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Degeneration from difference to differential Okamoto spaces for the sixth Painlevé equation ^(*)

THOMAS DREYFUS ⁽¹⁾ AND VIKTORIA HEU ⁽²⁾

ABSTRACT. — In the current paper we study the q -analogue introduced by Jimbo and Sakai of the well known Painlevé VI differential equation. We explain how it can be deduced from a q -analogue of Schlesinger equations and show that for a convenient change of variables and auxiliary parameters, it admits a q -analogue of Hamiltonian formulation. This allows us to show that Sakai's q -analogue of Okamoto space of initial conditions for qP_{VI} admits the differential Okamoto space *via* some natural limit process.

RÉSUMÉ. — Dans cet article, nous étudions le q -analogue de la sixième équation de Painlevé introduit par Jimbo et Sakai. Nous expliquons comment il peut être retrouvé à partir d'un q -analogue de l'équation de Schlesinger et nous montrons que, après un changement des paramètres, il admet une formulation en terme de q -système Hamiltonien. Cela nous permet nous prouver que le q -analogue de l'espace d'Okamoto des conditions initiales introduit par Sakai admet l'espace d'Okamoto différentiel comme limite lorsque q tend vers 1.

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Article proposé par Lucia Di Vizio.

Introduction

In [12], Jimbo and Sakai have introduced a q -analogue of the Painlevé VI equation, namely the following system of q -difference equations:

$$(qP_{\text{JS,VI}}) : \begin{cases} \mathbf{y} \cdot \sigma_{q,t} \mathbf{y} = \frac{\left(\sigma_{q,t} \mathbf{z} - t \frac{a_1 a_2}{\vartheta_1}\right) \left(\sigma_{q,t} \mathbf{z} - t \frac{a_1 a_2}{\vartheta_2}\right)}{a_3 a_4 \left(\sigma_{q,t} \mathbf{z} - \frac{1}{q \kappa_1}\right) \left(\sigma_{q,t} \mathbf{z} - \frac{1}{\kappa_2}\right)} \\ \frac{\mathbf{z} \cdot \sigma_{q,t} \mathbf{z}}{\frac{1}{q \kappa_1 \kappa_2}} = \frac{(\mathbf{y} - t a_1)(\mathbf{y} - t a_2)}{(\mathbf{y} - a_3)(\mathbf{y} - a_4)}, \end{cases}$$

where $\kappa_1, \kappa_2, \vartheta_1, \vartheta_2, a_1, a_2, a_3, a_4 \in \mathbb{C}^*$ are parameters subject to the relation

$$a_1 a_2 a_3 a_4 \kappa_1 \kappa_2 = \vartheta_1 \vartheta_2.$$

Here q is a complex parameter that is neither zero nor one, and $\sigma_{q,t}$ is the operator which to a function $f(t)$ associates $f(q \cdot t)$. The q -derivative

$$\partial_{q,t} := \frac{\sigma_{q,t} - 1}{(q - 1)t}$$

formally converges, when $q \rightarrow 1$, to the classical derivative ∂_t (differentiation with respect to t). It has been shown in [12] that the classical Painlevé VI equation may be obtained by some limit process, when q goes to 1, from its q -analogue. More precisely, by a series of changes of variables and parameters, $qP_{\text{JS,VI}}$ formally yields a certain system of differential equations with eight complex parameters, subject to one relation. As one can easily check, one can then further normalize these parameters to a quadruple $\Theta = (\theta_0, \theta_1, \theta_t, \theta_\infty)$ of complex parameters such that this system of differential equations is the non-autonomous Hamiltonian system

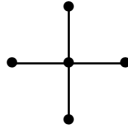
$$(P_{\text{VI}}) : \begin{cases} \partial_t \mathbf{y} = \partial_{\mathbf{Z}} H_{\text{VI}}^\Theta(\mathbf{y}, \mathbf{Z}, t) \\ \partial_t \mathbf{Z} = -\partial_{\mathbf{y}} H_{\text{VI}}^\Theta(\mathbf{y}, \mathbf{Z}, t), \end{cases}$$

where $H_{\text{VI}}^\Theta(\mathbf{y}, \mathbf{Z}, t)$ is given by

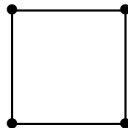
$$\frac{\mathbf{y}(\mathbf{y} - 1)(\mathbf{y} - t)}{t(t - 1)} \left(\mathbf{Z}^2 + \frac{\mathbf{Z}}{\mathbf{y} - t} \right) - \frac{1}{4} \left(\frac{(\theta_\infty - 1)^2 - 1}{t(t - 1)} \mathbf{y} + \frac{\theta_0^2}{(t - 1)\mathbf{y}} + \frac{\theta_t^2}{\mathbf{y} - t} - \frac{\theta_1^2}{t(\mathbf{y} - 1)} \right).$$

When reformulated as a single second order differential equation in \mathbf{y} , this non-autonomous Hamiltonian system (P_{VI}) yields the classical sixth Painlevé differential equation with auxiliary parameters Θ . Given a generic initial condition, i.e. $\mathbf{y}_0 \in \mathbb{C} \setminus \{0, 1, t_0\}$, and $\mathbf{Z}_0 \in \mathbb{C}$, Cauchy's theorem implies the existence and uniqueness of a germ at $t_0 \neq 0, 1$ of associated holomorphic solutions of (P_{VI}) . As shown in [20], it is moreover possible to give a

meaning to solutions including non-generic initial conditions. More precisely, Okamoto's space of initial conditions at a fixed time $t_0 \in \mathbb{C} \setminus \{0, 1\}$ is the second Hirzebruch surface \mathbb{F}_2 blown up in eight points, whose position is encoded by Θ and t_0 , minus a divisor formed by five irreducible components of self-intersection number (-2) related to each other according to the following intersection diagram:



Here each node represents an irreducible component, and nodes share a common edge if and only if they intersect each other. For each point in Okamoto's space of initial conditions at t_0 , there exists a unique associated germ (at t_0) of meromorphic solution of (P_{VI}) . A q -analogue of Okamoto's space for $(qP_{JS,VI})$ for some fixed generic time t_0 was found in [23], see also [14] for a very comprehensible presentation. It is given by $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, blown up in eight points, whose positions are encoded by $a_1, a_2, a_3, a_4, \kappa_1, \kappa_2, \vartheta_1, \vartheta_2$ and t_0 , minus a divisor formed of four irreducible components of self-intersection number (-2) , arranged according to the following intersection diagram:

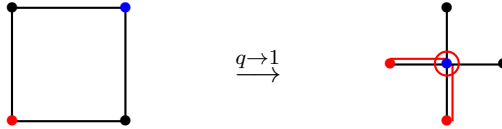


For each point in Sakai's q -analogue of Okamoto's space of initial conditions at t_0 , there exists a unique *discrete* solution of $(qP_{JS,VI})$, which, roughly speaking, encodes the values at $q^{\mathbb{Z}}t_0$ that a meromorphic solution with prescribed value at t_0 , if it exists, should interpolate. The questions addressed in the present paper are the following.

- (Q1) How can Okamoto's space of initial values at t_0 for (P_{VI}) be obtained *via* a natural limit process from its discrete analogue?
- (Q2) How can meromorphic solutions of (P_{VI}) be obtained *via* a natural limit process from their discrete analogue?

Let us first answer question (Q1) informally. What we will obtain in Section 4 is that one of the four irreducible components of the boundary of the q -Okamoto space at t_0 does degenerate at the limit $q \rightarrow 1$: for $q = 1$, it is no longer irreducible, but is itself the union of three irreducible components

of self-intersection (-2) , one of which coincides with the limit of a non-degenerating one, and the other two of which intersect only the latter. So in terms of intersection diagrams, the informal answer to question (Q1) is the following.



The key to this result (see Section 4.3 for a precise formulation) is to conveniently identify normalizations and changes of variables in $(qP_{JS,VI})$ before considering the limit process, such that the limit when $q \rightarrow 1$ is (P_{VI}) . To this end, we retrace, with some alterations, the method by which in [12], the q -difference equation $(qP_{JS,VI})$ has been obtained from a q -analogue of isomonodromic deformations. Here the isomonodromy condition to be considered concerns certain families, parametrized by a time variable t , of q -Fuchsian systems of rank 2, which for fixed q are of the form

$$\sigma_{q,x}Y(x,t) = \mathfrak{A}(x,t)Y(x,t), \quad \text{with} \quad \mathfrak{A}(x,t) = \mathfrak{A}_0(t) + x \frac{\mathfrak{A}_1(t)}{x-1} + x \frac{\mathfrak{A}_t(t)}{t(x-t)},$$

where x is the standard coordinate on $\mathbb{C} \subset \mathbb{P}^1$. As shown in [12], under certain generic conditions, for given spectral parameters $\kappa_1, \kappa_2, \vartheta_1, \vartheta_2, a_1, a_2, a_3, a_4 \in \mathbb{C}^*$ as above, such a family can be encoded by a triple of functions $(w(t), \mathbf{y}(t), \mathbf{z}(t))$. We show in Section 1.3 that a convenient change of variables and parameters (including a normalization) that is both compatible with the definition of the considered q -Fuchsian systems in [12] and that yields tracefree differential Fuchsian systems at the limit when $q \rightarrow 1$ is the following setting:

$$\lambda = \frac{\kappa_2 w}{q-1}, \quad \mathbf{Z} = \frac{\frac{(\mathbf{y}-ta_1)(\mathbf{y}-ta_2)}{q(\mathbf{y}-1)(\mathbf{y}-t)\mathbf{z}} - 1}{(q-1)\mathbf{y}}, \quad (0.1)$$

$$\begin{cases} a_1 = 1 + (q-1)\frac{\theta_t}{2}, & a_2 = \frac{1}{a_1}, & a_3 = 1 + (q-1)\frac{\theta_1}{2}, \\ a_4 = \frac{1}{a_3}, & \vartheta_1 = 1 + (q-1)\frac{\theta_0}{2}, & \vartheta_2 = \frac{1}{\vartheta_1}, \\ \kappa_1 = \frac{1}{\kappa_2}, & \kappa_2 = 1 + (q-1)\frac{\theta_\infty}{2}. \end{cases} \quad (0.2)$$

In Section 2.2, we both simplify and generalize the definition of q -isomonodromy in [12] into the requirement that there exists a certain matrix $\mathfrak{B}(x,t)$, depending rationally on x , such that the system of q -difference equations

$$\begin{cases} \sigma_{q,x}Y(x,t) = \mathfrak{A}(x,t)Y(x,t) \\ \sigma_{q,t}Y(x,t) = \mathfrak{B}(x,t)Y(x,t) \end{cases}$$

satisfies the q -analogue of integrability, namely $\sigma_{q,t}\mathfrak{A} \cdot \mathfrak{B} = \sigma_{q,x}\mathfrak{B} \cdot \mathfrak{A}$. We further introduce a sufficient condition for q -isomonodromy, which we call q -Schlesinger isomonodromy. We show that under some generic conditions such as the non-resonant condition $\frac{\vartheta_1}{\vartheta_2} \notin q^{\mathbb{Z}^*}$ and $\frac{\kappa_1}{\kappa_2} \notin q^{\mathbb{Z}^*}$, this q -Schlesinger isomonodromy of $\sigma_{q,x}Y = \mathfrak{A}Y$ is equivalent to the following q -Schlesinger equations

$$\begin{cases} \sigma_{q,t}\mathfrak{A}_0 = B_0\mathfrak{A}_0B_0^{-1} \\ \sigma_{q,t}\mathfrak{A}_1 = \frac{t-1}{qt-1} \frac{(qta_1-1)(qta_2-1)}{q(ta_1-1)(ta_2-1)} \cdot (qI_2 + B_0)\mathfrak{A}_1(I_2 + B_0)^{-1} \\ \sigma_{q,t}\mathfrak{A}_t = -B_0\mathfrak{A}_t \left(\frac{1}{a_1a_2}I_2 + qtB_0^{-1} \right) \\ \quad - \frac{t(t-1)}{(ta_1-1)(ta_2-1)} \cdot (qI_2 + B_0)\mathfrak{A}_t \left(I_2 + \frac{(qta_1-1)(qta_2-1)}{qt-1}(I_2 + B_0)^{-1} \right), \end{cases}$$

where

$$\begin{aligned} B_0 := -qt \left(\mathfrak{A}_0 + \mathfrak{A}_1 + \frac{1}{t}\mathfrak{A}_t + \frac{t-1}{(ta_1-1)(ta_2-1)}\mathfrak{A}_1 \right) \\ \times \left(\frac{1}{a_1a_2}\mathfrak{A}_0 + \frac{t(t-1)}{(ta_1-1)(ta_2-1)}\mathfrak{A}_1 \right)^{-1}. \end{aligned}$$

Moreover, we show in Section 2.3 that when the spectral values are functions of q given by (0.2), then these q -Schlesinger equations yield the usual (differential) Schlesinger equation at the limit $q \rightarrow 1$. When \mathfrak{A} is expressed with respect to $(w(t), \mathbf{y}(t), \mathbf{z}(t))$ and the spectral parameters, then these q -Schlesinger equations are (generically) equivalent to a system of q -difference equations given by $(qP_{\text{JS,VI}})$ and an additional equation for $\sigma_{q,t}w$. However, our weaker definition of q -isomonodromy is actually (generically) equivalent to $(qP_{\text{JS,VI}})$. We conclude that the change of variables and parameters (0.1) combined with (0.2) is a natural setting for the study of confluence of the q -Painlevé VI equation. And indeed, for generic values of q , the change of variable (0.1) defines a biregular transformation of Sakai's q -Okamoto space (see Section 4.2), such that the obtained modified q -Okamoto space yields, for spectral values (0.2), the differential Okamoto space when $q \rightarrow 1$ (see Section 4.3).

Under convenient assumptions on q and the spectral data, some meromorphic solutions of $(qP_{\text{JS,VI}})$ defined in a convenient sectorial neighborhood of $t = 0$ have been constructed in [18, 19]. On the other hand, contrary to the differential setting, the following question remains open: for a given generic value of t_0 and generic initial condition $(\mathbf{y}(t_0), \mathbf{z}(t_0))$, does there exist an associated meromorphic solution of $(qP_{\text{JS,VI}})$, defined on a connected subset of \mathbb{C}^* stable under multiplication by $q^{\pm 1}$? Of course the answer is likely to depend on particular choices of q and the spectral parameters. For example, when q is a n -th root of unity, then a necessary condition for the existence

of meromorphic solution is that the spectral parameters are chosen in a way such that the n -th iterate of $(qP_{\text{JS,VI}})$ is the identity. The question whether meromorphic solutions of $(qP_{\text{JS,VI}})$ over convenient set, parametrized by q , admit a limit when $q \rightarrow 1$ also seems difficult and remains open.

Much more abordable is the question of the existence of discrete solutions, essentially solved in [23, Prop. 1]. A discrete solution with initial value $(\mathbf{y}_0, \mathbf{z}_0) \in \mathbb{C}^* \times \mathbb{C}^*$ at $t_0 \in \mathbb{C}^*$ is a sequence $(\mathbf{y}_\ell, \mathbf{z}_\ell, t_\ell)_{\ell \in \mathbb{Z}}$ of points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{C}^*$ such that for $\ell \neq 0$, we have $t_\ell = q^\ell t_0$ and $(\mathbf{y}_\ell, \mathbf{z}_\ell) = (f_\ell(\mathbf{y}_0, \mathbf{z}_0, t_0, q), g_\ell(\mathbf{y}_0, \mathbf{z}_0, t_0, q))$, where f_ℓ, g_ℓ are the rational functions in variables $\mathbf{y}, \mathbf{z}, t, q$ such that the ℓ -th iterate of $(qP_{\text{JS,VI}})$ is of the form

$$\begin{cases} \sigma_{q,t}^\ell \mathbf{y} = f_\ell(\mathbf{y}, \mathbf{z}, t, q) \\ \sigma_{q,t}^\ell \mathbf{z} = g_\ell(\mathbf{y}, \mathbf{z}, t, q). \end{cases}$$

We refer to Section 3.2 for more details. It is shown in [23, Prop. 1] that discrete solutions with initial value in $\mathbb{C}^* \times \mathbb{C}^*$ at t_0 are well defined, in particular they exist and are unique, if t_0 and the spectral values are generic. Moreover, under this assumption, one can consider a space of initial values bigger than $\mathbb{C}^* \times \mathbb{C}^*$, namely the q -Okamoto space. We specify in Section 4.2 which are the special values for t_0 and the spectral parameters that need to be excluded here. Note that a discrete solution, as a sequence, does make sense even if q is a root of unity.

Of course there is an analogous notion of discrete solution for the *modified* q -Painlevé VI equation obtained by applying the change of variables and parameters (0.1), (0.2) to $(qP_{\text{JS,VI}})$. We prove in Section 3.3 that the thereby obtained system of q -difference equations may be interpreted as a q -analogue of an Hamiltonian system. More precisely, it is given by

$$(q\tilde{P}_{\text{VI}}) : \begin{cases} \partial_{q,t} \mathbf{y} = \partial_{q,\mathbf{Z}} H_{\text{VI}}^\ominus(\mathbf{y}, \mathbf{Z}, t) + (q-1)\mathcal{R}_1^\ominus(\mathbf{y}, \mathbf{Z}, t, q) \\ \partial_{q,t} \mathbf{Z} = -\partial_{q,\mathbf{y}} H_{\text{VI}}^\ominus(\mathbf{y}, \mathbf{Z}, t) + (q-1)\mathcal{R}_2^\ominus(\mathbf{y}, \mathbf{Z}, t, q), \end{cases}$$

where H_{VI}^\ominus is the Hamiltonian from the (differential) (P_{VI}) and for $i \in \{1, 2\}$, \mathcal{R}_i^\ominus is some rational function such that $\mathcal{R}_i^\ominus|_{q=1}$ is well defined and does not have poles outside the polar locus of H_{VI}^\ominus .

From this we deduce, also in Section 3.3, the answers to question (Q2). For $t_0 \in \mathbb{C}^*$ and $(\mathbf{y}_0, \mathbf{Z}_0) \in (\mathbb{C} \setminus \{0, 1, t_0\}) \times \mathbb{C}$, the sequence $(\mathbf{y}_\ell(q), \mathbf{Z}_\ell(q), q^\ell t)_{\ell \in \mathbb{N}}$ of triples defining the corresponding discrete solution, but seen as rational functions of q , is well defined, and encodes in some precise manner the Taylor series coefficients of the unique solution of (P_{VI}) with initial condition $(\mathbf{y}_0, \mathbf{Z}_0)$ at t_0 .

Each of the four sections following this introduction is decomposed into three parts. Each time, in the first part we briefly recall some notions and

known results in the differential case. In the second part, their q -analogues are discussed, and in the third part, confluence is addressed. Concerning the q -analogues, we usually recall some results from [12] and [23], complemented by some precisions that we deemed helpful, and to which we add new results. We finish by an appendix, see Section 5, explaining how our notion of q -isomonodromy is related to the one in [12].

We would like to emphasize that some results in the sequel require stronger assumptions on the complex variable q than $q \neq 0, 1$. These assumptions will of course be duly specified when needed. We choose not to accumulate these requirements along the way towards the q -Painlevé VI equation (which in and by itself is well-defined for $q \neq 0, 1$), in order to get the full picture of possible q -Okamoto spaces.

In this paper, we adopt the following (standard) notation.

$\mathrm{GL}_2(R)$	the ring of invertible 2×2 matrices with coefficients in a ring R .
$\mathrm{SL}_2(R)$	the ring of invertible 2×2 matrices of determinant 1 with coefficients in R .
$\mathrm{M}_2(R)$	the algebra of 2×2 matrices with coefficients in a ring R .
$\mathfrak{sl}_2(R)$	the ring of 2×2 matrices of trace 0 with coefficients in a ring R .
I_2	the identity matrix in $\mathrm{M}_2(R)$
$A^{(i,j)}$	the (i, j) -entry of a matrix A .
$\mathcal{O}(U)$	the ring of holomorphic functions on some complex domain $U \subset \mathbb{C}^n$.
$\mathcal{M}(U)$	the field of meromorphic functions on U .
$k[x_1, \dots, x_n]$	the ring of polynomials in n variables, named x_1, \dots, x_n , with coefficients in a field k .
$k(x_1, \dots, x_n)$	the fraction field of $k[x_1, \dots, x_n]$.

1. Fuchsian systems

1.1. Differential case

Let $\Theta = (\theta_0, \theta_1, \theta_t, \theta_\infty) \in \mathbb{C}^4$ with $\theta_\infty \neq 0$. We say that Θ satisfies the *non-resonant condition* if

$$\forall i \in \{0, 1, t, \infty\}, \quad \theta_i \notin \mathbb{Z}^*. \quad (1.1)$$

We consider a linear partial differential equation of the form

$$\partial_x Y(x, t) = A(x, t)Y(x, t), \quad \text{with} \quad A(x, t) = \frac{A_0(t)}{x} + \frac{A_1(t)}{x-1} + \frac{A_t(t)}{x-t}. \quad (1.2)$$

Here x is the standard coordinate on \mathbb{C} , seen as a subset of $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, and t is the standard coordinate on an open connected subset $U \subset \mathbb{C} \setminus \{0, 1\}$.

DEFINITION 1.1. — *We shall say that (1.2) is a family of \mathfrak{sl}_2 -Fuchsian systems with spectral data Θ if the following hold:*

- for each $i \in \{0, 1, t\}$, $A_i \in \mathfrak{sl}_2(\mathcal{O}(U))$,
- for all $i \in \{0, 1, t\}$ and all $t \in U$, we have

$$\text{Spec}(A_i(t)) = \left\{ \frac{1}{2}\theta_i, -\frac{1}{2}\theta_i \right\},$$

- the residue $A_\infty := -A_0 - A_1 - A_t$ at infinity is constant and normalized as follows:

$$A_\infty \equiv \begin{pmatrix} \frac{\theta_\infty}{2} & 0 \\ 0 & -\frac{\theta_\infty}{2} \end{pmatrix}.$$

With this normalization, the $(1, 2)$ entry of $x(x - 1)(x - t)A$, seen as an element of $\mathcal{O}(U)[x]$, is a polynomial of degree at most one. Let us assume that it has degree one and define a non-zero holomorphic function $\lambda(t) \in \mathcal{O}(U)$ and a meromorphic function $y(t) \in \mathcal{M}(U)$ by

$$A^{(1,2)}(x, t) = \frac{\lambda(t)(x - y(t))}{x(x - 1)(x - t)}. \tag{1.3}$$

Assuming moreover that $y(t) \neq 0, 1, t$, we may define a meromorphic function $Z(t) \in \mathcal{M}(U)$ by

$$A^{(1,1)}(y(t), t) = Z(t). \tag{1.4}$$

The next Lemma shows that the matrix A is determined by the triple (λ, y, Z) . In what follows, id denotes the function $t \mapsto t$.

LEMMA 1.2. — *If a family of \mathfrak{sl}_2 -Fuchsian systems (1.2) with spectral data Θ , with $\theta_\infty \neq 0$, gives rise to*

$$(\lambda, y, Z) \in (\mathcal{O}(U) \setminus \{0\}) \times (\mathcal{M}(U) \setminus \{0, 1, \text{id}\}) \times \mathcal{M}(U) \tag{1.5}$$

as above, then the coefficients of the matrix A necessarily are the following functions of λ, y, Z, x, t and Θ :

$$\begin{aligned}
 A^{(1,1)} &= \frac{(y-x)}{4\theta_\infty x(x-1)(x-t)} \left[y(y-1)(y-t) \left(2Z + \frac{\theta_\infty}{y-x} \right)^2 - a \right] \\
 &\quad - \frac{\theta_\infty}{4} \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t} + \frac{1}{y-x} \right). \\
 A^{(1,2)} &= \frac{\lambda(x-y)}{x(x-1)(x-t)} \\
 A^{(2,2)} &= -A^{(1,1)} \\
 A^{(2,1)} &= \frac{1}{\lambda} \left(\frac{c_0}{x} + \frac{c_1}{x-1} - \frac{c_0+c_1}{x-t} \right) \\
 c_0 &= \frac{y}{t\theta_\infty^2} \left((y-1)(y-t)(yZ + \theta_\infty)Z + \frac{(y-1-t)\theta_\infty^2 - a}{4} \right)^2 - \frac{t\theta_0^2}{4y} \\
 c_1 &= \frac{y-1}{(1-t)\theta_\infty^2} \left(y(y-t)((y-1)Z + \theta_\infty)Z + \frac{(y+1-t)\theta_\infty^2 - a}{4} \right)^2 \\
 &\quad - \frac{(1-t)\theta_1^2}{4(y-1)}.
 \end{aligned}$$

Here we denote

$$a := \frac{t\theta_0^2}{y} - \frac{(t-1)\theta_1^2}{y-1} + \frac{t(t-1)\theta_t^2}{y-t}.$$

Proof. — This lemma can be deduced from the formulas in [11, p. 443-444] by considering the tensor product of $\partial_x Y = AY$ with

$$\partial_x \zeta = \left(\frac{\theta_0}{2x} + \frac{\theta_1}{2(x-1)} + \frac{\theta_t}{2(x-t)} \right) \zeta. \quad \square$$

Remark 1.3. — If we have an arbitrary meromorphic triple (λ, y, Z) as in (1.5), then *via* the formulas in Lemma 1.2 we can associate a family of Fuchsian systems. Note however that the coefficients of the matrix functions A_0, A_1 and A_t are meromorphic functions of t . If one wants to obtain holomorphic coefficients, one might have to restrict to the complement of a discrete subset in U . Indeed, for example the product λy needs to be holomorphic.

Remark 1.4. — As explained in [17, §4], the condition $\theta_\infty \neq 0$ can actually be overcome if one works in a (conjugated) setting where A_∞ is assumed nonzero and normalized to $\begin{pmatrix} \theta_\infty/2 & 0 \\ \star & -\theta_\infty/2 \end{pmatrix}$, whereas λ is normalized to 1.

1.2. A discrete analogue

Let $q \in \mathbb{C} \setminus \{0, 1\}$. Let $\Theta := (\Theta_0, \Theta_1, \Theta_t, \Theta_\infty) \in (\mathbb{C}^*)^4$ and let $\bar{\Theta} := (\bar{\Theta}_0, \bar{\Theta}_1, \bar{\Theta}_t, \bar{\Theta}_\infty) \in (\mathbb{C}^*)^4$ with $\Theta_\infty \neq \bar{\Theta}_\infty$ be two quadrupels subject to the following relation:

$$\Theta_0 \bar{\Theta}_0 = \Theta_\infty \bar{\Theta}_\infty \Theta_t \bar{\Theta}_t \Theta_1 \bar{\Theta}_1. \tag{1.6}$$

Remark 1.5. — In order to motivate our choice of notation, let us indicate that with respect to confluence, will be led to consider Θ_i satisfying some relation with the θ_i from the differential context, and $\bar{\Theta}_i$ satisfying $\bar{\Theta}_i = 1/\Theta_i$ (see Section 1.3).

We say that $(\Theta, \bar{\Theta})$ satisfies the *non-resonant condition* if

$$\forall i \in \{0, 1, t, \infty\}, \quad \frac{\Theta_i}{\bar{\Theta}_i} \notin q^{\mathbb{Z}^*}. \tag{1.7}$$

If ζ is a standard coordinate in a complex domain that is stable under multiplication by q and $\frac{1}{q}$, then we define the following operators on functions of ζ :

$$\sigma_{q,\zeta} : f(\zeta) \mapsto f(q\zeta), \quad \partial_{q,\zeta} : f(\zeta) \mapsto \frac{f(q\zeta) - f(\zeta)}{(q-1)\zeta}.$$

We consider families of linear q -difference systems of the form

$$\begin{aligned} \sigma_{q,x} Y(x, t) &= \mathfrak{A}(x, t) Y(x, t), \\ \text{with } \mathfrak{A}(x, t) &= \mathfrak{A}_0(t) + x \frac{\mathfrak{A}_1(t)}{x-1} + x \frac{\mathfrak{A}_t(t)}{t(x-t)}, \end{aligned} \tag{1.8}$$

or, equivalently, by setting $\tilde{\mathfrak{A}}_0 = \frac{\mathfrak{A}_0 - I_2}{q-1}$, $\tilde{\mathfrak{A}}_1 = \frac{\mathfrak{A}_1}{q-1}$, $\tilde{\mathfrak{A}}_t = \frac{\mathfrak{A}_t}{t(q-1)}$,

$$\partial_{q,x} Y(x, t) = \tilde{\mathfrak{A}}(x, t) Y(x, t), \quad \text{with } \tilde{\mathfrak{A}}(x, t) = \frac{\tilde{\mathfrak{A}}_0(t)}{x} + \frac{\tilde{\mathfrak{A}}_1(t)}{x-1} + \frac{\tilde{\mathfrak{A}}_t(t)}{x-t}.$$

Here x is again the standard coordinate on \mathbb{C} , and t is the standard coordinate on an open connected subset \mathfrak{D} of \mathbb{C}^* .

DEFINITION 1.6. — *We shall say that (1.8) is a family of q -Fuchsian systems with spectral data $(\Theta, \bar{\Theta})$ if the following hold:*

- for each $i \in \{0, 1, t\}$, $\mathfrak{A}_i \in M_2(\mathcal{O}(\mathfrak{D}))$,
- for all $t \in \mathfrak{D}$, we have

$$\text{Spec}(\mathfrak{A}_0(t)) = \{\Theta_0, \bar{\Theta}_0\},$$

- we have $\det(\mathfrak{A}) = \frac{\Theta_\infty \bar{\Theta}_\infty \mathfrak{p}(x, t)}{(x-1)^2 (x-t)^2}$, where

$$\mathfrak{p}(x, t) := (x - t\Theta_t)(x - t\bar{\Theta}_t)(x - \Theta_1)(x - \bar{\Theta}_1), \tag{1.9}$$

- the matrix $\mathfrak{A}_\infty := \mathfrak{A}_0 + \mathfrak{A}_1 + \frac{1}{t}\mathfrak{A}_t$ is constant and normalized as follows:

$$\mathfrak{A}_\infty = \begin{pmatrix} \bar{\Theta}_\infty & 0 \\ 0 & \Theta_\infty \end{pmatrix}. \quad (1.10)$$

Remark 1.7. — The entries of the matrix $(x-1)(x-t)\mathfrak{A}(x,t) \in M_2(\mathcal{O}(\mathfrak{D})[x])$ have degree two. Hence the determinant of this matrix is a degree four polynomial. The coefficient of x^4 is $\det(\mathfrak{A}_\infty) = \Theta_\infty \bar{\Theta}_\infty$, which is coherent with $\det((x-1)(x-t)\mathfrak{A}(x,t)) = \Theta_\infty \bar{\Theta}_\infty \mathfrak{p}(x,t)$. On the other hand, the constant coefficient is $t \det(\mathfrak{A}_0(t)) = t^2 \Theta_0 \bar{\Theta}_0$. Note that (1.6) is then equivalent to $\Theta_\infty \bar{\Theta}_\infty \mathfrak{p}(0,t) = t^2 \Theta_0 \bar{\Theta}_0$. The assumption that two zeros of \mathfrak{p} are proportional to t , and the two others are independent of t will be needed for instance in the proof of Proposition 2.9.

With the normalization (1.10), the $(1,2)$ entry of $(x-1)(x-t)\mathfrak{A}$ is a polynomial of degree at most one in x . Let us assume that it has degree one and define a non-zero holomorphic function $\lambda(t) \in \mathcal{O}(\mathfrak{D})$ and a meromorphic function $\mathbf{y}(t) \in \mathcal{M}(\mathfrak{D})$ by

$$\mathfrak{A}^{(1,2)}(x,t) = \frac{(q-1)\lambda(t)(x-\mathbf{y}(t))}{(x-1)(x-t)}, \quad (1.11)$$

so that $\mathfrak{A}^{(1,2)}(\mathbf{y}(t),t) = 0$. Assuming moreover that $\mathbf{y}(t) \neq 0, 1, t$, we may define a meromorphic function $\mathbf{Z}(t) \in \mathcal{M}(\mathfrak{D})$ by

$$\mathfrak{A}^{(1,1)}(\mathbf{y}(t),t) = 1 + (q-1)\mathbf{y}(t)\mathbf{Z}(t). \quad (1.12)$$

Remark 1.8. — We chose here to slightly modify the notation from [12] because we are mainly interested at the limit when $q \rightarrow 1$. Towards this goal, it is worth mentioning that our variables satisfy $\mathfrak{A}^{(1,2)}(x,t) = \frac{\lambda(t)(x-\mathbf{y}(t))}{(x-1)(x-t)}$, and $\tilde{\mathfrak{A}}^{(1,1)}(\mathbf{y}(t),t) = \mathbf{Z}(t)$. More details are given in Section 1.3.

Analogously to the differential case of families of \mathfrak{sl}_2 -Fuchsian systems, we have the following lemma, which is a slight adaptation of the formulas in [12, p. 4].

LEMMA 1.9. — *If a family of q -Fuchsian systems (1.8) with spectral data $(\Theta, \bar{\Theta})$, with $\Theta_\infty \neq \bar{\Theta}_\infty$, gives rise to*

$$(\lambda, \mathbf{y}, \mathbf{Z}) \in (\mathcal{O}(\mathfrak{D}) \setminus \{0\}) \times (\mathcal{M}(\mathfrak{D}) \setminus \{0, 1, \text{id}\}) \times \mathcal{M}(\mathfrak{D})$$

as above, and if $1 + (q-1)\mathbf{y}(t)\mathbf{Z}(t)$ does not vanish identically, then the coefficients of the matrix \mathfrak{A} are necessarily the following functions of $\lambda, \mathbf{y}, \mathbf{Z}, x, t$

and $(\Theta, \bar{\Theta})$:

$$\mathfrak{A}(x) = \frac{1}{(x-1)(x-t)} \begin{pmatrix} \bar{\Theta}_\infty ((x-\mathbf{y})(x-\alpha) + z_1) (q-1)\boldsymbol{\lambda}(x-\mathbf{y}) \\ \Theta_\infty \bar{\Theta}_\infty \frac{(\gamma x + \delta)}{(q-1)\boldsymbol{\lambda}} \Theta_\infty ((x-\mathbf{y})(x-\beta) + z_2) \end{pmatrix} \quad (1.13)$$

with

$$\left\{ \begin{array}{l} z_1 = \frac{(\mathbf{y}-1)(\mathbf{y}-t)(1+(q-1)\mathbf{y}\mathbf{Z})}{\bar{\Theta}_\infty} \\ z_2 = \frac{\mathbf{p}(\mathbf{y})}{z_1} \\ \beta = -(\alpha-1) - (\mathbf{y}-t) + \frac{\mathbf{p}(0) - \mathbf{p}(\mathbf{y})}{t\mathbf{y}} + \frac{\mathbf{p}(t) - \mathbf{p}(\mathbf{y})}{t(t-1)(\mathbf{y}-t)} - \frac{\mathbf{p}(1) - \mathbf{p}(\mathbf{y})}{(t-1)(\mathbf{y}-1)} \\ \gamma = z_1 + z_2 + \alpha\beta + (\alpha-1+\beta)(\mathbf{y}-1) - \frac{\mathbf{p}(0) - \mathbf{p}(\mathbf{y})}{\mathbf{y}} + \frac{\mathbf{p}(1) - \mathbf{p}(\mathbf{y})}{\mathbf{y}-1} \\ \delta = \frac{\mathbf{p}(0) - (\alpha\mathbf{y} + z_1)(\beta\mathbf{y} + z_2)}{\mathbf{y}}, \end{array} \right. \quad (1.14)$$

and

$$\alpha = -\frac{t(\Theta_0 + \bar{\Theta}_0) - (\bar{\Theta}_\infty z_1 + \Theta_\infty z_2)}{(\Theta_\infty - \bar{\Theta}_\infty)\mathbf{y}} + \frac{\Theta_\infty}{\Theta_\infty - \bar{\Theta}_\infty} \left(1+t-\mathbf{y} + \frac{\mathbf{p}(0) - \mathbf{p}(\mathbf{y})}{t\mathbf{y}} + \frac{\mathbf{p}(t) - \mathbf{p}(\mathbf{y})}{t(t-1)(\mathbf{y}-t)} - \frac{\mathbf{p}(1) - \mathbf{p}(\mathbf{y})}{(t-1)(\mathbf{y}-1)} \right). \quad (1.15)$$

Recall that \mathbf{p} is defined in (1.9). Here we dropped the dependence on t in order to simplify the formulas, i.e. we write $\mathbf{p}(x)$ instead of $\mathbf{p}(x, t)$ and similarly $\mathbf{y} = \mathbf{y}(t)$, $\mathbf{Z} = \mathbf{Z}(t)$, $\boldsymbol{\lambda} = \boldsymbol{\lambda}(t)$, $\mathfrak{A}(x) = \mathfrak{A}(x, t)$.

