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NEW OPERATORS ON JET SPACES

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Résumé : Nous introduisons des opérations et des techniques fonctionnelles nouvelles dans l'étude des espaces de jets d'un espace fibré. Ces méthodes permettent d'obtenir en particulier une caractérisation canonique des transformations infinitésimales d'ordre quelconque et des prolongements de champs des vecteurs. Les expressions locales explicites des formules intrinsèques sont aussi présentées. Ces résultats peuvent être utilisés dans le cadre du calcul des variations d'ordre supérieur.

Summary : New operators and functorial techniques are introduced on the k-jet spaces of a fibered space, which lead to interesting characterizations of the infinitesimal contact transformations at any order. Canonical prolongations at al! orders of vector fields are obtained as an application useful in the calculus of variations. Global results as well as explicit local formulas are given.

INTRODUCTION

There is growing interest in jet spaces with respect to Mathematical-Physics, for they allow a structural and unifying analysis of differential equations. Jet spaces may be considered to be a natural framework for higher order field theories, in particular for the calculus of variations. We might also expect that they will play a role in the quantization via deformation. So we are concerned with all the prolongation techniques which naturally involve higher order jet spaces.

In this paper we introduce some new operators and techniques on jet spaces, and obtain results which are useful, for istance, in the calculus of variations [8] .

We will start with a systematic recall of functorial techniques -widely used in the paper-, giving intrinsic as well explicit local formulas. In particular we will make a broad analysis of affine structures involved in jet spaces (for the classical results see, for istance, [3]).

Then the contact form ϑ at any order will be investigated with respect to different possible global definitions (see also [12] and [4]) and to the functorial invariance properties. We will consider the kernel Δ of ϑ at any order, which is a non involutive distribution. For some pourposes it is more important than ϑ itself.

Then we will introduce a new operator r at any order, which will allow the exchange between jet and tangent spaces and maps and which will be shown to have analogies and relations with ϑ. We will investigate its functorial invariance properties and also the close connection with different affine structures.

At this point we will be in the position to introduce the set of infinitesimal contact transformations of any order, which are composed of the vector fields which preserve Δ. In this way we may avoid non-essential connections, which may be involved with the vector valued form ϑ (see, for istance, [2]). Moreover we can give several new characterizations of this set by means of the operator r and also explicit local formulas thus obtained.

For useful application -for istance in the calculus of variations [8]- we will show a canonical prolongation at all orders of any vector field, by using the jet functor and r. This new approach may also be applied to non-projectable vector fields and moreover will naturally produce explicit local formulas.

1. - THE JET FUNCTOR

We start with the basic notations on jet spaces and we give explicit formulas with respect to the functorial techniques which will be widely used in the following. All spaces and maps will be C^∞.

1.1 - Henceforth $p : E \to M$ is a fibered space (i.e. a surjective submersion), with

$$m \equiv \dim M \qquad \text{and} \qquad m + l \equiv \dim E.$$

The standard chart of E is denoted by (x^λ, y^i), with $1 \leqslant \lambda \leqslant m$, $1 \leqslant i \leqslant l$.

In some cases we will deal with a further fibered space $q : F \to N$, with $n \equiv \dim N$, whose standard chart is denoted by (z^α, w^j).

1.2 - J^kE is the k-jet space and $J^oE \equiv E$, with $0 \leqslant k$. It can be viewed naturally as a fibered space and a bundle

$$p^k : J^kE \to M \qquad \text{and} \qquad p^k_h : J^kE \to J^hE,$$

respectively, with $0 \leqslant h < k$. Moreover

$$p^k_h \circ p^h_l = p^k_l \qquad \text{and} \qquad p^k_h \circ p^h = p^k.$$

The standard chart of J^kE is denoted by (x^λ, y^i_Λ), with $0 \leqslant |\Lambda| \leqslant k$, where $\Lambda \equiv (\Lambda_1,\ldots,\Lambda_m)$ is a multi-index and $|\Lambda| \equiv \Lambda_1 + \ldots + \Lambda_m$. We put

$$0 \equiv (o,\ldots,o), \quad \Lambda + \lambda \equiv (\Lambda_1,\ldots,\Lambda_\lambda + 1,\ldots,\Lambda_m)$$

$$y^i \equiv y^i_0, \; y^i_\lambda \equiv y^i_{0+\lambda}, \; y^i_{\lambda\mu} \equiv y^i_{0+\lambda+\mu}, \ldots \; .$$

If $s : M \to E$ is a (local) section, then $j^k s : M \to J^kE$ is its k-lift, whose expression is $(x^\lambda, y^i_\Lambda) \circ j^k s = (x^\lambda, \partial_\Lambda s^i)$, where

$$s^i \equiv y^i \circ s \qquad \text{and} \qquad \partial_\Lambda s^i \equiv \partial_1^{\Lambda_1} \ldots \partial_m^{\Lambda_m} s^i.$$

Moreover whe have $p^k_h \circ j^k s = j^h s$.

