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Realisation of Abelian varieties as automorphism groups ^(*)

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ABSTRACT. — Let A be an Abelian variety over a field F . We show that A is isomorphic to the automorphism group scheme of a smooth projective F -variety if, and only if, $\text{Aut}_{gp}(\bar{A})$ is finite. This result was proved by Lombardo and Maffei [6] in the case $F = \mathbb{C}$, and recently by Blanc and Brion [1] in the case of an algebraically closed F .

RÉSUMÉ. — Soit A une variété abélienne sur un corps F . On montre que A est isomorphe au schéma en groupes des automorphismes d'une F -variété projective et lisse, si et seulement si le groupe des \bar{F} -automorphismes de A est fini. Ce résultat est dû à Lombardo et Maffei [6] lorsque $F = \mathbb{C}$. Il est dû à Blanc et Brion [1] lorsque $F = \bar{F}$.

1. Introduction

Let F be a field, with algebraic closure \bar{F} . Let X be a projective variety over F . The automorphism group functor $\mathbf{Aut}(X)$ is represented by a group scheme, locally of finite type over F ([8, Theorem 3.7]). Conversely, given a group scheme G , of finite type over F , it is natural to ask whether G can be realised as the automorphism group of such an X . When $G = A$ is an abelian variety, this question was recently considered in [6]. When $F = \mathbb{C}$, Lombardo and Maffei prove that A is the automorphism group of a projective smooth complex variety, if and only if $\text{Aut}_{gp}(A)$ is finite. They use analytic methods. Their result was extended to F algebraically closed of any characteristic in [1], using algebro-geometric techniques: blowups, Lie algebra computations and modding out actions of finite group schemes. Making a different use of these tools, we provide a generalisation of this result, to the case of all ground fields F .

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Article proposé par Damian Rössler.

1.1. Sketch of our construction

Let A/F be an abelian variety over a field F , such that $G := \text{Aut}_{gp}(\bar{A})$ is finite. We first introduce an integer $n \geq 1$, invertible in F , such that G acts faithfully on the n -torsion subgroup $A[n](\bar{F})$. Then, we pick an abelian variety B_1/F , enjoying the following properties.

- (1) The abelian varieties A and B_1 are “orthogonal”, in the sense that

$$\text{Hom}_{gp}(\bar{A}, \bar{B}_1) = 0,$$

where homomorphisms are taken over \bar{F} .

- (2) There exists an injection (of algebraic F -groups)

$$\iota : A[n] \hookrightarrow B_1.$$

We denote by B_2/F the abelian variety fitting into the diagonal extension

$$0 \longrightarrow A[n] \xrightarrow{a \mapsto (a, \iota(a))} A \times B_1 \xrightarrow{\pi} B_2 \longrightarrow 0.$$

Using point (1) above, we prove that automorphisms of (the variety) B_2 are diagonal: they come from automorphisms of $A \times B_1$, respecting orbits under the embedded $A[n]$. Next, we build an appropriate smooth closed F -subvariety $Y_2 \subset B_2$, stable by translations by $A \simeq \pi(A \times \{0\}) \subset B_2$.

We define a smooth F -variety X as the blowup

$$X := \text{Bl}_{Y_2} B_2.$$

The natural arrow

$$A \longrightarrow \mathbf{Aut}(B_2),$$

given by translations, lifts to an arrow

$$\tau : A \longrightarrow \mathbf{Aut}(X).$$

We show that τ is an isomorphism of algebraic groups over F .

2. Notation

2.1. Geometry over F

Let F be a field, with algebraic closure \bar{F} , and separable closure $F_s \subset \bar{F}$. We denote by $F[\epsilon]$, $\epsilon^2 = 0$, the F -algebra of dual numbers. We use it for differential calculus.

By a variety over F , we mean a separated F -scheme of finite type.

An algebraic F -group (or simply F -group) is an F -group scheme of finite type. It is often assumed to be reduced, hence smooth over F .

Let X be a variety over F . For a field extension E/F , we denote by $X_E := X \times_F E$ the E -variety obtained from X by extending scalars. We put $\bar{X} := X \times_F \bar{F}$.

If X is smooth over F , we denote by $TX \rightarrow X$ the tangent bundle of X . A global section of the tangent bundle is called a vector field on X .

We denote by $\text{Aut}(X)$ the (abstract) group of automorphisms of the F -variety X , and by $\text{Aut}(\bar{X})$ the group of automorphisms of the \bar{F} -variety \bar{X} .

If X/F is a projective variety, we denote by $\mathbf{Aut}(X)$ the F -group scheme of automorphisms of X ; it is locally of finite type over F . By [2, Lemma 3.1], there is a canonical isomorphism

$$H^0(X, TX) \xrightarrow{\sim} \text{Lie}(\mathbf{Aut}(X)).$$

If an abstract group G acts on a variety X , and if $Z \subset X$ is a closed subvariety, we denote by $\text{Stab}_G(Z) \subset G$, or simply by $\text{Stab}(Z) \subset G$ when no confusion arises, the subgroup of transformations leaving Z (globally) invariant.