Proof. — The general form of \mathfrak{A} in (1.13), together with the equation for z_1 in (1.14), is precisely what is needed in order for \mathfrak{A}_∞ to be of the required normalized form, and for $\boldsymbol{\lambda}, \mathbf{y}, \mathbf{Z}$ to satisfy the equalities (1.11) and (1.12). Via evaluation at $x = 0, 1, t, \mathbf{y}$, the equation

$$\det((x-1)(x-t)\mathfrak{A}(x)) = \Theta_\infty \bar{\Theta}_\infty \mathbf{p}(x)$$

with arbitrary α is equivalent to the remaining equations in (1.14). More precisely, the successive evaluations at $\mathbf{y}, 0, 1, t$ give the lines 2, 5, 4, and 3 of (1.14). In particular, using (1.6), we have

$$\det \mathfrak{A}_0 = \Theta_\infty \bar{\Theta}_\infty \Theta_t \bar{\Theta}_t \Theta_1 \bar{\Theta}_1 = \Theta_0 \bar{\Theta}_0.$$

Equation (1.15) then is equivalent to $\text{trace}(\mathfrak{A}_0) = \Theta_0 + \bar{\Theta}_0$. □

1.3. Confluence

The heuristic equality

$$\lim_{q \rightarrow 1} \partial_{q,x} = \partial_x$$

motivates the following definition.

DEFINITION 1.10. — *Let $f \in \mathbb{C}(g, x, q)$ such that $\{q = 1\}$ is not an irreducible component of the polar divisor of f , i.e. $f(g, x, 1)$ is a well defined rational function. Then we say that the q -difference equation $\partial_{q,x}g = f(g, x, q)$ discretises the differential equation $\partial_xg = f(g, x, 1)$.*

This definition generalizes in the obvious way to the case of systems of rational q -difference equations in several variables. It can also be generalized, in a more subtle way, to the case when the base field is not \mathbb{C} , but for example the field of meromorphic functions on some domain. The term *confluence* is used when the inverse phenomenon occurs: when objects associated to a discretized differential equation (most importantly, solutions), yield the corresponding object of the differential equation by some limit process as $q \rightarrow 1$. Confluence is widely studied, see for instance [3, 5, 6, 24, 25]. Before confluence can even be adressed, one of course needs to identify the appropriate discretization. The aim of the current section is to do so for families of \mathfrak{sl}_2 -Fuchsian systems $\partial_x Y = A(x, t)Y$ as in Definition 1.1. Note that the naive approach of setting $\mathfrak{A}(x, t, q) = I_2 + (q-1)x A(x, t)$ does in general not yield a q -Fuchsian system as in Definition 1.6. Instead, we will consider $\mathfrak{A}(x, t, q)$ given by a triple of meromorphic functions as in Lemma 1.9, but with an additional parameter q , and study when $\tilde{\mathfrak{A}} = \frac{\mathfrak{A} - I_2}{(q-1)x}$ admits a limit as $q \rightarrow 1$. Here, in a first step, we ignore the difficulty of the coefficients of \mathfrak{A} being meromorphic functions with respect to t , by simply considering $\lambda, \mathbf{y}, \mathbf{Z}$ as additional variables.

PROPOSITION 1.11. — *Let $\Theta = (\theta_0, \theta_1, \theta_t, \theta_\infty) \in \mathbb{C}^4$ with $\theta_\infty \neq 0$. Let $\Theta(q) = (\Theta_0, \Theta_1, \Theta_t, \Theta_\infty)(q)$ and $\bar{\Theta}(q) = (\bar{\Theta}_0, \bar{\Theta}_1, \bar{\Theta}_t, \bar{\Theta}_\infty)(q)$ be two quadruples of elements of $\mathbb{C}(q)$, i.e. rational functions in a complex variable q , such that $\Theta_\infty \neq \bar{\Theta}_\infty$ and such that (1.6) holds. Let*

$$\mathfrak{A} \in M_2(\mathbb{C}(x, \lambda, \mathbf{y}, \mathbf{Z}, t, q))$$

be the 2×2 -matrix with coefficients in $\mathbb{C}(x, \lambda, \mathbf{y}, \mathbf{Z}, t, q)$ (i.e. the set of rational functions in six complex variables named $x, \lambda, \mathbf{y}, \mathbf{Z}, t, q$) defined by the formulas in Lemma 1.9. Let $A \in \mathfrak{sl}_2(\mathbb{C}(x, \lambda, \mathbf{y}, \mathbf{Z}, t))$ be defined by formulas in Lemma 1.2. Denote $\tilde{\mathfrak{A}} = \frac{\mathfrak{A} - I_2}{(q-1)x}$. The following are equivalent.

- (1) The divisor $\{q = 1\}$ in $\mathbb{C}_{x,\lambda,\mathbf{y},\mathbf{Z},t,q}^6$ is not an irreducible component of the polar divisor of $\tilde{\mathfrak{A}}$ and the therefore well-defined rational matrix function $\tilde{\mathfrak{A}}|_{q=1}$ equals $A(x, \lambda, \mathbf{y}, \mathbf{Z}, t)$. In other words,

$$\lim_{q \rightarrow 1} \tilde{\mathfrak{A}}(x, \lambda, \mathbf{y}, \mathbf{Z}, t, q) = A(x, \lambda, \mathbf{y}, \mathbf{Z}, t).$$

- (2) Up to permuting the roles of Θ_i and $\bar{\Theta}_i$ for $i \in \{0, 1, t\}$, the following holds as $q \rightarrow 1$:

$$\begin{cases} \Theta_i(q) = 1 + (q-1)\frac{\theta_i}{2} + O((q-1)^2) & \forall i \in \{0, 1, t, \infty\} \\ \bar{\Theta}_i(q) = 1 - (q-1)\frac{\theta_i}{2} + O((q-1)^2) & \forall i \in \{0, 1, t, \infty\}. \end{cases} \quad (1.16)$$

Proof. — From the particular form of \mathfrak{A} it follows that \mathfrak{A} can be decomposed as $\mathfrak{A}_0 + \frac{x}{x-1}\mathfrak{A}_1 + \frac{x\mathfrak{A}_t}{t(x-t)}$ where $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_t$ do not depend on x . It follows that $\tilde{\mathfrak{A}}$ can be decomposed as $\frac{\tilde{\mathfrak{A}}_0}{x} + \frac{\tilde{\mathfrak{A}}_1}{x-1} + \frac{\tilde{\mathfrak{A}}_t}{x-t}$ where $\tilde{\mathfrak{A}}_0, \tilde{\mathfrak{A}}_1, \tilde{\mathfrak{A}}_t$ do not depend on x . Similarly, we may denote by A_0, A_1, A_t the residues of A with respect to $x = 0, 1, t$. We denote $\tilde{\mathfrak{A}}_\infty := -\lim_{x \rightarrow \infty} x\tilde{\mathfrak{A}} = \frac{1_2 - \mathfrak{A}_\infty}{q-1}$ and $A_\infty = -A_0 - A_1 - A_t$. Then (1) holds if and only if, for each $i \in \{0, 1, t, \infty\}$, we have

$$\lim_{q \rightarrow 1} \tilde{\mathfrak{A}}_i(x, \lambda, \mathbf{y}, \mathbf{Z}, t, q) = A_i(x, \lambda, \mathbf{y}, \mathbf{Z}, t).$$

Assume (1) holds. We have

$$\tilde{\mathfrak{A}}_\infty = \begin{pmatrix} \frac{1-\bar{\Theta}_\infty}{q-1} & 0 \\ 0 & \frac{1-\Theta_\infty}{q-1} \end{pmatrix}.$$

Since A_∞ is of normal form, we deduce the estimates for Θ_∞ and $\bar{\Theta}_\infty$. Since $\text{Spec}(\mathfrak{A}_0) = \{\Theta_0, \bar{\Theta}_0\}$, we have

$$\text{Spec}(\tilde{\mathfrak{A}}_0) = \left\{ \frac{\Theta_0 - 1}{q-1}, \frac{\bar{\Theta}_0 - 1}{q-1} \right\}.$$

Since $\text{Spec}(A_0) = \{\theta_0/2, -\theta_0/2\}$, we deduce the estimates for Θ_0 and $\bar{\Theta}_0$ as in the statement (up to interchanging their roles). Recall that $\det(\mathfrak{A}) = \frac{\Theta_\infty \bar{\Theta}_\infty (x-t\Theta_t)(x-t\bar{\Theta}_t)(x-\Theta_1)(x-\bar{\Theta}_1)}{(x-1)^2(x-t)^2}$. We therefore have

$$\begin{aligned} \det(\tilde{\mathfrak{A}}_1) &= \frac{\Theta_\infty \bar{\Theta}_\infty (1-t\Theta_t)(1-t\bar{\Theta}_t)(1-\Theta_1)(1-\bar{\Theta}_1)}{(q-1)^2(1-t)^2}, \\ \det(\tilde{\mathfrak{A}}_t) &= \frac{\Theta_\infty \bar{\Theta}_\infty t^2(1-\Theta_t)(1-\bar{\Theta}_t)(t-\Theta_1)(t-\bar{\Theta}_1)}{(q-1)^2(t-1)^2}. \end{aligned}$$

Since $\det(A_i) = -\theta_i^2/4$ for $i \in \{1, t\}$, we deduce the estimates for Θ_i and $\bar{\Theta}_i$ as in the statement (up to interchanging their roles). Hence (1) \Rightarrow (2).

Conversely, the above calculations show that if (2) holds and the limit $\lim_{q \rightarrow 1} \tilde{\mathfrak{A}}$ is a well defined element of $M_2(\mathbb{C}(x, \boldsymbol{\lambda}, \mathbf{y}, \mathbf{Z}, t))$, then this limit is of the required form. Moreover, it is straightforward to check (with some more effort), that if (2) holds, then the limit is well defined. Here one needs to use the Taylor series expansion of the Θ_i 's and $\bar{\Theta}_i$'s up to order $O((q-1)^3)$ and use the relation on the thereby appearing second order terms imposed by the equality $\Theta_0 \bar{\Theta}_0 = \Theta_\infty \bar{\Theta}_\infty \Theta_t \bar{\Theta}_t \Theta_1 \bar{\Theta}_1$. Hence (2) \Rightarrow (1). \square

Note that we have arranged the general definition of $(\boldsymbol{\lambda}, \mathbf{y}, \mathbf{Z})$ associated to a matrix $\mathfrak{A}(x, t)$ as in Definition 1.6, such that for the matrix $\tilde{\mathfrak{A}} = \frac{\mathfrak{A} - I_2}{(q-1)x}$, equations (1.11) and (1.12) may be written as

$$\tilde{\mathfrak{A}}^{(1,2)}(x, t) = \frac{\boldsymbol{\lambda}(t)(x - \mathbf{y}(t))}{(x-1)(x-t)}, \quad \tilde{\mathfrak{A}}^{(1,1)}(\mathbf{y}(t), t) = \mathbf{Z}(t).$$

This definition is analogous to the general definition in equations (1.3) and (1.4) of (λ, y, Z) associated to a matrix $A(x, t)$ as in Definition 1.1. Indeed, recall that these equations were given by

$$A^{(1,2)}(x, t) = \frac{\lambda(t)(x - y(t))}{(x-1)(x-t)}, \quad A^{(1,1)}(y(t), t) = Z(t).$$

Therefore, we expect $\partial_{q,x} Y = \mathfrak{A}(x, t, q)Y$ as in Definition 1.6, but with an additional parameter q , to be an appropriate discretization of family of \mathfrak{sl}_2 -Fuchsian systems $\partial_x Y = A(x, t)Y$ as in Definition 1.1, where moreover (λ, y, Z) are well defined, if the spectral data $(\Theta, \bar{\Theta})(q)$ satisfy (1.16) and if, in some convenient sense, we have

$$\lim_{q \rightarrow 1} (\boldsymbol{\lambda}, \mathbf{y}, \mathbf{Z})(t, q) = (\lambda, y, Z)(t). \tag{1.17}$$

Let us now explain what we shall mean by this convenient sense. Let $\Omega \subset \mathbb{C} \setminus \{0, 1\}$ be a connected, not necessarily open, subset, with 1 in its closure. Consider a connected open subset $\mathfrak{D} \subset \mathbb{C}^*$ and let $\mathbf{f}(t, q)$ be a function such that for each fixed $q \in \Omega$ sufficiently close to 1, we have a well-defined meromorphic function $t \mapsto \mathbf{f}(t, q)$ in $\mathcal{M}(\mathfrak{D})$.

DEFINITION 1.12. — *We say that $f \in \mathcal{M}(\mathfrak{D})$ is the limit of \mathbf{f} as $q \rightarrow 1$ if for generic values (i.e. outside a proper closed analytic subset) of $t \in \mathfrak{D}$, we have*

$$\lim_{\substack{q \rightarrow 1 \\ q \in \Omega}} \mathbf{f}(t, q) = f(t).$$

Analogously, if we have a reduced rational function

$$\phi(x, (\mathbf{f}_\ell)_{0 \leq \ell \leq m+n+1}) = \frac{\sum_{k=0}^n \mathbf{f}_k x^k}{\sum_{k=0}^m \mathbf{f}_{k+n+1} x^k},$$

where each coefficient $\mathbf{f}_\ell(t, q)$ is as above and $\lim_{q \rightarrow 1} \mathbf{f}_\ell(t, q) = f_\ell(t) \in \mathcal{M}(\mathfrak{D})$, then we say that $\phi(x, t) = \frac{\sum_{k=0}^n f_k x^k}{\sum_{k=0}^{m+n+1} f_k x^k}$ is the limit of $(x, t, q) \mapsto \phi(x, (\mathbf{f}_\ell(t, q))_{0 \leq \ell \leq m+n+1})$ as $q \rightarrow 1$.

PROPOSITION 1.13. — *Let \mathfrak{D} and \mathfrak{Q} be as above. Let $\Theta = (\theta_0, \theta_1, \theta_t, \theta_\infty) \in \mathbb{C}^4$ with $\theta_\infty \neq 0$. Let $\Theta(q)$ and $\bar{\Theta}(q)$ be two quadrupels of elements of $\mathbb{C}(q)$ such that $\Theta_\infty \neq \bar{\Theta}_\infty$ and such that equations (1.6) and (1.16) hold. Let $(\lambda(t, q), \mathbf{y}(t, q), \mathbf{Z}(t, q))$ be a triple of meromorphic functions in neighborhood of $\mathfrak{D} \times \mathfrak{Q} \subset \mathbb{C}^2$ such that $\lambda \mathbf{y}(\mathbf{y} - 1)(\mathbf{y} - t)(1 + (q - 1)\mathbf{y}\mathbf{Z})$ does not vanish identically on $\mathfrak{D} \times \mathfrak{Q}$ and let $(\lambda(t), y(t), Z(t))$ be a triple of meromorphic functions on \mathfrak{D} such that $\lambda y(y - 1)(y - t)$ does not vanish identically. Then for $\tilde{\mathfrak{A}} = \frac{\mathfrak{A} - I_2}{(q-1)x}$ and A as in Lemmas 1.9 and 1.2 respectively, we have*

$$\lim_{q \rightarrow 1} \tilde{\mathfrak{A}}(x, t, q) = A(x, t) \tag{1.18}$$

(in the sense of Definition 1.12) if and only if (1.17) holds.

Proof. — Let us first prove the “if” part of the statement. One the one hand, if (1.17) holds, then

$$\begin{aligned} \lim_{q \rightarrow 1} [(x, t, q) \mapsto A(x, \lambda(t, q), \mathbf{y}(t, q), \mathbf{Z}(t, q))] \\ = [(x, t) \mapsto A(x, \lambda(t), y(t), Z(t))]. \end{aligned} \tag{1.19}$$

On the other hand, let us consider the rational function

$$L := \frac{\tilde{\mathfrak{A}}(x, \lambda, \mathbf{y}, \mathbf{Z}, t) - A(x, \lambda, \mathbf{y}, \mathbf{Z}, t)}{q - 1} \in \mathbb{C}(x, \lambda, \mathbf{y}, \mathbf{Z}, t, q).$$

By definition of \mathfrak{A} , A and by Proposition 1.11, the affine part of the polar divisor of L is contained in

$$\begin{aligned} \{x \in \{0, 1, t\}\} \cup \{\lambda = 0\} \cup \{\mathbf{y} \in \{0, 1, t\}\} \cup \{1 + (q - 1)\mathbf{y}\mathbf{Z} = 0\} \\ \cup \{t \in \{0, 1\}\} \cup \{\Theta_\infty(q) = \bar{\Theta}_\infty(q)\} \end{aligned}$$

and does not contain $\{q = 1\}$. Hence if (1.17) holds, then

$$\lim_{q \rightarrow 1} [(x, t, q) \mapsto (q - 1)L(x, \lambda(t, q), \mathbf{y}(t, q), \mathbf{Z}(t, q), t, q)] = 0. \tag{1.20}$$

The addition of the limits (1.19) and (1.20) yields (1.18).

Let us now prove the “only if” part of the statement. If (1.18) holds, then the limit of the (1, 2) coefficient of $\tilde{\mathfrak{A}}(x, t, q)$ yields the (1, 2) coefficient of A as $q \rightarrow 1$. From the explicit formulas, we deduce

$$\lim_{q \rightarrow 1} \left[(x, t, q) \mapsto \frac{\lambda(t, q)(x - \mathbf{y}(t, q))}{(x - 1)(x - t)} \right] = \left[(x, t) \mapsto \frac{\lambda(t)(x - y(t))}{(x - 1)(x - t)} \right],$$

and thus $\lim_{q \rightarrow 1} (\mathbf{y}(t, q), \boldsymbol{\lambda}(t, q)) = (y(t), \lambda(t))$. By assumption, we have

$$\lim_{q \rightarrow 1} \left(\tilde{\mathfrak{A}}(x, t, q) - A(x, t) \right) = 0.$$

Since $\lim_{q \rightarrow 1} \mathbf{y}(t, q) = y(t)$, we deduce

$$\begin{aligned} 0 &= \lim_{q \rightarrow 1} \left(\left(\tilde{\mathfrak{A}}^{(1,1)}(x, t, q) - A^{(1,1)}(x, t) \right) |_{x=\mathbf{y}} \right) \\ &= \lim_{q \rightarrow 1} \left(\mathbf{Z}(t, q) - A^{(1,1)}(x, t) |_{x=\mathbf{y}} \right). \end{aligned} \quad (1.21)$$

On the other hand, again from $\lim_{q \rightarrow 1} \mathbf{y}(t, q) = y(t)$, we get

$$\begin{aligned} 0 &= \lim_{q \rightarrow 1} \left(A^{(1,1)}(x, t) |_{x=\mathbf{y}} - A^{(1,1)}(x, t) |_{x=y} \right) \\ &= \lim_{q \rightarrow 1} \left(A^{(1,1)}(x, t) |_{x=\mathbf{y}} - Z(t) \right). \end{aligned} \quad (1.22)$$

The addition of the limits (1.21) and (1.22) yields $\lim_{q \rightarrow 1} \mathbf{Z}(t, q) = Z(t)$. \square

According to the above proposition, under some generic hypotheses, for a convenient choice of spectral value functions $(\Theta, \bar{\Theta})(q)$, equation (1.16) provides a convenient setting for the discretization of families of \mathfrak{sl}_2 -Fuchsian systems as in Definition 1.1. Here the convenient conditions the spectral value functions must satisfy are the following (up to permutation of the roles of Θ_i and $\bar{\Theta}_i$ for $i \in \{0, 1, t\}$):

$$\left\{ \begin{array}{l} \Theta_i(q) = 1 + (q-1) \frac{\theta_i}{2} + O((q-1)^2) \quad \forall i \in \{0, 1, t, \infty\} \\ \bar{\Theta}_i(q) = 1 - (q-1) \frac{\theta_i}{2} + O((q-1)^2) \quad \forall i \in \{0, 1, t, \infty\} \\ \Theta_0 \bar{\Theta}_0 = \Theta_\infty \bar{\Theta}_\infty \Theta_t \bar{\Theta}_t \Theta_1 \bar{\Theta}_1 \\ \Theta_\infty(q) \neq \bar{\Theta}_\infty(q). \end{array} \right.$$

A simple way to achieve these conditions is to choose the following setting:

$$\left\{ \begin{array}{l} \Theta_i(q) = 1 + (q-1) \frac{\theta_i}{2} + O((q-1)^2) \quad \forall i \in \{0, 1, t, \infty\} \\ \bar{\Theta}_i = \frac{1}{\Theta_i} \quad \forall i \in \{0, 1, t, \infty\} \\ \Theta_\infty(q) \neq 1. \end{array} \right.$$

This convention $\Theta_i \bar{\Theta}_i = 1$ can be seen as a q -analogue of the tracefreeness of the differential Fuchsian systems we consider. Note that if $\Theta_i(q)$ is analytic in a neighborhood of 1 and $\Theta_i(q) = 1 + (q-1) \frac{\theta_i}{2} + O((q-1)^2)$, then the condition $\bar{\Theta}_i = \frac{1}{\Theta_i}$ implies that

$$\Theta_i(q) + \bar{\Theta}_i(q) = 2 + (q-1)^2 \frac{\theta_i^2}{4} + O((q-1)^3),$$

independently of the particular value of the second order term in the Taylor series expansion of Θ_i .

2. Schlesinger equations

2.1. Differential case

Let $A_0, A_1, A_t \in \mathfrak{sl}_2(\mathbb{C})$ and $t \in \mathbb{C} \setminus \{0, 1\}$. Consider the Fuchsian system

$$\partial_x Y(x) = A(x)Y(x) \quad \text{with} \quad A(x) = \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t}$$

over \mathbb{P}^1 . An important invariant of such a system is its monodromy, defined as follows.

In a neighborhood V of a point $x_0 \in \mathbb{C} \setminus \{0, 1, t\}$, this system admits a fundamental solution \mathcal{Y} , i.e. a holomorphic function $\mathcal{Y} : V \rightarrow \mathrm{SL}_2(\mathbb{C})$ satisfying $\mathcal{Y}' = A\mathcal{Y}$, yielding a group homomorphism

$$\rho : \begin{cases} \pi_1(\mathbb{P}^1 \setminus \{0, 1, t, \infty\}, x_0) \longrightarrow \mathrm{SL}_2(\mathbb{C}) \\ \gamma \longmapsto (\mathcal{Y}^\gamma)^{-1} \cdot \mathcal{Y}, \end{cases}$$

where \mathcal{Y}^γ denotes the analytic continuation of \mathcal{Y} along γ . If V is connected, any other fundamental solution on V is of the form $\mathcal{Y} \cdot M$ for some matrix $M \in \mathrm{SL}_2(\mathbb{C})$. Hence the conjugacy class

$$[\rho] := \{M^{-1}\rho M \mid M \in \mathrm{SL}_2(\mathbb{C})\} \subset \mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus \{0, 1, t, \infty\}, x_0), \mathrm{SL}_2(\mathbb{C}))$$

does not depend on the choice of the fundamental solution \mathcal{Y} near x_0 and is referred as the *monodromy* of the Fuchsian system. Note that the monodromy does not depend on the choice of the base point x_0 in the following sense. If $x_1 \in \mathbb{C} \setminus \{0, 1, t\}$, we may choose a path γ_1 from x_0 to x_1 in $\mathbb{C} \setminus \{0, 1, t\}$, yielding an isomorphism $\tau_{\gamma_1} : \pi_1(\mathbb{P}^1 \setminus \{0, 1, t, \infty\}, x_1) \xrightarrow{\sim} \pi_1(\mathbb{P}^1 \setminus \{0, 1, t, \infty\}, x_0)$. The representation $\rho_1 := \rho \circ \tau_{\gamma_1}$ then is the monodromy representation with respect to the fundamental solution \mathcal{Y}^{γ_1} , and the conjugacy class $[\rho_1]$ does not depend on the choice of the path γ_1 . We stress the fact that the most common definition of the monodromy is the inverse of this one, that is $\mathcal{Y}^{-1} \cdot \mathcal{Y}^\gamma$ but lead to an anti-representation, rather than a representation. One the other hand, both definitions exist in the literature and this choice has no influence in what follows.

Remark 2.1. — In general, it is not possible to compute explicitly the monodromy of the Fuchsian system associated to three matrices A_0, A_1, A_t as above. However, if $\gamma_0, \gamma_1, \gamma_t, \gamma_\infty$ denote the standard generators of $\pi_1(\mathbb{P}^1 \setminus \{0, 1, t, \infty\}, x_0)$ (each γ_i turning clockwise around i), and if the matrices

A_i are non-resonant, then the matrix $\rho(\gamma_i)$ is conjugated to the matrix $\exp(2\sqrt{-1}\pi A_i)$.

Let now

$$\partial_x Y(x, t) = A(x, t)Y(x, t) \tag{2.1}$$

be a family, parametrized by $t \in U$, of \mathfrak{sl}_2 -Fuchsian systems over \mathbb{P}^1 with spectral data Θ as in Definition 1.1. Here $U \subset \mathbb{C} \setminus \{0, 1\}$ is a connected open subset and $\theta_\infty \neq 0$. Let $t_0 \in U$ and let $\Delta \subset U$ be a small disc centered at t_0 such that $0, 1 \notin \Delta$. Let $x_0 \in \mathbb{P}^1 \setminus (\{0, 1, \infty\} \cup \Delta)$. Then for any $t \in \Delta$ we have a canonical isomorphism

$$\pi_1(\mathbb{P}^1 \setminus \{0, 1, t, \infty\}, x_0) \simeq \pi_1(\mathbb{P}^1 \setminus (\{0, 1, \infty\} \cup \Delta), x_0).$$

By the Cauchy–Kowalewskaja theorem on linear differential equations with parameters [10, Thm. 9.4.5, p. 348], see also [15, p. 14], if Δ is sufficiently small, there exists a neighborhood V of $x_0 \in \mathbb{P}^1 \setminus (\{0, 1, \infty\} \cup \Delta)$ such that there is a local holomorphic fundamental solution $\mathcal{Y} : V \times \Delta \rightarrow \mathrm{SL}_2(\mathbb{C})$ satisfying $\partial_x \mathcal{Y} = A\mathcal{Y}$. For any $t_1 \in \Delta$, this fundamental solution provides a group homomorphism

$$\rho_{t_1} : \begin{cases} \pi_1(\mathbb{P}^1 \setminus (\{0, 1, \infty\} \cup \Delta), x_0) \longrightarrow \mathrm{SL}_2(\mathbb{C}) \\ \gamma \longmapsto (\mathcal{Y}^\gamma)^{-1}(x_0, t_1) \cdot \mathcal{Y}(x_0, t_1). \end{cases}$$

This yields a holomorphic family $(\rho_t)_{t \in \Delta}$ of representations, and one may consider the induced family $([\rho_t])_{t \in \Delta}$ of conjugacy classes of representations of $\pi_1(\mathbb{P}^1 \setminus (\{0, 1, \infty\} \cup \Delta), x_0)$.

DEFINITION 2.2. — *We say that (2.1) is isomonodromic if one of the two following equivalent properties hold.*

- (1) *Any point $t_0 \in U$ admits a neighborhood Δ such that the associated family $([\rho_t])_{t \in \Delta}$ of monodromies is constant, i.e.*

$$[\rho_t] = [\rho_{t'}] \quad \forall t, t' \in \Delta.$$

- (2) *The system (2.1) can locally be completed into a Lax pair, i.e. any point $t_0 \in U$ admits a neighborhood Δ where there exists $B \in \mathfrak{sl}_2(\mathcal{O}(\Delta)(x))$ such that the following holds:*

- *the polar locus of B is contained in*

$$D := \{x = 0\} \cup \{x = 1\} \cup \{x = t\} \cup \{x = \infty\} \subset \mathbb{P}^1 \times \Delta,$$

- *and the system of differential equations*

$$\begin{cases} \partial_x Y(x, t) = A(x, t)Y(x, t) \\ \partial_t Y(x, t) = B(x, t)Y(x, t) \end{cases}$$

over $\mathbb{P}^1 \times \Delta$ satisfies the Lax equation

$$\partial_t A - \partial_x B = [B, A].$$

That these properties are indeed equivalent can be deduced from [2, Thm. 2]. Moreover, considering the special case of non-resonant spectral data in [2, Thm. 3], we have the following lemma.

LEMMA 2.3. — *Assume that the spectral data Θ satisfy the non-resonant condition (1.1). Under that condition, if the system (2.1) can be completed into a Lax pair over $\mathbb{P}^1 \times \Delta$ via a certain matrix function B , then this matrix is of the form*

$$B(x, t) = -\frac{A_t(t)}{x - t} + C(t),$$

where $C \in \mathfrak{sl}_2(\mathcal{O}(\Delta))$ is a tracefree diagonal matrix function.

A key ingredient in the proof of the above classical results are the well-known Schlesinger equations. They yield particular types of isomonodromic deformations.

DEFINITION 2.4. — *We say that (2.1) is Schlesinger isomonodromic if one of the two following equivalent properties hold.*

- (1) *The system (2.1) can be completed into a Lax pair via the matrix*

$$B(x, t) = -\frac{A_t(t)}{x - t}.$$

- (2) *The residues $A_i(t)$ of (2.1) satisfy the Schlesinger equations*

$$\begin{cases} A'_0(t) = \frac{-[A_0(t), A_t(t)]}{t} \\ A'_1(t) = \frac{[A_1(t), A_t(t)]}{1 - t} \\ A'_t(t) = \frac{[A_0(t), A_t(t)]}{t} - \frac{[A_1(t), A_t(t)]}{1 - t}. \end{cases} \quad (2.2)$$

In order to see that these properties are equivalent, it suffices to compare the residues at $x = 0, 1, t$ of the Lax equation in the case $B = -\frac{A_t(t)}{x-t}$. Of course Schlesinger isomonodromic families are isomonodromic. As is immediate to check, the converse holds locally up to conjugation:

LEMMA 2.5. — *Assume that (2.1) is isomonodromic and non-resonant. Let us consider $C(t) = \text{diag}(c(t), -c(t)) \in \mathfrak{sl}_2(\mathcal{O}(\Delta))$ be the tracefree diagonal matrix appearing in Lemma 2.3. Let $t_0 \in U$ and assume that Δ is a sufficiently small neighborhood of t_0 , such that there exists a non-vanishing holomorphic function $\mu \in \mathcal{O}(\Delta)$ such that $\mu'(t) = c(t)\mu(t)$. Then, the gauge transformation $Y = M\tilde{Y}$ with $M(t) = \text{diag}\left(\mu(t), \frac{1}{\mu(t)}\right)$ yields a family*

$$\partial_x \tilde{Y} = \tilde{A}\tilde{Y}$$

of \mathfrak{sl}_2 -Fuchsian systems which is Schlesinger isomonodromic.

By definition, the family (2.1) is Schlesinger isomonodromic if and only if the entries of A satisfy a certain list of differential equations. Since these entries may be written in terms of the triple (λ, y, Z) , see Lemma 1.2, we may translate the differential equations in terms of the entries of A into differential equations in terms of the triple (λ, y, Z) . This will lead to the sixth Painlevé equation, according to the following classical results, due to R. Fuchs [7].

PROPOSITION 2.6. — *Let $\partial_x Y = A(x, t)Y$ be a family of Fuchsian \mathfrak{sl}_2 -systems with spectral data Θ as in Definition 1.1, giving rise to a triple (λ, y, Z) as in (1.5). Then $\partial_x Y = AY$ is Schlesinger isomonodromic if and only if (y, Z) is a solution of*

$$\begin{cases} y'(t) = \frac{y(y-1)(y-t)}{t(t-1)} \left(2Z + \frac{1}{y-t} \right) \\ Z'(t) = \frac{-3y^2 + 2(t+1)y - t}{t(t-1)} Z^2 - \frac{2y-1}{t(t-1)} Z \\ \quad + \frac{1}{4} \left(\frac{(\theta_\infty - 1)^2 - 1}{t(t-1)} - \frac{\theta_0^2}{(t-1)y^2} - \frac{\theta_t^2}{(y-t)^2} + \frac{\theta_1^2}{t(y-1)^2} \right), \end{cases} \quad (2.3)$$

and λ is a solution of

$$\frac{\lambda'(t)}{\lambda(t)} = \frac{(\theta_\infty - 1)(y(t) - t)}{t(t-1)}. \quad (2.4)$$

COROLLARY 2.7. — *Assume that Θ satisfies the non-resonant condition (1.1). Let $\partial_x Y = AY$ be a family of Fuchsian \mathfrak{sl}_2 -systems with spectral data Θ as in Definition 1.1, parametrized by $t \in U \subset \mathbb{C} \setminus \{0, 1\}$, giving rise to a triple (λ, y, Z) as in (1.5). Define*

$$U^* := \{t \in U \mid \lambda(t) \neq 0, y(t) \neq \infty\}.$$

Then the family $\partial_x Y = AY$ parametrized by U^ is isomonodromic if and only if (y, Z) is a solution of (2.3).*

Proof. — If $\partial_x Y = AY$ is isomonodromic, then by Lemma 2.5, it is locally conjugated to a Schlesinger isomonodromic family *via* a diagonal gauge transformation. The latter does not affect (y, Z) . Hence by Proposition 2.6, (y, Z) is a solution of (2.3).

