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ALGEBRAIC POINTS OF ABELIAN FUNCTIONS IN TWO VARIABLES

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Résumé : On donne une mesure d'indépendance linéaire pour les coordonnées des points algébriques de fonctions abéliennes de deux variables. On en déduit un analogue abélien du théorème de Franklin-Schneider.

Summary : A linear independence measure is given for the coordinates of algebraic points of abelian functions in two variables. From this an abelian analogue of the Franklin-Schneider theorem is deduced.

Let A be a simple abelian variety defined over the field of algebraic numbers and let $\Theta : \mathbb{C}^2 \to A_{\mathbb{C}}$ be a normalised theta homomorphism (cf. [12], § 1.2). Let $\vartheta_0,\ldots,\vartheta_\nu$ be entire functions such that $(\vartheta_0(\underline{z}),\ldots,\vartheta_\nu(\underline{z}))$ forms a system of homogeneous coordinates for the point $\Theta(\underline{z})$ in projective ν-space. Put $f_i := \vartheta_i/\vartheta_0$. Assume that $\vartheta_0(\underline{0}) \neq 0$; then $f_i(\underline{0})$ is algebraic for all i. A point $\underline{u}$ in $\mathbb{C}^2$ with $\vartheta_0(\underline{u}) \neq 0$ is by definition an algebraic point of Θ if and only if $f_i(\underline{u})$ is algebraic for all i. The field of abelian functions associated with Θ is $\mathbb{C}(f_1,\ldots,f_\nu)$.

If (u_1,u_2) is a non-zero algebraic point of Θ, the coordinates u_1 and u_2 are linearly independent over the algebraic numbers (cf. [12], Théorème 3.2.1) ; the proof uses the Schneider-Lang criterion (cf. [5], Chapter III, Theorem 1). It is the main purpose of this paper to obtain, by means of Gel'fond's method, a quantitative refinement of this statement.

THEOREM 1. *For every compact subset* K *of* $\mathbb{C}^2 \setminus \{\underline{0}\}$ *that contains no zeros of* ϑ_0, *there exists an effectively computable* C *with the following property. Let* $\underline{u}$ *be an algebraic point of* Θ *that lies in* K, *and let* β *be an algebraic number. Let* A *be an upper bound for the (classical) heights of the numbers* $f_i(\underline{u})$, *let* B *be an upper bound for the height of* β *and take* $D := [\mathbb{Q}(f_1(\underline{u}),\dots,f_\nu(\underline{u}),\beta) : \mathbb{Q}]$; *assume* $A \geqslant e^e$, $B \geqslant e$. *Then*

$$(1) \qquad |\beta u_1 - u_2| > \exp(-CD^6 \log^2 A \log^4(DB \log A) \log^{-5}(D \log A)),$$

where $\underline{u} =: (u_1,u_2)$.

The dependence of this lower bound on B was first studied in [3]. Moreover, in an unpublished 1979 investigation, Y.Z. Flicker and D.W. Masser also studied the dependence on B and obtained $\log^4 B$ in the exponent. I wish to thank Dr. Masser for making available to me a report of this study, to which several improvements in the present paper are due.

The proof of Theorem 1 resembles that of Lemma 1 of [1] ; in parts where this resemblance is particularly strong, the exposition will be brief. The proof is preceded by a lemma that may be called, in Masser's terminology, a 'safe addition formula' for abelian functions.

LEMMA. *There exists an effectively computable* C' *with the following property. If* $\underline{w}_1$ *and* $\underline{w}_2$ *are points of* $\mathbb{C}^2$ *such that* $\vartheta_0(\underline{w}_1) \neq 0$, $\vartheta_0(\underline{w}_2) \neq 0$, $\vartheta_0(\underline{w}_1 + \underline{w}_2) \neq 0$, *then for every* i *in* $\{1,\dots,\nu\}$ *there exist polynomials* Φ_i, Φ_i^* *of total degree at most* C' *and a neighbourhood* N *of* $(\underline{w}_1,\underline{w}_2)$ *such that*

$$(2) \qquad f_i(\underline{z}_1 + \underline{z}_2) = \frac{\Phi_i^*}{\Phi_i}\,(f_1(\underline{z}_1),\dots,f_\nu(\underline{z}_1),f_1(\underline{z}_2),\dots,f_\nu(\underline{z}_2))$$

for all $(\underline{z}_1,\underline{z}_2)$ *in* N ; *the denominator is non-zero on* N. *The coefficients of these polynomials are algebraic integers in a field of degree at most* C'. *Their size (i.e., the maximum of the absolute values of their conjugates) is also bounded by* C'.

