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ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF A STRONGLY NONLINEAR PARABOLIC PROBLEM

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Résumé : Nous étudions le problème d'évolution $u_t + Au = 0$ dans $(0,T) \times R^N$, $u(0) = u_o$ dans IR^N, avec $N \geqslant 1$, $0 < T \leqslant \infty$, $Au = -\operatorname{div}(|Du|^{p-2} Du)$, Du étant le gradient de u, $1 < p < \infty$ et nous supposons que u_o appartient à un espace de fonctions intégrables. On prouve l'existence d'un temps fini d'extinction si $N \geqslant 2$ et $p < \frac{2N}{N+1}$. Dans le cas contraire (si $N = 1$ et $p > 1$ ou si $N \geqslant 2$ et $p \geqslant \frac{2N}{N+1}$) on prouve la loi de conservation : $\int_{IR^N} u(t,x)dx = \int_{IR^N} u_o(x)dx$ pour tout $t > 0$. On estime aussi la convergence vers zéro des intégrales $\int_{IR^N} |u(t,x)|^m dx$, $m > 1$ et on obtient certains effets régularisants.

Summary : The evolution problem $u_t + Au = 0$ in $(0,T) \times IR^N$, $u(0) = u_o$ in IR^N is considered where $N \geqslant 1$, $0 < T \leqslant \infty$, $Au = -\operatorname{div}(|Du|^{p-2} Du)$, with Du the gradient of u, $1 < p < \infty$ and u is supposed to belong to some integrable space. If $N \geqslant 2$ and $p < \frac{2N}{N+1}$ the existence of a finite extinction time is shown. On the contrary, if $N = 1$, $p > 1$ or $N \geqslant 2$, $p \geqslant \frac{2N}{N+1}$ conservation of total mass holds, i.e. $\int_{IR^N} u(t,x)dx = \int_{IR^N} u_o(x)dx$ for every $t > 0$. We prove also that the integrals $\int_{IR^N} |u(t,x)|^m dx$, $m > 1$ converge to zero as t goes to infinity, and some regularizing effects are shown.

INTRODUCTION AND PRELIMINARIES

We shall consider the asymptotic behaviour in time of the solutions of

$$(P)\quad \begin{cases} u_t + Au = 0 & \text{in} \quad (0,T) \times \mathrm{IR}^N \\ u(0) = u_o & \text{in} \quad \mathrm{IR}^N \end{cases}$$

with $N \geqslant 1$, $1 < p < \infty$ and $Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(|Du|^{p-2}Du)$ where $Du = \left(\frac{\partial u}{\partial x_i}\right)_i$ is the gradient of u. The operator A has been widely considered in the literature in P.D.E., and arises in several physical situations, such as one-dimensional non newtonian fluids and glaciology.

This behaviour depends strongly on p and N : in fact, if $p \geqslant \frac{2N}{N+1}$ we show that the total mass $\int_{\mathrm{IR}^N} u(t,x)dx$ is conserved, i.e, is independent of time. On the contrary if $p < \frac{2N}{N+1}$ we show that the solution corresponding to initial data $u_o \in L^m(\mathrm{IR}^N)$, $m = N(\frac{2}{p} - 1)$ vanishes in finite time. The existence of a finite extinction time was found by Bénilan and Crandall [2] for the equation (E) $u_t - \Delta u^m = 0$ in spatial domain IR^N (1) if and only if $0 < m < \frac{N-2}{N}$, $N \geqslant 3$. As it is noted in [2], equation (E) in bounded domains with homogeneous Dirichlet conditions has also that property if $0 < m < 1$. The case N = 1 was considered by Sabinina [8]. Several properties of solutions of (E) related to the ones we consider here can be found in Evans [5]. Finite extinction times for (E_β) $u_t - \Delta\beta(u) = 0$ with β maximal monotone graph and bounded domain are discussed in terms of β in [3].

We also consider the homogeneous Dirichlet problem

$$(P_\Omega)\quad \begin{cases} u_t - \operatorname{div}(|Du|^{p-2}Du) = 0 & \text{in} \quad (0,T) \times \Omega \\ u(x,t) = 0 & \text{in} \quad (0,T) \times \partial\Omega \\ u(x,0) = u_o(x) & \text{in} \quad \Omega \end{cases}$$

for $\Omega \subset \mathrm{IR}^N$ open and bounded. We show the existence of a finite extinction time if $p < 2$, $u_o \in L^m(\Omega)$, and m as above, completing a result of Bamberger [1] : he showed that effect for $\frac{2N}{N+2} \leqslant p < 2$ and $u_o \in L^2(\Omega)$. For $p \geqslant 2$ it is easy to see that solutions with positive initial data do not vanish.

