whose ranks with respect to the rank function coming from* X *are non-zero. Assume that the Chern characters of* $\mathscr{E}_1, \dots, \mathscr{E}_k$ *generate* $\mathrm{HH}_0(\mathscr{A})$*. Do the object* $\mathscr{E}_1, \dots, \mathscr{E}_k$ *split generate* $\mathscr{A}$*?*

The assumption that $\mathscr{A}$ is Calabi–Yau can not be withdrawn. Indeed, there are many known examples of derived categories of smooth projective general type surfaces for which the answer to the question is no. This relates to the existence of a phantom category inside their derived categories (see [8] for instance). On the other hand, it is a folklore conjecture that phantom categories do not exist inside Calabi–Yau categories. Furthermore, the Homological Mirror Symmetry conjecture and the truth of the statement corresponding to Question 3.13 in the context of Fukaya categories lead us to believe that it should have a positive answer.

The second item of Question 3.12 is inspired by known proofs of Homological Mirror Symmetry for elliptic curves (see [30, 36], see also [28]). Provided that the answer to the first item of Question 3.12 is positive, the third item of Question 3.12 is just a formulation of the Homological Mirror Symmetry conjecture for the cubic sevenfold and the rigid Calabi–Yau threefold Z_1.

Remark 3.14. — Let $T = E \times E \times E$ be the triple product of an elliptic curve E, given by the equation $\{Z_1^3 + Z_2^3 + Z_3^3 = 0\} \subset \mathbb{P}^2$ and let $\mathbb{Z}_3$ acts on T as follows:

$$(1).(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9) = (\alpha.z_1, z_2, z_3, \alpha.z_4, z_5, z_6, \alpha.z_7, z_8, z_9)$$

where α is a cubic root of unity. The quotient $T/\mathbb{Z}_3$ has a crepant resolution which is a Calabi–Yau threefold. We denote it by Z_2. The Hodge diamond of Z_2 is:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 36 & & 0 & \\ 1 & & 0 & & 0 & & 1 \\ & 0 & & 36 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

We consider the cubic sevenfolds having equations of type:

$$X_{\underline{b}} = \left\{ -z_1z_2z_3 - z_4z_5z_6 - z_7z_8z_9 + \sum_{i=1}^{9} b_i z_i^3 + \sum_{\substack{i\in[1,2,3]\\ j\in[4,5,6]\\ k\in[7,8,9]}} b_{ijk} z_i z_j z_k = 0 \right\},$$

where the $\underline{b} = (b_i, b_{ijk})$ is a vector of complex numbers. Let $\mathbb{Z}_3$ acts on $X_{\underline{b}}$ by:

$$\begin{aligned}(1).&(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9)\\ &= (\alpha.z_1, \alpha.z_2, \alpha.z_3, \alpha^2.z_4, \alpha^2.z_5, \alpha^2.z_6, z_7, z_8, z_9)\end{aligned}$$

The exists a semi-orthogonal decomposition:

$$\mathrm{D^b}(X_{\underline{b}})^{\mathbb{Z}_3} = \langle \mathscr{A}_{\underline{b}}, \mathscr{O}_{X_{\underline{b}}}, \ldots, \mathscr{O}_{X_{\underline{b}}}(5)\rangle,$$

where $\mathrm{D^b}(X_{\underline{b}})^{\mathbb{Z}_3}$ is the derived category of $\mathbb{Z}_3$-equivariant coherent sheaves on $X_{\underline{b}}$. In case $X_{\underline{b}}$ is smooth, the category $\mathscr{A}_{\underline{b}}$ is 3-Calabi–Yau. It is proved in [33] that (for the right choice of $\underline{b}$) there is an equivalence between $\mathscr{A}_{\underline{b}}$ and a version of the Fukaya category of Z_2. It would be interesting to check if their techniques can be applied to answer the third item of Question 3.12.

