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On the variational interpretation of local logarithmic Sobolev inequalities ^(*)

GAUTHIER CLERC ⁽¹⁾, GIOVANNI CONFORTI ⁽²⁾ AND IVAN GENTIL ⁽³⁾

ABSTRACT. — Otto calculus has established itself as a powerful tool for proving quantitative energy dissipation estimates and provides with an elegant geometric interpretation of certain functional inequalities such as the Logarithmic Sobolev inequality. However, the *local* versions of such inequalities, which can be proven by means of Bakry–Émery–Ledoux Γ -calculus, have not yet been given an interpretation in terms of this Riemannian formalism. In this short note we close this gap by explaining how Otto calculus applied to the Schrödinger problem yields a variations interpretation of local logarithmic Sobolev inequalities that could possibly unveil a novel class of local inequalities.

RÉSUMÉ. — Le calcul d’Otto est un outil puissant pour quantifier la dissipation d’énergie mais surtout il propose une interprétation géométrique simple de certaines inégalités fonctionnelles comme l’inégalité de Sobolev logarithmique. La version *locale* de ces inégalités, dont la démonstration repose sur le Γ -calcul développé par Bakry, Émery et Ledoux, n’avait pas encore une interprétation géométrique à la Otto. Dans cette courte note, nous comblons cette lacune et montrons comment le calcul d’Otto appliqué au problème de Schrödinger permet une interprétation variationnelle des inégalités de type Sobolev logarithmique locale, ce qui pourrait être un espoir pour l’élaboration de nouvelles d’inégalités locales.

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1. Introduction

Curvature dimension conditions

We are working on a (N, \mathfrak{g}) , a smooth, connected and compact Riemannian manifold without boundary. The heat equation, $\partial_t u = \Delta u$, starting from f is denoted $(t, x) \mapsto P_t f(x)$, where

$$P_t f(x) = \int f(y) p_t^x(y) dy, \quad x \in N,$$

and $p_t^x(y)$ is the density of the heat kernel with respect to the Riemannian measure dy . $\Delta = \nabla \cdot \nabla$ is the Laplace–Beltrami operator ($\nabla \cdot$ is the divergence operator whereas ∇ is the usual gradient operator). $(P_t)_{t \geq 0}$ is a reversible Markov semigroup in the sense of [7]. Next, we introduce the carré du champ operator Γ and its iterated form Γ_2 , they are defined by

$$\Gamma(f) = |\nabla f|^2, \quad \Gamma_2(f) = \frac{1}{2} \Delta \Gamma(f) - \Gamma(f, \Delta f).$$

In particular, Böchner’s formula yields the equivalent expression of Γ_2 in terms of Ricci’s curvature:

$$\Gamma_2(f) = \|\nabla f\|_{\text{H.S.}}^2 + \text{Ric}_{\mathfrak{g}}(\nabla f).$$

Following the seminal paper [6], we introduce for $\rho \in \mathbf{R}$, $n \in [1, \infty]$ the curvature-dimension condition $CD(\rho, n)$ by asking that

$$\Gamma_2(f) \geq \rho \Gamma(f) + \frac{1}{n} (\Delta f)^2,$$

holds for any smooth function f . In view of Böchner’s formula, the $CD(\rho, n)$ condition is equivalent to the fact that $\text{Ric}_{\mathfrak{g}} \geq \rho$ and $d \leq n$ where d is the dimension of N .

We also need to introduce the dual semigroup $(P_t^*)_{t \geq 0}$ acting on absolutely continuous measures $\mu \in \mathcal{P}(N)$ as follows

$$P_t^*(\mu) = P_t \left(\frac{d\mu}{dx} \right) dx. \tag{1.1}$$

For the sake of clarity, we limit the discussion in the present note to the Laplace–Beltrami operator in a compact Riemannian Manifold instead a general diffusion Markov operator. However, all of the results we are going to present, can be formulated in such general context following the paradigm of [7].

Bakry–Émery–Ledoux calculus and local logarithmic Sobolev inequalities

Bakry–Émery–Ledoux calculus, known as Γ calculus, has been used as a powerful tool to analyse the quantitative behaviour of the heat semigroup $(P_t)_{t \geq 0}$ in various ways. Among its main achievements are *local* logarithmic Sobolev inequalities, proven under $CD(\rho, \infty)$ and the $CD(0, n)$ conditions⁽¹⁾.