For $p > \frac{2N}{N+1}$ L. Véron [11] shows a smoothing and decay effect for the solutions

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(1) with $u_o \in L^\beta(\mathrm{IR}^N) \cap L^1(\mathrm{IR}^N)$ for a $\beta = \beta(m,N)$.

of (P_Ω) : in fact, if $N(\frac{2}{p}-1) < m_o < m \leqslant \infty$ and $u_o \in L^{m_o}(\Omega)$, then $u(t,.) \in L^m(\Omega)$ and in addition $\| u \| \leqslant Ct^{-\delta} . \| u_o \|^{\sigma}_{m_o}$ where δ, σ depend on m,m_o,p and N. We adapt his proof for (P) to get similar results. We know that for $m_o = N(\frac{2}{p}-1)$ solutions vanish. For $1 < m_o < N(\frac{2}{p}-1)$ we prove a «backwards» effect : for $t > 0$, $u(t,.) \in L^1(IR^N)$ and $\| u \|_1 \leqslant Ct^{-\delta} \| u_o \|^{\sigma}_{m_o}$ with $\delta, \sigma > 0$ as before.

We shall need some facts about the operator A in IR^N and in $\Omega \subset IR^N$ bounded with homogeneous Dirichlet conditions : First, if $J(u) = \frac{1}{p}\int_{IR^N} | Du |^p$ when $u \in L^2(IR^N)$ and $| Du | \in L^p(IR^N)$, $J(u) = +\infty$ otherwise, J is a convex l.s.c. proper functional in $L^2(IR^N)$ whose subdifferential A is defined as $Au = -\operatorname{div}(| Du |^{p-2} Du)$ in the domain $D(A) = \left\{ u \in L^2(IR^N) : | Du | \in L^p(IR^N), \operatorname{div}(| Du |^{p-2} Du) \in L^2(IR^N) \text{ and for every } v \in D(J), \int_{IR^N} Au . v = \int_{IR^N} | Du |^{p-2} Du . Dv \right\}$. If $p \geqslant 2$, the last condition may be omitted as it follows by density. A is accretive in $L^1(IR^N)$ and $L^\infty(IR^N)$, hence in every $L^p(IR^N)$, $1 \leqslant p \leqslant \infty$: in fact for $t > 0$ and $u_1,u_2 \in D(A) \cap L^p(IR^N)$, $\| [u_1(t,.) - u_2(t,.)]^+ \|_p \leqslant \| [u_1(0,.) - u_2(0,.)]^+ \|_p$ where $u^+ = \max(u,0)$. This implies a comparison principle that allows us to consider only nonnegative initial data and solutions ; for nonpositive data we consider $-u$ instead of u. Defining for $p \neq 2$, $A_p = A \cap (L^p(IR^N) \times L^p(IR^N))$ we may close A_p to find $\overline{A}_p$ m-accretive in $L^p(IR^N)$. As $\mathscr{D}(IR^N) \subset D(A_p)$, $\overline{D(A_p)}^{L^p} = L^p(IR^N)$.

The corresponding results for Ω bounded and homogeneous Dirichlet conditions are well known ; $Au = -\operatorname{div}(| Du |^{p-2} Du)$ and $D(A) = \left\{ u \in W_o^{1,p}(\Omega) \cap L^2(\Omega) : Au \in L^2(\Omega) \right\}$. On the other hand A_p is defined as m-accretive operator in $L^p(IR^N)$ by restriction if $p > 2$ and closure if $p < 2$.

We shall use the following inequality due to Nirenberg and Gagliardo (see [6] , Th. 9.3.).

LEMMA 0. *Let* q,r *be any numbers satisfyng* $1 \leqslant q,r \leqslant \infty$ *and* $u \in C_o^1(IR^N)$. *Then*

$$\| u \|_p \leqslant C \| Du \|^a_r \| u \|^{1-a}_q$$

where $\frac{1}{p} = a . \frac{1}{r^*} + (1-a)\frac{1}{q}$ *and* $\frac{1}{r^*} = \frac{1}{r} - \frac{1}{N}$ *for all a in the interval* $0 \leqslant a \leqslant 1$, *with* $C = C(N,q,r,a)$, *with the following exception :* $r = N$ *and* $a = 1$ *(hence* $p = \infty$*).*

We remark that by density the result remains true for $u \in L^q(IR^N) \cap L^p(IR^N)$ such that $Du \in L^r(IR^N)$ if $r,q < \infty$ and $q \leqslant r^*$ if r^* is positive. To show this, approach u by u^1 bounded, then convolve u^1 with a regular kernel to get $u^2 \in C^\infty(IR^N)$ and finally cut u^2 with a smooth function ζ_n which vanishes outside $B_{2n}(0)$ and is equal to 1 on $B_n(0)$; let us check this last step.