Conversely, if (y, Z) is a solution of (2.3), then locally in U^* one may choose a local non-vanishing solution $\tilde{\lambda}$ of (2.4). One obtains a family of Fuchsian \mathfrak{sl}_2 -systems $\partial_x \tilde{Y} = \tilde{A}\tilde{Y}$ given by $(\tilde{\lambda}, y, Z)$ as in Lemma 1.2. This family of systems is Schlesinger isomonodromic by Proposition 2.6 and conjugated to $\partial_x Y = AY$ by a diagonal gauge transformation of the form $Y = M\tilde{Y}$ with $M = \text{diag}(\mu, 1/\mu)$ satisfying $\tilde{\lambda}\mu^2 = \lambda$. Hence the initial system is isomonodromic. \square

2.2. A discrete analogue

We will now define a convenient q -analogue of isomonodromy, which will lead to a q -analogue of Schlesinger equations for families of Fuchsian linear q -difference systems. Analogously to the differential case, we will define q -isomonodromy by the existence of the q -analogue of a Lax pair. This definition differs from the approach to q -isomonodromy used for example in [12]. The relation between the two will be explained in Section 5. Let $q \in \mathbb{C} \setminus \{0, 1\}$. Let

$$\sigma_{q,x}Y(x, t) = \mathfrak{A}(x, t)Y(x, t),$$

$$\text{with } \mathfrak{A}(x, t) = \mathfrak{A}_0(t) + x \frac{\mathfrak{A}_1(t)}{x-1} + x \frac{\mathfrak{A}_t(t)}{t(x-t)}, \quad (2.5)$$

be a family of q -Fuchsian systems with spectral data $(\Theta, \bar{\Theta})$ as in Definition 1.6. From now on, we make the additional assumption that \mathfrak{D} is a connected open subset of \mathbb{C}^* , which is stable under multiplication by q and $\frac{1}{q}$.

DEFINITION 2.8. — *We say that the system (2.5) is q -isomonodromic if it can be completed into a q -Lax pair, i.e. there exists $\mathfrak{B} \in \text{GL}_2(\mathcal{M}(\mathfrak{D})(x))$ such that the system*

$$\begin{cases} \sigma_{q,x}Y = \mathfrak{A}(x, t)Y \\ \sigma_{q,t}Y = \mathfrak{B}(x, t)Y \end{cases}$$

satisfies the q -Lax equation

$$\mathfrak{A}(x, qt)\mathfrak{B}(x, t) = \mathfrak{B}(qx, t)\mathfrak{A}(x, t). \quad (2.6)$$

We will now establish the first step towards the q -analogue of Lemma 2.3: under the assumption that q is not a root of unity, if (2.5) is non-resonant and can be completed into a q -Lax pair *via* a matrix function \mathfrak{B} , then \mathfrak{B} has a very particular shape.

PROPOSITION 2.9. — *Assume that $q \notin e^{2i\pi\mathbb{Q}}$ and assume that $(\Theta, \bar{\Theta})$ satisfies the non-resonant condition (1.7). Let $\mathfrak{A} \in \text{GL}_2(\mathcal{O}(\mathfrak{D})(x))$ be as in (2.5) and let $\mathfrak{B} \in \text{GL}_2(\mathcal{M}(\mathfrak{D})(x))$ such that (2.6) holds. Then, there are matrices $\mathfrak{C}(t), B_0(t) \in \text{GL}_2(\mathcal{M}(\mathfrak{D}))$ with $\mathfrak{C}(t)$ diagonal, such that*

$$\mathfrak{B}(x, t) = \mathfrak{C}(t) \frac{(x-qt)(xI_2 + B_0(t))}{(x-qt\Theta_t)(x-qt\bar{\Theta}_t)}. \quad (2.7)$$

Moreover, $\text{Spec}(B_0(t)) = \{-qt\Theta_t, -qt\bar{\Theta}_t\}$.

Proof. — Let us write

$$\widehat{\mathfrak{A}}(x, t) := \frac{(x-t)(x-1)\mathfrak{A}(x, t)}{(x-t\Theta_t)(x-t\bar{\Theta}_t)}, \quad \mathfrak{B}(x, t) = \frac{(x-qt)\widehat{\mathfrak{B}}(x, t)}{(x-qt\Theta_t)(x-qt\bar{\Theta}_t)}.$$

With this notation, equation (2.6) reads

$$q\widehat{\mathfrak{A}}(x, qt)\widehat{\mathfrak{B}}(x, t) = \widehat{\mathfrak{B}}(qx, t)\widehat{\mathfrak{A}}(x, t). \quad (2.8)$$

From the expression of the determinant of $\mathfrak{A}(x, t)$ we have

$$\det\left(\widehat{\mathfrak{A}}(x, t)\right) = \Theta_\infty \bar{\Theta}_\infty \frac{(x - \Theta_1)(x - \bar{\Theta}_1)}{(x - t\Theta_t)(x - t\bar{\Theta}_t)}. \quad (2.9)$$

The determinant of $\widehat{\mathfrak{B}}(x, t)$ is an element of $\mathcal{M}(\mathfrak{D})(x)$. Let us write

$$\det\left(\widehat{\mathfrak{B}}(x, t)\right) = c(t) \frac{\prod_{i=1}^{n_1}(x - a_i(t))}{\prod_{j=1}^{n_2}(x - b_j(t))},$$

with $0 \neq c(t) \in \mathcal{M}(\mathfrak{D})$ and a_i, b_j being elements of the algebraic closure $\overline{\mathcal{M}(\mathfrak{D})}$ of $\mathcal{M}(\mathfrak{D})$ such that $a_i(t) \neq b_j(t)$ for all i, j . Applying the determinant in the both sides of (2.8), we find that

$$\begin{aligned} q^2 c(t) \frac{(x - \Theta_1)(x - \bar{\Theta}_1)}{(x - qt\Theta_t)(x - qt\bar{\Theta}_t)} \frac{\prod_{i=1}^{n_1}(x - a_i(t))}{\prod_{j=1}^{n_2}(x - b_j(t))} \\ = c(t) \frac{\prod_{i=1}^{n_1}(qx - a_i(t))}{\prod_{j=1}^{n_2}(qx - b_j(t))} \frac{(x - \Theta_1)(x - \bar{\Theta}_1)}{(x - t\Theta_t)(x - t\bar{\Theta}_t)}, \end{aligned}$$

which simplifies as follows:

$$\begin{aligned} \frac{1}{(x - qt\Theta_t)(x - qt\bar{\Theta}_t)} \frac{\prod_{i=1}^{n_1}(x - a_i(t))}{\prod_{j=1}^{n_2}(x - b_j(t))} \\ = \frac{1}{(qx - qt\Theta_t)(qx - qt\bar{\Theta}_t)} \frac{\prod_{i=1}^{n_1}(qx - a_i(t))}{\prod_{j=1}^{n_2}(qx - b_j(t))}. \end{aligned}$$

At $x = \infty$, the left hand side behaves like $x^{n_1 - n_2 - 2}$, while the right hand side behaves like $q^{n_1 - n_2 - 2} x^{n_1 - n_2 - 2}$. Since q is not a root of unity, we must have $n_1 = n_2 + 2$. We may thus rewrite the equality as

$$f(x) = f(qx),$$

$$\text{where } f : x \mapsto \frac{1}{(x - qt\Theta_t)(x - qt\bar{\Theta}_t)} \frac{\prod_{i=1}^{n_2+2}(x - a_i(t))}{\prod_{j=1}^{n_2}(x - b_j(t))} \in \mathcal{M}(\mathfrak{D})(x).$$

It follows easily that $f \in \mathcal{M}(\mathfrak{D})(x)$ is constant in x , forcing $n_2 = 0$ and, up to renumbering the a_i 's, that $a_1 = qt\Theta_t$ and $a_2 = qt\bar{\Theta}_t$. In particular,

$$\det(\widehat{\mathfrak{B}}(x, t)) = c(t)(x - qt\Theta_t)(x - qt\bar{\Theta}_t). \quad (2.10)$$

We may rewrite (2.8) as follows.

$$q\widehat{\mathfrak{B}}(x, t) = \widehat{\mathfrak{A}}(x, qt)^{-1}\widehat{\mathfrak{B}}(qx, t)\widehat{\mathfrak{A}}(x, t). \quad (2.11)$$

Let us now show that $\widehat{\mathfrak{B}}(x, t)$ is of the form $x \cdot \widehat{B}_\infty(t) + \widehat{B}_0(t)$, i.e. let us show that the function $x \mapsto \widehat{\mathfrak{B}}(x, t)$ has only one possible pole, at $x = \infty$, and that this is at most a simple pole.

Let us prove that $R_\infty(t)$ is diagonal. We have $\widehat{\mathfrak{B}}(x, t) = x^k R_\infty(t) + O(x^{k-1})$ as $x \rightarrow \infty$ for a certain $k \in \mathbb{Z}$ and a certain non-zero $R_\infty \in M_2(\mathcal{M}(\mathfrak{D}))$. Setting $\widehat{\mathfrak{A}}_\infty(t) := \widehat{\mathfrak{A}}(\infty, t)$, we have $\widehat{\mathfrak{A}}_\infty(t) = \mathfrak{A}_\infty(t)$ which is invertible. Moreover, $\mathfrak{A}_\infty(t) = \text{diag}(\overline{\Theta}_\infty, \Theta_\infty)$ is independent of t . From equation (2.11) we get

$$R_\infty(t) = q^{k-1} \text{diag}(\overline{\Theta}_\infty, \Theta_\infty)^{-1} R_\infty(t) \text{diag}(\overline{\Theta}_\infty, \Theta_\infty).$$

Taking the determinant on both sides yields $\det(R_\infty(t)) = \det(R_\infty(t))(q^{k-1})^2$. By (2.10) we find $\det(R_\infty(t)) = c(t) \neq 0$, so that $R_\infty(t)$ must actually be invertible and $1 = q^{2k-2}$. Since q is not a root of unity, we have $k = 1$. Hence $\widehat{\mathfrak{B}}(x, t)$ has a simple pole at $x = \infty$. Furthermore, since $\Theta_\infty \neq \overline{\Theta}_\infty$, we deduce that $R_\infty(t)$ is diagonal.

Let $\alpha(t) \in \mathcal{M}(\mathfrak{D})$ such that $\alpha(t) \notin \{0, \Theta_1, \overline{\Theta}_1, t\Theta_t, t\overline{\Theta}_t\} \cdot q^{\mathbb{Z}>0}$. Assume for a contradiction that $x = \alpha$ is a pole of $\widehat{\mathfrak{B}}(x, t)$. Then $x = \frac{1}{q}\alpha$ is a pole of $\widehat{\mathfrak{B}}(qx, t)$. From the particular form of $\widehat{\mathfrak{A}}$ and (2.9) we know that both $\widehat{\mathfrak{A}}(x, qt)^{-1}$ and $\widehat{\mathfrak{A}}(x, t)$ are finite and invertible at $x = \frac{1}{q}\alpha$, i.e. $\widehat{\mathfrak{A}}(\alpha/q, t), \widehat{\mathfrak{A}}(\alpha/q, qt)^{-1} \in \text{GL}_2(\mathcal{M}(\mathfrak{D}))$. Indeed, the only possible poles of $\widehat{\mathfrak{A}}(x, t)$ are $x = t\Theta_t$ and $x = t\overline{\Theta}_t$ and the only possible poles of $\widehat{\mathfrak{A}}(x, qt)^{-1}$ are $x = \Theta_1$ and $x = \overline{\Theta}_1$. Hence by (2.11), $\widehat{\mathfrak{B}}(x, t)$ has a pole at $x = \frac{1}{q}\alpha$ as well. By induction, $\widehat{\mathfrak{B}}(x, t)$ has a pole at $x = \frac{1}{q^n}\alpha$ for each $n \in \mathbb{N}$. Yet by assumption, $\widehat{\mathfrak{B}}(x, t)$ is a rational function of x and can therefore only have finitely many poles.

Let $\alpha(t) \in \{\Theta_1, \overline{\Theta}_1, t\Theta_t, t\overline{\Theta}_t\} \cdot q^{\mathbb{Z}>0}$. Assume for a contradiction that $x = \alpha$ is a pole of $\widehat{\mathfrak{B}}(x, t)$. Then $x = q\alpha$ is a pole of $\widehat{\mathfrak{B}}(x/q, t)$. We may rewrite equations (2.11) and (2.9) as

$$\begin{aligned} \det\left(\widehat{\mathfrak{A}}(x/q, t)\right) &= \frac{(x - q\Theta_1)(x - q\overline{\Theta}_1)}{(x - qt\Theta_t)(x - qt\overline{\Theta}_t)}, \\ \widehat{\mathfrak{B}}(x, t) &= q\widehat{\mathfrak{A}}(x/q, qt)\widehat{\mathfrak{B}}(x/q, t)\widehat{\mathfrak{A}}(x/q, t)^{-1}. \end{aligned}$$

The same reasoning as before shows that $\widehat{\mathfrak{B}}(x, t)$ then has a pole at $x = q\alpha$ as well. Again this leads by induction to an infinite number of poles, and therefore, a contradiction.

Finally, we treat the case $\alpha = 0$. We have $\widehat{\mathfrak{B}}(x, t) = x^k R_0(t) + O(x^{k+1})$ as $x \rightarrow 0$ for a certain $k \in \mathbb{Z}$ and a certain non-zero $R_0 \in M_2(\mathcal{M}(\mathfrak{D}))$.

Setting $\widehat{\mathfrak{A}}_0(t) := \widehat{\mathfrak{A}}(0, t)$, we have $\widehat{\mathfrak{A}}_0(t) = \frac{1}{t\Theta_t\bar{\Theta}_t}\mathfrak{A}_0(t)$ which is invertible. From equation (2.11) we get

$$R_0(t) = q^{k-1}\widehat{\mathfrak{A}}_0(qt)^{-1}R_0(t)\widehat{\mathfrak{A}}_0(t) = q^k\mathfrak{A}_0^{-1}(qt)R_0(t)\mathfrak{A}_0(t).$$

(2.10) yields $\det(R_0(t)) = q^2t^2c(t)\Theta_t\bar{\Theta}_t \neq 0$, so that R_0 is actually invertible. Since $\text{Spec}(\mathfrak{A}_0(t)) = \{\Theta_0, \bar{\Theta}_0\}$, we have $\det(\mathfrak{A}_0(qt)) = \det(\mathfrak{A}_0(t))$ and we may take the determinant on both sides of $R_0(t) = q^k\mathfrak{A}_0^{-1}(qt)R_0(t)\mathfrak{A}_0(t)$ and find $q^{2k} = 1$. Since q is not a root of unity, we deduce $k = 0$.

We have now proven that $\widehat{\mathfrak{B}}(x, t)$ has a simple pole at $x = \infty$ and is finite and non-zero everywhere else. More precisely, we have proven that

$$\widehat{\mathfrak{B}}(x, t) = \mathfrak{C}(t)(xI_2 + B_0(t))$$

holds for the diagonal matrix $\mathfrak{C}(t) := R_\infty(t) \in \text{GL}_2(\mathcal{M}(\mathfrak{D}))$ and the matrix $B_0 := \mathfrak{C}(t)^{-1}R_0(t) \in \text{GL}_2(\mathcal{M}(\mathfrak{D}))$. Moreover, from (2.10) we get $\det(xI_2 + B_0(t)) = (x - qt\Theta_t)(x - qt\bar{\Theta}_t)$, yielding the sought expression for the eigenvalues of B_0 . \square

We will see in the proof of the following proposition that the matrix B_0 , as well as the matrix \mathfrak{C} up to a scalar multiple, are uniquely defined by (2.6). However, once these matrices are obtained, (2.6) still imposes strong conditions on the matrix \mathfrak{A} , which will yield the q -Painlevé VI equation.

PROPOSITION 2.10. — *Assume that $q \notin e^{2i\pi\mathbb{Q}}$. Let $(\Theta, \bar{\Theta}) \in (\mathbb{C}^*)^4 \times (\mathbb{C}^*)^4$ such that $\Theta_\infty \neq \bar{\Theta}_\infty$ and such that the relation (1.6) as well as the non-resonant condition (1.7) hold. Let $(\lambda, \mathbf{y}, \mathbf{Z}) \in \mathcal{M}(\mathfrak{D})^3$. Denote*

$$X := \frac{(\mathbf{y} - 1)(\mathbf{y} - t)(1 + (q - 1)\mathbf{y}\mathbf{Z})}{\Theta_\infty\bar{\Theta}_\infty(\mathbf{y} - \Theta_1)(\mathbf{y} - \bar{\Theta}_1)} \in \mathcal{M}(\mathfrak{D})$$

and assume that $\mathbf{y}, (\mathbf{y} - 1), (\mathbf{y} - t), \lambda, X, \Theta_\infty X - 1, q\bar{\Theta}_\infty X - 1, \Theta_\infty X - \frac{(\mathbf{y}-t\Theta_t)(\mathbf{y}-t\bar{\Theta}_t)}{(\mathbf{y}-\Theta_1)(\mathbf{y}-\bar{\Theta}_1)}$ are all well defined meromorphic functions on \mathfrak{D} and are each not identically zero. Let $\mathfrak{A}(x, t) = \mathfrak{A}_0(t) + \frac{x}{x-1}\mathfrak{A}_1(t) + \frac{x}{t(x-t)}\mathfrak{A}_t(t)$ be defined by $(\lambda, \mathbf{y}, \mathbf{Z})$ as in Lemma 1.9 and assume that the coefficients of $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_t$ are holomorphic on a domain $\mathfrak{D}^* \subset \mathfrak{D}$ stable under multiplication by q and $1/q$. Let us denote $\mathfrak{A}_\infty = \mathfrak{A}_0 + \mathfrak{A}_1 + \frac{1}{t}\mathfrak{A}_t = \text{diag}(\bar{\Theta}_\infty, \Theta_\infty)$. The following are equivalent.

- (1) The family of q -Fuchsian systems $\sigma_{q,x}Y = \mathfrak{A}Y$, parametrized by \mathfrak{D}^* , is q -isomonodromic.
- (2) Denoting

$$B_0 := -qt \left(\mathfrak{A}_\infty + \frac{t-1}{(t\Theta_t-1)(t\bar{\Theta}_t-1)}\mathfrak{A}_1 \right) \left(\frac{1}{\Theta_t\bar{\Theta}_t}\mathfrak{A}_0 + \frac{t(t-1)}{(t\Theta_t-1)(t\bar{\Theta}_t-1)}\mathfrak{A}_1 \right)^{-1}$$

$$\mathfrak{C} := \text{diag}(c, 1), \quad \text{with } c = \frac{\sigma_{q,t}\lambda}{\lambda} \frac{\Theta_\infty X - 1}{q\bar{\Theta}_\infty X - 1},$$

the following equations hold:

$$\left\{ \begin{array}{l} \sigma_{q,t}\mathfrak{A}_0 = \mathfrak{C}B_0\mathfrak{A}_0B_0^{-1}\mathfrak{C}^{-1} \\ \sigma_{q,t}\mathfrak{A}_1 = \frac{t-1}{qt-1} \frac{(qt\Theta_t-1)(qt\bar{\Theta}_t-1)}{q(t\Theta_t-1)(t\bar{\Theta}_t-1)} \cdot \mathfrak{C}(qI_2+B_0)\mathfrak{A}_1(I_2+B_0)^{-1}\mathfrak{C}^{-1} \\ \sigma_{q,t}\mathfrak{A}_t = -\mathfrak{C}B_0\mathfrak{A}_0 \left(\frac{1}{\Theta_t\bar{\Theta}_t} I_2 + qtB_0^{-1} \right) \mathfrak{C}^{-1} \\ \quad - \frac{t(t-1)\mathfrak{C}(qI_2+B_0)\mathfrak{A}_1}{(t\Theta_t-1)(t\bar{\Theta}_t-1)} \left(I_2 + \frac{(qt\Theta_t-1)(qt\bar{\Theta}_t-1)}{qt-1} (I_2+B_0)^{-1} \right) \mathfrak{C}^{-1}. \end{array} \right.$$

(3) The pair (\mathbf{y}, \mathbf{Z}) is a solution of the following system of q -difference equations:

$$\left\{ \begin{array}{l} \sigma_{q,t}\mathbf{y} = \frac{\Theta_1\bar{\Theta}_1}{\mathbf{y}} \cdot \frac{\left(X - t\frac{\Theta_t\bar{\Theta}_t}{\Theta_0}\right) \left(X - t\frac{\Theta_t\bar{\Theta}_t}{\Theta_0}\right)}{\left(X - \frac{1}{\Theta_\infty}\right) \left(X - \frac{1}{q\bar{\Theta}_\infty}\right)} \\ \sigma_{q,t}\mathbf{Z} = \frac{1}{(q-1)\sigma_{q,t}\mathbf{y}} \left(\frac{(\sigma_{q,t}\mathbf{y} - qt\Theta_t)(\sigma_{q,t}\mathbf{y} - qt\bar{\Theta}_t)}{q(\sigma_{q,t}\mathbf{y} - 1)(\sigma_{q,t}\mathbf{y} - qt)X} - 1 \right). \end{array} \right. \quad (2.12)$$

Proof. — Let us first show that (1) \Leftrightarrow (2). Since q is not a root of unity, by Proposition 2.9 and the non-resonant assumption, (1) is equivalent to the existence of $B_0, \mathfrak{C} \in \text{GL}_2(\mathcal{M}(\mathfrak{D}))$ with \mathfrak{C} diagonal, such that for \mathfrak{B} given by (2.7), we have $(\sigma_{q,t}\mathfrak{A}(x, t)) \mathfrak{B}(x, t) = (\sigma_{q,x}\mathfrak{B}(x, t)) \mathfrak{A}(x, t)$. We may rewrite

$$\begin{aligned} & \frac{1}{x} (\sigma_{q,t}\mathfrak{A}(x, t)) \mathfrak{B}(x, t) - \frac{1}{x} (\sigma_{q,x}\mathfrak{B}(x, t)) \mathfrak{A}(x, t) \\ &= -\frac{1}{qt\Theta_t\bar{\Theta}_t} \frac{R_0(t)}{x} - \frac{qt-1}{(qt\Theta_t-1)(qt\bar{\Theta}_t-1)} \frac{R_1(t)}{x-1} \\ & \quad + \frac{(x-qt(\Theta_t+\bar{\Theta}_t))R_2(t) + qtR_3(t)}{(x-qt\Theta_t)(x-qt\bar{\Theta}_t)} - \mathfrak{C} \frac{(x-t(\Theta_t+\bar{\Theta}_t))R_4 - tR_5}{(x-t\Theta_t)(x-t\bar{\Theta}_t)}, \end{aligned}$$

where

$$\begin{aligned} R_0 &:= (\sigma_{q,t}\mathfrak{A}_0) \mathfrak{C}B_0 - \mathfrak{C}B_0\mathfrak{A}_0, \\ R_1 &:= (\sigma_{q,t}\mathfrak{A}_1) \mathfrak{C}(I_2+B_0) - \frac{q(t-1)}{qt-1} \frac{(qt\Theta_t-1)(qt\bar{\Theta}_t-1)}{q^2(t\Theta_t-1)(t\bar{\Theta}_t-1)} \mathfrak{C}(qI_2+B_0)\mathfrak{A}_1 \\ R_2 &:= \frac{1}{qt\Theta_t\bar{\Theta}_t} (\sigma_{q,t}\mathfrak{A}_0) \mathfrak{C}B_0 + \frac{qt-1}{(qt\Theta_t-1)(qt\bar{\Theta}_t-1)} (\sigma_{q,t}\mathfrak{A}_1) \mathfrak{C}(I_2+B_0) \\ & \quad + (\sigma_{q,t}\mathfrak{A}_\infty) \mathfrak{C} \\ R_3 &:= (\sigma_{q,t}\mathfrak{A}_0) \mathfrak{C} + \frac{qt-1}{qt} (\sigma_{q,t}\mathfrak{A}_1) \mathfrak{C} - \frac{(qt-1)(\sigma_{q,t}\mathfrak{A}_1) \mathfrak{C}(I_2+B_0)}{qt(qt\Theta_t-1)(qt\bar{\Theta}_t-1)} \\ & \quad - \frac{1}{qt} (\sigma_{q,t}\mathfrak{A}_\infty) \mathfrak{C}(qt(\Theta_t+\bar{\Theta}_t)I_2+B_0) \end{aligned}$$

$$\begin{aligned}
 R_4 &:= \frac{1}{qt\Theta_t\bar{\Theta}_t} B_0 \mathfrak{A}_0 + \frac{t-1}{q(t\Theta_t-1)(t\bar{\Theta}_t-1)} (qI_2 + B_0) \mathfrak{A}_1 + \mathfrak{A}_\infty \\
 R_5 &:= \mathfrak{A}_0 + \frac{t-1}{t} \mathfrak{A}_1 - \frac{(t-1)}{qt(t\Theta_t-1)(t\bar{\Theta}_t-1)} (qI_2 + B_0) \mathfrak{A}_1 \\
 &\quad - \frac{1}{qt} (qt(\Theta_t + \bar{\Theta}_t)I_2 + B_0) \mathfrak{A}_\infty.
 \end{aligned}$$

Hence (1) is equivalent to the vanishing $R_i = 0$ for all $i \in \{0, 1, 2, 3, 4, 5\}$ for some $B_0, \mathfrak{C} \in \text{GL}_2(\mathcal{M}(\mathfrak{D}))$ with \mathfrak{C} diagonal. Note that by Proposition 2.9, the matrix B_0 must have eigenvalues $-qt\Theta_t, -qt\bar{\Theta}_t$, so that $I_2 + B_0$ is invertible. If $R_0 = R_1 = R_4 = R_5 = 0$, then R_2 and R_3 are both equivalent to $\sigma_{q,t}\mathfrak{A}_\infty = \mathfrak{C}\mathfrak{A}_\infty\mathfrak{C}^{-1}$. We may therefore omit R_3 in the following. For $i \in \{0, 1\}$, the vanishing of R_i is equivalent to the equation for $\sigma_{q,t}\mathfrak{A}_i$ as in the statement, with general B_0, \mathfrak{C} . Substituting these equations into $R_2 = 0$ yields the equation for $\sigma_{q,t}\mathfrak{A}_t$ as in the statement, again with general B_0, \mathfrak{C} . However, $R_4 = 0$ is equivalent to B_0 being as in the statement. Note that this matrix B_0 is well-defined and invertible under the assumptions. As one can check by direct computation, this B_0 solves $R_5 = 0$. Hence (1) is equivalent to the existence of a diagonal matrix $\mathfrak{C} \in \text{GL}_2(\mathcal{M}(\mathfrak{D}))$ such that the equations in (2) hold for B_0 as in the statement. If \mathfrak{C} is such a convenient matrix, then for any $f \in \mathcal{M}(\mathfrak{D})$ non-vanishing, $f\mathfrak{C}$ is also convenient. Hence we may require that \mathfrak{C} is of the form $\text{diag}(c, 1)$. Since $(q-1)\boldsymbol{\lambda} = (1-t)\mathfrak{A}_1^{(1,2)} - t\mathfrak{A}_0^{(1,2)}$, we must have

$$(q-1)\sigma_{q,t}\boldsymbol{\lambda} = (1-qt)\sigma_{q,t}\mathfrak{A}_1^{(1,2)} - qt\sigma_{q,t}\mathfrak{A}_0^{(1,2)}.$$

With the equations in (2) for the $\sigma_{q,t}\mathfrak{A}_i$'s, this is equivalent to c being as in the statement. We conclude that (1) \Leftrightarrow (2).

Let us now show that (2) \Rightarrow (3). From $\mathbf{y} = -\frac{t}{(q-1)\boldsymbol{\lambda}}\mathfrak{A}_0^{(1,2)}$, we obtain

$$\sigma_{q,t}\mathbf{y} = -\frac{qt}{(q-1)\sigma_{q,t}\boldsymbol{\lambda}}\sigma_{q,t}\mathfrak{A}_0^{(1,2)} = -qt\frac{\sigma_{q,t}\mathfrak{A}_0^{(1,2)}}{(1-qt)\sigma_{q,t}\mathfrak{A}_1^{(1,2)} - qt\sigma_{q,t}\mathfrak{A}_0^{(1,2)}}.$$

Substituting the values of $\mathfrak{A}_0^{(1,2)}$ and $\mathfrak{A}_1^{(1,2)}$ from (2) then yields

$$\sigma_{q,t}\mathbf{y} = \frac{q}{\mathbf{y}} \cdot \frac{\Theta_1\bar{\Theta}_1\Theta_\infty\bar{\Theta}_\infty X^2 - t(\Theta_0 + \bar{\Theta}_0)X + t^2\Theta_t\bar{\Theta}_t}{(\Theta_\infty X - 1)(q\bar{\Theta}_\infty X - 1)}. \quad (2.13)$$

For the purpose of factorization, we use the equality

$$\Theta_0\bar{\Theta}_0 = \Theta_t\bar{\Theta}_t\Theta_1\bar{\Theta}_1\Theta_\infty\bar{\Theta}_\infty.$$

This yields the expression for $\sigma_{q,t}\mathbf{y}$ in the statement. Similarly, from

$$1 + (q-1)\mathbf{y}\mathbf{Z} = \mathfrak{A}_0^{(1,1)} + \frac{\mathbf{y}}{\mathbf{y}-1}\mathfrak{A}_1^{(1,1)} + \frac{\mathbf{y}}{\mathbf{y}-t}\left(\mathfrak{A}_\infty^{(1,1)} - \mathfrak{A}_0^{(1,1)} - \mathfrak{A}_1^{(1,1)}\right),$$

we obtain

$$\begin{aligned} & \sigma_{q,t}((\mathbf{y}-1)(\mathbf{y}-t)(1+(q-1)\mathbf{y}\mathbf{Z})) \\ &= (\sigma_{q,t}\mathbf{y}-1)\left(\bar{\Theta}_\infty\sigma_{q,t}\mathbf{y}-qt\sigma_{q,t}\mathfrak{A}_0^{(1,1)}\right)+(1-qt)\sigma_{q,t}\mathbf{y}\cdot\sigma_{q,t}\mathfrak{A}_1^{(1,1)}. \end{aligned}$$

Substituting the value of $\sigma_{q,t}\mathbf{y}$ from (2.13), the values of $\mathfrak{A}_0^{(1,1)}$ and $\mathfrak{A}_1^{(1,1)}$ from (2), as well as $\mathbf{Z} = \frac{1}{(q-1)\mathbf{y}} \left(\frac{\Theta_\infty\bar{\Theta}_\infty(\mathbf{y}-\Theta_1)(\mathbf{y}-\bar{\Theta}_1)}{(\mathbf{y}-1)(\mathbf{y}-t)} X - 1 \right)$, the right hand side simplifies to

$$\frac{(\sigma_{q,t}\mathbf{y}-qt\Theta_t)(\sigma_{q,t}\mathbf{y}-qt\bar{\Theta}_t)}{qX}.$$

Therefore, $\sigma_{q,t}\mathbf{Z}$ is as in the statement.

Let us now show that (3) \Rightarrow (2). Note that for each $i \in \{0, 1, t\}$, the matrices \mathfrak{A}_i may be expressed as functions of λ, X and \mathbf{y} . If (3) holds, then the matrices $\mathfrak{C}^{-1}\sigma_{q,t}\mathfrak{A}_i\mathfrak{C}$, with \mathfrak{C} as in (2), can also be expressed as functions of λ, X and \mathbf{y} . It is straightforward to check (with computer assistance) that the equations in (2) then are satisfied. \square

In analogy with the differential case, we give a name to the particular case when a family can be completed into a Lax pair *via* a matrix \mathfrak{B} as in (2.7) with $\mathfrak{C}(t) = \mathbf{I}_2$:

DEFINITION 2.11. — *We say that the family $\sigma_{q,x}Y = \mathfrak{A}(x,t)Y$ of Fuchsian systems is q -Schlesinger isomonodromic if it can be completed into a q -Lax pair via a matrix $\mathfrak{B} \in \text{GL}_2(\mathcal{M}(\mathfrak{D})(x))$ of the form*

$$\mathfrak{B}(x,t) = \frac{(x-qt)(x\mathbf{I}_2+B_0(t))}{(x-qt\Theta_t)(x-qt\bar{\Theta}_t)}. \tag{2.14}$$

Let us now say a few words about whether, analogously to the differential setting, a family of q -Fuchsian systems which is q -isomonodromic can be made q -Schlesinger isomonodromic *via* a gauge transformation. Let $\mathfrak{A} \in \text{GL}_2(\mathcal{O}(\mathfrak{D})(x))$ be as in (2.5) and assume that the family of q -Fuchsian systems $\partial_{q,x}Y = \mathfrak{A}(x,t)Y$ can be completed into a Lax pair *via* a matrix $\mathfrak{B} \in \text{GL}_2(\mathcal{M}(\mathfrak{D})(x))$ of the form (2.7), with $\mathfrak{C} \in \text{GL}_2(\mathcal{M}(\mathfrak{D}))$ diagonal. Assume there exists $\mathfrak{M} \in \text{GL}_2(\mathcal{M}(\mathfrak{D}))$ which is diagonal and solves the q -difference equation

$$\sigma_{q,t}\mathfrak{M}(t) = \mathfrak{C}(t)\mathfrak{M}(t).$$

Since \mathfrak{M} does not depend on x , performing the gauge transformation $Y = \mathfrak{M}\hat{Y}$ yields the family $\sigma_{q,x}\hat{Y} = \hat{\mathfrak{A}}(x,t)\hat{Y}$ given by

$$\hat{\mathfrak{A}}(x,t) = (\sigma_{q,x}\mathfrak{M}(t))^{-1}\mathfrak{A}(x,t)\mathfrak{M}(t) = \mathfrak{M}(t)^{-1}\mathfrak{A}(x,t)\mathfrak{M}(t).$$

Since \mathfrak{M} is diagonal, up to shrinking \mathfrak{D} to the domain of holomorphy of the coefficients of $\hat{\mathfrak{A}} \in \text{GL}(\mathcal{M}(\mathfrak{D})(x))$, this new family is still a family of

q -Fuchsian systems in the sense of Definition 1.6. Moreover, this new family can be completed into a Lax pair *via* the matrix

$$\widehat{\mathfrak{B}} := \mathfrak{M}^{-1} \mathfrak{C}^{-1} \mathfrak{B} \mathfrak{M} \in \mathrm{GL}_2(\mathcal{M}(\mathfrak{D}))(x).$$

Indeed, from the q -Lax equation for the initial family, we get

$$\begin{aligned} \widehat{\mathfrak{A}}(x, qt) \widehat{\mathfrak{B}}(x, t) &= \mathfrak{M}(qt)^{-1} \mathfrak{A}(x, qt) \mathfrak{M}(qt) \mathfrak{M}^{-1}(t) \mathfrak{C}^{-1}(t) \mathfrak{B}(x, t) \mathfrak{M}(t) \\ &= \mathfrak{M}(t)^{-1} \mathfrak{C}(t)^{-1} \mathfrak{A}(x, qt) \mathfrak{B}(x, t) \mathfrak{M}(t) \\ &= \mathfrak{M}(t)^{-1} \mathfrak{C}(t)^{-1} \mathfrak{B}(qx, t) \mathfrak{A}(x, t) \mathfrak{M}(t) \\ &= \widehat{\mathfrak{B}}(qx, t) \widehat{\mathfrak{A}}(x, t). \end{aligned}$$

Note that $\widehat{\mathfrak{B}}$ is given by

$$\widehat{\mathfrak{B}}(x, t) = \frac{(x - qt)(x\mathbf{I}_2 + \widehat{B}_0(t))}{(x - qt\Theta_t)(x - qt\overline{\Theta}_t)} \quad \text{with} \quad \widehat{B}_0 = \mathfrak{M}^{-1} B_0 \mathfrak{M}.$$

In other words, the conjugated family $\sigma_{q,x} \widehat{Y} = \widehat{\mathfrak{A}}(x, t) \widehat{Y}$ is q -Schlesinger isomonodromic. To find this conjugated family, we had to solve a diagonal system of q -difference equations, which boils down to solving two scalar linear q -difference equations. Contrarily to the differential case, the resolution of q -difference equations even of such simple form is not trivial, and does not seem to be known in full generality. However, if some strong assumptions on the domain of definition \mathfrak{D} are satisfied, one can use for example the following lemma.