- (1) *Local inequalities under $CD(\rho, \infty)$, cf. [3].* Under the $CD(\rho, \infty)$ condition the following inequalities are valid for all positive functions g and $T > 0$:

- The mild gradient commutation estimate

$$\frac{\Gamma(P_T g)}{P_T g} \leq e^{-2\rho T} P_T \left(\frac{\Gamma(g)}{g} \right). \quad (1.2)$$

- The local logarithmic Sobolev inequality

$$P_T(g \log g) - P_T g \log P_T g \leq \frac{1 - e^{-2\rho T}}{2\rho} P_T \left(\frac{\Gamma(g)}{g} \right). \quad (1.3)$$

- The reverse local logarithmic Sobolev inequality

$$P_T(g \log g) - P_T g \log P_T g \geq \frac{e^{2\rho T} - 1}{2\rho} \frac{\Gamma(P_T g)}{P_T g}. \quad (1.4)$$

- (2) *Local inequality under the $CD(0, n)$ condition, cf. [9].* Under the $CD(\rho, \infty)$ condition the following inequalities are valid for all positive functions g and $T > 0$:

- The local logarithmic Sobolev inequality

$$\begin{aligned} \frac{n}{2T} P_T g \exp \left[\frac{2}{nP_T g} (P_T(g \log g) - P_T g \log P_T g - TLP_T g) \right] \\ \leq P_T \left(\frac{\Gamma(g)}{g} \right) - \Delta P_T g + \frac{n}{2T} P_T g, \end{aligned} \quad (1.5)$$

and in particular

$$\frac{\Delta P_T g}{P_T g} - \frac{1}{P_T g} P_T \left(\frac{\Gamma(g)}{g} \right) \leq \frac{n}{2T} \quad (1.6)$$

⁽¹⁾ Extension to the general $CD(\rho, n)$ condition can be found in [5]. However, since $CD(\rho, \infty)$ and $CD(0, n)$ are the most relevant settings in applications, we limit the present discussion to these two scenarios.

- The reverse local logarithmic Sobolev inequality

$$\frac{n}{2T} P_T g \exp \left[-\frac{2}{nP_T g} (P_T(g \log g) - P_T g \log P_T g - TLP_T g) \right] \leq -\frac{\Gamma(P_T g)}{P_T g} + \Delta P_T g + \frac{n}{2T} P_T g, \quad (1.7)$$

which yields as a by product the celebrated Li-Yau inequality [9, 20].

$$\frac{\Gamma(P_T g)}{(P_T g)^2} - \frac{\Delta P_T g}{P_T g} \leq \frac{n}{2T}. \quad (1.8)$$

Remark 1.1.

- The main interest of such inequalities is that there are local, in the sense that they give estimates on the heat kernel itself instead of the entropy of the heat kernel. Moreover, in contrast with their integrated versions, these inequalities imply back the corresponding curvature-dimension conditions, i.e. inequalities (1.2)–(1.4) imply $CD(\rho, \infty)$, and (1.5) and (1.7) imply $CD(0, n)$.
- Inequalities (1.2)–(1.8) are also proved in [7, Theorem 5.5.2 and Theorem 6.7.3].
- Many others inequalities have been proved using the same kind of method. Unfortunately, it is actually not possible to reach it with Otto Calculus and the Schrödinger problem. Let mention for instance the isoperimetric inequality under the $CD(\rho, \infty)$ condition proved in [8], the local hypercontractivity inequality from [4] etc.

Otto calculus

The term Otto calculus, introduced by C. Villani in his book [23], refers to a Riemannian formalism on the space of probability measures $M = \mathcal{P}(N)$ that allows, among other things, to interpret the Wasserstein distance as a geodesic distance and the heat equation as the gradient flow of the entropy. Developed in the seminal papers [18, 21, 22], it has led to major advances in the quantifying trend to equilibrium of possibly non linear PDEs, to the discovery of new functional inequalities as well as to a new geometric interpretation of classical ones. We briefly present here the main ideas that lead to the definition of the Riemannian metric known as Otto metric. Our heuristic presentation is based on [14], to which we refer for more details (see also [13] for an informal presentation in French). However, given its importance, there are many other references that provide with an introduction to

Otto calculus, see e.g. the monograph [15] for rigorous constructions, [1] for a user-friendly presentation and the book [2, Chapter 1].

We begin by introducing the two main functionals of interest for this work: the Wasserstein distance and the relative entropy. The first one is defined for all $\mu, \nu \in \mathcal{P}_2(N)$ by

$$W_2(\mu, \nu) = \inf \sqrt{\iint d(x, y)^2 d\pi(x, y)},$$

where the infimum runs over all $\pi \in \mathcal{P}(N \times N)$ with marginals μ and ν and $\mathcal{P}_2(N)$ is the set of probability measures with finite second moment (since in our case N is compact, $\mathcal{P}_2(N) = \mathcal{P}(N)$). For any probability measure $\mu \in \mathcal{P}(N)$ the entropy with respect to the Riemannian measure is given by

$$\mathcal{F}(\mu) = \begin{cases} \int \log\left(\frac{d\mu}{dx}\right) d\mu, & \text{if } d\mu \ll dx, \\ +\infty, & \text{otherwise.} \end{cases} \quad (1.9)$$

We recall that a path $[0, 1] \ni t \mapsto \mu_t \in \mathcal{P}_2(N)$ is absolutely continuous if there exists a non negative function $l \in L^2([0, 1])$ such that for any $0 \leq s \leq t \leq 1$,

$$W_2(\mu_t, \mu_s) \leq \int_s^t l(r) dr.$$

In that case, one can define

$$|\dot{\mu}_t| := \limsup_{s \rightarrow t} \frac{W_2(\mu_t, \mu_s)}{|t - s|} \in L^2([0, 1]).$$