Assume $u \in C^\infty(IR^N)$ and put $u_n = u\,\zeta_n$, where $\zeta_n(x) = \zeta_o\left(\frac{|x|}{n}\right)$, $0 \leqslant \zeta_o \leqslant 1$, $\zeta_o(x) = 1$ if $|x| \leqslant 1$, $\zeta_o(x) = 0$ if $|x| \geqslant 2$ and $|D\zeta_n(x)| \leqslant C$. It is clear that $u_n \to u$ in $L^q(IR^N)$ and $L^p(IR^N)$. Also $D_{u_n} = Du\,.\,\zeta_n + u\,.\,D\zeta_n$. $Du\,.\,\zeta_n \to Du$ in $L^r(IR^N)$ and we have to prove that $u\,.\,D\zeta_n \to 0$ in $L^r(IR^N)$. Then, C representing different constants independent of n :

$$\| uD\zeta_n \|_r^r = \int_{IR^N} |u|^r \, |D\zeta_n|^r \leqslant \frac{c}{n^r} \int_{n \leqslant |x| \leqslant 2N} |u|^r$$

if $\quad q \leqslant r, \int |u|^r \leqslant \| u \|_\infty^{r-q} . \int |u|^q, \quad$ so $\quad \| uD\zeta_n \|_r^r \leqslant \dfrac{C \| u \|_\infty^{r-q}}{n^r} . \| u \|_q^q \to 0$;

if $\quad r < q \leqslant r^*, < q \leqslant r^*, \int |u|^r \leqslant \left(|u|^q \right)^{r/q} . \left(\int 1 \right)^{1-q/q}$, so :

$$\| u\, D\zeta_n \|_r^r \leqslant \frac{c}{n^r} . \| u \|^r_{L^q(n \leqslant |x| \leqslant 2n)} . n^{N(1-\frac{r}{q})} \to 0.$$

If $r^* < 0$ the previous proof applies as well for every q, $1 \leqslant q < \infty$.

Our plan is a follows : Sections 1, 2, 3 are devoted to problem (P). Section 1 studies the existence of a finite extinction time when $p < \frac{2N}{N+1}$, $u_o \in L^m(IR^N)$, $m = N(\frac{2}{p} - 1)$. Section 2 is devoted to conservation of mass and Section 3 to the regularizing effects and decay of the integral norms $\| u(t,.) \|_m$ as $t \to \infty$. Finally Section 4 gathers the results on (P_Ω), Ω open and bounded.

1. - FINITE EXTINCTION TIME

We obtain the following result

THEOREM 1. *Let* $N \geqslant 2$, $1 < p < \frac{2N}{N+1}$ *and let* $u_o \in L^m(IR^N)$ *where* $m = N(\frac{2}{p} - 1)$. *Then for every* $t > 0$ $u(t,.) \in L^\infty(IR^N)$ *and there exists* $t_o > 0$ *such taht* $u(t,.) = 0$ a.e. *if* $t \geqslant t_o$.

Proof. We may assume that $u_o(x)$, $u(t,x)$ are nonnegative. A formal proof to be justified later by discretization in time runs as follows : As $p < \frac{2N}{N+1}$ if $m = N(\frac{2}{p} - 1)$ we have $m > 1$. Let $p^* = \frac{Np}{N-p}$ and $q = \frac{m+p-2}{p}$: then $m = p^*q$. Also for $k \geqslant 0$ we write $(u-k)_+ = \max(u-k,0)$ and $v = v_k = (u-k)_+^q$. Multiply $u_t - \mathrm{div}(\,|Du|^{p-2} Du) = 0$ by $m(u-k)_+^{m-1}$ and integrate over IR^N to obtain :

$$(1.1) \quad \frac{d}{dt} \int_{IR^N} (u-k)_+^m = m \int_{IR^N} u_t (u-k)_+^{m-1} = m \int_{IR^N} \mathrm{div}(\,|Du|^{p-2} Du)(u-k)_+^{m-1}$$

Integration by parts and Sobolev's inequality give

$$(1.2)\qquad -\int_{\mathbb{R}^N} \operatorname{div}(|Du|^{p-2} Du)(u-k)_+^{m-1} = (m-1)\bar{q}^p \int_{\mathbb{R}^N} |Dv|^p \geqslant$$

$$\geqslant C_p(m-1)\bar{q}^p \left(\int_{\mathbb{R}^N} v^{p^*}\right)^{p/p^*}$$

Write $E_{m,k}(t) = \int_{\mathbb{R}^N} (u-k)_+^m \, dx$. (1.1) and (1.2) give

$$(1.3)\qquad \frac{d}{dt} E_{m,k}(t) + C_p\, m(m-1)\bar{q}^p E_{m,k}^{p/p^*}(t) \leqslant 0$$

Integrating (1.3) gives

$$(1.4)\qquad \begin{cases} E_{m,k}(t) \leqslant E_{m,k}(0) \left[1 - \dfrac{C_p m(m-1)p}{Nq^p(E_{m,k}(0))^{p/N}} \cdot t\right]^{\frac{N}{p}} & \text{for } 0 < t \leqslant t_{0,k} \\[2ex] E_{m,k}(t) = 0 & \text{for } t \geqslant t_{0,k} \end{cases}$$

where

$$t_{0,k} = \frac{N.q^p}{pC_p\, m(m-1)} E_{m,k}(0)^{p/N}$$

If we take $k = 0$ the existence of a finite extinction time $t_0 = t_{0,0}$ results. Given $\bar{t} > 0$, if we take $k > 0$ large enough extinction of $E_{m,k}(t)$ in time $t_{o,k} \leqslant \bar{t}$ may be obtained. Hence $u(t,.) \in L^\infty(\mathbb{R}^N)$ for $t > 0$, a regularizing effect.