LEMMA 2.12. — *Assume that $|q| > 1$ and \mathfrak{D} contains an annulus of the form*

$$\{t \in \mathbb{C} \mid a < |t| < b\}$$

for some real numbers $0 < a < b < \infty$. Let $c \in \mathcal{M}(\mathfrak{D}) \setminus \{0\}$. Then there exists a meromorphic solution $\mathbf{m}(t) \in \mathcal{M}(\mathfrak{D}) \setminus \{0\}$ of

$$\sigma_{q,t} \mathbf{m} = c \mathbf{m}.$$

Note that the assumption on \mathfrak{D} of the above lemma is satisfied if for instance $\mathfrak{D} = \mathbb{C}^* \setminus \bigcup_{\ell=1}^k a_\ell q^{\mathbb{Z}}$, for some $a_\ell \in \mathbb{C}^*$.

Proof. — Let us define

$$O_1 := \{t \in \mathbb{P}^1 \mid |t| < b\} \cup (\mathfrak{D} \cap \{t \in \mathbb{P}^1 \mid |t| \geq b\}),$$

$$O_2 := \{t \in \mathbb{P}^1 \mid |t| > a\} \cup (\mathfrak{D} \cap \{t \in \mathbb{P}^1 \mid |t| \leq a\}).$$

These are connected open sets satisfying $O_1 \cap O_2 = \mathfrak{D}$ and $O_1 \cup O_2 = \mathbb{P}^1$. By [1, Lem. 4.4], there exist $c_1 \in \mathcal{M}(O_1)$, and $c_2 \in \mathcal{M}(O_2)$ such that $c = c_1 c_2$. By construction, c_1 is a germ of meromorphic function at 0. By

Remark 5.4, there exists $0 \neq \mathbf{m}_1$ that is meromorphic on a punctured neighborhood of 0 in \mathbb{C}^* such that $\sigma_{q,t}\mathbf{m}_1 = c_1\mathbf{m}_1$. Using the functional equation and using the fact that \mathfrak{D} is stable by multiplication by q , we find that \mathbf{m}_1 may be continued into a meromorphic function on $qO_1 \setminus \{0\}$ where $qO_1 = \{qt, t \in O_1\}$. Similarly, we construct a non-zero meromorphic solution of $\sigma_{q,t}\mathbf{m}_2 = c_2\mathbf{m}_2$ that is meromorphic on $q^{-1}O_2 \setminus \{\infty\}$. Since $\mathfrak{D} \subset \mathbb{C}^*$ and $qO_1 \cap q^{-1}O_2 \supset O_1 \cap O_2 = \mathfrak{D}$ we find that a convenient solution is $\mathbf{m} = \mathbf{m}_1\mathbf{m}_2$. \square

2.3. Confluence

Let $\Theta(q) = (\Theta_0(q), \Theta_1(q), \Theta_t(q), \Theta_\infty(q))$ be a quadruple of rational functions in a complex variable q such that as $q \rightarrow 1$, we have

$$\Theta_i(q) = 1 + (q-1)\frac{\theta_i}{2} + O((q-1)^2) \quad \forall i \in \{0, 1, t, \infty\} \quad (2.15)$$

with $\theta_i \in \mathbb{C}$. We define $\bar{\Theta}$ by $\bar{\Theta}_i = \frac{1}{\Theta_i}$. Recall from Section 1.3 that these requirements on $(\Theta, \bar{\Theta})$ are a convenient setting for the discretization of \mathfrak{sl}_2 -Fuchsian systems with spectral data Θ (if $\theta_\infty \neq 0$). We shall now see under these requirements, the q -Schlesinger equations discretize the (differential) Schlesinger equations, and that the difference equation (2.12) generically characterizing q -isomonodromy discretizes the differential equation (2.3) generically characterizing isomonodromy.

2.3.1. The q -Schlesinger equations discretize the differential ones

The q -Schlesinger equations are obtained from the equations in point (2) of Proposition 2.10 by setting $\mathfrak{C} = \mathbf{I}_2$. With respect to the $\partial_{q,t}$ -operator and the matrices

$$\tilde{\mathfrak{A}}_0 = \frac{\mathfrak{A}_0 - \mathbf{I}_2}{q-1}, \quad \tilde{\mathfrak{A}}_1 = \frac{\mathfrak{A}_1}{q-1}, \quad \tilde{\mathfrak{A}}_t = \frac{\mathfrak{A}_t}{t(q-1)},$$

they read as follows:

$$\left\{ \begin{array}{l} \partial_{q,t}\tilde{\mathfrak{A}}_0 = \frac{B_0\tilde{\mathfrak{A}}_0B_0^{-1} - \tilde{\mathfrak{A}}_0}{(q-1)t} \\ \partial_{q,t}\tilde{\mathfrak{A}}_1 = \frac{1}{(q-1)t} \left(\frac{(t-1)}{qt-1} \frac{(qt\Theta_t-1)(qt\bar{\Theta}_t-1)}{q(t\Theta_t-1)(t\bar{\Theta}_t-1)} (q\mathbf{I}_2 + B_0) \tilde{\mathfrak{A}}_1 (\mathbf{I}_2 + B_0)^{-1} - \tilde{\mathfrak{A}}_1 \right) \\ \partial_{q,t}\tilde{\mathfrak{A}}_t = -\frac{1}{(q-1)^2t} \left(\mathbf{I}_2 + \frac{1}{qt\Theta_t\bar{\Theta}_t} B_0 \right) - \frac{1}{(q-1)t} \left(B_0\tilde{\mathfrak{A}}_0 \left(\frac{1}{qt\Theta_t\bar{\Theta}_t} \mathbf{I}_2 + B_0^{-1} \right) + \tilde{\mathfrak{A}}_t \right) \\ \quad - \frac{(t-1)(q\mathbf{I}_2+B_0)\tilde{\mathfrak{A}}_1}{q(q-1)t(t\Theta_t-1)(t\bar{\Theta}_t-1)} \cdot \left(\mathbf{I}_2 + \frac{(qt\Theta_t-1)(qt\bar{\Theta}_t-1)}{qt-1} (\mathbf{I}_2 + B_0)^{-1} \right), \end{array} \right. \quad (2.16)$$

where $B_0 = B_0(\tilde{\mathfrak{A}}_0, \tilde{\mathfrak{A}}_1, t, q)$ is the function with values in

$$G := \{M \in \mathrm{GL}_2(\mathbb{C}) \mid \det(M + \mathrm{I}_2) \neq 0\},$$

defined, on the complement of some proper Zariski closed subset of $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \times \mathbb{C} \times \mathbb{C}$, as $B_0(\tilde{\mathfrak{A}}_0, \tilde{\mathfrak{A}}_1, t, q)$ being given by

$$\begin{aligned} & -qt \left(\mathrm{I}_2 + (q-1) \left(\frac{t-1}{(t\Theta_t-1)(t\bar{\Theta}_t-1)} \tilde{\mathfrak{A}}_1 - \tilde{\mathfrak{A}}_\infty \right) \right) \\ & \quad \times \left(\mathrm{I}_2 + (q-1) \left(\tilde{\mathfrak{A}}_0 + \frac{t(t-1)}{(t\Theta_t-1)(t\bar{\Theta}_t-1)} \tilde{\mathfrak{A}}_1 \right) \right)^{-1}, \end{aligned}$$

where $\tilde{\mathfrak{A}}_\infty(q) = \mathrm{diag}\left(\frac{1-\bar{\Theta}_\infty}{q-1}, \frac{1-\Theta_\infty}{q-1}\right) = \mathrm{diag}\left(\frac{\theta_\infty}{2}, -\frac{\theta_\infty}{2}\right) + O(q-1)$. Since $\tilde{\mathfrak{A}}_\infty(q)$ is given in this context, we may write everything as a function of $\tilde{\mathfrak{A}}_0$ and $\tilde{\mathfrak{A}}_1$ by identifying $\tilde{\mathfrak{A}}_t := -(\tilde{\mathfrak{A}}_0 + \tilde{\mathfrak{A}}_1 + \tilde{\mathfrak{A}}_\infty)$. With this notation, as $q \rightarrow 1$, up to terms of order $O((q-1)^2)$, we have

$$\begin{aligned} & B_0(\tilde{\mathfrak{A}}_0, \tilde{\mathfrak{A}}_1, t, q) \\ & \sim -qt \left(\mathrm{I}_2 + (q-1) \left(\frac{1}{t-1} \tilde{\mathfrak{A}}_1 - \tilde{\mathfrak{A}}_\infty \right) \right) \left(\mathrm{I}_2 + (q-1) \left(\tilde{\mathfrak{A}}_0 + \frac{t}{t-1} \tilde{\mathfrak{A}}_1 \right) \right)^{-1} \\ & \sim -qt \left(\mathrm{I}_2 + (q-1) \left(\frac{1}{t-1} \tilde{\mathfrak{A}}_1 - \tilde{\mathfrak{A}}_\infty \right) \right) \left(\mathrm{I}_2 - (q-1) \left(\tilde{\mathfrak{A}}_0 + \frac{t}{t-1} \tilde{\mathfrak{A}}_1 \right) \right) \\ & \sim -qt \left(\mathrm{I}_2 + (q-1) \tilde{\mathfrak{A}}_t \right). \end{aligned}$$

Let f_0, f_1, f_t be the functions with values in $M_2(\mathbb{C})$, defined on some natural domain of definition inside $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \times \mathbb{C} \times (\mathbb{C} \setminus \{1\})$, that when evaluated in $(\tilde{\mathfrak{A}}_0, \tilde{\mathfrak{A}}_1, t, q)$, yield the right hand sides of the equations in (2.16). Then we have

$$\begin{aligned} f_0(\tilde{\mathfrak{A}}_0, \tilde{\mathfrak{A}}_1, t, q) &= \frac{\left(\mathrm{I}_2 + (q-1) \tilde{\mathfrak{A}}_t \right) \tilde{\mathfrak{A}}_0 \left(\mathrm{I}_2 - (q-1) \tilde{\mathfrak{A}}_t \right) - \tilde{\mathfrak{A}}_0}{(q-1)t} + O(q-1) \\ &= \frac{[\tilde{\mathfrak{A}}_0, \tilde{\mathfrak{A}}_t]}{0-t} + O(q-1), \end{aligned}$$

$$\begin{aligned}
 f_1(\tilde{\mathfrak{A}}_0, \tilde{\mathfrak{A}}_1, t, q) &= \frac{(\mathbf{I}_2 - (q-1)\frac{t}{1-t}\tilde{\mathfrak{A}}_t)\tilde{\mathfrak{A}}_1 \left(\mathbf{I}_2 - (q-1)\frac{t}{1-qt}\tilde{\mathfrak{A}}_t \right)^{-1} - \tilde{\mathfrak{A}}_1}{(q-1)t} + O(q-1) \\
 &= \frac{(\mathbf{I}_2 - (q-1)\frac{t}{1-t}\tilde{\mathfrak{A}}_t)\tilde{\mathfrak{A}}_1 \left(\mathbf{I}_2 + (q-1)\frac{t}{1-t}\tilde{\mathfrak{A}}_t \right) - \tilde{\mathfrak{A}}_1}{(q-1)t} + O(q-1) \\
 &= \frac{[\tilde{\mathfrak{A}}_1, \tilde{\mathfrak{A}}_t]}{1-t} + O(q-1).
 \end{aligned}$$

Using similar calculations, and the Taylor series expansion of B_0 until its second order term, one finds

$$f_t(\tilde{\mathfrak{A}}_0, \tilde{\mathfrak{A}}_1, t, q) = -\frac{[\tilde{\mathfrak{A}}_0, \tilde{\mathfrak{A}}_t]}{0-t} - \frac{[\tilde{\mathfrak{A}}_1, \tilde{\mathfrak{A}}_t]}{1-t} + O(q-1).$$

In summary, we have

$$\begin{cases}
 \partial_{q,t}\tilde{\mathfrak{A}}_0(t) = \frac{[\tilde{\mathfrak{A}}_0(t), \tilde{\mathfrak{A}}_t(t)]}{0-t} + O(q-1) \\
 \partial_{q,t}\tilde{\mathfrak{A}}_1(t) = \frac{[\tilde{\mathfrak{A}}_1(t), \tilde{\mathfrak{A}}_t(t)]}{1-t} + O(q-1) \\
 \partial_{q,t}\tilde{\mathfrak{A}}_t(t) = -\frac{[\tilde{\mathfrak{A}}_0(t), \tilde{\mathfrak{A}}_t(t)]}{0-t} - \frac{[\tilde{\mathfrak{A}}_1(t), \tilde{\mathfrak{A}}_t(t)]}{1-t} + O(q-1).
 \end{cases}$$

This proves that the q -Schlesinger equations (2.16) discretize the (differential) Schlesinger equations (2.2).

In order to complement this result, let us consider the function $\tilde{\mathfrak{B}}$ with values in $M_2(\mathbb{C})$, given, on its natural set of definition inside $GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \times \mathbb{C} \times \mathbb{C} \times (\mathbb{C} \setminus \{1\})$, by

$$\tilde{\mathfrak{B}}(\tilde{\mathfrak{A}}_0, \tilde{\mathfrak{A}}_1, x, t, q) := \frac{1}{(q-1)t} \left(\frac{(x-qt) \left(x\mathbf{I}_2 + B_0(\tilde{\mathfrak{A}}_0, \tilde{\mathfrak{A}}_1, t, q) \right)}{(x-qt\Theta_t)(x-qt\bar{\Theta}_t)} - \mathbf{I}_2 \right).$$

This function corresponds to the right hand side of the $\delta_{q,t}$ -version of $\sigma_{q,t}Y = \mathfrak{B}Y$ with \mathfrak{B} as in (2.14). It behaves, when $q \rightarrow 1$, as

$$\begin{aligned}
 &\tilde{\mathfrak{B}}(\tilde{\mathfrak{A}}_0, \tilde{\mathfrak{A}}_1, x, t, q) \\
 &\sim \frac{1}{(q-1)t} \left(\frac{(x-qt) \left((x-qt)\mathbf{I}_2 - (q-1)t\tilde{\mathfrak{A}}_t \right)}{(x-qt)^2} - \mathbf{I}_2 + O((q-1)^2) \right)
 \end{aligned}$$

$$\begin{aligned} &\sim \frac{1}{(q-1)t} \left(-(q-1)t \frac{\tilde{\mathfrak{A}}_t}{(x-qt)} + O((q-1)^2) \right) \\ &\sim -\frac{\tilde{\mathfrak{A}}_t}{x-t} + O(q-1). \end{aligned}$$

By the above estimates, $\tilde{\mathfrak{B}}$ can be continued analytically to $\{q=1\}$ and is there given by $-\frac{\tilde{\mathfrak{A}}_t}{x-t}$.

2.3.2. The q -Lax pairs discretize the differential ones

Let

$$\Omega \subset \mathbb{C}^* \setminus e^{2i\pi\mathbb{Q}}$$

be a connected subset with 1 in its closure. Let $\mathfrak{D} \subset \mathbb{C}^*$ be an open connected subset. We shall assume that the pair (\mathfrak{D}, Ω) satisfies the property that \mathfrak{D} is stable by multiplication by $q^{\pm 1}$, for every $q \in \Omega$. Note that unless $\mathfrak{D} = \mathbb{C}^*$, the subset Ω cannot be too large. Two examples of a convenient pair (\mathfrak{D}, Ω) with $\mathfrak{D} \subset \mathbb{C}^*$ are the following:

- $(\mathfrak{D}, \Omega) = (\mathbb{C}^* \setminus \{q_0^{\mathbb{R}}, q_0^{\mathbb{R}>0}\})$ where $q_0 \in \mathbb{C}$, with $|q_0| \neq 1$.
- \mathfrak{D} is an open sector with infinite radius centered at 0 and $\Omega =]1, +\infty[$.

In addition to our previous requirements on $(\Theta, \bar{\Theta})$, let us now moreover assume that the (differential) spectral values Θ satisfy the non-resonant condition $\theta_i \notin \mathbb{Z}^*$ for $i \in \{0, 1, t, \infty\}$ and $\theta_\infty \neq 0$. Note that for values of $q \in \Omega$ sufficiently close to 1, the (q -difference) non-resonant condition (1.7) then is automatically satisfied.

Let $A_0, A_1, A_t \in \text{GL}_2(\mathcal{O}(\mathfrak{D}))$ such that $\partial_x Y(x, t) = A(x, t)Y$ with $A = \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t}$ is a family of \mathfrak{sl}_2 -Fuchsian systems with spectral data Θ as in Definition 1.1. Let $\tilde{\mathfrak{A}}_0, \tilde{\mathfrak{A}}_1, \tilde{\mathfrak{A}}_t$ be holomorphic functions in a neighborhood of $\mathfrak{D} \times \Omega \subset \mathbb{C}^2$ with values in $\text{GL}_2(\mathbb{C})$, such that for each $q \in \Omega$, the q -difference equation $\partial_{q,x} Y = \tilde{\mathfrak{A}}(x, t, q)Y$ with $\tilde{\mathfrak{A}} = \frac{\tilde{\mathfrak{A}}_0}{x} + \frac{\tilde{\mathfrak{A}}_1}{x-1} + \frac{\tilde{\mathfrak{A}}_t}{x-t}$ yields, via $\mathfrak{A} = \text{I}_2 + x(q-1)\tilde{\mathfrak{A}}$, a family of q -Fuchsian systems with spectral data $(\Theta(q), \bar{\Theta}(q))$ as in Definition 1.6. By Proposition 1.13 it is convenient to assume that

$$\forall i \in \{0, 1, t\}, \quad \lim_{q \rightarrow 1} \tilde{\mathfrak{A}}_i(t, q) = A_i(t).$$

We shall moreover assume that

$$\forall i \in \{0, 1, t\}, \quad \lim_{q \rightarrow 1} \partial_{q,t} \tilde{\mathfrak{A}}_i(t, q) = A'_i(t)$$

and that the family $\partial_{q,x}Y = \tilde{\mathfrak{A}}(x, t, q)Y$ is q -isomonodromic for each $q \in \Omega$. By non-resonant, Section 2.3.1 and the proof of Proposition 2.10, this means that this family can be completed into a q -Lax pair

$$\begin{cases} \sigma_{q,x}Y = \left(\mathbf{I}_2 + (q-1)x\tilde{\mathfrak{A}}(x, t, q) \right) Y \\ \sigma_{q,t}Y = \mathfrak{C}(t, q) \left(\mathbf{I}_2 + (q-1)t\tilde{\mathfrak{B}}(x, t, q) \right) Y. \end{cases}$$

We then find

$$\begin{cases} \partial_{q,x}Y = \tilde{\mathfrak{A}}(x, t, q)Y \\ \partial_{q,t}Y = \left(\mathfrak{C}(t, q)\tilde{\mathfrak{B}}(x, t, q) + \frac{\mathfrak{C}(t, q) - \mathbf{I}_2}{(q-1)t} \right) Y, \end{cases} \quad (2.17)$$

where

$$\lim_{q \rightarrow 1} \tilde{\mathfrak{B}}(x, t, q) = -\frac{A_t(t)}{x-t} \quad \text{and} \quad \mathfrak{C}(t, q) = f(t, q) \operatorname{diag}(c_1(t, q), c_2(t, q)).$$

Here $f \in \mathcal{M}(\mathfrak{D} \times \Omega) \setminus \{0\}$ can be chosen arbitrarily and

$$c_1 = \frac{\sigma_{q,t} \left(t\tilde{\mathfrak{A}}_0^{(1,2)} + (t-1)\tilde{\mathfrak{A}}_1^{(1,2)} \right)}{t\tilde{\mathfrak{A}}_0^{(1,2)} + (t-1)\tilde{\mathfrak{A}}_1^{(1,2)}}, \quad c_2 = \frac{q\bar{\Theta}_\infty X - 1}{\Theta_\infty X - 1},$$

where X is some rational expression in terms of $q, t, \Theta_1, \Theta_\infty, \tilde{\mathfrak{A}}_0^{(1,1)}, \tilde{\mathfrak{A}}_0^{(1,2)}, \tilde{\mathfrak{A}}_1^{(1,1)}, \tilde{\mathfrak{A}}_1^{(1,2)}$ that can easily be made explicit. We assume these c_i and X to be well-defined and finite. Using the Taylor series expansion of $\Theta_1(q)$ and $\Theta_\infty(q)$, we readily compute that up to terms of order $O((q-1)^2)$, we have

$$\begin{aligned} \frac{1}{f} \mathfrak{C} &\sim \mathbf{I}_2 + \frac{(q-1)t}{t\tilde{\mathfrak{A}}_0^{(1,2)} + (t-1)\tilde{\mathfrak{A}}_1^{(1,2)}} \\ &\times \begin{pmatrix} t\partial_{q,t}\tilde{\mathfrak{A}}_0^{(1,2)} + (t-1)\partial_{q,t}\tilde{\mathfrak{A}}_1^{(1,2)} & 0 \\ 0 & (1-\theta_\infty)(\tilde{\mathfrak{A}}_0^{(1,2)} + \tilde{\mathfrak{A}}_1^{(1,2)}) \end{pmatrix}. \end{aligned}$$

Choosing f of the form $f(t, q) = 1 + (q-1)g(t)$ for some meromorphic function g , we can make sure that $\lim_{q \rightarrow 1} \mathfrak{C}(t, q) = \mathbf{I}_2$ and that the matrix

$$C(t) := \lim_{q \rightarrow 1} \frac{\mathfrak{C}(t, q) - \mathbf{I}_2}{(q-1)t}$$

exists and is tracefree. Consider a subset of \mathfrak{D} where $C(t)$ is holomorphic. With $\mathfrak{C} = \mathbf{I}_2 + (q-1)tC$, the q -Lax pair induced by (2.17) is

$$\begin{aligned} &\left(\mathbf{I}_2 + (q-1)x\sigma_{q,t}\tilde{\mathfrak{A}} \right) \left(\mathbf{I}_2 + (q-1)tC \right) \left(\mathbf{I}_2 + (q-1)t\tilde{\mathfrak{B}} \right) \\ &= \left(\mathbf{I}_2 + (q-1)tC \right) \left(\mathbf{I}_2 + (q-1)t\sigma_{q,x}\tilde{\mathfrak{B}} \right) \left(\mathbf{I}_2 + (q-1)x\tilde{\mathfrak{A}} \right). \end{aligned}$$

The term in $O(1)$ on both sides of the equality is $I_2 = I_2$. The term in $O(q-1)$ is

$$(q-1)x\sigma_{q,t}\tilde{\mathfrak{A}} + (q-1)t\tilde{\mathfrak{B}} = (q-1)t\sigma_{q,x}\tilde{\mathfrak{B}} + (q-1)x\tilde{\mathfrak{A}}.$$

Finally, the term in $O((q-1)^2)$ is

$$\begin{aligned} (q-1)^2xt\sigma_{q,t}(\tilde{\mathfrak{A}})C + (q-1)^2xt\sigma_{q,t}(\tilde{\mathfrak{A}})\tilde{\mathfrak{B}} + (q-1)^2t^2C\tilde{\mathfrak{B}} \\ = (q-1)^2xt\sigma_{q,x}(\tilde{\mathfrak{B}})\tilde{\mathfrak{A}} + (q-1)^2t^2C\sigma_{q,x}(\tilde{\mathfrak{B}}) + (q-1)^2xtC\tilde{\mathfrak{A}}. \end{aligned}$$

Then, dividing by $(q-1)^2xt$ and we obtain with $\partial_{q,x}C = 0$ that

$$\begin{aligned} \partial_{q,t}(\tilde{\mathfrak{A}}) - \partial_{q,x}(\tilde{\mathfrak{B}} + C) = \sigma_{q,x}(\tilde{\mathfrak{B}})\tilde{\mathfrak{A}} + x^{-1}tC\sigma_{q,x}(\tilde{\mathfrak{B}}) + C\tilde{\mathfrak{A}} \\ - \sigma_{q,t}(\tilde{\mathfrak{A}})C - \sigma_{q,t}(\tilde{\mathfrak{A}})\tilde{\mathfrak{B}} - x^{-1}tC\tilde{\mathfrak{B}} + O(q-1). \end{aligned}$$

With $\sigma_{q,t}(\tilde{\mathfrak{A}}) = \tilde{\mathfrak{A}} + O(q-1)$, $\sigma_{q,x}(\tilde{\mathfrak{B}}) = \tilde{\mathfrak{B}} + O(q-1)$, we obtain

$$\partial_{q,t}(\tilde{\mathfrak{A}}) - \partial_{q,x}(\tilde{\mathfrak{B}} + C) = [\tilde{\mathfrak{B}} + C, \tilde{\mathfrak{A}}] + O(q-1).$$

Since $\tilde{\mathfrak{A}} = A + O(q-1)$, $\tilde{\mathfrak{B}} = -\frac{A_t(t)}{x-t} + O(q-1)$, this shows that we obtain the confluence of the q -Lax pair (2.17) to the differential Lax pair given by

$$\begin{cases} \partial_x Y = A(x, t)Y \\ \partial_t Y = \left(-\frac{A_t(t)}{x-t} + C(t) \right) Y, \end{cases}$$

satisfying the Lax equation

$$\partial_t(A) - \partial_x \left(-\frac{A_t(t)}{x-t} + C \right) = \left[-\frac{A_t(t)}{x-t} + C, A \right].$$

2.3.3. The difference equations for triples $(\lambda, \mathbf{y}, \mathbf{Z})$ discretize the differential ones

Of particular interest for the results in this paper is the discretization of the characterization of Schlesinger isomonodromy in terms of triples $(\lambda(t), \mathbf{y}(t), \mathbf{Z}(t))$ given in Proposition 2.6, namely the system of differential equations given by (2.3) and (2.4). Proposition 2.10 suggests that a convenient q -analogue of this differential equation is given by

$$\begin{cases} \frac{\partial_{q,t}\lambda}{\lambda} = \frac{(\Theta_\infty - q\bar{\Theta}_\infty)X}{(q-1)t(1 - \Theta_\infty X)} \\ \partial_{q,t}\mathbf{y} = \frac{E - \mathbf{y}}{(q-1)t} \\ \partial_{q,t}\mathbf{Z} = \frac{1}{(q-1)t} \cdot \left(\frac{(E - qt\Theta_t)(E - qt\bar{\Theta}_t)}{q(q-1)E(E-1)(E-qt)X} - \frac{1}{(q-1)E} - \mathbf{Z} \right), \end{cases} \quad (2.18)$$

where $X(\mathbf{y}, \mathbf{Z}, t, q)$ and $E(\mathbf{y}, \mathbf{Z}, q, t)$ are defined respectively as

$$X := \frac{(\mathbf{y} - 1)(\mathbf{y} - t)(1 + (q - 1)\mathbf{y}\mathbf{Z})}{(\mathbf{y} - \Theta_1)(\mathbf{y} - \bar{\Theta}_1)}, \quad E := \frac{q}{\mathbf{y}} \cdot \frac{(X - t\Theta_0)(X - t\bar{\Theta}_0)}{(\Theta_\infty X - 1)(q\bar{\Theta}_\infty X - 1)}.$$

Here, as usual when considering confluence, we used our convention $\Theta_i \bar{\Theta}_i = 1$. Let us now show that (2.18) discretizes the system of differential equations given by equations (2.4) and (2.3).

Let $f_\lambda, f_{\mathbf{y}}, f_{\mathbf{Z}}$ be the rational functions in the variables $\mathbf{y}, \mathbf{Z}, t, q$ forming the right hand sides of the equations in (2.18). Using the estimates (2.15), we may compute the Taylor series expansion of $X(q) := X(\mathbf{y}, \mathbf{Z}, t, q)$ as $q \rightarrow 1$. Up to terms of order $O((q - 1)^3)$, we have

$$\begin{aligned} X(q) &= \frac{(\mathbf{y} - 1)(\mathbf{y} - t)(1 + (q - 1)\mathbf{y}\mathbf{Z})}{(\mathbf{y} - 1)^2 + \mathbf{y}(2 - \Theta_1 - \bar{\Theta}_1)} \sim \frac{\frac{\mathbf{y}-t}{\mathbf{y}-1}(1 + (q - 1)\mathbf{y}\mathbf{Z})}{1 - (q - 1)^2 \frac{\theta_1^2}{4} \frac{\mathbf{y}}{(\mathbf{y}-1)^2}} \\ &\sim \frac{\mathbf{y} - t}{\mathbf{y} - 1} (1 + (q - 1)\mathbf{y}\mathbf{Z}) \left(1 + (q - 1)^2 \frac{\theta_1^2}{4} \frac{\mathbf{y}}{(\mathbf{y} - 1)^2} \right) \\ &\sim \frac{\mathbf{y} - t}{\mathbf{y} - 1} \left(1 + (q - 1)\mathbf{y}\mathbf{Z} + (q - 1)^2 \frac{\theta_1^2}{4} \frac{\mathbf{y}}{(\mathbf{y} - 1)^2} \right). \end{aligned}$$

So in particular, we have $X(q) = \frac{\mathbf{y}-t}{\mathbf{y}-1} + O(q - 1)$. Since moreover $\Theta_\infty = 1 + O(q - 1)$ and $\Theta_\infty - q\bar{\Theta}_\infty = (q - 1)(\theta_\infty - 1) + O((q - 1)^2)$, we conclude that

$$f_\lambda(q) := f_\lambda(\mathbf{y}, \mathbf{Z}, t, q) = \frac{(\theta_\infty - 1)(\mathbf{y} - t)}{t(t - 1)} + O(q - 1).$$

In other words, the first difference equation in (2.18) discretizes (2.4). Similarly, up to order terms of order $O((q - 1)^3)$, we obtain

$$\begin{aligned} E(q) &= \frac{(X(q) - t\Theta_0) \left(X(q) - t\frac{1}{\Theta_0} \right)}{\mathbf{y} \left(X(q) - \frac{1}{\Theta_\infty} \right) \left(X(q) - \frac{\Theta_\infty}{q} \right)} \\ &\sim \frac{\frac{1}{\mathbf{y}} (X(q) - t)^2 - (q - 1)^2 \frac{\theta_0^2}{4} tX(q)}{(X(q) - 1)^2 - (q - 1) \frac{t-1}{\mathbf{y}-1} + (q - 1)^2 \left(\frac{t-1}{\mathbf{y}-1} - \frac{\mathbf{y}-t}{\mathbf{y}-1} \left(\frac{\theta_\infty^2 - 2\theta_\infty}{4} - \mathbf{y}\mathbf{Z} \right) \right)}. \end{aligned}$$

Substituting the Taylor expansion of $X(q)$ up to order $O((q-1)^3)$ yields $E(q)$

$$\begin{aligned} & \mathbf{y} \left(\left(1 - (q-1) \frac{\mathbf{y}-t}{t-1} \mathbf{Z} \right)^2 - (q-1)^2 \left(\frac{t\theta_0^2}{4} \frac{(\mathbf{y}-t)(\mathbf{y}-1)}{(t-1)^2 \mathbf{y}^2} + \frac{\theta_1^2 (\mathbf{y}-t)}{2(t-1)(\mathbf{y}-1)^2} \right) \right) \\ & \sim \frac{\left(1 - (q-1) \mathbf{y} \frac{\mathbf{y}-t}{t-1} \mathbf{Z} \right)^2 - (q-1) \frac{\mathbf{y}-1}{t-1} + (q-1)^2 \left(\frac{\mathbf{y}-1}{t-1} - \frac{(\mathbf{y}-t)(\mathbf{y}-1)}{(t-1)^2} \left(\frac{\theta_\infty^2 - 2\theta_\infty}{4} - \mathbf{y} \mathbf{Z} \right) - \frac{\theta_1^2 \mathbf{y}(\mathbf{y}-t)}{2(\mathbf{y}-1)^2(t-1)} \right)}{\mathbf{y} + (q-1) \frac{\mathbf{y}(\mathbf{y}-1)(2(\mathbf{y}-t)\mathbf{Z} + 1)}{t-1}} \\ & + (q-1)^2 \cdot \frac{\mathbf{y}(\mathbf{y}-1)(\mathbf{y}-t)}{t-1} \left(\frac{(3\mathbf{y}-1)(\mathbf{y}-t)\mathbf{Z}^2 + (3\mathbf{y}-2)\mathbf{Z}}{t-1} - \frac{t\theta_0^2}{4(t-1)\mathbf{y}^2} + \frac{\theta_1^2}{2(\mathbf{y}-1)^2} + \frac{(\theta_\infty-1)^2 + 3}{4(t-1)} \right). \end{aligned}$$

Already from the Taylor expansion of $E(q)$ up to order $O((q-1)^2)$, we deduce that

$$f_{\mathbf{y}}(q) = \frac{\mathbf{y}(\mathbf{y}-1)(2(\mathbf{y}-t)\mathbf{Z} + 1)}{t(t-1)} + O(q-1).$$

From the Taylor expansion of $E(q)$ and $X(q)$ up to order $O((q-1)^3)$, we deduce by a series of tedious but straightforward calculations that

$$\begin{aligned} f_{\mathbf{Z}}(q) &= -\frac{\mathbf{y}(\mathbf{y}-1)(\mathbf{y}-t)\mathbf{Z}^2}{t(t-1)} \left(\frac{1}{\mathbf{y}} + \frac{1}{\mathbf{y}-1} + \frac{1}{\mathbf{y}-t} \right) - \frac{(2\mathbf{y}-1)\mathbf{Z}}{t(t-1)} \\ &\quad - \frac{\theta_0^2}{4(t-1)\mathbf{y}^2} + \frac{\theta_1^2}{4t(\mathbf{y}-1)^2} - \frac{\theta_t^2}{4(\mathbf{y}-t)^2} + \frac{(\theta_\infty-1)^2 - 1}{4t(t-1)} + O(q-1). \end{aligned}$$

It follows that the second and third difference equation in (2.18) together discretize the system of differential equations (2.3).