Moreover, for any absolutely continuous path $(\mu_t)_{t \geq 0}$, there exists a unique vector field $(t, x) \mapsto V_t(x)$ such that $\int |V_t|^2 d\mu_t < \infty$ and $|\dot{\mu}_t|^2 = \int |V_t|^2 d\mu_t$ (a.e. in $[0, 1]$). The vector field V_t is in fact a limit in $L^2(\mu_t)$ of gradient of smooth functions in N . For every $t \in [0, 1]$ we set

$$\dot{\mu}_t := V_t, \quad (1.10)$$

and we call $\dot{\mu}_t$ the velocity of the path $(\mu_t)_{t \in [0, 1]}$ at time t . Finally, the vector field V_t is a weak solution of the continuity equation

$$\partial_t \mu_t + \nabla \cdot (\mu_t V_t) = 0. \quad (1.11)$$

To give an example, in the case of the heat equation $\partial_t \nu_t = \Delta \nu_t$, the velocity of the path $(\nu_t)_{t \geq 0}$ is

$$\dot{\nu}_t = -\nabla \log \frac{d\nu_t}{dx}. \quad (1.12)$$

The tangent space at $\mu \in \mathcal{P}_2(N)$ is identified with

$$T_\mu \mathcal{P}_2(N) = \overline{\{\nabla \varphi, \varphi : N \mapsto \mathbf{R}, \varphi \in C^\infty(N)\}}^{L_2(\mu)}.$$

The Riemannian metric on $T_\mu\mathcal{P}_2(N)$ is then defined via the scalar product $L^2(\mu)$,

$$\langle \nabla\varphi, \nabla\psi \rangle_\mu = \int \Gamma(\varphi, \psi) d\mu, \quad \nabla\varphi, \nabla\psi \in T_\mu\mathcal{P}_2(N).$$

Such metric is often referred to the Otto metric and it can be seen that the geodesics associated to the Otto metric are the displacement interpolations of optimal transport. Using this, a straightforward computation implies that the gradient of the entropy \mathcal{F} at μ is given by

$$\text{grad}_\mu \mathcal{F} = \nabla \log \left(\frac{d\mu}{dx} \right) \in T_\mu\mathcal{P}_2(N).$$

Accordingly, we can rewrite the Fisher information functional \mathcal{I} as

$$\mathcal{I}(\mu) := |\text{grad}_\mu \mathcal{F}|_\mu^2 =: \mathbf{\Gamma}(\mathcal{F})(\mu), \tag{1.13}$$

where $\mathbf{\Gamma}(\mathcal{F})$ can be interpreted as the carré du champ operator applied to the functional \mathcal{F} .

In light of (1.12), we can now view the semigroup $(P_t^*)_{t \geq 0}$ as the gradient flow of the function \mathcal{F} , that is to say

$$\dot{\nu}_t = -\text{grad}_{\nu_t} \mathcal{F}.$$

Now, we turn our attention to the second order calculus introducing covariant derivatives and Hessians. A remarkable fact is that the Hessian of the entropy \mathcal{F} can be expressed in terms of the Γ_2 operator. In fact, we have (see for instance [22] and [14, Section 3.3]),

$$\forall \mu \in \mathcal{P}_2(N), \nabla\varphi, \nabla\psi \in T_\mu\mathcal{P}_2(N), \quad \text{Hess}_\mu \mathcal{F}(\nabla\varphi, \nabla\psi) = \int \Gamma_2(\varphi, \psi) d\mu.$$

From the work of [12], the curvature-dimension condition $CD(\rho, n)$ ($\rho \in \mathbf{R}$, $n > 0$) is equivalent to the differential inequality

$$\forall \mu \in \mathcal{P}_2(N), \nabla\varphi \in T_\mu\mathcal{P}_2(N),$$

$$\text{Hess}_\mu \mathcal{F}(\nabla\varphi, \nabla\varphi) \geq \rho |\nabla\varphi|_\mu^2 + \frac{1}{n} \langle \text{grad}_\mu \mathcal{F}, \nabla\varphi \rangle_\mu. \tag{1.14}$$

As in a finite dimensional Riemannian manifold, the acceleration of a curve is defined as the covariant derivative of the velocity field along the curve itself. Recalling the definition of velocity $\dot{\mu}_t$ we gave at (1.10), it turns out that the acceleration, which we denote $\ddot{\mu}_t$ is given by

$$\ddot{\mu}_t = \nabla \left(\frac{d}{dt} \varphi_t + \frac{1}{2} |\nabla\varphi_t|^2 \right) \in T_{\mu_t}\mathcal{P}_2(N), \tag{1.15}$$

where $\dot{\mu}_t = \nabla\varphi_t$ in the above equation.