Proof. Let $(\underline{w}_1,\underline{w}_2)$ be any point in $\mathbb{C}^4$. Define $\sigma : \mathbb{C}^4 \to \mathbb{P}^{\nu^2+2\nu}(\mathbb{C})$ by $\sigma(\underline{z}_1,\underline{z}_2) := \psi(\Theta(\underline{z}_1),\Theta(\underline{z}_2))$, where ψ is the Segre embedding (cf. [9] , (2.12)) of $\mathbb{P}^\nu(\mathbb{C}) \times \mathbb{P}^\nu(\mathbb{C})$ into projective space. By the regularity of the addition in A, we find projective coordinates for $\Theta(\underline{z}_1 + \underline{z}_2)$ of the form

$$H_i(\Theta(\underline{z}_1), \Theta(\underline{z}_2)) \qquad (0 \leqslant i \leqslant \nu)$$

for all $(\underline{z}_1,\underline{z}_2)$ with the property that $\sigma(\underline{z}_1,\underline{z}_2)$ lies in a certain Zariski neighbourhood of $\sigma(\underline{w}_1,\underline{w}_2)$; here the polynomials H_i have algebraic coefficients. The continuity of σ now proves this for all $(\underline{z}_1,\underline{z}_2)$ in a neighbourhood of $(\underline{w}_1,\underline{w}_2)$. Let P be a fundamental region for $\mathbb{C}^2/\Omega$; covering the compact set P^2 with a finite number of these neighbourhoods shows that we can bound the degrees of the polynomials H_i , the sizes of their coefficients, the degree of the field generated by these coefficients and their common denominator independently of $(\underline{w}_1 , \underline{w}_2)$. In particular, it is no restriction to assume the coefficients to be algebraic integers.

Finally, if $\vartheta_0(\underline{w}_1) \neq 0$, $\vartheta_0(\underline{w}_2) \neq 0$, $\vartheta_0(\underline{w}_1 + \underline{w}_2) \neq 0$, these also hold on some neighbourhood of $(\underline{w}_1 , \underline{w}_2)$; hence

$$H_0(\Theta(\underline{z}_1), \Theta(\underline{z}_2)) \neq 0$$

on some neighbourhood of $(\underline{w}_1 , \underline{w}_2)$, which now proves (2). ■

Proof of Theorem 1. I. In this proof c_1 , c_2 ,... will denote effectively computable real numbers greater than 1 that depend only on Θ and K. Let x be some large real number ; further conditions on x will appear at later stages of the proof. Put $B' := xDB \log A$, $E := 4D^{1/2} \log^{1/2} A$ and assume

$$(3) \qquad |\beta u_1 - u_2| \leqslant \exp(-x^{16}D^6 \log^2 A \log^4 B' \log^{-5} E).$$

This will lead to a contradiction, which will prove (1).

The field $\mathbb{C}(f_1,...,f_\nu)$ has transcendence degree 2 over $\mathbb{C}$ (cf. [10] , § 6) ; assume, without loss of generality, that f_1 and f_2 are algebraically independent over $\mathbb{C}$. As in [8], § 4.2, we choose a system ξ_0 ,..., ξ_{D-1} of generators of $\mathbb{Q}(f_1(\underline{u}),...,f_\nu(\underline{u}),\beta)$ of the form

$$\xi_\delta = f_1^{j_1(\delta)} \cdots f_\nu^{j_\nu(\delta)}(\underline{u})\beta^{j_{\nu+1}(\delta)},$$

where the $j_i(\delta)$ are non-negative integers satisfying $j_1(\delta) + ... + j_{\nu+1}(\delta) \leqslant D-1$. Put