3. The sixth Painlevé equation

3.1. Differential case

Let $\Theta = (\theta_0, \theta_1, \theta_t, \theta_\infty) \in \mathbb{C}^4$. We define the rational function $H_{\text{VI}}^\Theta \in \mathbb{C}(y, Z, t)$ in three variables given by

$$\begin{aligned} H_{\text{VI}}^\Theta &:= \frac{\mathbf{y}(\mathbf{y}-1)(\mathbf{y}-t)}{t(t-1)} \left(Z^2 + \frac{Z}{\mathbf{y}-t} \right) \\ &\quad - \frac{1}{4} \left(\frac{(\theta_\infty-1)^2 - 1}{t(t-1)} \mathbf{y} + \frac{\theta_0^2}{(t-1)\mathbf{y}} + \frac{\theta_t^2}{\mathbf{y}-t} - \frac{\theta_1^2}{t(\mathbf{y}-1)} \right). \end{aligned}$$

Consider the non-autonomous Hamiltonian system defined by

$$\begin{cases} \mathbf{y}'(t) = \frac{\partial H_{\text{VI}}^\Theta}{\partial Z}(\mathbf{y}, Z, t) \\ \mathbf{Z}'(t) = -\frac{\partial H_{\text{VI}}^\Theta}{\partial \mathbf{y}}(\mathbf{y}, Z, t). \end{cases} \quad (3.1)$$

Explicitly, it is given by

$$\begin{cases} y'(t) = \frac{y(y-1)(y-t)}{t(t-1)} \left(2Z + \frac{1}{y-t} \right) \\ Z'(t) = \frac{-3y^2 + 2(t+1)y - t}{t(t-1)} Z^2 - \frac{2y-1}{t(t-1)} Z \\ \quad + \frac{1}{4} \left(\frac{(\theta_\infty - 1)^2 - 1}{t(t-1)} - \frac{\theta_0^2}{(t-1)y^2} - \frac{\theta_t^2}{(y-t)^2} + \frac{\theta_1^2}{t(y-1)^2} \right). \end{cases} \quad (3.2)$$

Recall from Corollary 2.7 that if for all $i \in \{0, 1, t, \infty\}$, we have $\theta_i \notin \mathbb{Z}^*$ and $\theta_\infty \neq 0$, then this system of differential equations characterizes isomonodromy for families of \mathfrak{sl}_2 -Fuchsian systems. Substituting $Z = \frac{t(t-1)y'(t)}{2y(y-1)(y-t)} - \frac{1}{2(y-t)}$ (from the first equation in (3.2)) into the second, we obtain the *sixth Painlevé equation* associated to the spectral data Θ :

$$P_{\text{VI}} : \begin{cases} y'' = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) y'^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) y' \\ \quad + \frac{y(y-1)(y-t)}{2t^2(t-1)^2} \left((\theta_\infty - 1)^2 + \frac{\theta_1^2(t-1)}{(y-1)^2} - \frac{\theta_0^2 t}{y^2} - \frac{(\theta_t^2 - 1)(t-1)t}{(y-t)^2} \right). \end{cases}$$

Conversely, given a meromorphic solution y of P_{VI} (we will see in the sequel that it exists), and assuming it is not identically equal to $0, 1, t$ (which is a trivially satisfied if $\theta_0\theta_1\theta_t \neq 0$), then the substitution formula yields a meromorphic function Z such that the pair (y, Z) is a meromorphic solution of (3.2).

Let us briefly recall the well-known results concerning the existence of analytic solutions of P_{VI} . By the Cauchy–Lipschitz theorem, for every $t_0 \in \mathbb{C} \setminus \{0, 1\}$ and every choice of $(y_0, y_1) \in (\mathbb{C} \setminus \{0, 1, t_0\}) \times \mathbb{C}$, there exists a unique holomorphic function $y(t)$ defined in a neighborhood of t_0 such that $y(t_0) = y_0$ and $y'(t_0) = y_1$, and such that y is a solution of the sixth Painlevé equation. Equivalently, for every $t_0 \in \mathbb{C} \setminus \{0, 1\}$ and every choice of $(y_0, Z_0) \in (\mathbb{C} \setminus \{0, 1, t_0\}) \times \mathbb{C}$, there exists a unique holomorphic solution $(y(t), Z(t))$ of the Hamiltonian system (3.2), defined in a neighborhood of t_0 , such that $(y(t_0), Z(t_0)) = (y_0, Z_0)$. By the so-called *Painlevé property*, any such germ of holomorphic solution can be meromorphically continued along any path in $\mathbb{C} \setminus \{0, 1\}$. In particular, on any simply connected subset U of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, there exists a unique meromorphic solution satisfying some initial condition as above at $t_0 \in U$ (see for instance [9, 13], see also Section 4.1 for some details).

3.2. A discrete analogue

Let us fix $q \in \mathbb{C} \setminus \{0, 1\}$ and let us consider the spectral data $(\Theta_0, \Theta_1, \Theta_t, \Theta_\infty, \bar{\Theta}_0, \bar{\Theta}_1, \bar{\Theta}_t, \bar{\Theta}_\infty) \in (\mathbb{C}^*)^8$ such that

$$\Theta_0 \bar{\Theta}_0 = \Theta_1 \bar{\Theta}_1 \Theta_t \bar{\Theta}_t \Theta_\infty \bar{\Theta}_\infty.$$

In [12] the q -Painlevé VI equation associated to such a spectral data was introduced. It is given by the following system of q -difference equations:

$$qP_{\text{JS,VI}}(\Theta, \bar{\Theta}) : \begin{cases} \frac{\mathbf{y} \cdot \sigma_{q,t} \mathbf{y}}{\Theta_1 \bar{\Theta}_1} = \frac{\left(\sigma_{q,t} \mathbf{z} - t \frac{\Theta_t \bar{\Theta}_t}{\Theta_0}\right) \left(\sigma_{q,t} \mathbf{z} - t \frac{\Theta_t \bar{\Theta}_t}{\bar{\Theta}_0}\right)}{\left(\sigma_{q,t} \mathbf{z} - \frac{1}{q \bar{\Theta}_\infty}\right) \left(\sigma_{q,t} \mathbf{z} - \frac{1}{\bar{\Theta}_\infty}\right)}, \\ \frac{\mathbf{z} \cdot \sigma_{q,t} \mathbf{z}}{\frac{1}{q \bar{\Theta}_\infty \Theta_\infty}} = \frac{(\mathbf{y} - t \Theta_t)(\mathbf{y} - t \bar{\Theta}_t)}{(\mathbf{y} - \Theta_1)(\mathbf{y} - \bar{\Theta}_1)}. \end{cases} \quad (3.3)$$

The auxiliary parameters in [12] bear other names, but we have written the equation in a way that the dictionary between the auxiliary parameters in [12] and the above $\Theta_i, \bar{\Theta}_i$ is obvious, see (0.2). This system of difference equations has been derived in [12], for $|q| \neq 1$, from the pseudo-constancy condition of the Birkhoff connection matrix for q -Fuchsian systems with non-resonant spectral data $(\Theta, \bar{\Theta})$. Note that the change of variable

$$\mathbf{z} = \frac{(\mathbf{y} - t \Theta_t)(\mathbf{y} - t \bar{\Theta}_t)}{q(\mathbf{y} - 1)(\mathbf{y} - t)(1 + (q - 1)\mathbf{y}\mathbf{Z})}$$

applied to (3.3) yields the q -difference system (2.12). In the case $q \notin e^{2i\pi\mathbb{Q}}$ and non-resonant $(\Theta, \bar{\Theta})$, solutions of the latter system have been shown in Proposition 2.10 to correspond (under some generic assumptions) to q -isomonodromic (in the sense of Definition 2.8) families of q -Fuchsian systems. Conversely, when starting with (2.12), the change of variable

$$\mathbf{Z} = \frac{\frac{(\mathbf{y} - t \Theta_t)(\mathbf{y} - t \bar{\Theta}_t)}{q(\mathbf{y} - 1)(\mathbf{y} - t)\mathbf{z}} - 1}{(q - 1)\mathbf{y}}$$

yields equation (3.3), which has a significantly shorter and more symmetric expression. Note that with this change of variable, one has $\sigma_{q,t} \mathbf{z} = X$, for X as in (2.12).

From now on, because we are ultimately interested in the behaviour under confluence, we will use our convention

$$\forall i \in \{0, 1, t, \infty\}, \quad \Theta_i \bar{\Theta}_i = 1,$$

by which equations (3.3) and (2.12) can obviously be simplified. In particular, from now on, the following system of q -difference equations will be referred

to as the q -Painlevé VI equation associated to spectral data $\Theta \in (\mathbb{C}^*)^4$:

$$qP_{\text{VI}}(\Theta) : \begin{cases} \mathbf{y} \cdot \sigma_{q,t} \mathbf{y} = \frac{(\sigma_{q,t} \mathbf{z} - t\Theta_0) \left(\sigma_{q,t} \mathbf{z} - t \frac{1}{\Theta_0} \right)}{\left(\sigma_{q,t} \mathbf{z} - \frac{\Theta_\infty}{q} \right) \left(\sigma_{q,t} \mathbf{z} - \frac{1}{\Theta_\infty} \right)}, \\ \mathbf{z} \cdot \sigma_{q,t} \mathbf{z} = \frac{1}{q} \frac{(\mathbf{y} - t\Theta_t) \left(\mathbf{y} - t \frac{1}{\Theta_t} \right)}{(\mathbf{y} - \Theta_1) \left(\mathbf{y} - \frac{1}{\Theta_1} \right)}. \end{cases} \quad (3.4)$$

Unfortunately, contrarily to the differential situation, the existence of a meromorphic solution having a prescribed value at a point $t_0 \in \mathbb{C}^*$ is in general not known. Let us now focus on discrete solutions, i.e. the sequence of values on $q^{\mathbb{Z}} t_0$ for some $t_0 \in \mathbb{C}^* \setminus q^{\mathbb{Z}}$ that a meromorphic solution defined on a domain containing the spiral $q^{\mathbb{Z}} t_0$ should interpolate. More precisely, a *discrete solution* of (3.4) is a sequence

$$(\mathbf{y}_\ell, \mathbf{z}_\ell, t_\ell)_{\ell \in \mathbb{Z}}$$

of points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{C}^*$, such that

- the sequence $(t_\ell)_{\ell \in \mathbb{Z}}$ is given by $t_\ell = q^\ell t_0$ for some $t_0 \in \mathbb{C}^*$
- the sequence $(\mathbf{y}_\ell, \mathbf{z}_\ell)_{\ell \in \mathbb{Z}}$ satisfies the following equations for each $\ell \in \mathbb{Z}$:

$$\begin{cases} \mathbf{y}_\ell \cdot \mathbf{y}_{\ell+1} = \frac{(\mathbf{z}_{\ell+1} - t_\ell \Theta_0) \left(\mathbf{z}_{\ell+1} - t_\ell \frac{1}{\Theta_0} \right)}{\left(\mathbf{z}_{\ell+1} - \frac{1}{\Theta_\infty} \right) \left(\mathbf{z}_{\ell+1} - \frac{\Theta_\infty}{q} \right)} \\ \mathbf{z}_\ell \cdot \mathbf{z}_{\ell+1} = \frac{1}{q} \frac{(\mathbf{y}_\ell - t_\ell \Theta_t) \left(\mathbf{y}_\ell - t_\ell \frac{1}{\Theta_t} \right)}{(\mathbf{y}_\ell - \Theta_1) \left(\mathbf{y}_\ell - \frac{1}{\Theta_1} \right)}, \end{cases} \quad (3.5)$$

- moreover, for each $\ell \in \mathbb{Z}$, the rational simplification of the relation implied by (3.5) between $(\mathbf{y}_0, \mathbf{z}_0, t_0)$ and $(\mathbf{y}_\ell, \mathbf{z}_\ell, q^\ell t_0)$ is also satisfied.

Note that a more intrinsic notion of discrete solution will appear in Section 4.2, where we briefly review the construction in [23] of a q -analogue of the Okamoto space. See also the pioneer work [8].

Let us explain why for discrete solutions as in the above definition we do not only take into account the relations between successive pairs. Let $t_0 \in \mathbb{C}^*$ and consider for example a pair $(\mathbf{y}_0, \mathbf{z}_0)$ where $\mathbf{y}_0 = \bar{\Theta}_1$ and where $\mathbf{z}_0 \in \mathbb{C}^*$. Then by the recurrence relation (3.5), we have $(\mathbf{y}_1, \mathbf{z}_1) = (\Theta_1, \infty)$. Then the second equation of the recurrence relation for $\ell = 1$ simply writes $\infty = \infty$. So a priori, \mathbf{z}_2 could take any value. But \mathbf{z}_2 is uniquely determined if we go back to $(\mathbf{y}_0, \mathbf{z}_0)$ and take into account rational simplifications. Indeed,

let us introduce an additional variable $\nu_\ell := \mathbf{z}_\ell \cdot (\mathbf{y}_\ell - \Theta_1)$. When we write ν_1 as a rational function of general $(\mathbf{y}_0, \mathbf{z}_0, t_0)$ via the recurrence relation, then one immediately checks that numerator and denominator can both be factorized by $(\mathbf{y}_0 - \bar{\Theta}_1)$. After this rational simplification, ν_1 is well-defined for $\mathbf{y}_0 = \bar{\Theta}_1$ and yields

$$\nu_1(\bar{\Theta}_1, \mathbf{z}_0, t_0) = \frac{(t_0\Theta_1 - \Theta_t)(t_0\Theta_1 - \bar{\Theta}_t)}{(\Theta_1 - \bar{\Theta}_1)q\mathbf{z}_0} + \Theta_1 \left(\left(\frac{\Theta_\infty}{q} + \bar{\Theta}_\infty \right) - t_0(\Theta_0 + \bar{\Theta}_0) \right).$$

On the other hand, the recurrence relation at level $\ell = 1$ may be written as

$$\begin{cases} \mathbf{y}_2 = \frac{1}{\mathbf{y}_1} \frac{(z_2 - qt_0\Theta_0)(z_2 - qt_0\bar{\Theta}_0)}{(z_2 - \bar{\Theta}_\infty) \left(z_2 - \frac{\Theta_\infty}{q} \right)} \\ \mathbf{z}_2 = \frac{1}{q} \frac{(\mathbf{y}_1 - qt_0\Theta_t)(\mathbf{y}_1 - qt_0\bar{\Theta}_t)}{\nu_1(\mathbf{y}_1 - \bar{\Theta}_1)}. \end{cases}$$

Substituting $\mathbf{y}_1 = \Theta_1$ and the value of ν_1 above yields

$$\mathbf{z}_2 = \frac{(\Theta_1 - qt_0\Theta_t)(\Theta_1 - qt_0\bar{\Theta}_t)}{\frac{(t_0\Theta_1 - \Theta_t)(t_0\Theta_1 - \bar{\Theta}_t)}{\mathbf{z}_0} + q\Theta_1(\Theta_1 - \bar{\Theta}_1) \left(\left(\frac{\Theta_\infty}{q} + \bar{\Theta}_\infty \right) - t_0(\Theta_0 + \bar{\Theta}_0) \right)}.$$

Hence $\mathbf{z}_2 \in \mathbb{P}^1$ is determined uniquely in terms of \mathbf{z}_0 . Moreover, since $\mathbf{y}_1 = \Theta_1$, \mathbf{y}_2 is determined uniquely in terms of \mathbf{z}_2 by the recurrence relation. So in summary, if $(\mathbf{y}_0, \mathbf{z}_0) = (\bar{\Theta}_1, \mathbf{z}_0)$ with $\mathbf{z}_0 \in \mathbb{C}^*$, then $(\mathbf{y}_1, \mathbf{z}_1)$ and $(\mathbf{y}_2, \mathbf{z}_2)$ are uniquely determined in terms of \mathbf{z}_0 . Conversely, from $(\mathbf{y}_2, \mathbf{z}_2)$ as above we can recover $(\bar{\Theta}_1, \mathbf{z}_0)$ under the condition that $(t_0\Theta_1 - \Theta_t)(t_0\Theta_1 - \bar{\Theta}_t)$ is non-zero.

More generally, [23, Prop. 1] ensures, in the case of sufficiently generic spectral data, the existence and uniqueness of discrete solutions with sufficiently generic prescribed initial data. These genericity conditions will be made precise in Section 4.2. What we will obtain is the following, see Remark 4.2.

PROPOSITION 3.1. — *Let $q \in \mathbb{C} \setminus \{0, 1\}$. Let $\Theta = (\Theta_0, \Theta_1, \Theta_t, \Theta_\infty) \in (\mathbb{C}^*)^4$ such that $1 \notin \{\Theta_0^2, \Theta_1^2, \Theta_t^2\}$ and such that $\Theta_\infty^2 \neq q$. Denote*

$$S_q := \{\Theta_1^{\varepsilon_1} \Theta_t^{\varepsilon_t}, \Theta_0^{\varepsilon_0} \Theta_\infty^{\varepsilon_\infty} \mid \varepsilon_0, \varepsilon_1, \varepsilon_t, \varepsilon_\infty \in \{-1, 1\}\} \cdot q^{\mathbb{Z}}. \quad (3.6)$$

Let

$$(\mathbf{y}_0, \mathbf{z}_0, t_0) \in \mathbb{C}^* \times \mathbb{C}^* \times (\mathbb{C}^* \setminus S_q).$$

Then, there exists a unique discrete solution $(\mathbf{y}_\ell, \mathbf{z}_\ell, t_\ell)_{\ell \in \mathbb{Z}}$ of (3.4) with initial value $(\mathbf{y}_0, \mathbf{z}_0, t_0)$.

3.3. Confluence

As usual for matters of confluence, in this section we will only consider spectral data $(\Theta, \bar{\Theta})$ related by $\Theta_i \bar{\Theta}_i = 1$. We will first establish that the sixth Painlevé equation, up to the change of variable and spectral data that we previously found to be convenient for confluence, admits a q -analogue of Hamiltonian formulation. From this, we will deduce the confluence of discrete and meromorphic solutions.

3.3.1. A q -analogue of Hamiltonian system

Let us apply the change of variable

$$z = \frac{(\mathbf{y} - t\Theta_t)(\mathbf{y} - t\bar{\Theta}_t)}{q(\mathbf{y} - 1)(\mathbf{y} - t)(1 + (q - 1)\mathbf{y}\mathbf{Z})}, \quad \mathbf{Z} = \frac{\frac{(\mathbf{y} - t\Theta_t)(\mathbf{y} - t\bar{\Theta}_t)}{q(\mathbf{y} - 1)(\mathbf{y} - t)\mathbf{z}} - 1}{(q - 1)\mathbf{y}}, \quad (3.7)$$

to (3.4). The resulting equation, which is the simplification of (2.12) by the convention $\bar{\Theta}_i = 1/\Theta_i$, is better adapted for questions of confluence. Indeed, roughly summarizing the result in Section 1.3, if a family of q -Fuchsian systems given by $(\lambda, \mathbf{y}, \mathbf{Z})$ discretizes a family of \mathfrak{sl}_2 -Fuchsian systems given by (λ, y, Z) , then we have $(\mathbf{y}, \mathbf{Z}) \rightarrow (y, Z)$ as $q \rightarrow 1$. On the other hand, the implied estimate $(\mathbf{y}, \mathbf{z}) \rightarrow (y, \frac{y-t}{y-1})$ as $q \rightarrow 1$ loses too much information. So we consider, for spectral data $\Theta = (\Theta_0, \Theta_1, \Theta_t, \Theta_\infty) \in (\mathbb{C}^*)^4$, the system of q -difference equations

$$q\tilde{P}_{\text{VI}}(\Theta) : \begin{cases} \sigma_{q,t}\mathbf{y} = \frac{1}{\mathbf{y}} \cdot \frac{\left(\frac{(\mathbf{y}-1)(\mathbf{y}-t)(1+(q-1)\mathbf{y}\mathbf{Z})}{(\mathbf{y}-\Theta_1)(\mathbf{y}-\bar{\Theta}_1)} - t\Theta_0\right) \left(\frac{(\mathbf{y}-1)(\mathbf{y}-t)(1+(q-1)\mathbf{y}\mathbf{Z})}{(\mathbf{y}-\Theta_1)(\mathbf{y}-\bar{\Theta}_1)} - t\bar{\Theta}_0\right)}{\left(\frac{(\mathbf{y}-1)(\mathbf{y}-t)(1+(q-1)\mathbf{y}\mathbf{Z})}{(\mathbf{y}-\Theta_1)(\mathbf{y}-\bar{\Theta}_1)} - \frac{\Theta_\infty}{q}\right) \left(\frac{(\mathbf{y}-1)(\mathbf{y}-t)(1+(q-1)\mathbf{y}\mathbf{Z})}{(\mathbf{y}-\Theta_1)(\mathbf{y}-\bar{\Theta}_1)} - \bar{\Theta}_\infty\right)} \\ \sigma_{q,t}\mathbf{Z} = \frac{\frac{(\sigma_{q,t}\mathbf{y} - qt\Theta_t)(\sigma_{q,t}\mathbf{y} - qt\bar{\Theta}_t)(\mathbf{y} - \Theta_1)(\mathbf{y} - \bar{\Theta}_1)}{q(\sigma_{q,t}\mathbf{y} - 1)(\sigma_{q,t}\mathbf{y} - qt)(\mathbf{y} - 1)(\mathbf{y} - t)(1 + (q - 1)\mathbf{y}\mathbf{Z})} - 1}{(q - 1)\sigma_{q,t}\mathbf{y}}, \end{cases} \quad (3.8)$$

where we denote $\bar{\Theta}_i := 1/\Theta_i$ for conciseness.

The following result states that this *modified* q -Painlevé VI equation $q\tilde{P}_{\text{VI}}(\Theta)$ is an appropriate q -analogue of a Hamiltonian system. First, let us introduce the Hamiltonian. To each datum $\Theta = (\theta_0, \theta_1, \theta_t, \theta_\infty) \in \mathbb{C}^4$, we

associate the rational function $H_{\text{VI}}^{\Theta} \in \mathbb{C}(\mathbf{y}, \mathbf{Z}, t)$ in three variables given by

$$H_{\text{VI}}^{\Theta}(\mathbf{y}, \mathbf{Z}, t) := \frac{\mathbf{y}(\mathbf{y} - 1)(\mathbf{y} - t)}{t(t - 1)} \left(\mathbf{Z}^2 + \frac{\mathbf{Z}}{\mathbf{y} - t} \right) - \frac{1}{4} \left(\frac{(\theta_{\infty} - 1)^2 - 1}{t(t - 1)} \mathbf{y} + \frac{\theta_0^2}{(t - 1)\mathbf{y}} + \frac{\theta_t^2}{\mathbf{y} - t} - \frac{\theta_1^2}{t(\mathbf{y} - 1)} \right). \quad (3.9)$$

Note this is nothing else than the Hamiltonian for the differential case. In the following, we denote abusively

$$1 + (q - 1) \frac{\Theta}{2} := \left(1 + (q - 1) \frac{\theta_0}{2}, 1 + (q - 1) \frac{\theta_1}{2}, 1 + (q - 1) \frac{\theta_t}{2}, 1 + (q - 1) \frac{\theta_{\infty}}{2} \right).$$

THEOREM 3.2. — *Let $\Theta \in \mathbb{C}^4$. Let $\mathcal{R}_1^{\Theta}, \mathcal{R}_2^{\Theta} \in \mathbb{C}(\mathbf{y}, \mathbf{Z}, t, q)$ be the (well-defined) rational functions in four variables such that the modified q -Painlevé VI equation (3.8) with spectral data given by*

$$\Theta := 1 + (q - 1) \frac{\Theta}{2}$$

reads

$$q\tilde{P}_{\text{VI}}(\Theta) : \begin{cases} \partial_{q,t}\mathbf{y} = \partial_{q,\mathbf{Z}}H_{\text{VI}}^{\Theta}(\mathbf{y}, \mathbf{Z}, t) + (q - 1)\mathcal{R}_1^{\Theta}(\mathbf{y}, \mathbf{Z}, t, q) \\ \partial_{q,t}\mathbf{Z} = -\partial_{q,\mathbf{y}}H_{\text{VI}}^{\Theta}(\mathbf{y}, \mathbf{Z}, t) + (q - 1)\mathcal{R}_2^{\Theta}(\mathbf{y}, \mathbf{Z}, t, q) \end{cases}$$

via (3.9) and the operator identity $1 + t(q - 1)\partial_{q,t} = \sigma_{q,t}$. Let $\mathcal{R} \in \{\mathcal{R}_1^{\Theta}, \mathcal{R}_2^{\Theta}\}$. The divisor $\{q = 1\}$ in $\mathbb{C}_{\mathbf{y}, \mathbf{Z}, t, q}^4$ is not an irreducible component of the polar divisor of \mathcal{R} . Moreover, the polar locus of the therefore well-defined rational function $\mathcal{R}|_{q=1}$ on $\mathbb{C}_{\mathbf{y}, \mathbf{Z}, t}^3$ is contained in the set

$$\mathcal{P} := \{\mathbf{y} = 0\} \cup \{\mathbf{y} = 1\} \cup \{\mathbf{y} = t\} \cup \{t = 0\} \cup \{t = 1\}.$$

Proof. — Let us first say some words about the well-definedness of $\mathcal{R}_1^{\Theta}, \mathcal{R}_2^{\Theta}$. For quadrupels Θ , there are well defined rational functions $f, g \in \mathbb{C}(\mathbf{y}, \mathbf{Z}, t, q, \Theta_0, \Theta_1, \Theta_t, \Theta_{\infty})$ such that $q\tilde{P}_{\text{VI}}(\Theta)$ can be written as

$$\begin{cases} \sigma_{q,t}\mathbf{y} = f(\mathbf{y}, \mathbf{Z}, t, q, \Theta) \\ \sigma_{q,t}\mathbf{Z} = g(\mathbf{y}, \mathbf{Z}, t, q, \Theta). \end{cases}$$

Indeed, it suffices to substitute the first equation in (3.8) into the second, so that the right hand sides only depends on the variables $\mathbf{y}, \mathbf{Z}, t, q, \Theta$. Note that $\partial_{q,\mathbf{Z}}H_{\text{VI}}^{\Theta}(\mathbf{y}, \mathbf{Z}, t)$ and $\partial_{q,\mathbf{y}}H_{\text{VI}}^{\Theta}(\mathbf{y}, \mathbf{Z}, t)$ can easily be calculated and are elements of $\mathbb{C}(\mathbf{y}, \mathbf{Z}, t, q)$. Then

$$\begin{cases} \mathcal{R}_1^{\Theta}(\mathbf{y}, \mathbf{Z}, t, q) := \frac{1}{q - 1} \left(\frac{f(\mathbf{y}, \mathbf{Z}, t, q, 1 + \frac{q-1}{2}\Theta) - \mathbf{y}}{(q - 1)t} - \partial_{q,\mathbf{Z}}H_{\text{VI}}^{\Theta}(\mathbf{y}, \mathbf{Z}, t) \right) \\ \mathcal{R}_2^{\Theta}(\mathbf{y}, \mathbf{Z}, t, q) := \frac{1}{q - 1} \left(\frac{g(\mathbf{y}, \mathbf{Z}, t, q, 1 + \frac{q-1}{2}\Theta) - \mathbf{Z}}{(q - 1)t} + \partial_{q,\mathbf{y}}H_{\text{VI}}^{\Theta}(\mathbf{y}, \mathbf{Z}, t) \right) \end{cases}$$

are indeed elements of $\mathbb{C}(\mathbf{y}, \mathbf{Z}, t, q)$ and are those required by the statement. Let us define

$$\begin{cases} \tilde{\mathcal{R}}_1^\Theta(\mathbf{y}, \mathbf{Z}, t, q) := \frac{1}{q-1} \left(\frac{f(\mathbf{y}, \mathbf{Z}, t, q, 1 + \frac{q-1}{2}\Theta) - \mathbf{y}}{(q-1)t} - \partial_{\mathbf{Z}} H_{\text{VI}}^\Theta(\mathbf{y}, \mathbf{Z}, t) \right) \\ \tilde{\mathcal{R}}_2^\Theta(\mathbf{y}, \mathbf{Z}, t, q) := \frac{1}{q-1} \left(\frac{g(\mathbf{y}, \mathbf{Z}, t, q, 1 + \frac{q-1}{2}\Theta) - \mathbf{Z}}{(q-1)t} + \partial_{\mathbf{y}} H_{\text{VI}}^\Theta(\mathbf{y}, \mathbf{Z}, t) \right). \end{cases}$$

Since $\frac{\partial}{\partial \mathbf{Z}} H_{\text{VI}}^\Theta$ and $\frac{\partial}{\partial \mathbf{y}} H_{\text{VI}}^\Theta$ are rational functions of $(\mathbf{y}, \mathbf{Z}, t)$, these $\tilde{\mathcal{R}}_1^\Theta, \tilde{\mathcal{R}}_2^\Theta$ are again rational functions in the variables $\mathbf{y}, \mathbf{Z}, t, q$. Denoting $\nabla_\star := \frac{\partial_{q,\star} - \partial_\star}{q-1}$, we have $\mathcal{R}_1^\Theta - \tilde{\mathcal{R}}_1^\Theta = -\nabla_{\mathbf{Z}}(H_{\text{VI}}^\Theta)$, and $\mathcal{R}_2^\Theta - \tilde{\mathcal{R}}_2^\Theta = \nabla_{\mathbf{y}}(H_{\text{VI}}^\Theta)$. In order to compute these differences, note that H^Θ is rational with only simple poles independent of \mathbf{Z} and for $\star \in \{\mathbf{y}, \mathbf{Z}\}$, the operator ∇_\star is $\mathbb{C}(t)$ -linear. So it suffices to compute $\nabla_x(x^n)$ for $n \in \mathbb{N}$ and $\nabla_x(\frac{1}{x-a})$ for a independent of x . We have $\nabla_x(1) = 0$, and for $n \in \mathbb{N}^*$, we find

$$\nabla_x(x^n) = x^{n-1} \left(\frac{q^n - 1}{(q-1)^2} - \frac{n}{q-1} \right) = x^{n-1} \left(\frac{\sum_{k=0}^{n-1} q^k - n}{q-1} \right) = x^{n-1} \sum_{k=0}^{n-1} [k]_q,$$

where $[k]_q = \frac{q^k - 1}{q-1} = \sum_{i=0}^{k-1} q^i$ with $[0]_q = 0$. In particular, we find $\nabla_x(x) = 0$, $\nabla_x(x^2) = x$, $\nabla_x(x^3) = (2+q)x^2$. For a independent of x , we find

$$\nabla_x \left(\frac{1}{x-a} \right) = \frac{-1}{(q-x)(x-a)(q-1)} + \frac{1}{(x-a)^2(q-1)} = \frac{x}{(q-x)(x-a)^2}.$$

We deduce

$$\begin{cases} -\nabla_{\mathbf{Z}}(H_{\text{VI}}^\Theta) = -\frac{\mathbf{y}(\mathbf{y}-1)(\mathbf{y}-t)}{t(t-1)} \mathbf{Z}, \\ \nabla_{\mathbf{y}}(H_{\text{VI}}^\Theta) = \frac{((q+2)\mathbf{y}-1-t)\mathbf{Z}+1}{t(t-1)} \mathbf{y} \mathbf{Z} \\ \quad - \frac{1}{4} \left(\frac{\theta_0^2}{q(t-1)\mathbf{y}^2} + \frac{\theta_t^2 \mathbf{y}}{(\mathbf{y}-t)^2(q\mathbf{y}-t)} - \frac{\theta_1^2 \mathbf{y}}{t(\mathbf{y}-1)^2(q\mathbf{y}-1)} \right). \end{cases}$$

Obviously, these differences do not have $\{q=1\}$ as an irreducible component of their respective polar divisors, and their restrictions to $q=1$ do not have poles outside \mathcal{P} . This means that the statement holds for $\mathcal{R} \in \{\mathcal{R}_1^\Theta, \mathcal{R}_2^\Theta\}$ if and only if it holds for $\mathcal{R} \in \{\tilde{\mathcal{R}}_1^\Theta, \tilde{\mathcal{R}}_2^\Theta\}$. But for the latter, we have already done most of the work. Indeed, the calculations in Section 2.3.3 at the end of Section 2.3 show that in restriction to any line $\{(\mathbf{y}, \mathbf{Z}, t) = (\mathbf{y}_0, \mathbf{Z}_0, t_0)\} \subset \mathbb{C}_{\mathbf{y}, \mathbf{Z}, t, q}^4$ with $(\mathbf{y}_0, \mathbf{Z}_0, t_0) \in \mathbb{C}^3 \setminus \mathcal{P}$, the two rational functions

$$\left((q-1)\tilde{\mathcal{R}}_1^\Theta \right) \Big|_{\{(\mathbf{y}, \mathbf{Z}, t) = (\mathbf{y}_0, \mathbf{Z}_0, t_0)\}}(q), \quad \left((q-1)\tilde{\mathcal{R}}_2^\Theta \right) \Big|_{\{(\mathbf{y}, \mathbf{Z}, t) = (\mathbf{y}_0, \mathbf{Z}_0, t_0)\}}(q)$$

vanish both at $q = 1$. It follows that $\{q = 1\}$ is an irreducible component of the zero divisor of both $((q-1)\tilde{\mathcal{R}}_2^\Theta)$ and $((q-1)\tilde{\mathcal{R}}_2^\Theta)$. In particular, $\{q = 1\}$ is not an irreducible component of the polar divisor of $\tilde{\mathcal{R}}_1^\Theta$ or $\tilde{\mathcal{R}}_2^\Theta$. Moreover, even though we did not push the Taylor series expansions in Section 2.3 far enough as to have an explicit expression for $\tilde{\mathcal{R}}_1^\Theta|_{q=1}$ and $\tilde{\mathcal{R}}_2^\Theta|_{q=1}$, it is still clear from the calculations that these functions cannot have poles outside \mathcal{P} . The result follows. \square

3.3.2. Confluence of discrete solutions

We will now see that discrete solutions of the modified q -Painlevé VI equation yield holomorphic solutions of the differential Painlevé VI equation by some limit process. The idea is that the successive $\partial_{q,t}$ -derivations should lead, when $q \rightarrow 1$, to the coefficients of the Taylor series expansion of the limit functions. Let us consider the operator

$$\delta_{q,t} := t\partial_{q,t} = \frac{\sigma_{q,t} - 1}{q - 1}.$$

As one can easily check, for each $n \in \mathbb{N}$, we have

$$\delta_{q,t}^n = \frac{1}{(q-1)^n} \cdot \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \sigma_{q,t}^k. \quad (3.10)$$

Here we use the convention $\delta_{q,t}^0 = \sigma_{q,t}^0 = 1$. We will prove the following.

THEOREM 3.3. — *Let $\Theta \in \mathbb{C}^4$. Let $t_0 \in \mathbb{C} \setminus \{0, 1\}$ and $(\mathbf{y}_0, \mathbf{Z}_0) \in (\mathbb{C} \setminus \{0, 1, t_0\}) \times \mathbb{C}$. Then, there exists a family $(\mathbf{y}_n, \mathbf{Z}_n)_{n \in \mathbb{N}}$ of pairs of rational functions $(\mathbf{y}_n, \mathbf{Z}_n) \in \mathbb{C}(q) \times \mathbb{C}(q)$ such that for generic values of q , the sequence $(\mathbf{y}_n(q), \mathbf{Z}_n(q), q^n t_0)_{n \in \mathbb{N}}$ is the positive part of the discrete solution of $q\tilde{P}_{\text{VI}}(1 + (q-1)\frac{\Theta}{2})$ with initial value $(\mathbf{y}_0, \mathbf{Z}_0, t_0)$. Consider the sequence $(a_n, b_n)_{n \in \mathbb{N}}$ of pairs of rational functions $(a_n, b_n) \in \mathbb{C}(q) \times \mathbb{C}(q)$ defined by*

$$\begin{cases} a_n(q) = \frac{1}{(q-1)^n} \cdot \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \mathbf{y}_k(q) \\ b_n(q) = \frac{1}{(q-1)^n} \cdot \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \mathbf{Z}_k(q). \end{cases}$$

Then for each $n \in \mathbb{N}$, neither $a_n(q)$ nor $b_n(q)$ has a pole at $q = 1$. Moreover, the power series

$$\sum_{n \geq 0} \frac{a_n(1)}{n!} (q-1)^n, \quad \sum_{n \geq 0} \frac{b_n(1)}{n!} (q-1)^n,$$

both converge and yield the pair of functions $q \mapsto (y(q \cdot t_0), Z(q \cdot t_0))$, where $(y(t), Z(t))$ is the unique solution of the Painlevé Hamiltonian system (3.1) with initial condition $(\mathbf{y}_0, \mathbf{Z}_0)$ at t_0 .

This result will be proven by the end of this section. As it turns out, rather than trying to calculate the limit for $q \rightarrow 1$ for the $(a_n(q), b_n(q))$ directly, it is easier to first construct a particular sequence of rational functions that are finite at $q = 1$ and then show that this sequence is actually the one from the statement. First we will need some general remarks.