2. Some simple inequalities on a toy model

In this section we show a series of inequalities inspired by a toy model proposed in [14]. This model, in a very simple way, captures some geometric features of the Schrödinger problem. Such inequalities will translate effortlessly to the true Schrödinger problem by means of Otto calculus, at least formally, and in this latter form they will be seen to be equivalent to local logarithmic inequalities, thus delivering the main message of this note. We shall do so in Section 3.

To build the toy model we consider a function $F : \mathbf{R}^d \rightarrow \mathbf{R}$ be a twice differentiable function with $d > 0$. We note F' (resp. F'') the gradient (resp. the Hessian) of F . For every $T > 0$ and $x, y \in \mathbf{R}^d$, the toy model is the following optimization problem

$$C_T(x, y) = \inf \left\{ \int_0^T [|\dot{\omega}_t|^2 + |F'(\omega_t)|^2] dt \right\}, \quad (2.1)$$

where the infimum taken over all smooth paths from $[0, T]$ to \mathbf{R}^d such that $\omega_0 = x$ and $\omega_T = y$. A standard variational argument shows that any minimizer $(X_t^T)_{t \in [0, T]}$ of (2.1) satisfies Newton's system

$$\begin{cases} \ddot{X}_t^T = \frac{1}{2}(|F'|^2)'(X_t^T) = F''(X_t^T)F'(X_t^T), \\ X_0^T = x, \quad X_T^T = y, \end{cases} \quad (2.2)$$

and is called an F -interpolation between x and y . From (2.2) we also deduce that

$$E_T(x, y) = |\dot{X}_t^T|^2 - |F'(X_t^T)|^2,$$

is conserved along optimal curves, i.e. it is a constant function of t . E_T plays the role of the conserved total (kinetic + potential) energy of a physical system. Mimicking (1.14), we say that F is (ρ, n) -convex if

$$F'' \geq \rho \text{Id} + \frac{1}{n} F' \otimes F'.$$

We proceed to show how the announced inequalities hold under either in the (ρ, ∞) or the $(0, n)$ setting. In both cases, they will be consequences of convexity estimates for the function

$$\lambda(t) = \int_0^t |\dot{X}_s^T + F'(X_s^T)|^2 ds,$$

where $(X_t^T)_{t \in [0, T]}$ is an F -interpolation. Note that $\lambda(T) = C_T(x, y) + 2(F(y) - F(x))$. Note that $\lambda(t)$ is a measure of how much the F -interpolation deviates from being a gradient flow.

- (1) *Estimates under (ρ, ∞) -convexity with $\rho \in \mathbf{R}$, cf. [11, 14].* A straightforward calculation using (ρ, ∞) -convexity and Newton's law yields that λ verifies the differential inequality

$$\frac{d^2}{dt^2}\lambda \geq 2\rho \frac{d}{dt}\lambda,$$

which can be integrated in various ways giving the following bounds

$$\frac{d}{dt}\lambda(0) \leq e^{-2\rho T} \frac{d}{dt}\lambda(T), \quad \lambda(T) - \lambda(0) \leq \frac{1 - e^{-2\rho T}}{2\rho} \frac{d}{dt}\lambda(T),$$

and

$$\lambda(T) - \lambda(0) \geq \frac{e^{2\rho T} - 1}{2\rho} \frac{d}{dt}\lambda(0).$$

We have therefore established the inequalities

$$\begin{aligned} |\dot{X}_0^T + F'(x)|^2 &\leq e^{-2\rho T} |\dot{X}_T^T + F'(y)|^2, \\ C_T(x, y) + 2(F(y) - F(x)) &\leq \frac{1 - e^{-2\rho T}}{2\rho T} |\dot{X}_T^T + F'(y)|^2, \\ C_T(x, y) + 2(F(y) - F(x)) &\geq \frac{e^{2\rho T} - 1}{2\rho T} |\dot{X}_0^T + F'(x)|^2. \end{aligned} \tag{2.3}$$

- (2) *Estimates under the $(0, n)$ -convexity with $n > 0$, cf. [10].* In this case we consider the map $\Phi(t) = \lambda(t) - tE_T(x, y)$ and observe that $(0, n)$ and Newton's law combined yield

$$\Phi'' \geq \frac{1}{2n} \Phi'^2,$$

which is equivalent to concavity of the map $[0, T] \ni t \mapsto \exp(-\frac{\Phi(t)}{2n})$. But then, using the basic convexity estimates

$$\begin{aligned} \exp\left(\frac{1}{2n}(\Phi(T) - \Phi(0))\right) &\leq 1 + \frac{1}{2n}T\Phi'(T) \\ \text{and } \exp\left(\frac{1}{2n}(\Phi(0) - \Phi(T))\right) &\leq 1 - \frac{1}{2n}T\Phi'(0), \end{aligned}$$

we obtain the inequalities

$$\begin{aligned} \exp\left(\frac{1}{2n}[C_T(x, y) + 2F(y) - 2F(x) - TE_T(x, y)]\right) \\ \leq 1 + \frac{T}{2n}(|\dot{X}_T^T + F'(y)|^2 - E_T(x, y)), \end{aligned}$$

$$\begin{aligned} \exp\left(-\frac{1}{2n}[C_T(x, y) + 2F(y) - 2F(x) - TE_T(x, y)]\right) \\ \leq 1 - \frac{T}{2n}(|\dot{X}_0^T + F'(x)|^2 - E_T(x, y)). \end{aligned}$$