$$L := [x^8D^3 \log A \log^2 B' \log^{-3} E]$$

and consider the auxiliary functions

$$(4) \qquad F(z) := \sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} \sum_{\delta=0}^{D-1} p(\lambda_1,\lambda_2,\delta)\xi_\delta f_1^{\lambda_1} f_2^{\lambda_2}(z,\beta z),$$

$$(5) \qquad F_s(z) := \sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} \sum_{\delta=0}^{D-1} p(\lambda_1,\lambda_2,\delta)\xi_\delta f_1^{\lambda_1} f_2^{\lambda_2}(z,\beta z-s\epsilon),$$

where $\epsilon := \beta u_1 - u_2$. As K is compact and the zero set of ϑ_0 is closed, these sets have a distance at least c_1^{-1}. The functions $f_1,...,f_\nu$ are continuous on the set K' of points $\underline{z}$ satisfying $\mathrm{dist}(\underline{z},K) \leqslant \frac{1}{2} c_1^{-1}$; hence their absolute values are bounded by some c_2 on K' and *a fortiori* on the ball U with radius $\frac{1}{4} c_1^{-1}$ centred at $\underline{u}$. Now put

$$S := [x^3 D \log B' \log^{-1} E].$$

As in § 4 of [6], an application of the box principle shows that there is a subset V of $\{1,\dots,S\}$ such that $\# V \geq c_3^{-1} S$ with the property that (su_1, su_2) and $(su_1, s\beta u_1)$ lie in $U + \Omega$ for all s in V, where Ω is the period lattice of Θ. Put

$$T := [x^{12} D^5 \log^2 A \log^3 B' \log^{-5} E]$$

and consider the system of linear equations

$$F_s^{(t)}(su_1) = 0 \qquad (s \in V, t = 0,\dots,T-1) \tag{6}$$

in the $p(\lambda_1, \lambda_2, \delta)$.

Take $1 \leq i \leq \nu$. Lemma 7.2 of [6], part of which remains valid without complex multiplication, states that for every integer s there exist polynomials $\Psi_{s,i}, \Psi^*_{s,i}$ of total degree $N_s \leq c_4 s^2$ such that, if $\vartheta_0(s\underline{u}) \neq 0$, then

$$f_i(s\underline{u}) = \frac{\Psi^*_{s,i}}{\Psi_{s,i}} (f_1(\underline{u}),\dots,f_\nu(\underline{u}))$$

and $\Psi_{s,i}(f_1(\underline{u}),\dots,f_\nu(\underline{u})) \neq 0$. The coefficients of these polynomials are algebraic numbers in a field of degree at most c_5, of size at most $c_6^{s^2}$ and with a common denominator at most $c_7^{s^2}$. According to the preceding Lemma, there also exist polynomials Φ_i, Φ^*_i of total degree at most c_8 and a neighbourhood N of the origin such that

$$f_i(\underline{u} + \underline{z}) = \frac{\Phi^*_i}{\Phi_i} (f_1(\underline{u}),\dots,f_\nu(\underline{u}), f_1(\underline{z}),\dots,f_\nu(\underline{z}))$$

for all $\underline{z}$ in N, with non-zero denominator, the coefficients are algebraic integers in a field of degree at most c_8, whose sizes are also bounded by c_8.

Now define

$$\Phi := \prod_{i=1}^{\nu} \Phi_i,$$

$$\varphi_{s,i}(\underline{z}) := \Phi^{N_s} (f_1(\underline{u}),\dots,f_\nu(\underline{u}), f_1(\underline{z}),\dots,f_\nu(\underline{z})) \Psi_{s,i}(f_1(\underline{u}+\underline{z}),\dots,f_\nu(\underline{u}+\underline{z})),$$

$$\psi_{s,i}(z) := \Phi^{N_s} (f_1(\underline{u}),\dots,f_\nu(\underline{u}), f_1(\underline{z}),\dots,f_\nu(\underline{z})) \Psi^*_{s,i}(f_1(\underline{u}+\underline{z}),\dots,f_\nu(\underline{u}+\underline{z})).$$