This formal proof can be made rigorous by means of the discrete scheme and Crandall-Liggett's results. Assume that $u_o \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, let $h > 0$ and define a discrete approximation to the solution of (P) thus : $u_{i+1} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is defined implicitely in terms of u_i by

$$(1.5)\qquad \frac{u_{i+1} - u_i}{h} + Au_{i+1} = 0$$

Now repeat the previous argument on (1.5) to obtain a discrete version of (1.3) and pass to the limit as $h \to 0$. The assumption on u_o can be weakened by approximation for $t_{0,k}$ depends only on $\| u_o \|_m$. The details repeat those in [2] for $u_t - \Delta u^m = 0$ and we omit them. Only the integration by parts needs some care : if $m \geqslant 2$, $u_o \in D(A) \cap L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then

$$(1.6)\qquad -\int_{\mathbb{R}^N} Au_i\, u_i^{m-1} + (m-1) \int_{\mathbb{R}^N} |Du_i|^p\, u_i^{m-2} = 0$$

by the characterization of $D(A)$. If $m < 2$ we have to linearize the function $\phi(u) = u^{m-1}$ near the origin to apply integration by parts. Passing to the limit it follows by Fatou in this case that

$$-\int_{\mathbb{R}^N} Au_i\, u_i^{m-1} + m\int_{\mathbb{R}^N} |Du_i|^p\, u_i^{m-2} \leqslant 0 \tag{1.7}$$

For u_0 as in the theorem the result follows by density for A is accretive #

2. - MASS CONSERVATION

We say that the mass conservation law (MCL) holds for (P) if for every $t > 0$ $\int_{\mathbb{R}^N} u(t,x)dx = \int_{\mathbb{R}^N} u_0(x)dx$.

In this section the validity of MCL is discussed in terms of p :

THEOREM 2. *MCL holds for* (P) *if and only if* $N = 1,\ p > 1$ *or* $N \geqslant 2,\ p \geqslant \frac{2N}{N+1}$.

In order to prove Theorem 2 we need some previous results. A variant of the following Lemma has been used in [10] :

LEMMA 1. *Let* $\Omega \subset \mathbb{R}^N$ *be an open set and let* $u \in L^2(\mathbb{R}^N)$ *be such that* $u \in D(A)$ *and* $-Au = u$ *a.e. in* Ω. *Let* $\eta \in C^\infty(\Omega)$ *be such that* $\mathrm{supp}(\eta) \subset\subset \Omega$, $\|\eta\|_\infty = 1$ *and let* χ *be the characteristic function of* $\mathrm{supp}(D\eta)$. *Then*

$$\|\eta\, Du\|_p \leqslant p\, \|D\eta\|_\infty \,.\, \|\chi u\|_p . \tag{2.1}$$

Proof. Multiply $u = Au$ by $u\eta^p$, integrate over $\mathbb{R}^N$, integrate by parts ($u \in D(A)$) and apply Hölder's inequality.

LEMMA 2. *Let* $\frac{2N}{N+1} \leqslant p \leqslant 2$ *and let* u *be a solution of* $Au + u = f$, $f \in L^1(\mathbb{R}^N)$. *Then* $\int_{\mathbb{R}^N} Au = 0$.

Proof. By accretivity of A in $L^1(\mathbb{R}^N)$, we may restrict ourselves to consider $f \in L_0^\infty(\mathbb{R}^N)$. We obtain first an estimate for $\|Du\|_p$ over the exterior of a ball : Assume $\mathrm{supp}(f) \subset B_R(0)$ and take $n > R$. Choose $\eta_n \in C^\infty(\mathbb{R}^N)$ such that $0 \leqslant \eta_n \leqslant 1$, $\eta_n = 0$ if $|x| \leqslant n$, $\eta_n = 1$ if $|x| \geqslant 2n$ and $\|D\eta_n\|_\infty \leqslant \frac{c_1}{n}$, $c_1 > 1$. Put $A_n = \{x \in \mathbb{R}^N : n \leqslant |x| \leqslant 2n\}$ and $D_n = \{x \in \mathbb{R}^N : |x| \geqslant n\}$. Then (2.1) gives in $\Omega = \mathbb{R}^N - B_R(0)$:

$$\|Du\|_{L^p(D_n)} \leqslant \frac{C}{n}\, \|\chi u\|_{L^p(\mathbb{R}^N)} \leqslant \frac{C}{n}\, \|u\|_{L^p(A_n)} \tag{2.2}$$

Hereafter C denotes several positive constants depending only on p and N and not on n.