Let G/F be a group scheme, locally of finite type. In the situation where G acts on X , we use the notation $\mathbf{Stab}_G(Z) \subset G$ for the closed F -subgroup scheme defined by

$$\mathbf{Stab}_G(Z)(A) = \{g \in G(A), g(Z_A) = Z_A\},$$

for all commutative F -algebras A . That it is representable follows from [4, II 1.3.6].

2.2. Frobenius and Verschiebung

If F has characteristic $p > 0$, we put

$$X^{(1)} := X \times_{\text{Frob}} F,$$

extension of scalars taken with respect to $\text{Frob} : F \xrightarrow{x \mapsto x^p} F$.

Recall the Frobenius homomorphism

$$\text{Frob}_X : X \rightarrow X^{(1)};$$

it is a morphism of F -varieties, functorial in X .

If X/F is an algebraic group, it is a group homomorphism.

If X is a commutative algebraic group, there is the Verschiebung homomorphism

$$\text{Ver}_X : X^{(1)} \longrightarrow X,$$

satisfying $(\text{Ver}_X \circ \text{Frob}_X) = p \text{Id}_X$.

If moreover X/F is a semi-abelian variety, Ver_X and Frob_X are isogenies.

2.3. Abelian varieties

If A and B are Abelian varieties over F , we denote by $\text{Hom}_{gp}(A, B)$ the group of homomorphisms of algebraic F -groups, from A to B . We denote by $\text{Hom}_{gp}(\bar{A}, \bar{B})$ the group of homomorphisms of algebraic \bar{F} -groups, from \bar{A} to \bar{B} . These are finite free \mathbb{Z} -modules. We adopt the similar notation for endomorphisms (End_{gp}) and automorphisms (Aut_{gp}). For an integer $n \geq 1$, we denote by $A[n]$ the n -torsion of A , seen as a finite group scheme over F .

2.4. Barycentric operations

Let A be an abelian variety over F . Then A comes naturally equipped with barycentric operations with integer coefficients. More precisely, for a positive integer n , denote by

$$\mathbf{Z}_1^n \subset \mathbf{Z}^n$$

the subset consisting of integers $\alpha = (\alpha_1, \dots, \alpha_n)$, with $\alpha_1 + \dots + \alpha_n = 1$.

For $\alpha \in \mathbf{Z}_1^n$, there is a barycentric operation

$$\mathcal{B}_\alpha : A^n \longrightarrow A,$$

$$(x_1, \dots, x_n) \mapsto \alpha_1 x_1 + \dots + \alpha_n x_n.$$

Associativity of the group law of A , provides natural associativity relations between the \mathcal{B}_α 's, for various α 's.

For instance, pick $\alpha = (\alpha_1, \alpha_2) \in \mathbf{Z}_1^2$ and $\gamma = (\gamma_1, \gamma_2) \in \mathbf{Z}_1^2$, and set

$$\delta := (\alpha_1 \gamma_1, \alpha_2 \gamma_1, \gamma_2) \in \mathbf{Z}_1^3.$$

Then, we have the associativity rule

$$\mathcal{B}_\gamma(\mathcal{B}_\alpha(x_1, x_2), x_3) = \mathcal{B}_\delta(x_1, x_2, x_3).$$

Remark 2.1. — More generally, these barycentric operations exist for *torsors* under commutative algebraic F -groups.

DEFINITION 2.2. — *Let $X \subset A$ be an F -subvariety. We say that X is stable under all barycentric operations, if the restriction*

$$(\mathcal{B}_\alpha)|_{X^n} : X^n \rightarrow A$$

factors through the closed immersion $X \hookrightarrow A$, for every $n \geq 2$ and every $\alpha \in \mathbf{Z}_1^n$.

In this case, we also say that X is barycentric.

Note that X is barycentric if and only if it is a translate of an algebraic F -subgroup $\overrightarrow{X} \subset A$. Checking this fact is left as an exercise for the reader. Of course, $X(F)$ might be empty. If X is geometrically reduced and geometrically connected, so is \overrightarrow{X} —hence \overrightarrow{X} is an abelian subvariety of A .

Let A and B be two abelian varieties over F . Recall the essential fact

$$\mathrm{Hom}_{\overline{F}\text{-var}}(\overline{A}, \overline{B}) = B(\overline{F}) \times \mathrm{Hom}_{gp}(\overline{A}, \overline{B}).$$

In particular, morphisms (of varieties) between abelian varieties commute with the barycentric operations \mathcal{B}_α .