Note that on a neighbourhood of the origin $\varphi_{s,i}$ and $\psi_{s,i}$ are holomorphic and $\varphi_{s,i}$ is non-zero. As

$$F_s^{(t)}(su_1) = \sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} \sum_{\delta=0}^{D-1} p(\lambda_1,\lambda_2,\delta)\xi_\delta\, s^{-t} \frac{d^t}{dz^t} \left(\varphi_{s,1}^{-\lambda_1} \psi_{s,1}^{\lambda_1} \varphi_{s,2}^{-\lambda_2} \psi_{s,2}^{\lambda_2} (z,\beta z)\right)\Bigg|_{z=0},$$

Leibniz' rule shows that we have found a solution of (6) if we choose the $p(\lambda_1,\lambda_2,\delta)$ in such a way that

$$(7) \qquad f_{s,t} = 0 \qquad (s \in V,\ t = 0,\dots,T-1)\,,$$

where

$$f_{s,t} := \sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} \sum_{\delta=0}^{D-1} p(\lambda_1,\lambda_2,\delta)\xi_\delta \frac{d^t}{dz^t} \left(\varphi_{s,1}^{L-\lambda_1} \psi_{s,1}^{\lambda_1} \varphi_{s,2}^{L-\lambda_2} \psi_{s,2}^{\lambda_2} (z,\beta z)\right)\Bigg|_{z=0}.$$

The number of equations in (7) is at most

$$ST \leqslant c_9\, x^{15}\, D^6 \log^2 A \log^4 B' \log^{-6} E,$$

while the number of unknowns is

$$(L+1)^2 D \geqslant c_{10}^{-1}\, x^{16}\, D^7 \log^2 A \log^4 B' \log^{-6} E.$$

From the above estimates it follows that $\psi_{s,i}^{\lambda_i}(\underline{z})$ can be written as a polynomial in $f_1(\underline{u}),\dots,f_\nu(\underline{u}),f_1(\underline{z}),\dots,f_\nu(\underline{z})$ of total degree at most $c_{11}\,\lambda_i\, s^2$; the coefficients are algebraic numbers in a field of degree at most c_{12}, whose sizes and common denominator are bounded by $c_{13}^{\lambda_i s^2}$. With the aid of Lemma 5.1 of [6] it is now easy to see that the expression

$$\frac{d^t}{dz^t}\, \psi_{s,i}^{\lambda_i}(z,\beta z)\Bigg|_{z=0}$$

is a polynomial in $f_1(\underline{u}),\dots,f_\nu(\underline{u})$ of total degree at most $c_{14}(\lambda_i s^2 + t)$; the coefficients are algebraic numbers in a field of degree at most c_{16} over $\mathbb{Q}(\beta)$, whose sizes and common denominator are bounded by $c_{16}^{\lambda_i s^2 + t \log t + t \log B}$. A similar statement holds for

$$\frac{d^t}{dz^t}\, \varphi_{s,i}^{L-\lambda_i}(z,\beta z)\Bigg|_{z=0}.$$

Thus the coefficients of the system of linear equations (7) lie in a field of degree at most $c_{17}\, D$ and their size and common denominator are bounded by

$$c_{18}^{T\log T+T\log B}\prod_{i=1}^{\nu}(H(f_i(\underline{u}))+1)^{c_{19}(D+LS^2)}\leqslant \exp(c_{20}\,x^{14}\,D^5\log^2 A\log^4 B'\log^{-5}E).$$

According to Lemme 1.3.1 of [11], if $x > 2c_9c_{10}$, this implies the existence of rational integers $p(\lambda_1,\lambda_2,\delta)$, not all zero, such that (7) and thereby (6) hold, while

$$P := \max |p(\lambda_1,\lambda_2,\delta)| \leqslant \exp(c_{21}\,x^{14}\,D^5\log^2 A\log^4 B'\log^{-5}E).$$