By virtue of [9] , Corollary 2, the following estimate applies to u(x), for $|x| > R$:

$$u(x) \leqslant C\,|x|^{-\frac{p}{2-p}} \tag{2.3}$$

Also by accretivity $\|u\|_1 \leqslant \|f\|_1$, so that

$\|u\|^p_{L^p(A_n)} \leqslant \|u\|_{L^1(A_n)} \cdot \|u\|^{p-1}_{L^\infty(An)} = o(1)^{(1)} \cdot n^{-\frac{p(p-1)}{2-p}}$. It follows that

$\|Du\|_{L^p(D_n)} = o(1) \cdot n^{-\frac{1}{2-p}}$. Putting $\zeta_n(x) = 1 - \eta_n(x)$ we have

$$\left|\int Au \cdot \zeta_n\right| \leqslant \int |Du|^{p-1}\,|D\zeta_n| \leqslant \frac{o(1)}{n} \cdot n^{-\frac{p-1}{2-p}} \cdot n^{N/p} = 0(n) \cdot n^{\frac{N}{p}-\frac{1}{2-p}} \tag{2.4}$$

Since $\int_{IR^N} Au = \lim_{n\to\infty} \int_{IR^N} Au\,\zeta_n$, the desired result follows whenever $\frac{N}{p} - \frac{1}{2-p} \leqslant 0$ i.e. $p \geqslant \frac{2N}{N+1}$ #

We say that the finite propagation property (PF) holds for (P) if for every admissible initial datum $u_o(x)$ having compact support in IR^N, the corresponding solution u(t,x) is such that for every $t > 0$ u(t,.) has compact support in IR^N. It is know that (PF) holds for (P) if and only if $p > 2$ (see [4]). There exists a simple relation between (FP) and (MCL) :

LEMMA 3. *If* $p > 2$, *then* (MCL) *holds.*

Proof. Let u(x,t) be a solution of (P) such that $u(x,0) = u_o(x)$ has compact support. If $t > 0$ we know that there exists n such that supp $u(t') \subset B_n(0)$ for $0 \leqslant t' \leqslant t$. Take ζ_n as before. Then for t' fixed :

$$\int_{IR^N} Au = \int_{IR^N} Au \cdot \zeta_n = \int_{IR^N} |Du|^{p-2}\,Du \cdot D\zeta_n = 0$$

Hence $\int_{IR^N} u_t\,dx = 0$ and it follows that $\int_{IR^N} u(t,x) = \int_{IR^N} u_o(x)$. This last assertion can be justified by means of the discrete scheme as before.

If $supp(u_o)$ is not compact, approximate u_o by $\{u_{o_n}\}$, a sequence of initial data with compact support #

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(1) Here o(1) denotes a quantity that goes to 0 as $n \to \infty$.

Proof (of Theorem 2). If $N = 1$, $p > 1$ or $N \geqslant 2$, $2 > p \geqslant \frac{2N}{N+1}$ the result follows from Lemma 2 applied to the discrete scheme

$$\frac{u_{1+1} - u_1}{h} + Au_{i+1} = 0$$

for then $\int_{IR^N} u_i = \int_{IR^N} u_{i+1}$. If $p > 2$ it follows from Lemma 3 in the same way. The case $p = 2$ is classic (and it falls within the scope of [2]).

For the negative part it is sufficient to remind Theorem 1, for (MCL) is incompatible with extinction #

3. DECAY OF THE INTEGRAL NORMS. REGULARIZING EFFECT

Our first result is the extension to IR^N of the work of L. Véron [11] for the case Ω bounded.

THEOREM 3. *Let* $p > \frac{2N}{N+m_o}$, $u_o \in L^{m_o}(IR^N)$ *with* $m_o \geqslant 1$. *If* $t > 0$, $u(t,.) \in L^m(IR^N)$ *for every* m *such that* $m_o \leqslant m \leqslant \infty$. *In addition, the following estimate holds :*

(3.1) $$\| u(t,.) \|_m \leqslant \frac{C}{t^\delta} . \| u_o \|^\sigma_{m_o}$$ for some constant $C = C(m,m_o,N,p)$, where

(3.2)
$$\begin{cases} \delta = \dfrac{N(m - m_o)}{m(m_o p + N(p-2))} \quad \text{if } m < +\infty, \ \delta = \dfrac{N}{m_o p + N(p-2)} \quad \text{if } m = +\infty. \\[2ex] \sigma = \dfrac{m_o(mp + N(p-2))}{m(m_o p + N(p-2))} \quad \text{if } m < +\infty, \ \sigma = \dfrac{m_o p}{m_o p + N(p-2))} \quad \text{if } m = +\infty. \end{cases}$$

Proof. The case $m = m_o$ follows from the accretivity property ; it suffices to show the case $m = + \infty$, the intermediate cases being obtained from these by interpolation. Assume (for simplicity) that $u \geqslant 0$; for $p \leqslant N$ we adapt the iterative procedure of L. Véron [11] as follows. Define the sequences m_n , r_n by :

(3.3) $$m_n = \gamma^n . m_o \quad \text{with} \quad 1 < \gamma < \frac{N}{N-1}, \quad m_o\left(\frac{\gamma p}{N(\gamma-1)} - 1\right) > \frac{1}{\gamma-1}$$

(3.4) $$\frac{r_n + p-2}{m_n} = \frac{r_n}{m_{n-1}} - \frac{p}{N}$$

Note that from (3.3) and (3.4) it follows :

(3.5) $$r_n = \frac{\gamma p}{N(\gamma-1)} m_{n-1} + \frac{p-2}{\gamma-1} = \frac{\gamma^n p}{N(\gamma-1)} m_o + \frac{p-2}{\gamma-1}.$$