If $X \subset A$ is a geometrically reduced closed F -subvariety, the smallest geometrically reduced barycentric F -subvariety containing X is called the barycentric envelope of X . We denote it by $\mathcal{E}(X)$.

Assume now that X is geometrically reduced and geometrically connected. Pick $n \geq 1$ and $\alpha \in \mathbf{Z}_1^n$. Consider $\mathcal{B}_\alpha(X^n) \subset A$ as a geometrically reduced and geometrically connected closed subvariety of A . Then, if n and α are chosen so that $\mathcal{B}_\alpha(X^n)$ is of maximal dimension, we have $\mathcal{B}_\alpha(X^n) = \mathcal{E}(X)$. Thus, $\mathcal{E}(X)$, being geometrically connected and geometrically reduced, is a translate of an abelian subvariety of A .

3. Statement of the theorem

THEOREM 3.1. — *Let A be an Abelian variety, over a field F . The following are equivalent:*

- (1) *The group $G := \mathrm{Aut}_{gp}(\overline{A})$ is finite.*
- (2) *There exists a smooth projective F -variety X , such that A is isomorphic to $\mathbf{Aut}(X)$ (as algebraic groups over F).*

Note that (2) \Rightarrow (1) can be checked over \overline{F} , which follows from [1, Theorem A]. Our task in this paper is to prove the converse implication.

4. Auxiliary results

4.1. Blowups

This section contains two elementary lemmas on automorphisms of blowups, which we provide with short proofs. A good recent reference on this topic, also containing more advanced material, is [9, Section 2].

LEMMA 4.1. — *Let $Y \hookrightarrow D$ be a closed immersion of smooth F -varieties, such that all connected components of Y have codimension ≥ 2 in D .*

Denote by $\beta : X := \text{Bl}_Y(D) \rightarrow D$ the blowup of Y inside D .

The F -variety X is smooth.

Let f be an automorphism of the F -variety D . Then, f lifts via β to an automorphism of X , if and only if $f(Y) = Y$.

Proof. — If $f(Y) = Y$, then f lifts to an automorphism of X by the universal property of the blowup.

Conversely, assume that f lifts to an automorphism ϕ of X , so that we have a commutative square

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X \\ \downarrow & & \downarrow \\ D & \xrightarrow{f} & D. \end{array}$$

To check that $f(Y) = Y$, can assume that $F = \bar{F}$. It then suffices to prove that $Y \subset D$ and $f(Y) \subset D$ have the same set of F -rational points. This is clear, since the fiber of β over a point $x \in D(F)$ is either a point if $x \notin Y(F)$, or a projective space of dimension ≥ 1 if $x \in Y(F)$. \square

Lemma 4.1 has an infinitesimal analogue, as follows.

LEMMA 4.2. — *Let $Y \hookrightarrow D$ be a closed immersion of smooth F -varieties, such that all connected components of Y have codimension ≥ 2 in D .*

Denote by $\beta : X := \text{Bl}_Y(D) \rightarrow D$ the blowup of Y inside D .

Let $s : D \rightarrow TD$ be a vector field on D . Then, s lifts to a vector field on X , if and only if $s|_Y$ takes values in TY .

Proof. — Denote by $i : E \hookrightarrow X$ the exceptional divisor. The restriction

$$\beta|_{X-E} : X - E \rightarrow D - Y$$

is an isomorphism.

We thus have a natural injective F -linear arrow

$$\rho : H^0(X, TX) \longrightarrow H^0(D - Y, TD) = H^0(D, TD),$$

$$\sigma \mapsto \sigma|_{X-E}.$$

Note that the equality $H^0(D - Y, TD) = H^0(D, TD)$ follows from the fact that $Y \subset D$ has codimension ≥ 2 . On E , we have a natural extension of vector bundles

$$0 \longrightarrow TE \longrightarrow i^*(TX) \longrightarrow N_{E/X} \longrightarrow 0,$$

where $N_{E/X} \simeq \mathcal{O}_E(-1)$ is the normal bundle of E in X . Since Y has codimension ≥ 2 in D , we have $H^0(E, \mathcal{O}_E(-1)) = 0$. This can be checked on the fibers of β over geometric points of Y , which are projective spaces of dimension ≥ 1 . Hence, $\sigma|_E$ takes values in TE . Consequently, $\rho(\sigma)|_Y$ takes values in TY .

Conversely, let $s : D \longrightarrow TD$ be a vector field on D . Then s corresponds to an automorphism ψ of the $F[\epsilon]$ -scheme $D \times_F F[\epsilon]$, reducing to the identity at $\epsilon = 0$. Assume that $s|_Y$ takes values in TY . Then, ψ restricts to an automorphism of the closed subscheme $Z \times_F F[\epsilon] \subset D \times_F F[\epsilon]$. By the universal property (and compatibility with base change) of the blowup, ψ lifts, via $\beta \times_F F[\epsilon]$, to an automorphism of $X \times_F F[\epsilon]$. Equivalently, s lifts, via β , to a vector field on X . \square

4.2. Hypersections in projective space

We could not find a reference in the literature for the following result, so that we provide it with a proof.