The $\delta_{q,t}$ -operator on a field of functions with complex variable t is additive and satisfies the following algebraic properties:

$$\begin{cases} \delta_{q,t}(fg) = (\delta_{q,t}f)g + f\delta_{q,t}g + (q-1)\delta_{q,t}(f)\delta_{q,t}(g), \\ \delta_{q,t}\left(\frac{1}{f}\right) = \frac{-\delta_{q,t}f}{f(f+(q-1)\delta_{q,t}f)}. \end{cases} \quad (3.11)$$

In particular, for any rational function $F \in \mathbb{C}(\mathbf{y}, \mathbf{Z}, t, q)$ we may define, by treating \mathbf{y} and \mathbf{Z} like functions of t , a rational function Δ_F with two additional variables such that

$$\delta_{q,t}F(\mathbf{y}, \mathbf{Z}, t, q) = \Delta_F(\mathbf{y}, \mathbf{Z}, t, q, \delta_{q,t}\mathbf{y}, \delta_{q,t}\mathbf{Z}).$$

Let $\delta_t = t\partial_t$ be the formal limit of $\delta_{q,t}$ when q goes to 1. The q -analogue of the chain rule that we will need is the following.

LEMMA 3.4. — *Let $F \in \mathbb{C}(\mathbf{y}, \mathbf{Z}, t, q)$ and $\Delta_F \in \mathbb{C}(\mathbf{y}, \mathbf{Z}, t, q, \delta_{q,t}\mathbf{y}, \delta_{q,t}\mathbf{Z})$ be as above. Define $R_F \in \mathbb{C}(\mathbf{y}, \mathbf{Z}, t, q, \delta_{q,t}\mathbf{y}, \delta_{q,t}\mathbf{Z})$ by*

$$R_F := \frac{\Delta_F - \left(\frac{\partial F}{\partial \mathbf{y}} \cdot \delta_t \mathbf{y} + \frac{\partial F}{\partial \mathbf{Z}} \cdot \delta_t \mathbf{Z} + \frac{\partial F}{\partial t} \cdot t \right)}{q-1}.$$

If $\{q = 1\}$ is not an irreducible component of the polar locus of F , then it is not an irreducible component of the polar locus of Δ_F and R_F . Moreover, when F is seen as an element of $\mathbb{C}(\mathbf{y}, \mathbf{Z}, t, q, \partial_{q,t}\mathbf{y}, \partial_{q,t}\mathbf{Z})$, then the affine parts of the polar locus of $\Delta_F|_{q=1}$ and $R_F|_{q=1}$ are contained in the polar locus of $F|_{q=1}$.

Proof. — Let $P, Q \in \mathbb{C}[\mathbf{y}, \mathbf{Z}, t, q]$, $0 \neq Q$, such that $F = P/Q$. We use (3.11) to compute successively $\Delta_{Q^{-1}}$, $R_{Q^{-1}}$ and $\Delta_{P/Q}$, $R_{P/Q}$. We have

$$\Delta_{Q^{-1}} = \frac{-\Delta_Q}{Q(Q+(q-1)\Delta_Q)} = \frac{-\Delta_Q}{Q^2} + \frac{(q-1)\Delta_Q^2}{Q^2(Q+(q-1)\Delta_Q)}.$$

Then $(q-1)R_{Q^{-1}}$ is given by

$$\begin{aligned} \Delta_{Q^{-1}} + \frac{1}{Q^2} \left(\frac{\partial Q}{\partial \mathbf{y}} \cdot \delta_t \mathbf{y} + \frac{\partial Q}{\partial \mathbf{Z}} \cdot \delta_t \mathbf{Z} + \frac{\partial Q}{\partial t} \cdot t \right) \\ = -\frac{(q-1)R_Q}{Q^2} + \frac{(q-1)\Delta_Q^2}{Q^2(Q+(q-1)\Delta_Q)}. \end{aligned}$$

We have

$$\begin{aligned} \Delta_F = \Delta_{P/Q} &= \frac{\Delta_P}{Q} + P\Delta_{Q^{-1}} + (q-1)\Delta_P\Delta_{Q^{-1}} \\ &= \frac{\Delta_P}{Q} + P \left(\frac{-\Delta_Q}{Q^2} + \frac{(q-1)\Delta_Q^2}{Q^2(Q+(q-1)\Delta_Q)} \right) - (q-1)\Delta_P \frac{\Delta_Q}{Q(Q+(q-1)\Delta_Q)}. \end{aligned}$$

Finally,

$$\begin{aligned} R_F = R_{P/Q} &= R_P Q^{-1} + P R_{Q^{-1}} + \Delta_P \Delta_{Q^{-1}} \\ &= \frac{R_P}{Q} + P \left(\frac{-R_Q}{Q^2} + \frac{\Delta_Q^2}{Q^2(Q+(q-1)\Delta_Q)} \right) - \frac{\Delta_P \Delta_Q}{Q(Q+(q-1)\Delta_Q)}. \end{aligned}$$

This proves that $\{q=1\}$ is not an irreducible component of the polar locus of Δ_F and R_F . Note that $\Delta_P|_{q=1}, \Delta_Q|_{q=1}, R_P|_{q=1}, R_Q|_{q=1} \in \mathbb{C}[\mathbf{y}, \mathbf{Z}, t, \delta_{q,t}\mathbf{y}, \delta_{q,t}\mathbf{Z}]$. Hence the affine parts of the polar loci of $\Delta_F|_{q=1}$ and $R_F|_{q=1}$ are contained in the zero locus of $Q|_{q=1}$. This concludes the proof. \square

Let $H_1, H_2 \in \mathbb{C}(\mathbf{y}, \mathbf{Z}, t)$ and $R_1, R_2 \in \mathbb{C}(\mathbf{y}, \mathbf{Z}, t, q)$ be rational functions in three and four complex variables respectively such that

- the affine part of the polar locus of each of the functions H_i with $i \in \{1, 2\}$ is contained in the subset $\mathcal{P} \subset \mathbb{C}_{\mathbf{y}, \mathbf{Z}, t}^3$ given by $\mathcal{P} := \{\mathbf{y} = 0\} \cup \{\mathbf{y} = 1\} \cup \{\mathbf{y} = t\} \cup \{t = 0\} \cup \{t = 1\}$.
- for each $i \in \{1, 2\}$, the polar locus of R_i does not contain $\{q = 1\}$, and the affine part of the polar locus of $R_i|_{\{q=1\}}$ is contained in \mathcal{P} .

Consider the system of q -difference equations

$$\begin{cases} \delta_{q,t}\mathbf{y} = H_1(\mathbf{y}, \mathbf{Z}, t) + (q-1)R_1(\mathbf{y}, \mathbf{Z}, t, q) \\ \delta_{q,t}\mathbf{Z} = H_2(\mathbf{y}, \mathbf{Z}, t) + (q-1)R_2(\mathbf{y}, \mathbf{Z}, t, q). \end{cases} \quad (3.12)$$

Applying the operator $\delta_{q,t}$ on both sides and then substituting the values of $\delta_{q,t}\mathbf{y}, \delta_{q,t}\mathbf{Z}$ imposed by this system yields a second order relation. There exist rational functions $R_1^{(1)}, R_2^{(1)} \in \mathbb{C}(\mathbf{y}, \mathbf{Z}, t, q)$ such that this second order system is of the form

$$\begin{cases} \delta_{q,t}^2 \mathbf{y} = H_1^{(1)}(\mathbf{y}, \mathbf{Z}, t) + (q-1)R_1^{(1)}(\mathbf{y}, \mathbf{Z}, t, q) \\ \delta_{q,t}^2 \mathbf{Z} = H_2^{(1)}(\mathbf{y}, \mathbf{Z}, t) + (q-1)R_2^{(1)}(\mathbf{y}, \mathbf{Z}, t, q), \end{cases}$$

where $H_1^{(1)}, H_2^{(1)} \in \mathbb{C}(\mathbf{y}, \mathbf{Z}, t)$ and $R_1^{(1)}, R_2^{(1)} \in \mathbb{C}(\mathbf{y}, \mathbf{Z}, t, q)$ are given for $i \in \{1, 2\}$ by

$$\begin{aligned} H_i^{(1)} &= \frac{\partial H_i}{\partial \mathbf{y}} \cdot \delta_t \mathbf{y} + \frac{\partial H_i}{\partial \mathbf{Z}} \cdot \delta_t \mathbf{Z} + \frac{\partial H_i}{\partial t} \cdot t \\ R_i^{(1)} &= \Delta_{R_i}(\mathbf{y}, \mathbf{Z}, t, q, H_1, H_2) \\ &\quad + \frac{\Delta_{H_i}(\mathbf{y}, \mathbf{Z}, t, q, H_1, H_2) - \left(\frac{\partial H_i}{\partial \mathbf{y}} \cdot \delta_t \mathbf{y} + \frac{\partial H_i}{\partial \mathbf{Z}} \cdot \delta_t \mathbf{Z} + \frac{\partial H_i}{\partial t} \cdot t \right)}{q - 1}. \end{aligned}$$

By Lemma 3.4, for $i \in \{1, 2\}$, the polar locus of $R_i^{(1)}$ does not contain $\{q = 1\}$, and the affine part of the polar locus of $R_i^{(1)}|_{\{q=1\}}$ is contained in \mathcal{P} .

We may apply this discussion to the modified q -Painlevé VI equation (3.8) with spectral data given by $\Theta := 1 + (q - 1)\frac{\Theta}{2}$ and begin the proof of Theorem 3.3. Let us fix $\Theta \in \mathbb{C}^4$ and set

$$\begin{aligned} H_1 &:= H_1^{(0)} := t \frac{\partial}{\partial \mathbf{Z}} H_{\text{VI}}^\Theta, & R_1 &:= t \widetilde{\mathcal{R}}_1^\Theta, \\ H_2 &:= H_2^{(0)} := -t \frac{\partial}{\partial \mathbf{y}} H_{\text{VI}}^\Theta, & R_2 &:= t \widetilde{\mathcal{R}}_2^\Theta, \end{aligned}$$

where $H_{\text{VI}}^\Theta(\mathbf{y}, \mathbf{Z}, t)$ is given by (3.9) and where for $i \in \{1, 2\}$, $\widetilde{\mathcal{R}}_i^\Theta(\mathbf{y}, \mathbf{Z}, t, q)$ is as in the proof of Theorem 3.2. With this convention, by Theorem 3.2, the modified q -Painlevé VI equation with spectral data $1 + (q - 1)\frac{\Theta}{2}$ is, when expressed with respect to the $\delta_{q,t}$ -operator, given by (3.12). Moreover, by induction on $n \in \mathbb{N}$, the associated system of order $n + 1$ is of the form

$$\begin{cases} \delta_{q,t}^{n+1} \mathbf{y} = H_1^{(n)}(\mathbf{y}, \mathbf{Z}, t) + (q - 1)R_1^{(n)}(\mathbf{y}, \mathbf{Z}, t, q) \\ \delta_{q,t}^{n+1} \mathbf{Z} = H_2^{(n)}(\mathbf{y}, \mathbf{Z}, t) + (q - 1)R_2^{(n)}(\mathbf{y}, \mathbf{Z}, t, q), \end{cases}$$

for some well defined rational functions $H_1^{(n)}, H_2^{(n)} \in \mathbb{C}(\mathbf{y}, \mathbf{Z}, t)$ and $R_1^{(n)}, R_2^{(n)} \in \mathbb{C}(\mathbf{y}, \mathbf{Z}, t, q)$ such that

- $H_i^{(n)} = \frac{\partial H_i^{(n-1)}}{\partial \mathbf{y}} \cdot \delta_t \mathbf{y} + \frac{\partial H_i^{(n-1)}}{\partial \mathbf{Z}} \cdot \delta_t \mathbf{Z} + \frac{\partial H_i^{(n-1)}}{\partial t} \cdot t$ and
- the polar locus of $R_i^{(n)}$ does not contain $\{q = 1\}$, and the affine part of the polar locus of $R_i^{(n)}|_{\{q=1\}}$ is contained in \mathcal{P} .

Now let us fix

$$t_0 \in \mathbb{C} \setminus \{0, 1\}, \quad (\mathbf{y}_0, \mathbf{Z}_0) \in (\mathbb{C} \setminus \{0, 1, t_0\}) \times \mathbb{C}.$$

Note that if the line $\{(\mathbf{y}, \mathbf{Z}, t) = (\mathbf{y}_0, \mathbf{Z}_0, t_0)\} \subset \mathbb{C}_{\mathbf{y}, \mathbf{Z}, t, q}^4$ is contained in the polar divisor of $R_i^{(n)}$ for some n , then $R_i^{(n)}|_{q=1}$ would have a pole at $(\mathbf{y}_0, \mathbf{Z}_0, t_0)$. But this cannot happen because $(\mathbf{y}_0, \mathbf{Z}_0, t_0) \notin \mathcal{P}$. Therefore, we

may define a sequence of pairs of rational functions $(\tilde{a}_n, \tilde{b}_n)_{n \in \mathbb{N}} \in (\mathbb{C}(q) \times \mathbb{C}(q))^{\mathbb{N}}$ as follows. We take the initial functions to be the constants $\tilde{a}_0(q) := \mathbf{y}_0, \tilde{b}_0(q) := \mathbf{Z}_0$, and for $n \in \mathbb{N}$, we set

$$\begin{cases} \tilde{a}_{n+1}(q) = H_1^{(n)}(\mathbf{y}_0, \mathbf{Z}_0, t_0) + (q-1)R_1^{(n)}(\mathbf{y}_0, \mathbf{Z}_0, t_0, q) \\ \tilde{b}_{n+1}(q) = H_2^{(n)}(\mathbf{y}_0, \mathbf{Z}_0, t_0) + (q-1)R_2^{(n)}(\mathbf{y}_0, \mathbf{Z}_0, t_0, q). \end{cases}$$

Note that by construction, for each $n \in \mathbb{N}$, the pair $(\tilde{a}_n, \tilde{b}_n)$ is well defined and finite when evaluated at $q = 1$. If $(y(t), Z(t))$ is the unique solution of the Painlevé Hamiltonian system (3.1) with initial condition $(\mathbf{y}_0, \mathbf{Z}_0)$ at t_0 , then its successive derivations with respect to the differential operator δ_t satisfy precisely $((\delta_t^{n+1}y)(t_0), (\delta_t^{n+1}Z)(t_0)) = (H_1^{(n)}(\mathbf{y}_0, \mathbf{Z}_0, t_0), H_2^{(n)}(\mathbf{y}_0, \mathbf{Z}_0, t_0))$. On the other hand, the successive δ_t -derivatives of $(y(t), Z(t))$ evaluated at t_0 coincide with the evaluation at $q = 1$ of the successive ∂_q -derivatives of the holomorphic functions

$$q \mapsto y(q \cdot t_0), \quad q \mapsto Z(q \cdot t_0).$$

Therefore, the power series $\sum_{n \geq 0} \frac{\tilde{a}_n(1)}{n!} (q-1)^n, \sum_{n \geq 0} \frac{\tilde{b}_n(1)}{n!} (q-1)^n$ both converge and yield the pair of functions $q \mapsto (y(qt_0), Z(qt_0))$, where $(y(t), Z(t))$ is this unique solution. It remains to prove that the sequence $(\tilde{a}_n, \tilde{b}_n)_{n \in \mathbb{N}}$ coincides with the $(a_n, b_n)_{n \in \mathbb{N}}$ defined in the statement of Theorem 3.3. On the other hand, since $\sigma_{q,t} = 1 + (q-1)\delta_{q,t}$, for each $n \in \mathbb{N}$ we have

$$\sigma_{q,t}^n = (1 + (q-1)\delta_{q,t})^n = \sum_{k=0}^n \binom{n}{k} (q-1)^k \delta_{q,t}^k. \quad (3.13)$$

We are therefore inclined to define the sequence $(\mathbf{y}_n, \mathbf{Z}_n)_{n \in \mathbb{N}} \in (\mathbb{C}(q) \times \mathbb{C}(q))^{\mathbb{N}}$ given by

$$\begin{cases} \mathbf{y}_n(q) = \sum_{k=0}^n \binom{n}{k} (q-1)^k \tilde{a}_k(q) \\ \mathbf{Z}_n(q) = \sum_{k=0}^n \binom{n}{k} (q-1)^k \tilde{b}_k(q). \end{cases}$$

By construction, the elements of the sequence $(\mathbf{y}_n, \mathbf{Z}_n, q^n t_0)_{n \in \mathbb{N}} \in (\mathbb{C}(q) \times \mathbb{C}(q) \times \mathbb{C}(q))^{\mathbb{N}}$ are related to $(\mathbf{y}_0, \mathbf{Z}_0, t_0)$ by the same rational relation as those of a discrete solution with initial value $(\mathbf{y}_0, \mathbf{Z}_0, t_0)$ of the modified q -Painlevé VI equation (3.8) with spectral data $\Theta = 1 + (q-1)\frac{\Theta}{2}$. Therefore, the sequence of rational functions $(\mathbf{y}_n, \mathbf{Z}_n, q^n t_0)_{n \in \mathbb{N}}$ is the (positive part of) the solution with initial value $(\mathbf{y}_0, \mathbf{Z}_0, t_0)$, seen as a rational function of the variable q .

The process in (3.13) allowing to recover $\sigma_{q,t}^n$ from $1, \delta_{q,t}, \dots, \delta_{q,t}^n$ is inverse to the process in (3.10) allowing to recover $\delta_{q,t}^n$ from $\sigma_{q,t}^0, \dots, \sigma_{q,t}^n$. Therefore,

the sequence $(\tilde{a}_n, \tilde{b}_n)_{n \in \mathbb{N}}$ constructed above coincides with the $(a_n, b_n)_{n \in \mathbb{N}}$ defined in the statement of Theorem 3.3. This concludes the proof of Theorem 3.3.

Remark 3.5. — Note that with respect to the notation in Theorem 3.3, for each $n \in \mathbb{N}$, as $q \rightarrow 1$, we have

$$\mathbf{y}_n(q) - y(q^n t_0) = O(q - 1) \quad \text{and} \quad \mathbf{Z}_n(q) - Z(q^n t_0) = O(q - 1).$$

Indeed, we have

$$\begin{aligned} \mathbf{y}_n(q) &= \sum_{k=0}^n \binom{n}{k} (q-1)^k a_k(q) = a_0(q) + O(q-1) \\ &= y(t_0) + O(q-1) = y(q^n t_0) + O(q-1). \end{aligned}$$

The argument for $(\mathbf{Z}_n(q) - Z(q^n t_0)) = O(q - 1)$ is identical.

COROLLARY 3.6. — *Let $\Theta \in \mathbb{C}^4$. Let $\Omega \subset \mathbb{C} \setminus \{0, 1\}$ be a subset with 1 in its closure. Let $t_0 \in \mathbb{C} \setminus \{0, 1\}$. Let $(\mathbf{y}_0(q), \mathbf{Z}_0(q))$ be a pair of continuous functions $\Omega \rightarrow \mathbb{C}$ such that the limit*

$$(\mathbf{y}_0, \mathbf{Z}_0) := \lim_{\substack{q \rightarrow 1 \\ q \in \Omega}} (\mathbf{y}_0(q), \mathbf{Z}_0(q))$$

exists in \mathbb{C}^2 and satisfies $\mathbf{y}_0 \notin \{0, 1, t_0\}$. Then, there exists a family $(\mathbf{y}_n(q), \mathbf{Z}_n(q))_{n \in \mathbb{N}} \in \mathbb{C}(\mathbf{y}_0(q), q) \times \mathbb{C}(\mathbf{Z}_0(q), q)$ of pairs of continuous functions such that for generic values of q , the sequence $(\mathbf{y}_n(q), \mathbf{Z}_n(q), q^n t_0)_{n \in \mathbb{N}}$ is the positive part of the discrete solution of $q \tilde{P}_{\text{VI}}(1 + (q-1)\frac{\Theta}{2})$ with initial value $(\mathbf{y}_0(q), \mathbf{Z}_0(q), t_0)$. Moreover, for each $n \in \mathbb{N}$, as $q \rightarrow 1$, we have

$$\mathbf{y}_n(q) - \mathbf{y}_0(q) = O(q - 1), \quad \mathbf{Z}_n(q) - \mathbf{Z}_0(q) = O(q - 1).$$

Proof. — In the proof of Theorem 3.3, we may replace $(\mathbf{y}_0, \mathbf{Z}_0)$ by $(\mathbf{y}_0(q), \mathbf{Z}_0(q))$ in the definition of $(a_n(q), b_n(q))_{n \in \mathbb{N}}$. Note that the latter then is a sequence of pairs of rational functions, each evaluated in a pair of continuous functions in q which, as $q \rightarrow 1$, admit a finite limit which is not in the polar locus of the restriction to $q = 1$ of these rational functions. We conclude that for each $n \in \mathbb{N}$, the pair $(a_n(q), b_n(q))$ may be continued to a continuous function on $\Omega \cup \{1\}$ with finite value at $q = 1$. Moreover, as before, we have the relation

$$\begin{aligned} (\mathbf{y}_n(q), \mathbf{Z}_n(q)) &= \sum_{k=0}^n \binom{n}{k} (q-1)^k (a_k(q), b_k(q)) \\ &= (a_0(q), b_0(q)) + O(q-1) \\ &= (\mathbf{y}_0(q), \mathbf{Z}_0(q)) + O(q-1). \end{aligned} \quad \square$$

4. Okamoto's space of initial conditions

4.1. Differential case

Let us review the construction in [20] of a convenient space of initial conditions for the Painlevé VI equation, and recall why it proves the Painlevé property. Let $\Theta = (\theta_0, \theta_1, \theta_t, \theta_\infty) \in \mathbb{C}^4$. Recall the Hamiltonian system (3.1) associated to the Painlevé VI differential equation with spectral data Θ :

$$\begin{cases} y'(t) = \frac{\partial H_{\text{VI}}^\Theta}{\partial Z}(y, Z, t) \\ Z'(t) = -\frac{\partial H_{\text{VI}}^\Theta}{\partial y}(y, Z, t). \end{cases}$$

Let us fix a time $t_0 \in \mathbb{C} \setminus \{0, 1\}$. Recall from Section 3.1 that for any initial value $(y_0, Z_0) \in (\mathbb{C} \setminus \{0, 1, t_0\}) \times \mathbb{C}$, there exists a unique germ of holomorphic solution $(y(t), Z(t))$ of (3.1) such that $(y(t_0), Z(t_0)) = (y_0, Z_0)$. Since we have $y_0 \notin \{0, 1, t_0\}$ here, we may equivalently consider the space of initial conditions $(u_0, v_0) \in (\mathbb{C} \setminus \{0, 1, t_0\}) \times \mathbb{C}$, where we identify

$$(u_0, v_0) = (y_0, y_0(y_0 - 1)(y_0 - t_0)Z_0). \quad (4.1)$$

We compactify this space of initial values to the second Hirzebruch surface \mathbb{F}_2 , using the following coordinate charts of \mathbb{C}^2 -spaces, endowed with their obvious rational transition maps:

$$\begin{aligned} (u_0, v_0) &=: (u, v), & (u_1, v_1) &= \left(u, \frac{1}{v}\right), \\ (u_2, v_2) &= \left(\frac{1}{u}, \frac{v}{u^2}\right), & (u_3, v_3) &= \left(\frac{1}{u}, \frac{u^2}{v}\right). \end{aligned}$$

Here what we have added by the compactification is the union of the horizontal line

$$\mathcal{H} := \{v_1 = 0\} \cup \{v_3 = 0\}$$

and the four vertical lines given by

$$\mathcal{D}_i := \{u_0 = i\} \cup \{u_1 = i\} \quad \text{for } i \in \{0, 1, t_0\}, \quad \mathcal{D}_\infty := \{u_2 = 0\} \cup \{u_3 = 0\}.$$

Now the Hamiltonian system (3.1) defines a meromorphic vector field on $\mathbb{F}_2 \times (\mathbb{C} \setminus \{0, 1\})$. Explicitly, it is given with respect to the coordinates (u, v, t)

by

$$\left\{ \begin{array}{l} t' = 1 \\ u' = \frac{1}{t(t-1)} (2v + u(u-1)) \\ v' = \frac{1}{4} \left(\frac{4v^2 - t^2\theta_0^2}{t(t-1)u} + \frac{4v^2 - (t-1)^2\theta_1^2}{t(t-1)(u-1)} + \frac{4v^2 - t^2(t-1)^2\theta_t^2}{t(t-1)(u-t)} \right) \\ \quad + \frac{\theta_\infty(\theta_\infty - 2)u(u-1)(u-t)}{4t(t-1)} \\ \quad + \frac{1}{4} \left(-\frac{\theta_0^2(u-1-t)}{t-1} + \frac{\theta_1^2(u+1-t)}{t} - \frac{(t(t-1)\theta_t^2 - 4v)(u-1+t)}{t(t-1)} \right). \end{array} \right.$$

One realizes that the vector field is infinite on the set given for each fixed $t = t_0$ by

$$\mathcal{H} \cup \mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_t \cup \mathcal{D}_\infty.$$

More precisely, it is infinite or undetermined (of the form “ $\frac{0}{0}$ ”) precisely there. These indeterminacy points will be called *base points* in the following. If we assume that

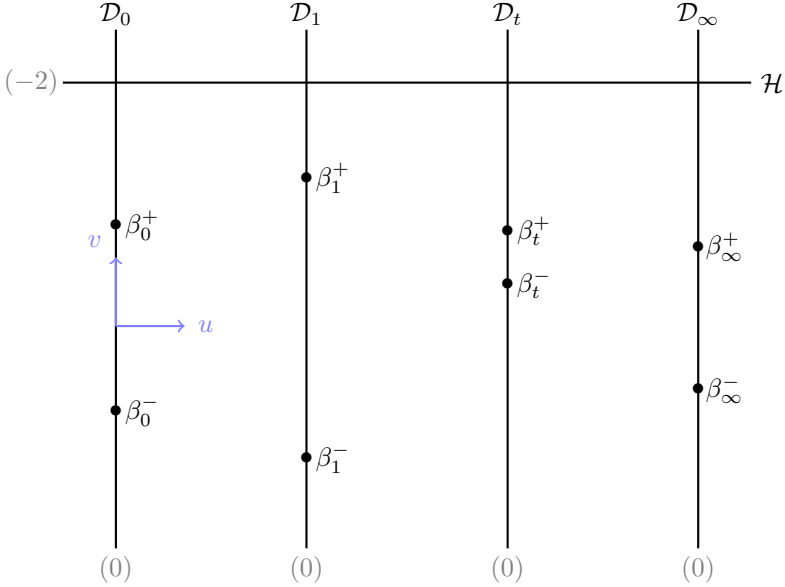
$$\theta_0 \neq 0, \quad \theta_1 \neq 0, \quad \theta_t \neq 0, \quad \theta_\infty \neq 1, \quad (4.2)$$

then there are precisely eight such base points. With respect to the four charts of \mathbb{F}_2 , these base points, each possibly visible in several charts, are precisely the following:

	(u_0, v_0)	(u_1, v_1)	(u_2, v_2)	(u_3, v_3)
β_0^\pm	$(0, \pm \frac{t\theta_0}{2})$	$(0, \pm \frac{2}{t\theta_0})$		
β_1^\pm	$(1, \pm \frac{(t-1)\theta_1}{2})$	$(1, \pm \frac{2}{(t-1)\theta_1})$	$(1, \pm \frac{(t-1)\theta_1}{2})$	$(1, \pm \frac{2}{(t-1)\theta_1})$
β_t^\pm	$(t, \pm \frac{t(t-1)\theta_t}{2})$	$(t, \pm \frac{2}{t(t-1)\theta_t})$	$(\frac{1}{t}, \pm \frac{(t-1)\theta_t}{2t})$	$(\frac{1}{t}, \pm \frac{2t}{(t-1)\theta_t})$
β_∞^+			$(0, \frac{\theta_\infty - 2}{2})$	$(0, \frac{2}{\theta_\infty - 2})$
β_∞^-			$(0, -\frac{\theta_\infty}{2})$	$(0, -\frac{2}{\theta_\infty})$

In the following discussion, we assume (4.2). For fixed t , the Hirzebruch surface, as well as the configuration of particular lines and base points, are

resumed in the following picture. Here “(n)” indicates “self-intersection number equal to n”.



For any fixed t , let us denote by $\widehat{\mathbb{F}}_2^t$ the result of the above Hirzebruch surface \mathbb{F}_2 after blow up of the eight base points. For each $i \in \{0, 1, t, \infty\}$, we denote by \mathcal{D}_i^{**} the strict transform of \mathcal{D}_i after blow up of β_i^\pm , i.e. the closure of $\mathcal{D}_i \setminus \{\beta_i^\pm\}$ in $\widehat{\mathbb{F}}_2^t$. Note that each \mathcal{D}_i^{**} has self-intersection number -2 . The *Okamoto space of initial values* at the time t for the sixth Painlevé equation with spectral data Θ is by definition

$$\text{Oka}_t := \widehat{\mathbb{F}}_2^t \setminus \mathcal{I}^t, \quad \text{where } \mathcal{I}^t := \mathcal{D}_0^{**} \cup \mathcal{D}_1^{**} \cup \mathcal{D}_t^{**} \cup \mathcal{D}_\infty^{**} \cup \mathcal{H}.$$

For example in order to blow up β_0^- , one replaces a neighborhood of β_0^- containing none of the other seven base points, by the corresponding neighborhood in the spaces $\mathbb{C}_{u_01, v_01}^2$ and $\mathbb{C}_{u_02, v_02}^2$ related to $\mathbb{C}_{u, v}^2$ according to the following transition maps:

$$(u_{01}, v_{01}) = \left(u, \frac{2v + t\theta_0}{2u} \right), \quad (u_{02}, v_{02}) = \left(\frac{2u}{2v + t\theta_0}, v + \frac{t\theta_0}{2} \right).$$

Note that $(u_{02}, v_{02}) = \left(\frac{1}{v_{01}}, u_{01}v_{01} \right)$ whenever $v_{01} \neq 0$. In these two new charts, what in $\mathbb{C}_{u, v}^2$ was the point β_0^- now corresponds to the exceptional line, isomorphic to \mathbb{P}^1 , given by

$$\mathcal{E}_0^- := \{u_{01} = 0\} \cup \{v_{02} = 0\}.$$

The complementary of \mathcal{E}_0^- however is in biholomorphic correspondence with the corresponding open subset of $\mathbb{C}_{u,v}^2$. The vector field in these new charts is given as follows:

$$\left\{ \begin{array}{l} u'_{01} = \frac{1}{t(t-1)} \left(2 \left(u_{01} v_{01} - \frac{t\theta_0}{2} \right) + u_{01} (u_{01} - 1) \right) \\ v'_{01} = -\frac{1}{t(t-1)} \left(v_{01}^2 - v_{01} + \frac{t\theta_0}{2} \right) - \frac{t+1}{t^2(t-1)} (u_{01} v_{01} - t\theta_0) v_{01} + \frac{v_{01}}{t} \\ \quad + \frac{1}{4} \left(-\frac{\theta_0^2}{t-1} + \frac{\theta_1^2}{t} - \theta_t^2 \right) \\ \quad + \frac{1}{4} \left(\frac{(2u_{01}v_{01}-t\theta_0)^2 - (t-1)^2\theta_1^2}{t(t-1)(u_{01}-1)} \right. \\ \quad \quad \left. + \frac{(2u_{01}v_{01}-t\theta_0)^2 - t^2(t-1)^2\theta_t^2}{t^2(t-1)(u_{01}-t)} + \frac{\theta_\infty(\theta_\infty-2)(u_{01}-1)(u_{01}-t)}{t(t-1)} \right), \\ \\ u'_{02} = \frac{1}{t(t-1)} \left(1 - u_{02} + \frac{t\theta_0}{2} u_{02}^2 \right) + \frac{t+1}{t^2(t-1)} (v_{02} - t\theta_0) u_{02} \\ \quad - \frac{u_{02}}{t} - \frac{u_{02}^2}{4} \left(-\frac{\theta_0^2}{t-1} + \frac{\theta_1^2}{t} - \theta_t^2 \right) \\ \quad - \frac{u_{02}^2}{4} \left(\frac{(2v_{02}-t\theta_0)^2 - (t-1)^2\theta_1^2}{t(t-1)(u_{02}v_{02}-1)} \right. \\ \quad \quad \left. + \frac{(2v_{02}-t\theta_0)^2 - t^2(t-1)^2\theta_t^2}{t^2(t-1)(u_{02}v_{02}-t)} + \frac{\theta_\infty(\theta_\infty-2)(u_{02}v_{02}-1)(u_{02}v_{02}-t)}{t(t-1)} \right) \\ v'_{02} = \frac{v_{02}-t\theta_0}{t(t-1)u_{02}} + \frac{1}{4} \left(\frac{(2v_{02}-t\theta_0)^2 - (t-1)^2\theta_1^2}{t(t-1)(u_{02}v_{02}-1)} + \frac{(2v_{02}-t\theta_0)^2 - t^2(t-1)^2\theta_t^2}{t(t-1)(u_{02}v_{02}-t)} \right) \\ \quad + \frac{\theta_\infty(\theta_\infty-2)u_{02}v_{02}(u_{02}v_{02}-1)(u_{02}v_{02}-t)}{4t(t-1)} \\ \quad + \frac{1}{4} \left(-\frac{\theta_0^2(u_{02}v_{02}-1-t)}{t-1} + \frac{\theta_1^2(u_{02}v_{02}+1-t)}{t} \right. \\ \quad \quad \left. - \frac{(t(t-1)\theta_t^2 - 2(2v_{02}-t\theta_0))(u_{02}v_{02}-1+t)}{t(t-1)} \right) + \frac{\theta_0}{2}. \end{array} \right.$$

We see that there is no base point on $\mathcal{E}_0^- = \{u_{01} = 0\} \cup \{v_{02} = 0\}$. In other words, the blow up was sufficient to *resolve* the base point β_0^- . Moreover, we see that the vector field is finite on all points of \mathcal{E}_0^- except the one given by the intersection with the strict transform of \mathcal{D}_0 , which is visible in these charts as $\{u_{02} = 0\}$. As shown in [20], this actually holds true for all eight base points, i.e. on $\text{Ok}a_t$, the vector field is everywhere finite and free of base points.

As explained in [16], this situation can be conveniently reformulated as follows. On $\mathbb{F}_2 \times (\mathbb{C} \setminus \{0, 1\})$, the meromorphic vector field given by (3.1) defines a singular holomorphic foliation. The singular locus corresponds to the eight families (parametrized by $t \in \mathbb{C} \setminus \{0, 1\}$) of base points. After blowing up the singular locus, the induced foliation on

$$\bigcup_{t \in \mathbb{C} \setminus \{0, 1\}} \widehat{\mathbb{F}}_2^t$$

is non-singular. Moreover, it is transversal to $\{t = cst\}$ on the complementary of $\bigcup_{t \in \mathbb{C} \setminus \{0,1\}} \mathcal{I}^t$, i.e. on

$$\text{Oka} := \bigcup_{t \in \mathbb{C} \setminus \{0,1\}} \text{Oka}_t.$$

This implies that for any $t_0 \in \mathbb{C} \setminus \{0,1\}$ and any initial condition given by a point in Oka_{t_0} , in turn given by a point in some chart $\mathbb{C}_{u_{ij}, v_{ij}}^2$ of $\widehat{\mathbb{F}}_2^{t_0}$, one obtains a unique germ of holomorphic integral curve of the form $(u_{ij}(t), v_{ij}(t), t)$. Translated back into the variables (y, Z) , this yields a meromorphic solution y of the sixth Painlevé equation, associated to this initial condition. Moreover, since $(u_{ij}(t), v_{ij}(t), t)$ parametrizes a germ of leaf of the Painlevé foliation on Oka , and this foliation is transversal to $\{t = cst\}$, this parametrization of a germ of leaf can be analytically continued in Oka along any path with starting point t_0 . In other words, the meromorphic solution y of the Painlevé equation can be meromorphically continued along any path in $\mathbb{C} \setminus \{0,1\}$ with starting point t_0 . As for the usual analytic continuation, this meromorphic continuation has no reason to be uniform. But we obtain a well-defined meromorphic function on every simply connected subset of $\mathbb{C} \setminus \{0,1\}$ containing t_0 . This phenomenon, which is also observed for the other five Painlevé equations, is also known as the *Painlevé property* of solutions of Painlevé equations.