Now we claim that, if we write $v = u^{q_{n-1}}$ with $q_n = \frac{m_n+p-2}{p}$, Nirenberg-Gagliardo's inequality applies to v. Namely one has :

(3.6) $$\| v \|_{\frac{m_n}{q_{n-1}}}^{\frac{r_n+p-2}{q_{n-1}}} \leqslant C \,.\, \| Dv \|_p^p \,.\, \| v \|_{\frac{m_{n-1}}{q_{n-1}}}^{\frac{r_n-m_{n-1}}{q_{n-1}}}$$

That is a consequence of the following facts : i) As it was pointed out in Theorem 1, we can suppose $u \in D(A) \cap L^1(IR^N) \cap L^\infty(IR^N)$ hence $v \in L^{\frac{m_n}{q_{n-1}}}(IR^N) \cap L^{\frac{m_{n-1}}{q_{n-1}}}(IR^N)$ for each $m_{n-1} > 1$, for then $\frac{m_{n-1}}{q_{n-1}}$ is always greater than one, ii) $Dv \in L^p(IR^N)$ as a consequence of the validity of formula (1.7), iii) Nirenberg-Gagliardo's inequality (Lemma 0) applies with the present regularity, as it was observed at the introduction.

We shall give a formal proof, just as at the first part of Theorem 1 (rigorous justification by means of the discrete schema approximation is made in the same way as there). Assume first $p < N$. Multiply the equation $u_t - Au = 0$ by $m_{n-1}\, u^{m_{n-1}-1}$ and integrate over IR^N to get

(3.7) $$\frac{d}{dt}\left(\int_{IR^N} v^{\frac{m_{n-1}}{q_{n-1}}}\right) + C_{m,n}\left(\int_{IR^N} | Dv |^p\right) \leqslant 0$$

Next multiply (3.7) by $\| u \|_{m_{n-1}}^{r_n-m_{n-1}}$ and use (3.6). It follows that

(3.8) $$\| u \|_{m_{n-1}}^{r_n-m_{n-1}} \,.\, \frac{d}{dt}\left(\| u \|_{m_{n-1}}^{m_{n-1}}\right) + C \,\| u \|_{m_n}^{r_n+p-2} \leqslant 0$$

where C involves $C_{m,n}$ and the constant in (3.6), which depends only on N and p. Take $t_n = t(1 - \frac{1}{2^n})$ and integrate (3.8) in $[t_{n-1},t_n]$. In this way we obtain :

(3.9) $$\| u(t_n) \|_{m_n}^{r_n+p-2} \leqslant \frac{2^n}{C.t} \| u(t_{n-1}) \|_{m_{n-1}}^{r_n}$$

The previous argument remains true if we replace u by $u_k = (u-k)_+$ for some $k > 0$. But then $|\Omega_{k,t}| = \text{meas}\,\{x : u_k(t) > 0\}$ is finite and

$$\| u_k(t) \|_\infty = \lim_{m_n \to \infty} \sup \| u_k(t) \|_{m_n} \leqslant \lim_{m_n \to \infty} \sup \| u_k(t_n) \|_{m_n}.$$

Now (3.1), (3.2) follow from two facts : a) $\lim_{m_n \to \infty} \sup \| u_k(t_n) \|_{m_n}$ can be evaluated now just in the same way as in [11], which implies estimates (3.1) (3.2) for u_k. b) These estimates do not depend on k, and consequently we can pass to the limit and obtain the desired results for k = 0.

When p = N, choose $\{\beta_n\}$ such that

$$\beta_n = q_n + m_n \left(1 - \frac{1}{N}\right) \tag{3.10}$$

Write $w_{n-1} = u^{\beta_{n-1}}$. Then $D\left(u^{q_{n-1}}\right) = \frac{q_{n-1}}{\beta_{n-1}} \cdot w_{n-1}^{\frac{q_{n-1}-\beta_{n-1}}{\beta_{n-1}}} \cdot Dw_{n-1}$, i.e,

$Dw_{n-1} = \frac{\beta_{n-1}}{q_{n-1}} \cdot D\left(u^{q_{n-1}}\right) \cdot w^{\frac{\beta_{n-1}-q_{n-1}}{\beta_{n-1}}}$. Now by Hölder

$$\left(\int_{IR^N} | Dw_{n-1} |\right)^N \leqslant \left(\frac{\beta_{n-1}}{q_{n-1}}\right)^N \cdot \left(\int_{IR^N} | D\left(u^{q_{n-1}}\right) |^N\right) \cdot \left(\int_{IR^N} u^{m_{n-1}}\right)^{N-1} \tag{3.11}$$

On the other hand, by Sobolev

$$\left(\int_{IR^N} | Dw_{n-1} |\right)^N \geqslant C_N \left(\int_{IR^N} u^{\frac{N\beta_{n-1}}{N-1}}\right)^{N-1} \tag{3.12}$$

Now multiply (3.7) by $\| u \|_{m_{n-1}}^{r_n - m_{n-1}}$, use (3.11), (3.12) and a standard interpolation argument to get :

$$\| u \|_{m_{n-1}}^{r_n - m_{n-1}} \frac{d}{dt} \left[\| u \|_{m_{n-1}}^{m_{n-1}} \right] + C \cdot \| u \|_{m_n}^{r_n + N - 2} \leqslant 0 \tag{3.13}$$

where $C = C_{m,n} \cdot \left(\frac{\beta_{n-1}}{q_{n-1}}\right)^N \cdot C_N$. (3.13) is the analogous of (3.8) and we can now argue as in the previous case.