PROPOSITION 4.3. — *Let S be a geometrically irreducible smooth projective F -variety, of dimension ≥ 2 . Let $m \geq 1$ be an integer. Then, S contains a geometrically irreducible smooth projective F -curve, of genus $g \geq m$.*

Proof. — Pick a projective embedding $S \subset \mathbb{P}^n$ (everything is over F). Let $d \geq 1$ be an integer. Let $H \subset \mathbb{P}^n$ be a degree d hypersurface, given by $h \in H^0(\mathbb{P}^n, \mathcal{O}(d))$. By Bertini's theorem, for d large enough and h general, $S \cap H$ is smooth and geometrically irreducible, of dimension one less than S . This version of Bertini's theorem works over any F (see [10, 3] for the delicate case where F is finite). Proceeding by induction, we reduce to the case where S is a surface.

We then take $C := S \cap H$, and show that $g(= h^1(C, \mathcal{O}_C))$ goes to infinity with d . To do so, consider the exact sequence of coherent $\mathcal{O}_{\mathbb{P}^n}$ -modules

$$0 \rightarrow \mathcal{O}_S(-d) \xrightarrow{\times h} \mathcal{O}_S \rightarrow \mathcal{O}_C \rightarrow 0.$$

Taking Euler characteristics, we get

$$g - 1 = -\chi(\mathcal{O}_C) = \chi(\mathcal{O}_S(-d)) - \chi(\mathcal{O}_S).$$

We conclude using the following fact, applied to $X = S$.

For a closed m -dimensional F -subvariety $X \subset \mathbb{P}^n$, the association

$$d \mapsto \chi(\mathcal{O}_X(-d))$$

is a degree m polynomial function of d . A classical proof is by induction on $m \geq 0$. \square

4.3. (Semi-)abelian varieties

The next Lemma is borrowed from [2, Lemma 5.3]. We provide here a different proof. In practice, we will apply it to abelian varieties, in which case it is due to Chow.

LEMMA 4.4. — *Assume that F has characteristic $p > 0$.*

Let A, B be semi-abelian varieties over F . Then, all elements of $\text{Hom}_{gp}(\overline{A}, \overline{B})$ are defined over the separable closure $F_s \subset \overline{F}$.

Proof. — We have to show the following. Let E/F be a purely inseparable algebraic extension. Let $g : A_E \rightarrow B_E$ be a homomorphism of algebraic groups over E . Then g is defined over F . Without loss of generality, we can assume that E/F is finite. By induction, we reduce to the case where $E = F(\sqrt[p]{a})/F$ is a primitive purely inseparable extension of height one. Note that $\text{Frob} : E \rightarrow E$ takes values in F . Hence, $g^{(1)} : A_E^{(1)} \rightarrow B_E^{(1)}$ is defined over F . The Frobenius homomorphism

$$\text{Frob}_A : A \rightarrow A^{(1)}$$

presents $A^{(1)}$ as a quotient of A , by a finite (characteristic) sub- F -group $\mu_A \subset A$.

From the relation

$$\text{Ver}_A \circ \text{Frob}_A = p \text{Id}_A,$$

we deduce $\mu_A \subset A[p]$. Same holds for B .

Combining these facts, we get that the E -morphism

$$A/\mu_A \rightarrow B/\mu_B$$

induced by g , is defined over F . Modding out further, we get that the E -morphism

$$A/A[p] \rightarrow B/B[p],$$

induced by g , is defined over F .

Via the iso

$$\begin{aligned} A/A[p] &\xrightarrow{\sim} A \\ \bar{a} &\longmapsto pa, \end{aligned}$$

this isomorphism is actually g itself. The Lemma is proved. \square

LEMMA 4.5. — *For each $n \geq 2$, there exists an (absolutely) simple n -dimensional abelian variety A over F_s .*

Proof. — Since F_s is separably closed, “simple” is the same as “absolutely simple”, for abelian varieties over F_s (use Lemma 4.4). Without loss of generality, we assume that F_s is the algebraic closure of its prime subfield. Over $\overline{\mathbb{Q}}$, we can then use the existence of abelian surfaces with a prescribed CM type. Over $\overline{\mathbb{F}}_p$, we can use Honda–Tate theory. For concrete constructions, and more general results, we refer to [7, Theorem 1] (where $F_s = \overline{\mathbb{Q}}$), and [5, Theorem 2] (where $F_s = \overline{\mathbb{F}}_p$). \square

LEMMA 4.6. — *Let B be an abelian variety over F , whose simple factors (over \overline{F}) are of dimensions ≥ 2 . (Equivalently: all \overline{F} -homomorphisms from an elliptic curve to \overline{B} are constant.)*

Then, there exists a smooth F -subvariety $Y \subset B$, which is a disjoint union of smooth F -curves, and of a separable closed point, such that

$$\mathbf{Stab}(Y) = \{\mathrm{Id}\} \subset \mathbf{Aut}(B).$$

Proof. — Assume first that B is F -simple, in the sense that it has no non-trivial proper abelian F -subvariety. By Proposition 4.3, we can pick a geometrically irreducible smooth F -curve $C \subset B$, of arbitrarily large genus $g \geq 2$.