In particular,

$$\begin{aligned} E_T(x, y) &\leq \frac{2n}{T} + |\dot{X}_0^T + F'(x)|^2, \\ E_T(x, y) &\geq -\frac{2n}{T} + |\dot{X}_T^T + F'(y)|^2. \end{aligned}$$

3. The Schrödinger problem

We now move on to analyse the Schrödinger problem which we obtain from the toy model discussed above by replacing the underlying space is $\mathcal{P}_2(N)$ and equipping it with the Otto metric, see [14, 16]. We remark that this is not the original formulation of the Schrödinger problem which, by means of Sanov's theorem and large deviations theory is usually cast as an entropy minimization problem on path space, see the survey [19].

DEFINITION 3.1 (Schrödinger problem). — *For any probability measures $\mu, \nu \in \mathcal{P}_2(N)$, we define the entropic cost $\mathcal{C}_T(\mu, \nu)$ as*

$$\mathcal{C}_T(\mu, \nu) = \inf \left\{ \int_0^T [|\dot{\mu}_s|^2_{\mu_s} + \mathbf{\Gamma}(\mathcal{F})(\mu_s)] ds \right\} \in [0, \infty], \quad (3.1)$$

where the infimum runs over all absolutely continuous paths $(\mu_s)_{s \in [0, T]}$ satisfying $\mu_0 = \mu$ and $\mu_T = \nu$ and $\mathbf{\Gamma}(\mathcal{F})$ has been defined in (1.13). Minimizers of $\mathcal{C}_T(\mu, \nu)$ are called entropic interpolations between μ and ν .

Let us now recall some fundamental properties of entropic interpolations. For a detailed account, we refer to [19].

- For any $\mu, \nu \in \mathcal{P}(N)$, absolutely continuous with a positive \mathcal{C}^∞ density with respect to dx , then there exists $f, g \in \mathcal{C}^\infty(N, (0, \infty))$ such that minimizers of (3.1) $(\mu_t^T)_{t \in [0, T]}$ are given by,

$$\mu_t^T = P_t f P_{T-t} g dx. \quad (3.2)$$

Moreover, the couple (f, g) is unique up to a multiplicative constant and the velocity field of the entropic interpolation is given by

$$\dot{\mu}_t^T = \nabla \log \frac{P_{T-t} g}{P_t f}. \quad (3.3)$$

- Entropic interpolations are solution to a second order equation, akin to Newton's law (2.2). This fact has been shown in [11, Theorem 1.2], providing a geometric interpretation of earlier results of Gigli and Tamanini [17]. In the work [14, Section 3.3], the considerations made in the above works, limited to the classical version of the Schrödinger

problem, have been extended to a much broader class of problems by means of an heuristic argument based on Otto calculus. In a nutshell, the entropic interpolation $(\mu_t^T)_{t \in [0, T]}$ is a solution of

$$\ddot{\mu}_t^T = \frac{1}{2} \operatorname{grad}_{\mu_t^T} \Gamma(\mathcal{F}) = \operatorname{Hess}_{\mu_t^T} \mathcal{F}(\operatorname{grad}_{\mu_t^T} \mathcal{F}) \in T_{\mu_t^T} \mathcal{P}_2(N), \quad (3.4)$$

where $(\dot{\mu}_t^T)_{t \in [0, T]}$ is the acceleration defined in (1.15), that is completely analogous to Newton's law (2.2). Moreover, the quantity

$$\mathcal{E}_T(\mu, \nu) = |\dot{\mu}_t^T|_{\mu_t^T}^2 - \Gamma(\mathcal{F})(\mu_t^T) = |\dot{\mu}_t^T|_{\mu_t^T}^2 - |\operatorname{grad}_{\mu_t^T} \mathcal{F}|_{\mu_t^T}^2 \quad (3.5)$$

is conserved, i.e. it doesn't depends on $t \in [0, T]$.

It remains to write the inequalities derived for the toy model in terms of entropic interpolations. By replacing the function $\lambda(t)$ used in section 2 with the function

$$\Lambda(t) = \int_0^t |\dot{\mu}_s^T + \operatorname{grad}_{\mu_s^T} \mathcal{F}|_{\mu_s^T}^2 ds,$$

where $(\mu_s^T)_{t \in [0, T]}$ is an entropic interpolation. We obtain, arguing exactly as before but working on $\mathcal{P}_2(N)$ and using the Riemannian formalism associated with the Otto metric the following estimates