When p > N we do not need to use the iterative procedure. For note that Nirenberg-Gagliardo's inequality reads :

$$\| v \|_\infty \leqslant C \| Dv \|_p^a \cdot \| v \|_{m/q}^{1-a} \quad \text{where} \quad a = \frac{N(m+p-2)}{mp+N(p-2)}, \quad m > 1 \tag{3.14}$$

(3.14) and (3.7) give

$$\| u \|_m^{\frac{(1-a)pq}{a}} \frac{d}{dt} \left(\| u \|_m^m \right) + C_m \left(\frac{1}{c}\right)^{\frac{p}{a}} \| u^q \|_\infty^{\frac{p}{a}} \leqslant 0, \; q = \frac{m+p-2}{p} \tag{3.15}$$

Now note that from the inequality

$$\phi(t)^{\omega}\frac{d}{dt}\phi(t)+k\,\psi(t)^{\theta}\leqslant 0$$

it follows, integrating between 0 and t

(3.16) $$\psi(t)\leqslant\left(\frac{1}{kt}\right)^{1/\theta}\cdot\frac{(\phi(0))^{\frac{\omega+1}{\theta}}}{\omega+1}$$

Use (3.16) with $\psi(t)=\|u^q\|_\infty$, $\phi(t)=\|u\|_m^m$, $\omega=\frac{(1-a)pq}{a}$, $\theta=\frac{p}{a}$ and (3.1), (3.2) follow. Note that this argument includes the case N = 1 which was discarded in [11] #

When $1<m_o<N\left(\frac{2}{p}-1\right)$ we have the following result, concerning a «backwards regularizing effect.

THEOREM 4. *Let* $1\leqslant m_o<N\left(\frac{2}{p}-1\right)$, $u_o\in L^{m_o}(IR^N)$. *If* $t>0$, $u(t,.)\in L^m(IR^N)$ *for every* m *such that* $1\leqslant m\leqslant m_o$. *In addition the following estimate holds :*

(3.17) $$\begin{cases}\|u(t,.)\|_m\leqslant\dfrac{c}{t^{\delta}}\|u_o\|_{m_o}^{\sigma} & \textit{for some constant } C=C(m,m_o,N,p), \textit{ where}\\[2ex] \delta=\dfrac{N(m_o-m)}{m[N(2-p)-m_op]}, & \sigma=\dfrac{m_o[N(2-p)-mp]}{m[N(2-p)-m_op]}\end{cases}$$

Proof. Let us see first that $u(t,.)\in L^m(IR^N)$ for each m such that $1<m<m_o$ (the case $m=m_o$ follows by accretivity). Remark that

(3.18) $$\|v\|_{m/q}\leqslant C\,\|Dv\|_p^a\,.\,\|v\|_{m_o/q}^{1-a},$$

where v, q are as in the last part of Theorem 3, the validity of (3.18) is justified as there, and $a=\frac{N(m_o-m)(m+p-2)}{m[m_o(N-p)-N(m+p-2)]}$. Arguing as in Theorem 1 (with k = 0), we arrive at

(3.19) $$\frac{d}{dt}E_m(t)+k\,E_m(t)^{\frac{pq}{am}}\leqslant 0,\quad E_m(t)=\int_{IR^N}u^m(t,x)dx.$$

Now notice that solutions of the inequality $f'+\alpha f^{\gamma}\leqslant 0$ with $\gamma>1$ satisfy $f\leqslant\frac{1}{((\gamma-1)\alpha t)^{\frac{1}{\gamma-1}}}$. This gives (3.17) .

The case $u(t,.)\in L^1(IR^N)$ is obtained by modifyng slightly the previous argument :

instead of (3.18) write

$$\|v\|_{1/q} \leqslant C\,\|Dv\|_p^a \cdot \|v\|_{m/q}^{1-a} \quad \text{with} \quad 1 < m < 3 - p\left(1 + \frac{1}{N}\right), \tag{3.20}$$

$a = \dfrac{N(m-1)(p-1)}{[m(N-p)-N(p-1)]}$. Corresponding to (3.19) we have

$$\|u\|_m^{(1-a)\frac{pq}{a}} \cdot \frac{d}{dt}\left(\|u\|_n^m\right) + C_m\left(\frac{1}{c}\right)^{\frac{p}{a}} \|u\|_1^{\frac{pq}{a}} \leqslant 0. \tag{3.21}$$

Now integrate (3.21) between 0 and t and use the fact that $\|u(t)\|_m$ is not increasing in t to get the result #

4. - BOUNDED DOMAINS

Concerning (P_Ω) with Ω bounded, it is known that there is a finite extinction time if $u_o \in L^2(\Omega)$ and $\frac{2N}{N+2} \leqslant p < 2$ ([1]). In that paper, extinction of the L^2 norm of the solution implies this result. The method of the proof of Theorem 1, based on the extinction of the L^m norm of solutions for some $m > 1$, enables us to extend the above mentioned result to get the following complete picture.