The group $\mathbf{Aut}(\overline{C})$ is finite. Indeed, $\mathrm{Lie}(\mathbf{Aut}(C))$ is the space of vector fields on C , which vanishes since $g \geq 2$.

Let us show that $\mathcal{E}(C) = B$. The barycentric envelope $\mathcal{E}(C)$ is a translate of an abelian subvariety $B' \subset B$. Since B is F -simple, we get $B' = B$, hence $\mathcal{E}(C) = B$.

Now, let $g \in \mathbf{Aut}(B)(\overline{F}[\epsilon]) = B(\overline{F}[\epsilon]) \times \mathrm{Aut}_{gp}(\overline{B})$ be such that

$$g_{|_{C \times_F \overline{F}[\epsilon]}} = \mathrm{Id}_{|_{C \times_F \overline{F}[\epsilon]}}.$$

Because g commutes to barycentric operations, g acts as the identity on the closed subscheme

$$\mathcal{E}(C) \times_F \overline{F}[\epsilon] \subset B \times_F \overline{F}[\epsilon].$$

Since $\mathcal{E}(C) = B$, it follows that $g = \mathrm{Id}$. Thus, we get a natural embedding of F -group schemes

$$H := \mathbf{Stab}_{\mathbf{Aut}(B)}(C) \hookrightarrow \mathbf{Aut}(C).$$

In particular, H is finite étale over F . Let E/F be a finite separable field extension, such that $H(E) = H(\overline{F})$. Denote by

$$\Phi := \bigcup_{h \in H(E), h \neq e} \overline{B}^h \subset \overline{B}$$

be the (strict) closed subscheme, consisting of points fixed by at least one non-trivial element $h \in H(E)$. It is defined over F by Galois descent. There exists a finite separable field extension L/E , and a point $b \neq 0 \in B(L)$, which does not lie in $\Phi(L)$, nor in $C(L)$. We then have a separable zero-cycle $[b]$ in the F -variety B , of degree $[L : F]$. Define $Y \subset B$ as the disjoint union of $[b]$ and C . We claim that Y has the required property. Indeed, let $f \in \mathbf{Aut}(B)(\overline{F}[\epsilon])$ be an automorphism stabilizing Y - or more accurately, $Y \times_F \overline{F}[\epsilon] \subset B \times_F \overline{F}[\epsilon]$. Then, f permutes the two connected components of the scheme $Y \times_F \overline{F}[\epsilon]$. For dimension reasons, it preserves $C \times_F \overline{F}[\epsilon]$ on the one hand, and $[b] \times_F \overline{F}[\epsilon]$ on the other hand. From the first fact, we know that f belongs to $H(\overline{F})$; in particular, it is defined over E , hence over L . From the latter fact, we get $f(b) = b$, hence $f = \text{Id}$. The Lemma is proved in this case.

Assume now that $B = B_1 \times \dots \times B_n$, where the B_i 's are F -simple abelian varieties. We can then adapt the preceding proof, as follows. For each i , let $C_i \subset B_i$, $L_i/E_i/F$ and $b_i \in B(L_i)$ be as in the first part of the proof. We can fulfill the extra requirements that no C_i passes through 0, and that the C_i 's are of different genus (using Proposition 4.3). In particular, when $i \neq j$, \overline{C}_i is not \overline{F} -isomorphic to \overline{C}_j . We can also assume that $L_i = L$ and $E_i = E$ are independent of i . Set

$$b := (b_1, \dots, b_n) \in B(L).$$

Define Y to be the disjoint union of $[b]$, and of the n curves

$$C_i \simeq \{0\} \times \dots \times \{0\} \times C_i \times \{0\} \times \dots \times \{0\} \hookrightarrow B.$$

It is not hard to see, that Y enjoys the required property.

In general, write $B = (\prod_1^r B_j)/\mu$, where S_1, \dots, S_r are F -simple abelian varieties, and where μ is a finite F -subgroup, intersecting trivially each coordinate axis. We can choose

$$Y \hookrightarrow B_1 \times \dots \times B_n$$

as in the previous part of the proof, and such that the composite

$$Y \hookrightarrow B_1 \times \dots \times B_n \xrightarrow{\text{can}} \left(\prod_1^r B_j \right) / \mu = B$$

is a closed immersion, identifying Y to a smooth closed subvariety of B .