Take $s\in V$, $\eta\in\mathbb{R}$, $z\in\mathbb{C}$ such that $|z-su_1|=\eta$. Then the distance between $(z,\beta z)$ and $(su_1,s\beta u_1)$ is bounded by $2B\eta$; if $\eta=(8c_1B)^{-1}$, it follows that $(z,\beta z)$ lies in $U'+\Omega$, where U' is the ball with radius $\frac{1}{2}c_1^{-1}$ centred at $\underline{u}$. Similarly $(z,\beta z - s\epsilon)\in U'$. Note that $U'\subset K'$ and therefore $|f_i(\underline{z})|\leqslant c_2$ for all $\underline{z}$ in U'. Comparison of the definitions of F and F_s now gives

$$\sup_{|z-su_1|=\eta} |F(z)-F_s(z)| \leqslant Pc_{22}^{D+L}\,S\,|\epsilon|.$$

By Cauchy's inequality this implies

$$|F^{(t)}(su_1)-F_s^{(t)}(su_1)| \leqslant t^{c_{23}t}\,B^t\,Pc_{24}^{D+L}\,S\,|\epsilon|.$$

If $t\leqslant T-1$, it now follows from (6) that

$$(8)\qquad |F^{(t)}(su_1)| \leqslant \exp(-c_{25}^{-1}\,x^{16}\,D^6\log^2 A\log^4 B'\log^{-5}E).$$

Define the entire function G by

$$G(z) := g(z)F(zu_1),$$

where

$$g(z) := \vartheta_0^{2L}(zu_1,\beta zu_1).$$

By Lemma 1 of [7], the function g satisfies

$$(9)\qquad |g(z)| \leqslant \exp(c_{26}L\,|z|^2)\ ;$$

also the definition of V gives

$$(10)\qquad |g(s)| \geqslant \exp(-c_{27}\,LS^2) \qquad (s\in V).$$

Formulas (8), (9) and (10) form the starting-point for an extrapolation procedure on G, analogous to that in [1], which yields

$$(11)\qquad F_s^{(t)}(su_1)=0 \qquad (s\in V,\ t=0,\dots,T'-1),$$

where $T' := [x^2 T]$.

II. By Proposition 1.2.3 of [12], the partial derivatives of $f_1,\dots,f_\nu$ are polynomials in $f_1,\dots,f_\nu$. Therefore there exist polynomials $P_1,\dots,P_\nu$ such that the functions $h_{i,s}$, defined by

$$h_{i,s}(z) := f_i(z + su_1, \beta z + su_2)$$

satisfy

$$h'_{i,s} = P_i(h_{1,s},\dots,h_{\nu,s})$$

and

$$h_{i,s}(0) = f_i(su_1, su_2).$$

Define

$$Q_1(X_1,\dots,X_\nu) := \sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} \sum_{\delta=0}^{D-1} p(\lambda_1,\lambda_2,\delta)\xi_\delta X_1^{\lambda_1} X_2^{\lambda_2}.$$

As

$$h_{i,s}^{(t)}(0) = \frac{d^t}{dz^t} f_i(z,\beta z - s\epsilon) \Bigg|_{z=su_1},$$

(11) shows

$$\frac{d^t}{dz^t} Q_1(h_{1,s}(z),\dots,h_{\nu,s}(z)) \Bigg|_{z=0} = 0 \qquad (s \in V,\ t = 0,\dots,T'-1),$$

i.e.

(12) $$\sum_{s\in V} \operatorname*{ord}_{z=0} Q_1(h_{1,s}(z),\dots,h_{\nu,s}(z)) \geqslant c_3^{-1} ST' \geqslant c_{28}^{-1} x^{17} D^6 \log^2 A \log^4 B' \log^{-6} E.$$

Let $Q_2,\dots,Q_n$ be generators of the ideal of $\mathbb{C}[X_1,\dots,X_\nu]$ corresponding to the affine part of A. Then

(13) $$Q_j(f_1(\underline{w}),\dots,f_\nu(\underline{w})) = 0 \qquad (j = 2,\dots,n)$$

for every $\underline{w}$ that is not a zero of ϑ_0; thus in particular

(14) $$\operatorname*{ord}_{z=0} Q_j(h_{1,s}(z),\dots,h_{\nu,s}(z)) = \infty \qquad (s \in V,\ j = 2,\dots,n).$$