- (1) *Estimates under $CD(\rho, \infty)$ with $\rho \in \mathbf{R}$, cf. [11, 14].* In this case we have that if $(\mu_t^T)_{t \in [0, T]}$ is the entropic interpolation between μ and ν :

$$|\dot{\mu}_0^T + \operatorname{grad}_{\mu} \mathcal{F}|_{\mu}^2 \leq e^{-2\rho T} |\dot{\mu}_T^T + \operatorname{grad}_{\nu} \mathcal{F}|_{\nu}^2, \quad (3.6)$$

$$\mathcal{C}_T(\mu, \nu) + 2(\mathcal{F}(\nu) - \mathcal{F}(\mu)) \leq \frac{1 - e^{-2\rho T}}{2\rho} |\dot{\mu}_T^T + \operatorname{grad}_{\nu} \mathcal{F}|_{\nu}^2, \quad (3.7)$$

$$\mathcal{C}_T(\mu, \nu) + 2(\mathcal{F}(\nu) - \mathcal{F}(\mu)) \geq \frac{e^{2\rho T} - 1}{2\rho} |\dot{\mu}_0^T + \operatorname{grad}_{\mu} \mathcal{F}|_{\mu}^2. \quad (3.8)$$

- (2) *Estimates under the $(0, n)$ -convexity with $n > 0$, cf. [10].* In this case we have that if $(\mu_t^T)_{t \in [0, T]}$ is the entropic interpolation between μ and ν :

$$\begin{aligned} & \exp\left(\frac{1}{2n} [\mathcal{C}_T(\mu, \nu) + 2\mathcal{F}(\nu) - 2\mathcal{F}(\mu) - T\mathcal{E}_T(\mu, \nu)]\right) \\ & \leq 1 + \frac{T}{2n} (|\dot{\mu}_T^T + \operatorname{grad}_{\nu} \mathcal{F}|_{\nu}^2 - \mathcal{E}_T(\mu, \nu)), \quad (3.9) \end{aligned}$$

$$\begin{aligned} & \exp\left(-\frac{1}{2n} [\mathcal{C}_T(\mu, \nu) + 2\mathcal{F}(\nu) - 2\mathcal{F}(\mu) - T\mathcal{E}_T(\mu, \nu)]\right) \\ & \leq 1 - \frac{T}{2n} (|\dot{\mu}_0^T + \operatorname{grad}_{\mu} \mathcal{F}|_{\mu}^2 - \mathcal{E}_T(\mu, \nu)). \quad (3.10) \end{aligned}$$

In particular,

$$\begin{aligned} \mathcal{E}_T(\mu, \nu) &\leq \frac{2n}{T} + |\dot{\mu}_T^T + \text{grad}_\nu \mathcal{F}|_\nu^2, \\ \mathcal{E}_T(\mu, \nu) &\geq -\frac{2n}{T} + |\dot{\mu}_0^T + \text{grad}_\mu \mathcal{F}|_\mu^2. \end{aligned} \tag{3.11}$$

4. Local Logarithmic Sobolev inequalities via Schrödinger problem and Otto calculus

The main contribution of this note is to show how the Bakry–Émery–Ledoux estimates (1.2) to (1.8) are in fact equivalent to the inequalities derives in Section 3 along entropic interpolations. The key to do this is the product formula (3.2).

The mild gradient commutation estimate (1.2)

Let us investigate in details the first inequality (3.6). Using (3.2) we find

$$\begin{aligned} |\dot{\mu}_t^T + \text{grad}_{\mu_t^T} \mathcal{F}|_{\mu_t^T}^2 &= \int \Gamma \left(\log \frac{P_{T-t}g}{P_t f} + \log(P_{T-t}g P_t f) \right) P_{T-t}g P_t f \, dx \\ &= 4 \int \frac{\Gamma(P_{T-t}g)}{P_{T-t}g} P_t f \, dx. \end{aligned} \tag{4.1}$$

Then, by reversibility of the semigroup $(P_t)_{t \geq 0}$, inequality (3.6) can be written as

$$\int \frac{\Gamma(P_T g)}{P_T g} f \, dx \leq e^{-2\rho T} \int P_T \left(\frac{\Gamma(g)}{g} \right) f \, dx. \tag{4.2}$$

But, for any positive functions f, g , the path

$$[0, T] \ni t \mapsto \frac{P_t f P_{T-t} g \, dx}{\int f P_T g \, dx} \tag{4.3}$$

is an entropic interpolation, then the previous inequality (4.2) is valid for any functions $f, g > 0$. If we now let $f \, dy \rightarrow \delta_x$ weakly, we get

$$\frac{\Gamma(P_T g)}{P_T g}(y) \leq e^{-2\rho T} P_T \left(\frac{\Gamma(g)}{g} \right)(y),$$

which is precisely (1.2).