An automorphism of B stabilizing $Y \subset B$ then lifts, via the quotient *can*, to an automorphism of $B_1 \times \dots \times B_n$ stabilizing $Y \subset B_1 \times \dots \times B_n$. We conclude as before. \square

5. Proof of the implication (1) \Rightarrow (2)

Let A/F be an abelian variety, such that $G := \text{Aut}(\bar{A})$ is finite. We give a construction of a smooth projective F -variety X , such that $A = \mathbf{Aut}(X)$, in several steps.

5.1. Construction of X

Denote by g the dimension of A .

Let $n \geq 1$ be an integer, invertible in F , such that the action of G on $A[n](F_s) \simeq (\mathbb{Z}/n)^{2g}$ is faithful. Such an n exists: use that G is finite, and that torsion points of order prime to $\text{char}(F)$ in $A(\bar{F})$ are Zariski-dense in A .

Let B_s be an abelian variety over F_s , of dimension $g' \geq g$, such that

$$\text{Hom}_{gp}(\bar{A}, \bar{B}) = \text{Hom}_{gp}(\bar{B}, \bar{A}) = 0.$$

Since \bar{A} has a finite number of simple components (up to isogeny), which are all defined over F_s by Lemma 4.4, the existence of B_s follows from Lemma 4.5. For example, take for B_s a product of simple abelian varieties, of dimensions greater than that of the simple components of \bar{A} .

Let E/F , be the finite Galois extension, with group Γ , which is minimal w.r.t. the following properties.

- (1) The extension E/F splits the F -group of multiplicative type $A[n]$. In other words, $A[n](E) \simeq (\mathbb{Z}/n)^{2g}$.
- (2) The abelian variety B_s is defined over E : there exists an abelian E -variety B_E , such that $B_E \times_E F_s \simeq B_s$.
- (3) Same as (1), for B_E : we have $B_E[n](E) \simeq (\mathbb{Z}/n)^{2g'}$.

Using (1), we view $A[n](E)$ as a $(\mathbb{Z}/n)[\Gamma]$ -module.

Introduce the Weil restriction of scalars

$$B_1 := R_{E/F}(B_E).$$

Geometrically, we have $\bar{B}_1 \simeq \bar{B}_s^m$, where m is the cardinality of Γ .

We have

$$B_1[n] = R_{E/F}((\mathbb{Z}/n)^{2g'}),$$

so that E/F splits $B_1[n]$, and $B_1[n](E)$ is a free $(\mathbb{Z}/n)[\Gamma]$ -module of rank $2g'$.

LEMMA 5.1. — *There exists an embedding of $(\mathbb{Z}/n\mathbb{Z})[\Gamma]$ -modules*

$$A[n](E) \hookrightarrow B_1[n](E);$$

that is to say, an embedding of finite étale F -group schemes

$$\iota : A[n] \hookrightarrow B_1[n].$$

Proof. — We give two (seemingly) different proofs.

The first one uses the perfect duality

$$(\cdot)^\vee := \text{Hom}(\cdot, \mathbb{Z}/n),$$

in the category of $(\mathbb{Z}/n)[\Gamma]$ -modules. Pick a generating set $t_1, \dots, t_{2g'}$ of the \mathbb{Z}/n -module $A[n](E)^\vee$ which is free of rank $2g \leq 2g'$. Introduce the surjection of $(\mathbb{Z}/n)[\Gamma]$ -modules

$$\begin{aligned} (\mathbb{Z}/n)[\Gamma]^{2g'} &\longrightarrow A[n](E)^\vee, \\ e_i &\mapsto t_i, \end{aligned}$$

where e_i denotes the i -th element of the canonical basis. Dualizing it yields an injection of $(\mathbb{Z}/n)[\Gamma]$ -modules

$$\iota : A[n](E) \longrightarrow (\mathbb{Z}/n)[\Gamma]^{2g'} \simeq B_1[n],$$

concluding the construction.

The second proof is more conceptual. Choose an embedding of constant E -group schemes

$$(\mathbb{Z}/n)^{2g} \simeq A_E[n] \hookrightarrow B_E[n] \simeq (\mathbb{Z}/n)^{2g'},$$

which exists simply because $g \leq g'$.

Applying $R_{E/F}$ yields an embedding of F -group schemes

$$R_{E/F}(A_E)[n] \hookrightarrow R_{E/F}(B_E)[n] = B_1[n].$$

Composing it with the natural embedding of F -groups

$$A[n] \hookrightarrow R_{E/F}(A_E)[n],$$

arising by adjunction from the identity of $A_E[n]$, we get the desired ι . \square

Form the exact sequence of algebraic F -groups

$$0 \longrightarrow A[n] \xrightarrow{a \mapsto (a, \iota(a))} A \times B_1 \xrightarrow{\pi} B_2 \longrightarrow 0.$$

Its cokernel B_2 is an abelian variety over F .