Put $W := \{\Theta(z,\beta z) \mid z \in \mathbb{C}\}$. Then W, with the addition of A, forms a subgroup of A; it follows that the Zariski closure of W, with the addition of A, forms an algebraic subgroup of A. Small values of z are separated, thus W is infinite. As A is simple, this implies that $\overline{W} = A_{\mathbb{C}}$. Therefore the Zariski closure of

$$\left\{ \Theta(z + su_1, \beta z + su_2) \mid z \in \mathbb{C} \, , \vartheta_0(z + su_1, \beta z + su_2) \neq 0 \right\}$$

is also equal to $A_{\mathbb{C}}$. Now suppose for a moment that

$$\operatorname*{ord}_{z=0} \; Q_1(h_{1,s}(z),\dots,h_{\nu,s}(z)) = \infty$$

for some s in V. By continuity, this implies that (13) also holds if $j = 1$. But that contradicts either the algebraic independence of f_1 and f_2 or the linear independence of $\xi_0,\dots,\xi_{D-1}$. Thus

$$\operatorname*{ord}_{z=0} \; Q_1(h_{1,s}(z),\dots,h_{\nu,s}(z)) < \infty \qquad (s \in V). \tag{15}$$

The set of common zeros of $Q_2,\dots,Q_n$ has algebraic dimension two (cf. [9] , (2.7)). As, by (14) and (15), Q_1 is not in the ideal generated by $Q_2,\dots,Q_n$, the set of common zeros of $Q_1,\dots,Q_n$ has algebraic dimension at most one (cf. [9] , (1.14)). It is no restriction to assume $n > \nu$. Then the Main Theorem of [2] implies that either

$$\sum_{s \in V} \operatorname*{ord}_{z=0} \; Q_1(h_{1,s}(z),\dots,h_{\nu,s}(z)) \leqslant$$

$$c_{29} L^2 + c_{30} LS \leqslant \exp(c_{31} \, x^{16} \, D^6 \log^2 A \log^4 B' \log^{-6} E,$$

which contradicts (12) if $x > c_{28} \, c_{31}$, or the points $\Theta(s\underline{u})$ are not all different. As Θ induces an isomorphism between $\mathbb{C}^2 / \Omega$ and $A_{\mathbb{C}}$, the equality of $\Theta(s\underline{u})$ and $\Theta(s'\underline{u})$, say, shows that there is an $\underline{\omega} \in \Omega$ with

$$s\underline{u} = s'\underline{u} + \underline{\omega}.$$

Therefore we have now proved the theorem under the hypothesis

$$\forall_{m \leqslant S} \; m\underline{u} \notin \Omega.$$

III. It now remains to prove the theorem in the case where $m\underline{u} \in \Omega$ for some $m \leqslant S$. In particular, let m be the smallest positive integer with this property ; then the points $\Theta(\underline{u}), \Theta(2\underline{u}),\dots, \Theta(m\underline{u})$ are all different. As before, we can choose a subset V' of $\{1,\dots,m\}$ such that $\# V' \geqslant c_{32}^{-1} m$ with the property that (su_1, su_2) and $(su_1, s\beta u_1)$ lie in $U + \Omega$ for all s in V'. Put

$$L := [x^5\, mD^2 \log A \log B' \log^{-2} E],$$

where E, B' retain their earlier meaning, and let F and F_s be defined again by (4) and (5). Put

$$T := [x^9\, mD^4 \log^2 A \log^2 B' \log^{-4} E]$$

and consider the system of linear equations

$$F_s^{(t)}(su_1) = 0 \qquad (s \in V',\ t = 0,\dots,T-1). \tag{16}$$

By the same method used earlier, it is proved that the coefficients $p(\lambda_1,\lambda_2,\delta)$ may be chosen in such a way that they are not all zero and (16) holds. Now let V be the set of all $s \in \{1,\dots,S\}$ that differ by a multiple of m from an element of V' ; here S has the same meaning as before. Then $\# V \geq c_{33}^{-1} S$; as $m\underline{u}$ is a period of every f_i, (16) implies

$$F_s^{(t)}(su_1) = 0 \qquad (s \in V,\ t = 0,\dots,T-1).$$

Repeating the extrapolation procedure gives

$$F_s^{(t)}(su_1) = 0 \qquad (s \in V,\ t = 0,\dots,T'-1)$$

where $T' := [x^2 T]$. Define Q_1 and $h_{i,s}$ as before ; then

$$\sum_{s \in V'} \operatorname*{ord}_{z=0} Q_1(h_{1,s}(z),\dots,h_{\nu,s}(z)) \geq c_{32}^{-1} mT' \geq c_{34}^{-1} x^{11} m^2 D^4 \log^2 A \log^2 B' \log^{-4} E.$$