THEOREM 5. *Assume that Ω is bounded and regular. Let $u_o \in L^m(\Omega)$ where $m \geqslant \max\left\{N\left(\frac{2}{p} - 1\right), 1\right\}$ and $p < 2$. The corresponding solution of (P_Ω) vanishes in a finite time t_o. If $p \geqslant 2$ there are, for $u_o \in C^\infty(\Omega)$ and $u_o > 0$, solutions which are strictly positive for every $t > 0$.*

Proof. Let $m > N\left(\frac{2}{p} - 1\right)$ $\left(\text{the case } m = N\left(\frac{2}{p} - 1\right) \text{ is an easy modification of the proof in Theorem 1}\right)$. We write again $q = \frac{m+p-2}{p}$, $v = u^q$. By Hölder

$$\left(\int_\Omega u^m\right) \leqslant \left(\int_\Omega u^{p^*q}\right)^{\frac{m}{p^*q}} \cdot |\Omega|^{\frac{p^*q-m}{p^*q}}, \text{ where } p^* = \frac{Np}{N-p},\ |\Omega| = \text{meas}(\Omega). \tag{4.1}$$

Starting as in Theorem 1 (with $k = 0$) we arrive at

$$\frac{d}{dt}\left(\int_\Omega u^m\right) + \frac{m(m-1)}{q^p}\left(\int_\Omega |Dv|^p\right) \leqslant 0 \tag{4.2}$$

Next use Sobolev ($\|Dv\|_p \geqslant c\,\|v\|_{p^*}$) and (4.1) to obtain

$$(4.3)\qquad \frac{d}{dt}\left(\int_\Omega u^m\right) + \frac{cm(m-1)}{p^q\,|\Omega|^\omega}\left(\int_\Omega u^m\right)^{\frac{pq}{m}} \leqslant 0\,,\ \text{with}\ \omega = \frac{N-p}{N^2}\cdot\frac{N(p-2)+mp}{(m+p-2)}$$

From (4.3) we conclude that u vanishes at most at t_o, where

$$t_o = \frac{(2-p)q^p}{cm^2(m-1)}\,.\,|\Omega|^\omega\,.\,\|u_o\|_m^{2-p}$$

Assume now that Ω is connected [(1)]. When p = 2 the fact that for $u_o \geqslant 0$, $u_o \neq 0$ and $t > 0$, $u(t,.) > 0$ follows from the strong maximum principle of L. Nirenberg (see [7]). If $p > 2$ take $\Omega = B_R(0)$ and g a positive eigenfunction corresponding to the first eigenvalue λ of $-\Delta$ in $B_R(0)$ with homogeneous Dirichlet conditions ; g is radially symmetric, C^∞ and $Ag \leqslant Cg$ for some $C > 0$. To check this last assertion, note that

$$-g'' - \frac{N-1}{r}g' = g \text{ and hence } Ag = -\lambda(p-1)\,|g'|^{p-2}g'' - \frac{N-1}{r}|g'|g'^{\,p-2} =$$

$$= \lambda(p-1)\,|g'|g'^{\,p-2} + (p-1)\frac{N-1}{r}|g'|g'^{\,p-2} - \frac{N-1}{r}|g'|g'^{\,p-2} \leqslant \lambda(p-1)\,|g'|g'^{\,p-2} \leqslant Cg.$$

Now try as a subsolution $\overline{v}(t,x) = T(t)\,g(x)$, where $T(t) = \dfrac{T_o}{(1+c(p-2)T_o^{p-2}\,t)^{1/p-2}}$ solves $T'(t) + CT(t)^{p-1} = 0$. It follows from the maximum principle that if $u_o(x) \geqslant T_o\,g(x)$, the corresponding $u(t,x)$ is greater or equal than $\overline{v}(t,x)$ for each $t > 0$ #

Remark. Observe that as a consequence of the decay of some m-norm, $m > 1$ and Ω being bounded, MCL never holds. When $p \geqslant 2$ we have shown that for smooth initial data there is a retention property : if $u_o > 0$ in some $\tilde{\Omega} \subset \Omega$, $u(t,x) > 0$ in $\overline{\tilde{\Omega}}$ for each $t > 0$.

We conclude by noting that the results of this paper are valid when Au is replaced by other similar nonlinear.

$$\Delta_p u = \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)$$

As a natural generalization we may consider operators like

$$Bu = \sum_{i=1}^N \frac{\partial}{\partial x_i}\,\beta_i\left(\frac{\partial u}{\partial x_i}\right)$$

(1) For general Ω argue on each connected component.

where $\sum_{i=1}^{N} s\ \beta_i(s_i) \geqslant c\,|s|^p$ with $s=(s_1,\ldots,s_N)$.

Some of the previous results have immediate counterparts. In particular Theorem 1 remains valid unchanged.

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