We have F -embeddings

$$A \xrightarrow{a \mapsto (a,0)} B_2$$

and

$$B_1 \xrightarrow{b_1 \mapsto (0,b_1)} B_2.$$

Introduce the quotient

$$q : B_2 \longrightarrow B_3 := B_2/A \simeq B_1/\iota(A[n]).$$

Let $Y_3 \subset B_3$ be a smooth F -subvariety, enjoying the properties of Lemma 4.6, where we take B to be our B_3 , and set $Y_3 := Y$.

Put

$$Y_2 := q^{-1}(Y_3).$$

The restriction

$$q|_{Y_2} : Y_2 \longrightarrow Y_3$$

is an A -torsor.

We now define

$$X := \text{Bl}_{Y_2}(B_2)$$

to be the blowup of Y_2 in B_2 .

5.2. Proof that $\mathbf{Aut}(X) \simeq A$

Translating by elements of A inside B_2 yields a natural arrow

$$A \longrightarrow \mathbf{Aut}(B_2).$$

Since $Y_2 \subset B_2$ is stable by these translations, we get an induced arrow of F -group schemes

$$\tau : A \longrightarrow \mathbf{Aut}(X).$$

It is clear that τ is an embedding. We are going to show that it is an isomorphism.

Let us first check that it induces a bijection

$$A(\bar{F}) \xrightarrow{\sim} \mathbf{Aut}(\bar{X}) = \mathbf{Aut}(X)(\bar{F}).$$

Pick $\phi \in \mathbf{Aut}(\bar{X})$. It induces a birational isomorphism f_2 of the \bar{F} -variety \bar{B}_2 , which is a regular isomorphism since B_2 is an abelian variety. Thus, we get a commutative diagram

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\phi} & \bar{X} \\ \downarrow & & \downarrow \\ \bar{B}_2 & \xrightarrow{f_2} & \bar{B}_2, \end{array}$$

where the vertical arrows are the structure morphism of the blowup.

Using Lemma 4.1, we get $f_2(\bar{Y}_2) = \bar{Y}_2$. We know that

$$f_2(x) = g_2(x) + t_2,$$

where $g \in \text{Aut}_{gp}(\bar{B}_2)$, and $t_2 \in B_2(\bar{F})$. We have to show that $g_2 = \text{Id}$ and $t_2 \in A(\bar{F})$. To do so, we can assume without loss of generality that $t_2 \in B_1(\bar{F})$.

We then have to prove $g_2 = \text{Id}$ and

$$t_2 \in A(\bar{F}) \cap B_1(\bar{F}) = \iota(A[n])(\bar{F}).$$

Geometrically, $\bar{B}_1 \simeq \bar{B}_s^m$. Since $\text{Hom}_{gp}(\bar{A}, \bar{B}_s) = \text{Hom}_{gp}(\bar{B}_s, \bar{A}) = 0$, we get

$$\text{Hom}_{gp}(\bar{A}, \bar{B}_1) = \text{Hom}_{gp}(\bar{B}_1, \bar{A}) = 0.$$

Therefore g_2 leaves $\bar{A} \subset \bar{B}_2$ and $\bar{B}_1 \subset \bar{B}_2$ stable.

We infer that g_2 lifts, via $\bar{\pi}$, to a diagonal group automorphism

$$\delta = (h, g_1)$$

of $\bar{A} \times \bar{B}_1$, which automatically leaves the diagonally embedded $\bar{A}[n]$ stable.

Consider the automorphism of \bar{B}_1 given by

$$f_1(b_1) := g_1(b_1) + t_2,$$

and the diagonal automorphism of $\bar{A} \times \bar{B}_1$ given by

$$\Delta(a, b_1) := (h(a), f_1(b_1)).$$

Since δ leaves $\bar{A} \times \iota(\bar{A}[n]) \subset \bar{A} \times \bar{B}_1$ stable, there exists $f_3 \in \text{Aut}(\bar{B}_3)$ such that the diagram

$$\begin{array}{ccc} \bar{A} \times \bar{B}_1 & \xrightarrow{\Delta} & \bar{A} \times \bar{B}_1 \\ \downarrow \bar{\pi} & & \downarrow \bar{\pi} \\ \bar{B}_2 & \xrightarrow{f_2} & \bar{B}_2 \\ \downarrow \bar{q} & & \downarrow \bar{q} \\ \bar{B}_3 & \xrightarrow{f_3} & \bar{B}_3 \end{array}$$

commutes.