Let us illustrate the above by an example. Consider a germ of solution y of P_{VI} , associated to an initial condition on $\mathcal{E}_0^- \setminus \mathcal{D}_0^{**}$ at $t = t_0$. That is, we have $(u_{01}, v_{01})(t_0) = (0, \alpha)$ for some $\alpha \in \mathbb{C}$. Hence $y = u_{01}$ is holomorphic near t_0 . From the explicit formula of the vector field, we readily calculate the first terms of the Taylor series expansion of y :

$$y(t) = -\frac{\theta_0}{(t_0 - 1)}(t - t_0) - \frac{\theta_0(2\alpha - 1 - t_0)}{2t_0(t_0 - 1)^2}(t - t_0)^2 + O((t - t_0)^3).$$

Note that y has a simple zero at t_0 , and its Taylor series expansion at t_0 up to order two is uniquely determined by α .

4.2. A discrete analogue

Let us first review the construction in [23] of a convenient q -analogue of the Okamoto space for the q -Painlevé VI equation, which proves the existence of discrete solutions, et then adapt this discussion to our modified q -Painlevé VI equation.

Let $q \in \mathbb{C} \setminus \{0,1\}$. Let $\Theta = (\Theta_0, \Theta_1, \Theta_t, \Theta_\infty) \in (\mathbb{C}^*)^4$ and denote, as usual, $\overline{\Theta}_i := 1/\Theta_i$ for each $i \in \{0, 1, t, \infty\}$. Consider the rational functions

$f, g \in \mathbb{C}(\mathbf{y}, \mathbf{z}, t)$ in three complex variables given by

$$\left\{ \begin{aligned} g(\mathbf{y}, \mathbf{z}, t) &:= \frac{1}{q} \frac{(\mathbf{y} - t\Theta_t)(\mathbf{y} - t\bar{\Theta}_t)}{\mathbf{z}(\mathbf{y} - \Theta_1)(\mathbf{y} - \bar{\Theta}_1)}, \\ f(\mathbf{y}, \mathbf{z}, t) &:= \frac{(g(\mathbf{y}, \mathbf{z}, t) - t\Theta_0)(g(\mathbf{y}, \mathbf{z}, t) - t\bar{\Theta}_0)}{\mathbf{y} \left(g(\mathbf{y}, \mathbf{z}, t) - \frac{\Theta_\infty}{q} \right) (g(\mathbf{y}, \mathbf{z}, t) - \bar{\Theta}_\infty)} \\ &= \frac{\left(\frac{1}{q} \frac{(\mathbf{y} - t\Theta_t)(\mathbf{y} - t\bar{\Theta}_t)}{\mathbf{z}(\mathbf{y} - \Theta_1)(\mathbf{y} - \bar{\Theta}_1)} - t\Theta_0 \right) \left(\frac{1}{q} \frac{(\mathbf{y} - t\Theta_t)(\mathbf{y} - t\bar{\Theta}_t)}{\mathbf{z}(\mathbf{y} - \Theta_1)(\mathbf{y} - \bar{\Theta}_1)} - t\bar{\Theta}_0 \right)}{\mathbf{y} \left(\frac{1}{q} \frac{(\mathbf{y} - t\Theta_t)(\mathbf{y} - t\bar{\Theta}_t)}{\mathbf{z}(\mathbf{y} - \Theta_1)(\mathbf{y} - \bar{\Theta}_1)} - \frac{\Theta_\infty}{q} \right) \left(\frac{1}{q} \frac{(\mathbf{y} - t\Theta_t)(\mathbf{y} - t\bar{\Theta}_t)}{\mathbf{z}(\mathbf{y} - \Theta_1)(\mathbf{y} - \bar{\Theta}_1)} - \bar{\Theta}_\infty \right)}. \end{aligned} \right. \quad (4.3)$$

Note that with this notation, the sixth q -Painlevé equation (3.4) with spectral data Θ writes

$$\sigma_{q,t}\mathbf{y} = f(\mathbf{y}, \mathbf{z}, t), \quad \sigma_{q,t}\mathbf{z} = g(\mathbf{y}, \mathbf{z}, t).$$

For each fixed $t = t_0 \in \mathbb{C}^*$ such that

$$t \neq \Theta_0^{\pm 1} \Theta_\infty / q, \quad t \neq \Theta_0^{\pm 1} \bar{\Theta}_\infty, \quad t \neq \Theta_t^{\pm 1} \Theta_1, \quad t \neq \Theta_t^{\pm 1} \bar{\Theta}_1, \quad (4.4)$$

the expressions of $f(\mathbf{y}, \mathbf{z}, t_0)$ and $g(\mathbf{y}, \mathbf{z}, t_0)$ in (4.3) are reduced, i.e. they admit no common factor in nominator and denominator. Now choose any $t = t_0 \in \mathbb{C}^*$ satisfying (4.4) and consider the rational map

$$\mathfrak{S}_t : \begin{cases} \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \\ (\mathbf{y}, \mathbf{z}) \mapsto (f(\mathbf{y}, \mathbf{z}, t_0), g(\mathbf{y}, \mathbf{z}, t_0)). \end{cases} \quad (4.5)$$

Note that for any point $(\mathbf{y}_0, \mathbf{z}_0) \in \mathbb{C}^* \times \mathbb{C}^* \subset \mathbb{P}^1 \times \mathbb{P}^1$, the image $\mathfrak{S}_t(\mathbf{y}_0, \mathbf{z}_0)$ is a well defined point in $\mathbb{P}^1 \times \mathbb{P}^1$. However, on the complement of $\mathbb{C}^* \times \mathbb{C}^*$ in $\mathbb{P}^1 \times \mathbb{P}^1$, there are some points for which the image under \mathfrak{S}_t is undetermined, i.e. at least one of the coordinates of the image contains, even after switching to homogeneous coordinates in the source, an expression of the form “ $\frac{0}{0}$ ”. These points will we called *critical points* in the following. Let us assume

$$\Theta_0^2 \neq 1, \quad \Theta_1^2 \neq 1, \quad \Theta_t^2 \neq 1, \quad \Theta_\infty^2 \neq q. \quad (4.6)$$

Then there are precisely eight critical points, which are given, for $t = t_0$, as follows:

$$\begin{aligned} \gamma_0^-(t) : (\mathbf{y}, \mathbf{z}) &= (0, t\Theta_0/q), & \gamma_0^+(t) : (\mathbf{y}, \mathbf{z}) &= (0, t\bar{\Theta}_0/q), \\ \gamma_1^-(t) : (\mathbf{y}, \mathbf{z}) &= (\bar{\Theta}_1, \infty), & \gamma_1^+(t) : (\mathbf{y}, \mathbf{z}) &= (\Theta_1, \infty), \\ \gamma_t^-(t) : (\mathbf{y}, \mathbf{z}) &= (t\Theta_t, 0), & \gamma_t^+(t) : (\mathbf{y}, \mathbf{z}) &= (t\bar{\Theta}_t, 0), \\ \gamma_\infty^-(t) : (\mathbf{y}, \mathbf{z}) &= (\infty, \bar{\Theta}_\infty), & \gamma_\infty^+(t) : (\mathbf{y}, \mathbf{z}) &= (\infty, \Theta_\infty/q). \end{aligned}$$

Let us denote by $\text{Bl}(\mathbb{P}^1 \times \mathbb{P}^1)_{x_1, \dots, x_k}$, the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the k distinct points x_1, \dots, x_k . For each $t \in \mathbb{C}^*$ satisfying (4.4), let us define

$$\mathfrak{P}_t := \text{Bl}(\mathbb{P}^1 \times \mathbb{P}^1)_{\gamma_0^-(t), \gamma_0^+(t), \gamma_1^-(t), \gamma_1^+(t), \gamma_t^-(t), \gamma_t^+(t), \gamma_\infty^-(t), \gamma_\infty^+(t)}. \quad (4.7)$$

Here we continue to assume (4.6), so that these are indeed eight distinct points, the blow up is well defined, and does moreover not depend on the order in which the points are successively blown up. As a slight improvement to [23, Proposition 1], we obtain the following.

PROPOSITION 4.1. — *Let $q \in \mathbb{C} \setminus \{0, 1\}$. Let $\Theta \in (\mathbb{C}^*)^4$ such that (4.6) holds. Denote*

$$S_q := \{\Theta_1^{\varepsilon_1} \Theta_t^{\varepsilon_t}, \Theta_0^{\varepsilon_0} \Theta_\infty^{\varepsilon_\infty} \mid \varepsilon_0, \varepsilon_1, \varepsilon_t, \varepsilon_\infty \in \{-1, 1\}\} \cdot q^{\mathbb{Z}}. \quad (4.8)$$

Then for any $t \in \mathbb{C}^* \setminus S_q$, the map

$$\mathfrak{S}_t : \mathfrak{P}_t \dashrightarrow \mathfrak{P}_{qt}$$

induced by (4.5), via pre-composition and post-composition with the blow ups in (4.7), is biregular. That is, this rational map contains no indeterminacy points and is bijective. In particular, it is a biholomorphism.

Proof. — We simply follow the proof in [23], where generic t, Θ , were considered, and make sure that it goes through for fixed t, Θ as in the statement. In order to make the following argumentation shorter, let us first consider an example.

Consider the rational map φ from $\mathbb{P}^1 \times \mathbb{P}^1$ with standard coordinates (\mathbf{y}, \mathbf{z}) to $\mathbb{P}^1 \times \mathbb{P}^1$ with standard coordinates $(\widehat{\mathbf{y}}, \widehat{\mathbf{z}})$ defined by $(\mathbf{y}, \mathbf{z}) \mapsto (\widehat{\mathbf{y}}, \widehat{\mathbf{z}}) = (\mathbf{y}, \mathbf{z}(\mathbf{y} - \Theta_1))$. This map is actually a so-called *elementary transformation* with respect to the ruling $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ given by $(\mathbf{y}, \mathbf{z}) \mapsto \mathbf{y}$. In the complement of the fiber $\{y = \Theta_1\}$ of this ruling, it defines a biholomorphism onto its image. Moreover, this map has an indeterminacy point at $(\mathbf{y}, \mathbf{z}) = (\Theta_1, \infty)$. With the exception of this indeterminacy point, every point in the fiber $\{y = \Theta_1\}$ is mapped to the point $(\widehat{\mathbf{y}}, \widehat{\mathbf{z}}) = (\Theta_1, 0)$. Conversely, the inverse rational map $(\widehat{\mathbf{y}}, \widehat{\mathbf{z}}) \mapsto (\mathbf{y}, \mathbf{z}) = \left(\widehat{\mathbf{y}}, \frac{\widehat{\mathbf{z}}}{\widehat{\mathbf{y}} - \Theta_1}\right)$ has an indeterminacy point at $(\widehat{\mathbf{y}}, \widehat{\mathbf{z}}) = (\Theta_1, 0)$ and maps the rest of the fiber $\{\widehat{y} = \Theta_1\}$ to the point $(\mathbf{y}, \mathbf{z}) = (\Theta_1, \infty)$. However, as one can easily check, the rational map obtained by considering the composition

$$\text{Bl}(\mathbb{P}^1 \times \mathbb{P}^1)_{(\mathbf{y}, \mathbf{z}) = (\Theta_1, \infty)} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\varphi} \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \text{Bl}(\mathbb{P}^1 \times \mathbb{P}^1)_{(\widehat{\mathbf{y}}, \widehat{\mathbf{z}}) = (\Theta_1, 0)},$$

is biregular. So in summary, the elementary transformation φ blows up the point $(\mathbf{y}, \mathbf{z}) = (\Theta_1, \infty)$ and contracts the strict transform of the line $\{y = \Theta_1\}$, and becomes biregular when pre- and post-composed with the blow-ups of $(\mathbf{y}, \mathbf{z}) = (\Theta_1, \infty)$ and $(\widehat{\mathbf{y}}, \widehat{\mathbf{z}}) = (\Theta_1, 0)$ respectively.

The key is now to use elementary transformations in order to decompose the map \mathfrak{S}_t into a sequence of biregular isomorphisms.

- Consider the rational map from $\mathbb{P}^1 \times \mathbb{P}^1$ with standard coordinates (\mathbf{y}, \mathbf{z}) to $\mathbb{P}^1 \times \mathbb{P}^1$ with standard coordinates $(\mathbf{y}, \widehat{\mathbf{z}})$ defined by $(\mathbf{y}, \mathbf{z}) \mapsto (\mathbf{y}, \widehat{\mathbf{z}}) = \left(\mathbf{y}, \mathbf{z} \frac{(\mathbf{y}-\Theta_1)(\mathbf{y}-\bar{\Theta}_1)}{(\mathbf{y}-t\Theta_t)(\mathbf{y}-t\bar{\Theta}_t)} \right)$. This map can be seen as the composition of four (commuting) elementary transformations with respect to the ruling $(\mathbf{y}, \mathbf{z}) \mapsto \mathbf{y}$. Note that by assumption, the set $\{\Theta_1, \bar{\Theta}_1, t\Theta_t, t\bar{\Theta}_t\}$ has cardinality four, which implies that no two of these elementary transformations cancel each other out. Therefore, the considered rational map induces a biregular isomorphism from \mathfrak{P}_t to the surface $\mathfrak{P}_t^{(1)}$, where $\mathfrak{P}_t^{(1)}$ denotes the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ with standard coordinates $(\mathbf{y}, \widehat{\mathbf{z}})$ at the eight points given, with respect to these coordinates, by

$$\begin{aligned} & \left(0, \frac{\Theta_0}{qt} \right), \left(0, \frac{\bar{\Theta}_0}{qt} \right), \quad (\bar{\Theta}_1, 0), (\Theta_1, 0), \\ & (t\Theta_t, \infty), (t\bar{\Theta}_t, \infty), \quad (\infty, \bar{\Theta}_\infty), (\infty, \Theta_\infty/q). \end{aligned}$$

- Consider the biregular map from $\mathbb{P}^1 \times \mathbb{P}^1$ with standard coordinates $(\mathbf{y}, \widehat{\mathbf{z}})$ to $\mathbb{P}^1 \times \mathbb{P}^1$ with standard coordinates $(\mathbf{y}, \widetilde{\mathbf{z}})$ defined by $(\mathbf{y}, \widehat{\mathbf{z}}) \mapsto (\mathbf{y}, \widetilde{\mathbf{z}}) = \left(\mathbf{y}, \frac{1}{q\widehat{\mathbf{z}}} \right)$. This map induces a biregular isomorphism from $\mathfrak{P}_t^{(1)}$ to the surface $\mathfrak{P}_t^{(2)}$, where $\mathfrak{P}_t^{(2)}$ denotes the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ with standard coordinates $(\mathbf{y}, \widetilde{\mathbf{z}})$ at the eight points given, with respect to these coordinates, by

$$\begin{aligned} & (0, t\bar{\Theta}_0), (0, t\Theta_0), \quad (\bar{\Theta}_1, \infty), (\Theta_1, \infty), \\ & (t\Theta_t, 0), (t\bar{\Theta}_t, 0), \quad (\infty, \Theta_\infty/q), (\infty, \bar{\Theta}_\infty). \end{aligned}$$

- Consider the rational map from $\mathbb{P}^1 \times \mathbb{P}^1$ with standard coordinates $(\mathbf{y}, \widetilde{\mathbf{z}})$ to $\mathbb{P}^1 \times \mathbb{P}^1$ with standard coordinates $(\widehat{\mathbf{y}}, \widetilde{\mathbf{z}})$ defined by $(\mathbf{y}, \widetilde{\mathbf{z}}) \mapsto (\widehat{\mathbf{y}}, \widetilde{\mathbf{z}}) = \left(\mathbf{y} \frac{(\widetilde{\mathbf{z}}-\Theta_\infty/q)(\widetilde{\mathbf{z}}-\bar{\Theta}_\infty)}{(\widetilde{\mathbf{z}}-t\Theta_0)(\widetilde{\mathbf{z}}-t\bar{\Theta}_0)}, \widetilde{\mathbf{z}} \right)$. This map can be seen as the composition of four (commuting) elementary transformations with respect to the ruling $(\mathbf{y}, \widetilde{\mathbf{z}}) \mapsto \widetilde{\mathbf{z}}$. Note that by assumption, the set $\{t\Theta_0, t\bar{\Theta}_0, \Theta_\infty/q, \bar{\Theta}_\infty\}$ has cardinality four, which implies that the considered rational map induces a biregular isomorphism from $\mathfrak{P}_t^{(2)}$ to the surface $\mathfrak{P}_t^{(3)}$, where $\mathfrak{P}_t^{(3)}$ denotes the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ with standard coordinates $(\widehat{\mathbf{y}}, \widetilde{\mathbf{z}})$ at the eight points given,

with respect to these coordinates, by

$$(\infty, t\bar{\Theta}_0), (\infty, t\Theta_0), \quad (\bar{\Theta}_1, \infty), (\Theta_1, \infty), \\ \left(\frac{\Theta_t}{qt}, 0\right), \left(\frac{\bar{\Theta}_t}{qt}, 0\right), \quad (0, \Theta_\infty/q), (0, \bar{\Theta}_\infty).$$

- Consider the biregular map from $\mathbb{P}^1 \times \mathbb{P}^1$ with standard coordinates $(\hat{\mathbf{y}}, \hat{\mathbf{z}})$ to $\mathbb{P}^1 \times \mathbb{P}^1$ with standard coordinates $(\tilde{\mathbf{y}}, \tilde{\mathbf{z}})$ defined by $(\hat{\mathbf{y}}, \hat{\mathbf{z}}) \mapsto (\tilde{\mathbf{y}}, \tilde{\mathbf{z}}) = \left(\frac{1}{\hat{\mathbf{y}}}, \hat{\mathbf{z}}\right)$. This map induces a biregular isomorphism from $\mathfrak{P}_t^{(3)}$ to the surface $\mathfrak{P}_t^{(4)}$, where $\mathfrak{P}_t^{(4)}$ denotes the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ with standard coordinates $(\tilde{\mathbf{y}}, \tilde{\mathbf{z}})$ at the eight points given, with respect to these coordinates, by

$$(0, t\bar{\Theta}_0), (0, t\Theta_0), \quad (\Theta_1, \infty), (\bar{\Theta}_1, \infty), \\ (qt\bar{\Theta}_t, 0), (qt\Theta_t, 0), \quad (\infty, \Theta_\infty/q), (\infty, \bar{\Theta}_\infty).$$

It now suffices to see that $\mathfrak{P}_t^{(4)} = \mathfrak{P}_{qt}$ and that the biregular map $\mathfrak{P}_t \xrightarrow{\sim} \mathfrak{P}_t^{(4)}$, obtained by composing all of the above, coincides with \mathfrak{S}_t . \square

Remark 4.2. — The above proposition implies that if (4.6) holds, then for each $t \in \mathbb{C}^* \setminus S_q$, for each $n \in \mathbb{N}$, we have biregular maps

$$\mathfrak{S}_t^{(n)} := \mathfrak{S}_{q^{nt}} \circ \cdots \circ \mathfrak{S}_{qt} \circ \mathfrak{S}_t : \mathfrak{P}_t \rightarrow \mathfrak{P}_{q^{nt}}, \\ \mathfrak{S}_t^{(-n)} := \left(\mathfrak{S}_{q^{-nt}}\right)^{-1} : \mathfrak{P}_t \rightarrow \mathfrak{P}_{q^{-nt}}.$$

In particular, if $t_0 \in \mathbb{C}^* \setminus S_q$, and $(\mathbf{y}_0, \mathbf{z}_0) \in \mathbb{C}^* \times \mathbb{C}^* \subset \mathfrak{P}_{t_0}$, then for each $n \in \mathbb{Z}$, we obtain a well-defined element

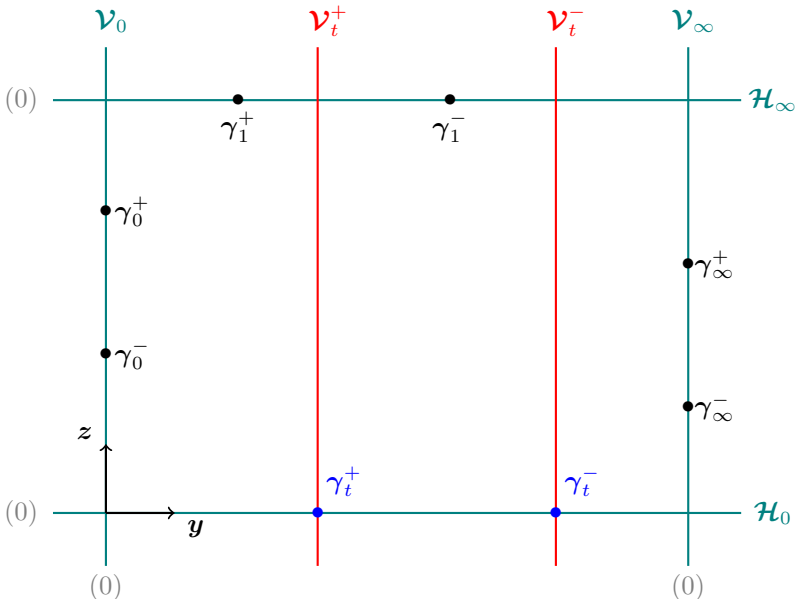
$$(\tilde{\mathbf{y}}_n, \tilde{\mathbf{z}}_n) := \mathfrak{S}_{t_0}^{(n)}(\mathbf{y}_0, \mathbf{z}_0) \in \mathfrak{P}_{q^{nt_0}}$$

and a well-defined element $(\mathbf{y}_n, \mathbf{z}_n) \in \mathbb{P}^1 \times \mathbb{P}^1$ obtained by projecting *via* the natural regular map $\mathfrak{P}_{q^{nt_0}} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. But the fact that the association $(\mathbf{y}_0, \mathbf{z}_0) \rightarrow (\mathbf{y}_n, \mathbf{z}_n)$ is well-defined for each $n \in \mathbb{Z}$ is simply a reformulation of the statement of Proposition 3.1. So the above proposition proves the latter.

The q -analogue of an Okamoto space for fixed $t \in \mathbb{C}^* \setminus S_q$ will be a certain Zariski-open subset of \mathfrak{P}_t . In order to define it, let us go back to $\mathbb{F}_0 := \mathbb{P}^1 \times \mathbb{P}^1$ with standard coordinates (\mathbf{y}, \mathbf{z}) , endowed with the eight distinct points γ_i^\pm and identify some particular components of this space, namely, the following vertical and horizontal lines:

$$\mathcal{H}_0 : \{z = 0\}, \quad \mathcal{H}_\infty : \{z = \infty\}, \quad \mathcal{V}_0 : \{y = 0\}, \quad \mathcal{V}_\infty : \{y = \infty\}, \\ \mathcal{V}_t^+ : \{y = t\bar{\Theta}_t\}, \quad \mathcal{V}_t^- : \{y = t\Theta_t\}.$$

This configuration of points and lines is illustrated in the following picture. Here the colors can be ignored for now, their use will become clear later, see Remark 4.5.



Recall that \mathfrak{P}_t is the blow up of \mathbb{F}_0 in the eight points γ_i^\pm with $i \in \{0, 1, t, \infty\}$. In \mathfrak{P}_t , we denote by

$$\mathcal{H}_0^{**}, \mathcal{H}_\infty^{**}, \mathcal{V}_0^{**}, \mathcal{V}_\infty^{**}, \mathcal{V}_t^{+*}, \mathcal{V}_t^{-*}$$

the strict transforms of the corresponding projective lines in \mathbb{F}_0 . We denote

$$\mathcal{J}^t := \mathcal{H}_0^{**} \cup \mathcal{H}_\infty^{**} \cup \mathcal{V}_0^{**} \cup \mathcal{V}_\infty^{**}.$$

Under the map $\mathfrak{S}_t : \mathfrak{P}_t \rightarrow \mathfrak{P}_{qt}$, the set \mathcal{J}^t is mapped to the set \mathcal{J}^{qt} . So in some sense, the set \mathcal{J}^t can be seen as invariant under the q -Painlevé map \mathfrak{S}_t . Moreover, a discrete solution $(\mathbf{y}_n, \mathbf{z}_n, q^n t_0)_{n \in \mathbb{Z}}$ given by an initial condition in \mathcal{J}^t is not very interesting in the sense that both $(\mathbf{y}_n)_{n \in \mathbb{Z}}$ and $(\mathbf{z}_n)_{n \in \mathbb{Z}}$ simply oscillate between 0 and ∞ . The Okamoto space $q\text{Oka}_t$ for fixed $t \in \mathbb{C}^* \setminus S_q$ as introduced in [23] is by definition the complement of \mathcal{J}^t in \mathfrak{P}_t . So we define

$$q\text{Oka}_t := \mathfrak{P}_t \setminus \mathcal{J}^t.$$

Note that the strict transforms $\mathcal{V}_t^{\pm*}$ of the vertical lines \mathcal{V}_t in \mathbb{F}_0 are not contained in \mathcal{J}^t . They do however play a particular role in the relation to the construction of the *modified* Okamoto space $q\widetilde{\text{Oka}}_t$ that we will now define. The letter will be better suited for the confluence problem.

Let us recall that $\mathbf{Z} = \frac{(\mathbf{y}-t\Theta_t)(\mathbf{y}-t\Theta_t)z-1}{q(\mathbf{y}-1)(\mathbf{y}-t)z}$. Motivated by (4.1), we apply the change of variable

$$\begin{aligned} (\mathbf{u}, \mathbf{v}) &= (\mathbf{y}, \mathbf{y}(\mathbf{y}-1)(\mathbf{y}-t)\mathbf{Z}) \\ &= \left(\mathbf{y}, \frac{(\mathbf{y}-t\Theta_t)(\mathbf{y}-t/\Theta_t)}{q(q-1)z} - \frac{(\mathbf{y}-1)(\mathbf{y}-t)}{(q-1)} \right), \end{aligned} \quad (4.9)$$

to the modified q -Painlevé VI equation (3.8) with spectral data Θ . There are well-defined rational functions $\tilde{f}, \tilde{g} \in \mathbb{C}(\mathbf{u}, \mathbf{v}, t)$ such that with respect to these variables, (3.8) is of the form

$$\sigma_{q,t}\mathbf{u} = \tilde{f}(\mathbf{u}, \mathbf{v}, t), \quad \sigma_{q,t}\mathbf{v} = \tilde{g}(\mathbf{u}, \mathbf{v}, t).$$

More precisely, we have

$$\left\{ \begin{aligned} \tilde{f}(\mathbf{u}, \mathbf{v}, t) &= \frac{\left(\frac{(\mathbf{u}-1)(\mathbf{u}-t)+(q-1)\mathbf{v}}{(\mathbf{u}-\Theta_1)(\mathbf{u}-\frac{1}{\Theta_1})} - t\Theta_0 \right) \left(\frac{(\mathbf{u}-1)(\mathbf{u}-t)+(q-1)\mathbf{v}}{\mathbf{u}(\mathbf{u}-\Theta_1)(\mathbf{u}-\frac{1}{\Theta_1})} - t\frac{1}{\Theta_0} \right)}{\left(\frac{(\mathbf{u}-1)(\mathbf{u}-t)+(q-1)\mathbf{v}}{(\mathbf{u}-\Theta_1)(\mathbf{u}-\frac{1}{\Theta_1})} - \frac{1}{\Theta_\infty} \right) \left(\frac{(\mathbf{u}-1)(\mathbf{u}-t)+(q-1)\mathbf{v}}{(\mathbf{u}-\Theta_1)(\mathbf{u}-\frac{1}{\Theta_1})} - \frac{\Theta_\infty}{q} \right)} \\ \tilde{g}(\mathbf{u}, \mathbf{v}, t) &= \frac{(\mathbf{u}-\Theta_1) \left(\mathbf{u} - \frac{1}{\Theta_1} \right) \left(\tilde{f}(\mathbf{u}, \mathbf{v}, t) - qt\Theta_t \right) \left(\tilde{f}(\mathbf{u}, \mathbf{v}, t) - qt\frac{1}{\Theta_t} \right)}{q(q-1)(\mathbf{u}-1)(\mathbf{u}-t) + q(q-1)^2\mathbf{v}} \\ &\quad - \frac{\left(\tilde{f}(\mathbf{u}, \mathbf{v}, t) - 1 \right) \left(\tilde{f}(\mathbf{u}, \mathbf{v}, t) - qt \right)}{(q-1)}. \end{aligned} \right.$$

We consider the second Hirzebruch surface \mathbb{F}_2 with \mathbb{C}^2 -charts $\mathbb{C}_{\mathbf{u}_0, \mathbf{v}_0}^2, \mathbb{C}_{\mathbf{u}_1, \mathbf{v}_1}^2, \mathbb{C}_{\mathbf{u}_2, \mathbf{v}_2}^2, \mathbb{C}_{\mathbf{u}_3, \mathbf{v}_3}^2$ glued together along their $\mathbb{C}^* \times \mathbb{C}^*$ -subsets according to the transition maps given by

$$\begin{aligned} (\mathbf{u}_0, \mathbf{v}_0) &= (\mathbf{u}, \mathbf{v}), & (\mathbf{u}_1, \mathbf{v}_1) &= \left(\mathbf{u}, \frac{1}{\mathbf{v}} \right), \\ (\mathbf{u}_2, \mathbf{v}_2) &= \left(\frac{1}{\mathbf{u}}, \frac{\mathbf{v}}{\mathbf{u}^2} \right), & (\mathbf{u}_3, \mathbf{v}_3) &= \left(\frac{1}{\mathbf{u}}, \frac{\mathbf{u}^2}{\mathbf{v}} \right). \end{aligned} \quad (4.10)$$

Here we will consider (\mathbf{u}, \mathbf{v}) as the standard coordinates, with respect to which we will define rational maps such as the following. For each fixed $t = t_0 \in \mathbb{C}^*$ satisfying (4.4), we obtain a rational map

$$\tilde{\mathfrak{S}}_t : \begin{cases} \mathbb{F}_2 \dashrightarrow \mathbb{F}_2 \\ (\mathbf{u}, \mathbf{v}) \mapsto \left(\tilde{f}(\mathbf{u}, \mathbf{v}, t), \tilde{g}(\mathbf{u}, \mathbf{v}, t) \right). \end{cases} \quad (4.11)$$

LEMMA 4.3. — *Let $\Theta \in (\mathbb{C}^*)^4$ such that (4.6) holds. Let $t \in \mathbb{C}^* \setminus S_q$, where S_q is defined in (4.8). The indeterminacy points of the rational map $\tilde{\mathfrak{S}}_t$*

defined in (4.11) are precisely the following eight points (in the source \mathbb{F}_2).

$$\begin{aligned} \beta_0^-(t) : (\mathbf{u}, \mathbf{v}) &= \left(0, \frac{t(\bar{\Theta}_0 - 1)}{q - 1}\right), \\ \beta_0^+(t) : (\mathbf{u}, \mathbf{v}) &= \left(0, \frac{t(\Theta_0 - 1)}{q - 1}\right), \\ \beta_1^-(t) : (\mathbf{u}, \mathbf{v}) &= \left(\bar{\Theta}_1, \frac{(\bar{\Theta}_1 - 1)(t - \bar{\Theta}_1)}{q - 1}\right), \\ \beta_1^+(t) : (\mathbf{u}, \mathbf{v}) &= \left(\Theta_1, \frac{(\Theta_1 - 1)(t - \Theta_1)}{q - 1}\right), \\ \beta_t^-(t) : (\mathbf{u}, \mathbf{v}) &= \left(t\Theta_t, -\frac{t(\Theta_t - 1)(t\Theta_t - 1)}{q - 1}\right), \\ \beta_t^+(t) : (\mathbf{u}, \mathbf{v}) &= \left(t\bar{\Theta}_t, -\frac{t(\bar{\Theta}_t - 1)(t\bar{\Theta}_t - 1)}{q - 1}\right), \\ \beta_\infty^-(t) : (\mathbf{u}_2, \mathbf{v}_2) &= \left(0, \frac{\bar{\Theta}_\infty - 1}{q - 1}\right), \\ \beta_\infty^+(t) : (\mathbf{u}_2, \mathbf{v}_2) &= \left(0, \frac{\Theta_\infty - q}{q(q - 1)}\right). \end{aligned}$$

Proof. — The statement can easily be verified by direct computation. Note however that this lemma can also be deduced, with much less computation, from Proposition 4.4 below. \square

In addition to the eight indeterminacy points β_i^\pm for $i \in \{0, 1, t, \infty\}$, we identify the following particular projective lines in \mathbb{F}_2 :

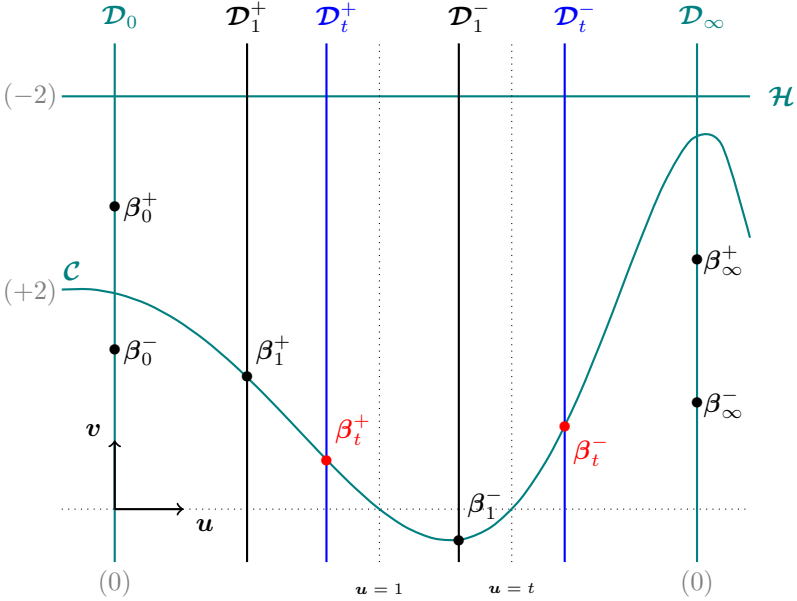
$$\begin{aligned} \mathcal{H} &:= \{\mathbf{v}_1 = 0\} \cup \{\mathbf{v}_3 = 0\}, \\ \mathcal{D}_0 &:= \{\mathbf{u} = 0\} \cup \{\mathbf{u}_1 = 0\}, & \mathcal{D}_\infty &:= \{\mathbf{u}_2 = 0\} \cup \{\mathbf{u}_3 = 0\}, \\ \mathcal{D}_1^+ &:= \{\mathbf{u} = \Theta_1\} \cup \{\mathbf{u}_1 = \Theta_1\}, & \mathcal{D}_1^- &:= \{\mathbf{u} = \bar{\Theta}_1\} \cup \{\mathbf{u}_1 = \bar{\Theta}_1\}, \\ \mathcal{D}_t^+ &:= \{\mathbf{u} = t\bar{\Theta}_t\} \cup \{\mathbf{u}_1 = t\bar{\Theta}_t\}, & \mathcal{D}_t^- &:= \{\mathbf{u} = t\Theta_t\} \cup \{\mathbf{u}_1 = t\Theta_t\}. \end{aligned}$$

Moreover, we introduce the following curve (that corresponds to $\mathcal{H}_0 : \{\mathbf{z} = 0\}$):

$$\begin{aligned} \mathcal{C} &:= \{(\mathbf{u} - 1)(\mathbf{u} - t) = (1 - q)\mathbf{v}\} \cup \{(\mathbf{u}_1 - 1)(\mathbf{u}_1 - t)\mathbf{v}_1 = 1 - q\} \\ &\quad \cup \{(1 - \mathbf{u}_3)(1 - t\mathbf{u}_3)\mathbf{v}_3 = 1 - q\}. \end{aligned}$$

The configuration of these points, lines and the curve \mathcal{C} in \mathbb{F}_2 is illustrated in the following figure. Here the grey numbers indicate the self-intersection number of the corresponding curve. The use of colors will become clear later,

see Remark 4.5.