3.3.2. Double quartic fivefold

The story for the double quartic fivefold is very similar to that of the cubic sevenfold. Namely, let $T = E \times E \times E$ be the triple product of an elliptic curve E, given by the equation $\{Z_1^4 + Z_2^4 + Z_3^2 = 0\} \subset \mathbb{P}(1,1,2)$ and let $\mathbb{Z}_4 \times \mathbb{Z}_4$ acts on T as follows:

$$(1,0).(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9,) = (-i.z_1, z_2, -z_3, z_4, z_5, z_6, i.z_7, z_8, -z_9)$$
$$(0,1).(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9) = (z_1, z_2, z_3, -i.z_4, z_5, -z_6, i.z_7, z_8, -z_9),$$

where i is a square root of -1. The quotient $T/\mathbb{Z}_4 \times \mathbb{Z}_4$ has a crepant resolution which is a Calabi–Yau threefold. We denote it by Z_3. The Hodge diamond of Z_3 is:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 90 & & 0 & \\ 1 & & 0 & & 0 & & 1 \\ & 0 & & 90 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

As explained in [5, 6, 31], the mirror of Z_3 ought to be a Landau–Ginzburg model related to a smooth double quartic fivefold. Let $X_3 \subset \mathbb{P}(1,1,1,1,1,1,2)$ be a smooth double quartic fivefold. We have a semi-orthogonal decomposition:

$$\mathrm{D}^{\mathrm{b}}(X_3) = \langle \mathscr{A}_3, \mathscr{O}_{X_1}, \dots, \mathscr{O}_{X_1}(3)\rangle,$$

where $\mathscr{A}_3$ is a 3-Calabi–Yau category. It follows from [26] that $\mathscr{A}_3$ is the homotopy category of the DG-category of graded matrix factorizations of the equation of X_1. Therefore, the category $\mathscr{A}_3$ can be interpreted as a Landau–Ginzburg model for the double quartic fivefold X_3. We found out in Example 3.5 that the Hodge diamond of $\mathscr{A}_3$ is:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & 90 & & 90 & & 1 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

which is again a favorable presage as far as mirror symmetry is concerned.

We have shown in Example 2.9 that there exists a 3-spherical bundle $\mathscr{F}_{X_3}$ on X_3 such that $\mathscr{F}_{X_3}(-1)$ and $\mathscr{F}_{X_3}(-2)$ are in $\mathscr{A}_3$ and that the Chern characters of these two bundles generate $\mathrm{HH}_0(\mathscr{A}_3)$. We ask for X_3 the analogous question to 3.12:

Question 3.15. —

(1) *Do the objectis $\mathscr{F}_{X_3}(-1)$ and $\mathscr{F}_{X_3}(-2)$ split-generate the category $\mathscr{A}_3$?*
(2) *For $n \geqslant 0$, denote by $T^n_{\mathscr{F}_{X_3}(-2)}$ the n-th composition with itself of the spherical twist along $\mathscr{F}_{X_3}(-2)$. Can we reconstruct a smooth elliptic curve, say E, and an action of $\mathbb{Z}_4 \times \mathbb{Z}_4$ on $E \times E \times E$ from the ring $\bigoplus_{n \geqslant 0} \operatorname{Hom}(\mathscr{F}_{X_3}(-1), T^n_{\mathscr{F}_{X_3}(-2)}(\mathscr{F}_{X_3}(-1)))$?*

(3) *Is there a smooth double quartic fivefold X_3 such that the Fukaya category of Z_3 is equivalent to the category of $\mathscr{A}_\infty$-modules over the $\mathscr{A}_\infty$-algebra* $\mathrm{RHom}(\mathscr{F}_{X_3}(-1) \oplus \mathscr{F}_{X_3}(-2), \mathscr{F}_{X_3}(-1) \oplus \mathscr{F}_{X_1}(-2))$*?*

It would be again interesting to know if the techniques developed in [33] could be used to answer the third item of Question 3.15.

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