Because $f_2(\bar{Y}_2) = \bar{Y}_2$, we get $f_3(\bar{Y}_3) = \bar{Y}_3$. By Lemma 4.6, we conclude that $f_3 = \text{Id}$. Hence, we have $t_2 \in \iota(A[n])(\bar{F})$ and $g_2 = \text{Id}$. Since δ preserves the diagonally embedded $\bar{A}[n]$, we get that h , restricted to $\bar{A}[n] \subset \bar{A}$, is the identity. Since G acts faithfully on $A[n]$, we conclude that $h = \text{Id}$. Hence, $g_2 = \text{Id}$ as well, and our job is done.

We have proved that τ induces a bijection on \overline{F} -points. If F has characteristic zero, this is enough to conclude that τ is an isomorphism of algebraic F -groups. In general, it remains to check that the F -linear map on tangent spaces

$$d_e(\tau) : \mathrm{Lie}(A) \longrightarrow \mathrm{Lie}(\mathbf{Aut}(X))$$

is bijective. Recall that $\mathrm{Lie}(\mathbf{Aut}(X))$ is the space of vector fields on X ; that is, global section of the tangent bundle $TX \rightarrow X$. Let

$$s : X \longrightarrow TX$$

be such a section. Restricting s to the complement of the exceptional divisor, we get a global section σ' of the tangent bundle of $B_2 - Y_2$. Since B_2 is an abelian variety, its tangent bundle is trivial, so that σ' is given by an arrow of F -varieties

$$\sigma' : B_2 - Y_2 \longrightarrow \mathbb{A}(\mathrm{Lie}(B_2)),$$

with target an affine space of dimension $\dim(B_2)$. Since Z_2 has codimension ≥ 2 in B_2 , σ' extends to a morphism

$$\sigma : B_2 \longrightarrow \mathbb{A}(\mathrm{Lie}(B_2)),$$

which is constant because B_2/F is proper. Write $\sigma = t$, with $t \in \mathrm{Lie}(B_2)$. To conclude, we have to show $t \in \mathrm{Lie}(A)$.

For $y \in B_2(\overline{F})$, denote by

$$\alpha_y : \overline{B}_2 \longrightarrow \overline{B}_2$$

the \overline{F} -morphism given by

$$x \longmapsto x + y.$$

Recall that the linear isomorphisms

$$d_y \alpha_{-y} : T_y(\overline{B}_2) \xrightarrow{\sim} \mathrm{Lie}(B_2) \otimes_F \overline{F}$$

are used to trivialize the tangent bundle of B_2 .

Since σ lifts to a section of the tangent bundle of the blowup $\mathrm{Bl}_{Y_2}(B_2)$, Lemma 4.2 implies, when $y \in Y_2(\overline{F})$, that t belongs to

$$d_y \alpha_{-y}(T_y(\overline{Y}_2)) \subset \mathrm{Lie}(B_2) \otimes_F \overline{F}.$$

Taking a y lying above (via \tilde{q}) an isolated separable point of \overline{Y}_3 , we conclude that $t \in \mathrm{Lie}(A)$, as desired.

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Bibliography

- [1] J. BLANC & M. BRION, “Abelian varieties as automorphism groups of smooth projective varieties in arbitrary characteristics”, preprint, <http://www-fourier.univ-grenoble-alpes.fr/~mbrion/abelian.pdf>.
- [2] M. BRION, “Homomorphisms of algebraic groups: representability and rigidity”, *Mich. Math. J.* **72** (2022), p. 51-76.
- [3] F. CHARLES & B. POONEN, “Bertini irreducibility theorems over finite fields”, *J. Am. Math. Soc.* **29** (2016), no. 1, p. 81-94.
- [4] M. DEMAZURE & P. GABRIEL, *Groupes algébriques*, Masson, 1970.
- [5] E. W. HOWE & H. J. ZHU, “On the Existence of Absolutely Simple Abelian Varieties of a Given Dimension over an Arbitrary Field”, *J. Number Theory* **92** (2002), no. 1, p. 139-163.
- [6] D. LOMBARDO & A. MAFFEI, “Abelian varieties as automorphism groups of smooth projective varieties”, *Int. Math. Res. Not.* **2020** (2020), no. 7, p. 1921-1932.
- [7] G. MARTIN, “Infinitesimal automorphisms of algebraic varieties and vector fields on elliptic surfaces”, 2020, <https://arxiv.org/abs/2004.07227>.
- [8] H. MATSUMURA & F. OORT, “Representability of group functors, and automorphisms of algebraic schemes”, *Invent. Math.* **4** (1967), p. 1-25.
- [9] S. MORI, “The endomorphism rings of some abelian varieties”, *Jpn. J. Math.* **3** (1977), p. 105-109.
- [10] B. POONEN, “Bertini theorems over finite fields”, *Ann. Math.* **160** (2005), no. 3, p. 1099-1127.