Note that the points β_1^\pm , respectively β_t^\pm , are precisely the intersection between \mathcal{C} and \mathcal{D}_1^\pm , respectively \mathcal{C} and \mathcal{D}_t^\pm . Note further that $\mathcal{C} \cap \mathcal{H} = \emptyset$, that $\beta_0^\pm \notin \mathcal{H} \cup \mathcal{C}$ because $\Theta_0, \bar{\Theta}_0 \neq 0$ and that $\beta_\infty^\pm \notin \mathcal{H} \cup \mathcal{C}$ because $\Theta_\infty, \bar{\Theta}_\infty \neq 0$.

Now let us denote, for each $t \in \mathbb{C}^*$ satisfying (4.4), by

$$\tilde{\mathfrak{P}}_t := \text{Bl}(\mathbb{P}^1 \times \mathbb{P}^1)_{\beta_0^-(t), \beta_0^+(t), \beta_1^-(t), \beta_1^+(t), \beta_t^-(t), \beta_t^+(t), \beta_t^-(t), \beta_\infty^-(t), \beta_\infty^+(t)}$$

the blow up of \mathbb{F}_2 at the eight points $\beta_i^\pm(t)$. Here we continue to assume (4.6). In $\tilde{\mathfrak{P}}_t$, we denote by

$$\mathcal{H}, \mathcal{D}_0^{**}, \mathcal{D}_\infty^{**}, \mathcal{D}_1^{+*}, \mathcal{D}_1^{-*}, \mathcal{D}_t^{+*}, \mathcal{D}_t^{-*}, \mathcal{C}^{****}$$

the strict transforms of the corresponding projective lines/curves in \mathbb{F}_2 . We define

$$q\widetilde{\text{Oka}}_t := \tilde{\mathfrak{P}}_t \setminus \mathbf{I}^t, \text{ where } \mathbf{I}^t := \mathcal{H} \cup \mathcal{D}_0^{**} \cup \mathcal{D}_\infty^{**} \cup \mathcal{C}^{****}.$$

As we shall see, this is an alternative q -Okamoto space of initial values of qP_{VI} , and $q\widetilde{\text{Oka}}_t$ is convenient for the study of confluence. Before formulating the equivalence of $q\text{Oka}_t$ and $q\widetilde{\text{Oka}}_t$, let us give a name to the exceptional curves. We denote, for each $i \in \{0, 1, t, \infty\}$, by \mathcal{F}_i^\pm the exceptional lines in $\tilde{\mathfrak{P}}_t$ corresponding to blow up of γ_i^\pm and by \mathcal{E}_i^\pm the exceptional lines in $\tilde{\mathfrak{P}}_t$ corresponding to blow up of β_i^\pm .

PROPOSITION 4.4. — *Let $q \in \mathbb{C} \setminus \{0, 1\}$. Let $\Theta \in (\mathbb{C}^*)^4$ such that (4.6) holds. Let $t \in \mathbb{C}^* \setminus S_q$, where S_q is defined in (4.8). Consider the birational map given, with respect to the standard coordinates and the above notation, by*

$$\varphi : \begin{cases} \mathfrak{P}_t \dashrightarrow \widetilde{\mathfrak{P}}_t \\ (\mathbf{y}, \mathbf{z}) \mapsto (\mathbf{u}, \mathbf{v}) = \left(\mathbf{y}, \frac{(\mathbf{y} - t\Theta_t)(\mathbf{y} - t\bar{\Theta}_t) - q(\mathbf{y} - 1)(\mathbf{y} - t)\mathbf{z}}{q(q - 1)\mathbf{z}} \right). \end{cases}$$

This map is biregular and induces bijections

$$\mathcal{V}_t^{\pm*} \simeq \mathcal{E}_t^{\pm}, \quad \mathcal{F}_t^{\pm} \simeq \mathcal{D}_t^{\pm*} \quad \text{and} \quad \mathcal{F}_i^{\pm} \simeq \mathcal{E}_i^{\pm} \quad \forall i \in \{0, 1, \infty\}.$$

Moreover, it induces a bijection $\mathbf{J}^t \xrightarrow{\sim} \mathbf{I}^t$ and therefore provides an isomorphism

$$q\text{Oka}_t \xrightarrow{\sim} q\widetilde{\text{Oka}}_t.$$

Proof. — First, consider the rational map $\mathbb{F}_0 \dashrightarrow \mathbb{F}_2$ given, with respect to the standard coordinates, by the same formula as φ . We will abusively denote it again by φ . Note that φ preserves the fibers of the rulings $\mathbb{F}_2 \rightarrow \mathbb{P}^1$ and $\mathbb{F}_0 \rightarrow \mathbb{P}^1$ given by $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u}$ and $(\mathbf{y}, \mathbf{z}) \mapsto \mathbf{y}$. Hence we may restrict and corestrict φ to a map $\varphi^\circ : \mathbb{F}_0 \setminus \mathcal{V}_t^{\pm} \dashrightarrow \mathbb{F}_2 \setminus \mathcal{D}_t^{\pm}$. It is however immediate to check that φ° is regular and a bijection, with inverse map given by

$$\psi : \begin{cases} \mathbb{F}_2 \setminus \mathcal{D}_t^{\pm} \longrightarrow \mathbb{F}_0 \setminus \mathcal{V}_t^{\pm} \\ (\mathbf{u}, \mathbf{v}) \mapsto (\mathbf{y}, \mathbf{z}) = \left(\mathbf{u}, \frac{1}{q} \cdot \frac{(\mathbf{u} - t\Theta_t)(\mathbf{u} - t\bar{\Theta}_t)}{(\mathbf{u} - 1)(\mathbf{u} - t) + (q - 1)\mathbf{v}} \right). \end{cases}$$

Moreover, it is immediate to check that φ° maps the points γ_i^{\pm} to the points β_i^{\pm} for each $i \in \{0, 1, \infty\}$, and that it induces bijections

$$\mathcal{V}_0 \simeq \mathcal{D}_0, \quad \mathcal{V}_\infty \simeq \mathcal{D}_\infty, \quad \mathcal{H}_0 \setminus \{\gamma_t^{\pm}\} \simeq \mathcal{H}_\infty \setminus \mathcal{D}_t^{\pm}, \quad \mathcal{H} \setminus \mathcal{V}_t^{\pm} \simeq \mathcal{C} \setminus \{\beta_t^{\pm}\}.$$

To conclude, we use the same argument as in the proof of Proposition 4.1. Namely, in a neighborhood of the fibers \mathcal{V}_t^{\pm} and \mathcal{D}_t^{\pm} , φ is an elementary transformation blowing up the point γ_t^{\pm} and contracting the strict transform of the fiber \mathcal{V}_t^{\pm} onto the point β_t^{\pm} . Therefore, the induced map $\varphi : \text{Bl}(\mathbb{F}_0)_{\gamma_t^{\pm}} \rightarrow \text{Bl}(\mathbb{F}_2)_{\beta_t^{\pm}}$ is biregular. The result follows. \square

Remark 4.5. — As shown in the above proof, the rational map $\varphi : \mathbb{F}_2 \dashrightarrow \mathbb{F}_0$ corresponding to the change of variable (4.9) composed with the change of variable (3.7) (which relates $qP_{\text{VI}}(\Theta)$ to the modified $q\widetilde{P}_{\text{VI}}(\Theta)$), respects the scheme of colors in the diagrams representing the particular lines in \mathbb{F}_2 and \mathbb{F}_0 .

4.3. Confluence

In this section, we will see that the differential Okamoto space can be obtained from the second version of the q -difference one by a limit process. More precisely, we will show that they smoothly fit together into a family of Okamoto-spaces, parametrized by a neighborhood of $q = 1$ in \mathbb{C} .

Let $\Theta = (\theta_0, \theta_1, \theta_t, \theta_\infty) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times (\mathbb{C} \setminus \{1\})$, and consider, as in Section 3.3, the quadrupel of rational functions

$$\Theta(q) = 1 + \frac{q-1}{2}\Theta. \tag{4.12}$$

Let $t \in \mathbb{C}^*$. Let us consider the second Hirzebruch surface \mathbb{F}_2 with coordinates $(\mathbf{u}, \mathbf{v}), (\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2), (\mathbf{u}_3, \mathbf{v}_3)$ as in (4.10). For each $i \in \{0, 1, t, \infty\}$, we may define meromorphic functions in the variable $q \in \mathbb{C}$, holomorphic in a neighborhood of $\{q = 1\}$, of the form

$$\beta_i^\pm : \mathbb{C} \rightarrow \mathbb{F}_2,$$

given, for each fixed $q \in \mathbb{C}$, as follows.

$$\begin{aligned} \beta_0^-(q) : (\mathbf{u}, \mathbf{v}) &= \left(0, \frac{-t\theta_0/2}{\Theta_0(q)}\right), \\ \beta_0^+(q) : (\mathbf{u}, \mathbf{v}) &= \left(0, \frac{t\theta_0}{2}\right), \\ \beta_1^-(q) : (\mathbf{u}, \mathbf{v}) &= \left(\frac{1}{\Theta_1(q)}, \frac{-\theta_1/2 \left(t - \frac{1}{\Theta_1(q)}\right)}{\Theta_1(q)}\right), \\ \beta_1^+(q) : (\mathbf{u}, \mathbf{v}) &= \left(\Theta_1(q), \frac{\theta_1(t - \Theta_1(q))}{2}\right), \\ \beta_t^-(q) : (\mathbf{u}, \mathbf{v}) &= \left(t\Theta_t(q), -\frac{t\theta_t}{2} (t\Theta_t(q) - 1)\right), \\ \beta_t^+(q) : (\mathbf{u}, \mathbf{v}) &= \left(\frac{t}{\Theta_t(q)}, \frac{t(t - \Theta_t(q))\theta_t/2}{\Theta_t(q)^2}\right), \\ \beta_\infty^-(q) : (\mathbf{u}_2, \mathbf{v}_2) &= \left(0, \frac{-\theta_\infty/2}{\Theta_\infty(q)}\right), \\ \beta_\infty^+(q) : (\mathbf{u}_2, \mathbf{v}_2) &= \left(0, \frac{\theta_\infty/2 - 1}{q}\right). \end{aligned}$$

Note that on the one hand, for generic values of q and t , these correspond in the confluence setting (4.12) to the values of the β_i^\pm in Lemma 4.3. On the other hand, for $q = 1$ and $t \neq 1$, we have $\beta_i^\pm(1) = \beta_i^\pm$ with β_i^\pm as

in Section 4.1. Now we may see these functions as the parametrized curves $\{(q, \beta_i^\pm) \mid q \in \mathbb{C}^*\}$ in the product

$$\mathbb{C}^* \times \mathbb{F}_2.$$

We will abusively call them the curves β_i^\pm . Moreover, in this product space, we may identify the following planes

$$\begin{aligned} \mathcal{H} &:= \{v_1 = 0\} \cup \{v_3 = 0\}, \\ \mathcal{D}_0 &:= \{u = 0\} \cup \{u_1 = 0\}, \quad \mathcal{D}_\infty := \{u_2 = 0\} \cup \{u_3 = 0\} \end{aligned}$$

and the surface

$$\begin{aligned} \mathcal{C} &:= \{(u-1)(u-t) = (1-q)v\} \cup \{(u_1-1)(u_1-t)v_1 = 1-q\} \\ &\quad \cup \{(1-u_3)(1-tu_3)v_3 = 1-q\}. \end{aligned}$$

Note that the curve defined by the restriction of \mathcal{C} to $\{q = 1\}$ is degenerate: it has three irreducible components, given, with respect to the notation in Section 4.1, by \mathcal{H} , \mathcal{D}_1 and \mathcal{D}_t respectively. Now denote by

$$\Omega_t := \text{Bl}(\mathbb{C}^* \times \mathbb{F}_2)_{\beta_i^\pm} \setminus \mathcal{J} \quad \text{where} \quad \mathcal{J} := \mathcal{H} \cup \mathcal{C}^{****} \cup \mathcal{D}_0^{**} \cup \mathcal{D}_\infty^{**}$$

the blow up of this product space along the eight curves β_i^\pm minus the strict transforms of the mentioned particular surfaces.⁽¹⁾ Then we have

$$\begin{cases} \Omega_t|_{q=1} = \text{Oka}_t & \text{for all } t \in \mathbb{C} \setminus \{0, 1\} \\ \Omega_t|_{q=q_0} = q \widetilde{\text{Oka}}_t(q_0) & \text{for all } (t, q_0) \in \mathbb{C}^* \times (\mathbb{C} \setminus \{0, 1\}) \\ & \text{such that (4.6) holds and } t \notin S_{q_0}. \end{cases}$$

Recall that S_{q_0} was defined in (4.8). Let us see how these two conditions of validity fit together in the family

$$\Omega := \bigcup_{t \in \mathbb{C} \setminus \{0, 1\}} \Omega_t.$$

The condition (4.6) is vacuous for q_0 sufficiently close to 1, because we assumed (4.2). However, in the parameter space $\mathbb{C}^* \times \mathbb{C}^*$ with coordinates (t, q) of Ω , the set $\{(t, q) \mid t \in S_q, q \neq 1\}$ decomposes as an infinite union of curves of the form

$$\{\Theta_1^{\varepsilon_1} \Theta_t^{\varepsilon_t} q^k \mid \varepsilon_0, \varepsilon_1 \in \{-1, 1\}, k \in \mathbb{Z}\} \quad \text{and} \quad \{\Theta_0^{\varepsilon_0} \Theta_\infty^{\varepsilon_\infty} q^k \mid \varepsilon_t, \varepsilon_\infty \in \{-1, 1\}, k \in \mathbb{Z}\}.$$

By (4.12), the adherence of each of these curves at $\{q = 1\}$ is given by $\{q = 1, t = 1\}$.

⁽¹⁾ Strictly speaking, here one has to choose an order for the eight curves to be blown up in order to obtain a well-defined result in restriction to those $q \in \mathbb{C}^*$ where the curves intersect. We will however neglect these values of q anyway afterwards, because they are not close to 1.

Remark 4.6. — The behavior of the whole set S_q when q goes to 1 may be wild. This can be fixed as in [24], by requiring that q tends to 1 along a q -spiral. More precisely, let $t \in \mathbb{C} \setminus \{0, 1\}$ and fix $|q_0| > 1$ with $t \notin q_0^{\mathbb{R}}$. We have $q_0^\varepsilon \rightarrow 1$, when $\varepsilon > 0$ is a real number going to 0 and for $\varepsilon > 0$ sufficiently close to 0, we find $t \notin S_{q_0^\varepsilon}$.

5. Appendix: The relation between two notions of q -isomonodromy

Some authors interpret the pseudo-constancy of the Birkhoff connection matrix as a suitable discrete analogue for the isomonodromy of families of Fuchsian systems. We will explain here how this is related to our notion of q -isomonodromy (see Section 2.2). This Birkhoff connection matrix is defined *via* certain fundamental solutions of the family of q -Fuchsian systems parameterized by $t \in \mathfrak{D}$. Therefore, we shall first recall from [24] the construction of fundamental solutions (see also [4, 21, 22] for constructions in some more general settings). Note that these fundamental solutions will be meromorphic matrix functions on $\mathbb{C}^* \times \mathfrak{D}$. In particular, they are uniform in t , which is one of the reasons why the definition of monodromy in the differential case should not be translated literally to the q -difference setting.

Let q be a complex number with $|q| > 1$. Let \mathfrak{D} be open connected subset of \mathbb{C}^* . Let

$$\sigma_{q,x} Y(x, t) = \mathfrak{A}(x, t) Y(x, t),$$

$$\text{with } \mathfrak{A}(x, t) = \mathfrak{A}_0(t) + x \frac{\mathfrak{A}_1(t)}{x-1} + x \frac{\mathfrak{A}_t(t)}{t(x-t)} \quad (5.1)$$

be a family of q -Fuchsian systems as in Definition 1.6 with *non-resonant* spectral data $(\Theta, \bar{\Theta})$. Note that in particular, we assume that for each $i \in \{0, 1, t\}$, we have $\mathfrak{A}_i \in M_2(\mathcal{O}(\mathfrak{D}))$, i.e. these matrices have holomorphic entries. Moreover, by the requirements of Definition 1.6,

$$\text{Spec}(\mathfrak{A}_0(t)) = \{\bar{\Theta}_0, \Theta_0\} \quad \text{and} \quad \mathfrak{A}_\infty = \begin{pmatrix} \bar{\Theta}_\infty & 0 \\ 0 & \Theta_\infty \end{pmatrix},$$

where $\mathfrak{A}_\infty = \mathfrak{A}_0 + \mathfrak{A}_1 + \frac{\mathfrak{A}_t}{t}$. Note that we may have $\bar{\Theta}_0 = \Theta_0$ so that the matrix $\mathfrak{A}_0(t)$ may be not diagonalisable. Let us choose $P \in \text{GL}_2(\mathcal{M}(\mathfrak{D}))$ such that

$$P(t)\mathfrak{A}_0(t)P(t)^{-1} = J_0, \quad (5.2)$$

where J_0 is in Jordan normal form, with eigenvalues $\bar{\Theta}_0, \Theta_0$.

LEMMA 5.1. — *For each $i \in \{0, \infty\}$, there exists a matrix $L_i(x) \in \mathrm{GL}_2(\mathcal{O}(\mathbb{C}^*))$ satisfying the q -difference equation*

$$\sigma_{q,x}L_0 = J_0L_0, \quad \sigma_{q,x}L_\infty = \begin{pmatrix} \bar{\Theta}_\infty & 0 \\ 0 & \Theta_\infty \end{pmatrix} L_\infty.$$

Proof. — As explained for example in [24, p. 1024], for any $a \in \mathbb{C}$, there exists a meromorphic function $\epsilon_{q,a}(x)$ on \mathbb{C}^* satisfying the q -difference equation

$$\sigma_{q,x}\epsilon_{q,a} = a\epsilon_{q,a}.$$

Indeed, one may set $\epsilon_{q,a}(x) := \frac{\vartheta_q(x)}{\vartheta_q(x/a)}$, where $\vartheta_q(x) = \sum_{n \in \mathbb{Z}} q^{\frac{-n(n+1)}{2}} x^n$ is the Jacobi theta function satisfying $\sigma_{q,x}\vartheta_q(x) = x\vartheta_q(x)$. We may now choose $L_\infty(x) := \mathrm{diag}\left(\epsilon_{q,\bar{\Theta}_\infty}(x), \epsilon_{q,\Theta_\infty}(x)\right)$. If J_0 is diagonal, the construction of L_0 is similar. Let us assume that $J_0 = \begin{pmatrix} \Theta_0 & 1 \\ 0 & \Theta_0 \end{pmatrix}$ is not diagonal. Let us introduce the q -logarithm $\ell_q(x) = \frac{x\partial_x\vartheta_q(x)}{\vartheta_q(x)}$, that satisfies $\sigma_{q,x}(\ell_q) = \ell_q + 1$. Then, we may take

$$L_0(x) = \begin{pmatrix} \epsilon_{q,\Theta_0}(x) & \frac{\epsilon_{q,\Theta_0}(x)\ell_q(x)}{\Theta_0} \\ 0 & \epsilon_{q,\Theta_0}(x) \end{pmatrix}. \quad \square$$

Remark 5.2. — Of course the choice of the matrices L_i in the above lemma is not unique. Moreover, some authors prefer to replace $\epsilon_{q,a}(x)$ and $\ell_q(x)$ respectively by

$$a^{\frac{\ln(x)}{\ln(q)}} \quad \text{and} \quad \frac{\ln(x)}{\ln(q)},$$

which satisfy the same q -difference equations, and yield matrices L_i whose entries are defined no longer on \mathbb{C}^* , but on the Riemann surface of the complex logarithm.

PROPOSITION 5.3. — *Let L_0, L_∞ be as in Lemma 5.1. Let $P \in \mathrm{GL}_2(\mathcal{M}(\mathfrak{D}))$ such that (5.2) holds. There exists a unique pair $(\mathfrak{H}_0, \mathfrak{H}_\infty)$ of meromorphic matrix functions satisfying the following:*

- $\mathfrak{H}_0(x, t), \mathfrak{H}_\infty(x^{-1}, t) \in \mathrm{GL}_2(\mathcal{M}(\mathbb{C} \times \mathfrak{D}))$,
- $\mathfrak{H}_0(0, t) = \mathfrak{H}_\infty(\infty, t) = \mathrm{I}_2$,
- the matrix functions $\mathfrak{U}_0, \mathfrak{U}_\infty \in \mathrm{GL}_2(\mathcal{M}(\mathbb{C}^* \times \mathfrak{D}))$ defined by

$$\mathfrak{U}_0(x, t) := \mathfrak{H}_0(x, t)P(t)^{-1}L_0(x) \quad \mathfrak{U}_\infty(x, t) := \mathfrak{H}_\infty(x, t)L_\infty(x)$$

are both solutions of (5.1).

Proof. — We will closely follow [24, p. 1034], where an analogous result for fixed t has been established, but we also need to take the t -dependency

into account. We focus on the existence of \mathfrak{H}_0 as in the statement; the construction of \mathfrak{H}_∞ is analogous. The change of variable $Y = \mathfrak{H}_0 P^{-1} L_0$ leads us to the q -difference equation

$$\sigma_{q,x}(\mathfrak{H}_0(x,t))\mathfrak{A}_0(t) = \mathfrak{A}(x,t)\mathfrak{H}_0(x,t). \quad (5.3)$$

It suffices to show that for each $t_0 \in \mathfrak{D}$, there exists a unique germ of holomorphic solution \mathfrak{H}_0 of (5.3) with $\mathfrak{H}_0(0,t) = \mathbf{I}_2$ defined in a neighborhood $U \times \Delta$ of $(x,t) = (0,t_0) \in \mathbb{C} \times \mathfrak{D}$. Indeed, the functional equation (5.3) then allows to extend this holomorphic solution to a meromorphic solution on $\mathbb{C} \times \Delta$. By uniqueness, we obtain a unique meromorphic solution on $\mathbb{C} \times \mathfrak{D}$.

Let $t_0 \in \mathfrak{D}$ and let $\Delta \subset \mathfrak{D}$ be a sufficiently small disc with center t_0 . Recall that $\mathfrak{D} \subset \mathbb{C}^*$, so that we may assume there exists some $R \in]0, 1[$ such that $|t| > R$ for each $t \in \Delta$. Let $\mathbb{D}(0; R) = \{x \in \mathbb{C} \mid |x| < R\}$. Moreover, we may assume that on $\mathbb{D}(0; R) \times \Delta$, the matrix function $\mathfrak{A}(x,t)$ is holomorphic and given as the sum of a normally convergent series $\sum_{i,j \geq 0} A_{ij} x^i (t - t_0)^j$ with $A_{ij} \in M_2(\mathbb{C})$. In particular, on this product we may write $\mathfrak{A}(x,t) = \sum_{i,j \geq 0} A_i(t) x^i$ with $A_i(t) \in M_2(\mathcal{O}_b(\Delta))$, such that the power series

$$\alpha(x) := \sum_{i \geq 0} \alpha_i x^i, \quad \text{where} \quad \alpha_i := \|A_i(t)\|_\infty := \sup_{t \in \Delta} \|A_i(t)\|,$$

converges on $\mathbb{D}(0; R)$. Here $\|\cdot\|$ is some submultiplicative norm on $M_2(\mathbb{C})$ and $\mathcal{O}_b(\Delta)$ denotes the ring of uniformly bounded holomorphic functions on Δ . For $\lambda \in \mathbb{C}^*$, let us consider the map

$$\Psi_\lambda : \begin{cases} M_2(\mathcal{O}_b(\Delta)) & \longrightarrow M_2(\mathcal{O}_b(\Delta)) \\ X & \longmapsto \lambda X \mathfrak{A}_0(t) - \mathfrak{A}_0(t) X. \end{cases}$$

As we may see for instance in [24, p. 1033], the set of eigenvalues of Ψ_λ is given by

$$\text{Spec}(\Psi_\lambda) = \{\lambda \Theta_0 - \Theta_0, \lambda \bar{\Theta}_0 - \bar{\Theta}_0, \lambda \bar{\Theta}_0 - \Theta_0, \lambda \bar{\Theta}_0 - \bar{\Theta}_0\}.$$

In particular, by the non-resonant assumption, for every $n \in \mathbb{N}_{>0}$, the endomorphism Ψ_{q^n} is invertible. If we write $\mathfrak{H}_0(x,t) = \sum_{i=0}^\infty H_i(t) x^i$, with $H_0(t) = \mathbf{I}_2$, then equation (5.3) is formally equivalent to

$$\forall n \in \mathbb{N}_{>0}, \quad H_n(t) = \Psi_{q^n}^{-1} \left(\sum_{i=1}^n A_i(t) H_{n-i}(t) \right).$$

In particular, $H_n(t)$ is uniquely determined from the lower order terms. As shown in [24, p. 1034], the inverse endomorphism $\Psi_{q^n}^{-1}$ tends to zero as $n \rightarrow \infty$, and there exists a bound β such that

$$\forall n \in \mathbb{N}, \quad \forall X \in M_2(\mathcal{O}_b(\Delta)), \quad \|\Psi_{q^n}^{-1}(X)\|_\infty \leq \beta \|X\|_\infty.$$

By induction, one deduces that for each $n \in \mathbb{N}$, the value of $\|H_n(t)\|_\infty$ is less or equal to the n -th coefficient in the power series expansion of

$$\frac{\|H_0(t)\|_\infty}{1 - \beta \sum_{k=1}^{\infty} \alpha_k x^k}.$$

Since this power series has positive radius of convergence, we obtain that $\mathfrak{H}_0(x, t)$ is holomorphic on $U \times \Delta$, where U is a neighborhood of $x = 0$ in \mathbb{C} . \square

Remark 5.4. — We may also solve order one equations having only meromorphic coefficients. More precisely, let $\mathbb{C}(\{x\})$ be the field of germs of meromorphic functions at $x = 0$. Let $0 \neq c \in \mathbb{C}(\{x\})$, let v be its valuation, and let $c_0 \in \mathbb{C}^*$ such that $c = c_0 x^v + \dots$. By [24, p. 1034], there exists $0 \neq \mathfrak{m} \in \mathbb{C}(\{x\})$ solution of $\sigma_{q,x} \mathfrak{m} = c c_0^{-1} x^{-v} \mathfrak{m}$. Consider the Jacobi theta function ϑ_q and \mathfrak{e}_{q,c_0} , that are defined in the proof of Lemma 5.1. Then, $\mathfrak{e}_{q,c_0} \vartheta_q^v \mathfrak{m}$ satisfies $\sigma_{q,x}(\mathfrak{e}_{q,c_0} \vartheta_q^v \mathfrak{m}) = c \mathfrak{e}_{q,c_0} \vartheta_q^v \mathfrak{m}$ and is meromorphic on a punctured neighborhood of 0 in \mathbb{C}^* .

PROPOSITION 5.5. — *Let \mathfrak{U}_∞ be as in Proposition 5.3. The following are equivalent.*

- (1) *The family (5.1) is q -Schlesinger isomonodromic.*
- (2) *The matrix $\mathfrak{B}_\infty := \sigma_{q,t} \mathfrak{U}_\infty \cdot \mathfrak{U}_\infty^{-1} \in \mathrm{GL}_2(\mathcal{M}(\mathbb{C}^* \times \mathfrak{D}))$ is rational in x :*

$$\mathfrak{B}_\infty \in \mathrm{GL}_2(\mathcal{M}(\mathfrak{D})(x)).$$

Proof. — By definition, we have $\mathfrak{B}_\infty = \sigma_{q,t} \mathfrak{H}_\infty \cdot \sigma_{q,t} L_\infty \cdot L_\infty^{-1} \cdot \mathfrak{H}_\infty^{-1}$. Since L_∞ does not depend on t , we actually have $\mathfrak{B}_\infty = \sigma_{q,t} \mathfrak{H}_\infty \cdot \mathfrak{H}_\infty^{-1} \in \mathrm{GL}_2(\mathcal{M}(\mathbb{P}^1 \setminus \{0\}) \times \mathfrak{D})$ and $\mathfrak{B}_\infty(\infty, t) = \mathrm{I}_2$. Moreover, using $\sigma_{q,x} \sigma_{q,t} \mathfrak{U}_\infty = \sigma_{q,t} \sigma_{q,x} \mathfrak{U}_\infty$ and the definition of \mathfrak{B}_∞ , we find that $X = \mathfrak{B}_\infty$ solves the q -difference equation

$$\sigma_{q,t} \mathfrak{A} \cdot X = \sigma_{q,x} X \cdot \mathfrak{A}. \tag{5.4}$$

It follows from [24, §1.1.3] and the non-resonant condition on the eigenvalues of \mathfrak{A}_∞ , that this q -difference equation (5.4) admits a unique formal solution $X = \sum_{n \geq 0} \xi_n(t) x^{-n} \in \mathrm{GL}_2(\mathcal{M}(\mathfrak{D})[[x^{-1}]])$ with $\xi_0 = \mathrm{I}_2$. Hence the power series expansion of \mathfrak{B}_∞ at $x = \infty$ coincides with this unique formal solution.

On the other hand, by Proposition 2.9, the (non-resonant) family (5.1) is q -Schlesinger isomonodromic if and only if there exists a solution $\mathfrak{B} \in \mathrm{GL}_2(\mathcal{M}(\mathfrak{D})(x))$ of the q -Lax equation (5.4) which satisfies $\mathfrak{B}(\infty, t) = \mathrm{I}_2$. By uniqueness of the formal solution of (5.4), the equality holds $\mathfrak{B} = \mathfrak{B}_\infty$. The result follows. \square

Given $\mathfrak{U}_0, \mathfrak{U}_\infty$ as in Proposition 5.3, we may define the *Birkhoff connection matrix* $\mathfrak{P}(x, t)$ by

$$\mathfrak{P} := \mathfrak{U}_\infty^{-1} \cdot \mathfrak{U}_0 \in \mathrm{GL}_2(\mathcal{M}(\mathbb{C}^* \times \mathfrak{D})). \tag{5.5}$$

PROPOSITION 5.6. — *Let $\mathfrak{U}_0, \mathfrak{U}_\infty, \mathfrak{P}$ be as above. The following are equivalent.*

- (1) *The Birkhoff connection matrix \mathfrak{P} is pseudo constant, i.e. $\sigma_{q,t}\mathfrak{P} = \mathfrak{P}$.*
- (2) *For $\mathfrak{B}_0, \mathfrak{B}_\infty \in \mathrm{GL}_2(\mathcal{M}(\mathbb{C}^* \times \mathfrak{D}))$ defined by $\mathfrak{B}_i := \sigma_{q,t}\mathfrak{U}_i \cdot \mathfrak{U}_i^{-1}$, we have $\mathfrak{B}_0 = \mathfrak{B}_\infty$.*

Proof. — We have

$$\begin{aligned} \mathfrak{P}^{-1} \cdot \sigma_{q,t}\mathfrak{P} &= \mathfrak{U}_0^{-1} \cdot \mathfrak{U}_\infty \cdot \sigma_{q,t}\mathfrak{U}_\infty^{-1} \cdot \sigma_{q,t}\mathfrak{U}_0 \\ &= \mathfrak{U}_0^{-1} \cdot \mathfrak{B}_\infty^{-1} \cdot \sigma_{q,t}\mathfrak{U}_0 \\ &= \mathfrak{U}_0^{-1} \cdot \mathfrak{B}_\infty^{-1} \cdot \mathfrak{B}_0 \cdot \mathfrak{U}_0. \end{aligned}$$

The result follows. □

COROLLARY 5.7. — *If the Birkhoff connection matrix \mathfrak{P} given in (5.5) is pseudo constant, then the family (5.1) is q -Schlesinger isomonodromic.*

Proof. — Recall from the proof of Proposition 5.5 that $\mathfrak{B}_\infty = \sigma_{q,t}\mathfrak{H}_\infty \cdot \mathfrak{H}_\infty^{-1}$. Similarly, we obtain $\mathfrak{B}_0 = \sigma_{q,t}\mathfrak{H}_0 \cdot \sigma_{q,t}P^{-1} \cdot P \cdot \mathfrak{H}_0^{-1}$, where P is as in (5.2). Note that $\mathfrak{B}_\infty(\infty, t) = \mathrm{I}_2$ and $\mathfrak{B}_0(0, t) = P(qt)^{-1}P(t)$. By assumption and Proposition 5.6, we have $\mathfrak{B}_\infty = \mathfrak{B}_0$. It follows that $\mathfrak{B}_\infty \in \mathrm{GL}_2(\mathcal{M}(\mathbb{C}^* \times \mathfrak{D}))$ can be meromorphically continued to $x = 0$ and $x = \infty$. Hence \mathfrak{B}_∞ is rational in x . We conclude by Proposition 5.5. □

The above corollary establishes the sought relation between the two notions of q -isomonodromy (in the non-resonant case with $|q| > 1$): pseudo-constancy of the Birkhoff connection matrix is a stronger requirement than q -isomonodromy as in Section 2.2. Note that the Birkhoff connection matrix in (5.5) is not canonically defined: it depends on the choices of L_0, L_∞ and P . However, for any choice, its pseudo-constancy implies q -Schlesinger isomonodromy. As a final remark, we indicate that for example in [6, 12], another type of (again non-canonical) Birkhoff connection matrix has been considered, namely

$$\tilde{\mathfrak{P}} := \mathfrak{U}_\infty^{-1} \cdot \tilde{\mathfrak{U}}_0,$$

with $\tilde{\mathfrak{U}}_0 := \mathfrak{U}_0 \cdot P$. It has been shown in [6, Proposition 1], see also [12, Thm. 3], that the analogue of Proposition 5.6 for $\tilde{\mathfrak{U}}_0, \mathfrak{U}_\infty, \tilde{\mathfrak{P}}$ holds under the additional assumption that $\mathfrak{A}_0(t)$ is either constant or proportional to t . The choice of \mathfrak{U}_0 in the above exposition was made to circumvent this assumption.

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