The local logarithmic Sobolev inequalities (1.3)–(1.4) under $CD(\rho, \infty)$

Let us have a look to the inequality (3.7). First, since $(\mu_t^T)_{t \in [0, T]}$ is the entropic interpolation between μ and ν , then

$$\mathcal{C}_T(\mu, \nu) + 2(\mathcal{F}(\nu) - \mathcal{F}(\mu)) = \int_0^T |\dot{\mu}_s^T + \text{grad}_{\mu_s^T} \mathcal{F}|_{\mu_s^T}^2 ds,$$

and from (4.1),

$$\begin{aligned} \int_0^T |\dot{\mu}_t^T + \text{grad}_{\mu_t^T} \mathcal{F}|_{\mu_t^T}^2 dt &= 4 \int_0^T \int \frac{\Gamma(P_{T-t}g)}{P_{T-t}g} P_t f \, dx dt \\ &= 4 \int_0^T \int P_t \left(\frac{\Gamma(P_{T-t}g)}{P_{T-t}g} \right) f \, dx dt. \end{aligned}$$

We know that if $\Phi(t) = P_t(P_{T-t}g \log P_{T-t}g)$, ($t \in [0, T]$), then

$$\Phi'(t) = P_t \left(\frac{\Gamma(P_{T-t}g)}{P_{T-t}g} \right).$$

So, at the end,

$$\begin{aligned} \mathcal{C}_T(\mu, \nu) + 2(\mathcal{F}(\nu) - \mathcal{F}(\mu)) &= 4 \int \int_0^T \Phi'(t) dt f \, dx \\ &= 4 \int [P_T(g \log g) - P_T g \log P_T g] f \, dx. \end{aligned} \tag{4.4}$$

By using the path defined in (4.3), one can conclude that inequality (3.7) can be written as

$$\int [P_T(g \log g) - P_T g \log P_T g] f \, dx \leq \frac{1 - e^{-2\rho T}}{2\rho} \int P_T \left(\frac{\Gamma(g)}{g} \right) f \, dx,$$

for any functions $f, g > 0$. Again, if $f \, dx \rightarrow \delta_y$ ($y \in N$) weakly, the previous inequality becomes

$$P_T(g \log g)(y) - P_T g(y) \log P_T g(y) \leq \frac{1 - e^{-2\rho T}}{2\rho} P_T \left(\frac{\Gamma(g)}{g} \right)(y),$$

which is exactly the local logarithmic Sobolev inequality (1.3). Arguing similarly, inequality (3.8) is seen to be equivalent to the local reverse logarithmic Sobolev inequality (1.4).

Local logarithmic Sobolev inequalities (1.5)–(1.6) under $CD(0, n)$ and the Li-Yau inequality (1.8).

To understand the last inequalities, is it enough to compute the conserved quantity. We have,

$$\begin{aligned} \mathcal{E}_T(\mu, \nu) &= \int \left[\Gamma \left(\log \frac{P_{T-t}g}{P_t f} \right) - \Gamma(\log P_{T-t}g P_t f) \right] P_{T-t}g P_t f \, dx \\ &= -4 \int \Gamma(\log P_{T-t}g, \log P_t f) P_{T-t}g P_t f \, dx \\ &= -4 \int \Gamma(P_{T-t}g, P_t f) \, dx = 4 \int \Delta P_T g f \, dx. \end{aligned} \tag{4.5}$$

From the identities (4.4), (4.1) and (4.5), inequalities (3.9) becomes,

$$\begin{aligned} \exp \left(\frac{1}{2n} \left[4 \int [P_T(g \log g) - P_T g \log P_T g] f \, dx - 4T \int \Delta P_T g f \, dx \right] \right) \\ \leq 1 + \frac{T}{2n} \left(4 \int P_T \left(\frac{\Gamma(g)}{g} \right) f \, dx - 4 \int \Delta P_T g f \, dx \right). \end{aligned}$$

Using the path of probability measures (4.3), for any functions $f, g > 0$, we have

$$\begin{aligned} \exp \left(\frac{2}{n \int P_T g f \, dx} \left[\int [P_T(g \log g) - P_T g \log P_T g] f \, dx - T \int \Delta P_T g f \, dx \right] \right) \\ \leq 1 + \frac{2T}{n \int P_T g f \, dx} \left(\int P_T \left(\frac{\Gamma(g)}{g} \right) f \, dx - \int \Delta P_T g f \, dx \right). \end{aligned}$$

Again, if we let $f dx \rightarrow \delta_y$ ($y \in N$), we get inequality (1.5). A similar argument can be used for the equivalence between (3.10) and the reverse local inequality (1.7) and to show the equivalence of (3.11) and the Li-Yau inequality (1.8).

Remark 4.1. — When $\mu = \delta_y$ for some $y \in N$ and $\nu \in P(N)$, the entropic interpolation is explicit and given by the following path

$$[0, T] \ni t \mapsto \mu_t^T = p_t^y P_{T-t} \left(\frac{d\nu}{dx} \frac{1}{p_T^y} \right) dx. \tag{4.6}$$

It is interesting to notice that it is in fact, the path used in the computations of this note.

Conclusion

We have demonstrated how local logarithmic Sobolev inequalities (1.2)–(1.7) admit a nice geometrical interpretation via inequalities (3.6)–(3.10) by

means of Otto calculus and the Schrödinger problem. Such interpretation was first given for the integrated version of the logarithmic Sobolev inequality in [18] and our results are the first, to the best of our knowledge to cover the local versions. We hope that our considerations will be of help to understand local inequalities for general functionals satisfying the curvature dimension conditions beyond, thus going beyond the case of the Boltzmann entropy.