Another application of the Main Theorem of [2] gives the desired contradiction. Note that for this special case of the theorem we may replace (1) with

$$|\beta u_1 - u_2| > \exp(-CmD^5 \log^2 A \log^3 (DB \log A) \log^{-4} (D \log A)),$$

which is sharper if m is small compared to S. ∎

As a corollary to Theorem 1, an abelian analogue of the Franklin-Schneider theorem is easily obtained. It should be noted that the assumption as to the nature of β, necessary in the exponential and elliptic versions of this result (cf. [1]) does not occur here.

THEOREM 2. *For every point* $\underline{a}$ *in* $\mathbb{C}^2 \setminus \{0\}$ *such that* $\vartheta_0(\underline{a}) \neq 0$, *there exists an effectively computable* C'' *with the following property. Let* $\alpha_1,\dots,\alpha_\nu,\beta$ *be algebraic numbers, let* $A \geq e^e$ *be an upper bound for the heights of* $\alpha_1,\dots,\alpha_\nu$ *and let* $B \geq e$ *be an upper bound for the height of* β.

Then if $D = [\mathbb{Q}(\alpha_1,\dots,\alpha_\nu,\beta) : \mathbb{Q}]$, we have

$$(17) \quad \sum_{i=1}^{\nu} |f_i(\underline{a}) - \alpha_i| + |\beta a_1 - a_2| > \exp(-C''D^6 \log^2 A \log^4(DB \log A) \log^{-5}(D \log A)).$$

Proof. Let $Q_2,\dots,Q_n$ be generators of the ideal of $\mathbb{C}[X_1,\dots,X_\nu]$ corresponding to the affine part of A. If $Q_j(\alpha_1,\dots,\alpha_\nu) \neq 0$ for some j with $2 \leqslant j \leqslant n$, then the result is trivial, as $Q_j(f_1(\underline{a}),\dots,f_\nu(\underline{a})) = 0$. Thus we may assume $(\alpha_1,\dots,\alpha_\nu)$ to be on the affine part of A. By the smoothness of A at $\Theta(a)$, the matrix of partial derivatives of $(f_1,\dots,f_\nu)$ at a has rank 2. Thus there exist k and ℓ such that the matrix of partial derivates of (f_k,f_ℓ) at $\underline{a}$ has rank 2. According to Theorem 7.4 in Chapter I of [4] , there are open neighbourhoods U of $\underline{a}$ and V of $(f_k(\underline{a}), f_\ell(\underline{a}))$ such that (f_k,f_ℓ) induces a biholomorphic mapping from U onto V. If C" is sufficiently large, the negation of (17) implies that $(f_k(\underline{u}), f_\ell(\underline{u}))$ belongs to V for some $\underline{u} \in U$ and

$$|\underline{a} - \underline{u}| \leqslant c \exp(-C''D^6 \log^2 A \log^4(DB \log A) \log^{-5}(D \log A))$$

for some c that depends only on $\underline{a}$ and Θ. Thus

$$(18) \quad |\beta u_1 - u_2| \leqslant |\beta a_1 - \beta u_1| + |a_2 - u_2| + |\beta a_1 - a_2| \leqslant$$

$$(|\beta| c + c + 1) \exp(-C''D^6 \log^2 A \log^4(DB \log A) \log^{-5}(D \log A)).$$

Let K be a compact subset of $\mathbb{C}^2 \setminus \{\underline{0}\}$ containing a neighbourhood of $\underline{a}$ but no zeros of ϑ_0 ; by Theorem 1, (18) is impossible if C" is sufficiently large in terms of c and K. ■

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