Bibliography

- [1] L. AMBROSIO & N. GIGLI, “A user’s guide to optimal transport”, in *Modelling and optimisation of flows on networks*, Lecture Notes in Mathematics, vol. 2062, Springer, 2013, p. 1-155.
- [2] L. AMBROSIO, N. GIGLI & G. SAVARÉ, *Gradient flows in metric spaces and in the space of probability measures*, 2nd ed., Birkhäuser, 2008, vii+334 pages.
- [3] D. BAKRY, “On Sobolev and logarithmic Sobolev inequalities for Markov semigroups”, in *New trends in stochastic analysis (Charingworth, 1994)*, World Scientific, 1997, p. 43-75.
- [4] D. BAKRY, F. BOLLEY & I. GENTIL, “Dimension dependent hypercontractivity for Gaussian kernels”, *Probab. Theory Relat. Fields* **154** (2012), no. 3-4, p. 845-874.
- [5] ———, “The Li–Yau inequality and applications under a curvature-dimension condition”, *Ann. Inst. Fourier* **67** (2017), no. 1, p. 397-421.
- [6] D. BAKRY & M. ÉMERY, “Diffusions hypercontractives”, in *Séminaire de probabilités XIX, Univ. Strasbourg 1983/84*, Lecture Notes in Mathematics, vol. 1123, Springer, 1985, p. 177-206.
- [7] D. BAKRY, I. GENTIL & M. LEDOUX, *Analysis and geometry of Markov diffusion operators*, Grundlehren der Mathematischen Wissenschaften, vol. 348, Springer, 2014, xx+552 pages.
- [8] D. BAKRY & M. LEDOUX, “Lévy–Gromov’s isoperimetric inequality for an infinite-dimensional diffusion generator”, *Invent. Math.* **123** (1996), no. 2, p. 259-281.
- [9] ———, “A logarithmic Sobolev form of the Li–Yau parabolic inequality”, *Rev. Mat. Iberoam.* **22** (2006), no. 2, p. 683-702.
- [10] G. CLERC, G. CONFORTI & I. GENTIL, “Long-time behaviour of entropic interpolations”, *Potential Anal.* **59** (2023), no. 1, p. 65-95.
- [11] G. CONFORTI, “A second order equation for Schrödinger bridges with applications to the hot gas experiment and entropic transportation cost”, *Probab. Theory Relat. Fields* **174** (2019), no. 1-2, p. 1-47.
- [12] M. ERBAR, K. KUWADA & K.-T. STURM, “On the equivalence of the entropic curvature-dimension condition and Bochner’s inequality on metric measure spaces”, *Invent. Math.* **201** (2015), no. 3, p. 993-1071.
- [13] I. GENTIL, “L’entropie, de Clausius aux inégalités fonctionnelles”, *Gaz. Math., Soc. Math. Fr.* **168** (2021), p. 15-23.
- [14] I. GENTIL, C. LÉONARD & L. RIPANI, “Dynamical aspects of the generalized Schrödinger problem via Otto calculus – A heuristic point of view”, *Rev. Mat. Iberoam.* **36** (2020), no. 4, p. 1071-1112.
- [15] N. GIGLI, *Second order analysis on $(\mathcal{P}_2(M), W_2)$* , Memoirs of the American Mathematical Society, vol. 1018, American Mathematical Society, 2012, 154 pages.
- [16] N. GIGLI & L. TAMANINI, “Benamou–Brenier and duality formulas for the entropic cost on $\text{RCD}^*(K, N)$ spaces”, *Probab. Theory Relat. Fields* **176** (2020), no. 1-2, p. 1-34.

- [17] ———, “Second order differentiation formula on $\text{RCD}^*(K, N)$ spaces”, *J. Eur. Math. Soc.* **23** (2021), no. 5, p. 1727-1795.
- [18] R. JORDAN, D. KINDERLEHRER & F. OTTO, “The variational formulation of the Fokker-Planck equation”, *SIAM J. Math. Anal.* **29** (1998), no. 1, p. 1-17.
- [19] C. LÉONARD, “A survey of the Schrödinger problem and some of its connections with optimal transport”, *Discrete Contin. Dyn. Syst.* **34** (2014), no. 4, p. 1533-1574.
- [20] P. LI & S. T. YAU, “On the parabolic kernel of the Schrödinger operator”, *Acta Math.* **156** (1986), p. 153-201.
- [21] F. OTTO, “The geometry of dissipative evolution equations: The porous medium equation”, *Commun. Partial Differ. Equations* **26** (2001), no. 1-2, p. 101-174.
- [22] F. OTTO & C. VILLANI, “Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality”, *J. Funct. Anal.* **173** (2000), no. 2, p. 361-400.
- [23] C. VILLANI, *Optimal transport. Old and new.*, Springer, 2009, xxii+973 pages.