1.3 - PROPOSITION. *Let* $H : E \to F$ *be a morphism over the diffeomorphism* $h : M \to N$. *Then there is a unique morphism* $J^kH : J^kE \to J^kF$ *over* $h : M \to N$, *such that, for each (local) section* $s : M \to E$, *the following diagram commutes*

$$\begin{array}{ccc} J^kE & \xrightarrow{\quad J^kH \quad} & J^kF \\ {\scriptstyle j^k s}\uparrow & & \uparrow{\scriptstyle j^k(H^*s)} \\ M & \xrightarrow{\quad h \quad} & N \end{array}$$

where H^*s *is the section* $H^*s \equiv H \circ s \circ h^{-1} : N \to F$.

Its expression is given by

$$z^\alpha \circ J^kH = h^\alpha,$$

$$w^j_A \circ J^kH = \partial_\Lambda \, \partial_I H^j \, y^1_{M(1,1)} \cdots y^1_{M(1,I_1)} \cdots y^l_{M(l,1)} \cdots y^l_{M(l,I_l)}$$

$$\partial_{B(1,1)} g^1 \circ h \ldots \partial_{B(1,b(1))} g^1 \circ h \ldots \partial_{B(m,1)} g^m \circ h \ldots \partial_{B(m,b(m))} g^m \circ h,$$

where $h^\alpha \equiv z^\alpha \circ h$, $H^j \equiv w^j \circ H$, $g^\lambda \equiv x^\lambda \circ h^{-1}$,

$$\begin{array}{lll} \Lambda,\ M^{(1,1)},\dots,M^{(l,I_l)} & \textit{are multi-indices in} & (1,\dots,m) \\ B^{(1,1)},\dots,B^{(m,b(m))} & \text{"} \quad \text{"} \quad \text{"} & (1,\dots,n) \\ I & \textit{is a multi-index in} & (1,\dots,l) \end{array}$$

and where the summation is extended to all the multi-indices whith the conditions

$$b(\mu) = \Lambda_\mu + M_\mu^{(1,1)} + \dots + M_\mu^{(l,I_l)}, \quad 1 \leqslant \mu \leqslant m,$$

$$A = B^{(1,1)} + \dots + B^{(m,b(m))}$$

$$I_h > 0 \Rightarrow |M^{(h,1)}|,\dots,|M^{(h,I_h)}| > 0, \quad 1 \leqslant h \leqslant l,$$

$$b(\mu) > 0 \Rightarrow |B^{(\mu,1)}|,\dots,|B^{(\mu,b(\mu))}| > 0.$$

Proof. It will follow by induction on $|A|$ such that $1 \leqslant |A| \leqslant k$ (after a long calculation!).

If H is over M, then we have a simpler expression.

COROLLARY. *Let* $H : E \to F$ *be a morphism over* $M \equiv N$. *Then there is a unique morphism* $J^kH : J^kE \to J^kF$ *over* M, *such that, for each (local) section* $s : M \to E$, *the following diagram commutes*

$$\begin{tikzcd} J^kE \arrow[rr, "J^kH"] & & J^kF \\ & M \arrow[ul, "j^ks"] \arrow[ur, "j^k(H \circ s)"'] & \end{tikzcd}$$

Its expression is

$$(x^\lambda, w^j_A) \circ J^kH = (x^\lambda, \partial_\Lambda \partial_I H^j \, y^1_{M^{(1,1)}} \cdots y^l_{M^{(l,I_l)}}),$$

where the summation is extended to all the multi-indices with the conditions

$$\Lambda + M^{(1,1)} + \dots + M^{(l,I_l)} = A,$$

$$I_h > 0 \Rightarrow |M^{(h,1)}|,\dots,|M^{(h,I_h)}| > 0.$$

1.4 - Coming back to the general case, we can see that the following diagram commutes, for $0 \leqslant h \leqslant k$,

$$\begin{tikzcd} J^kE \arrow[r, "J^kH"] \arrow[d, "p^k_h"'] & J^kF \arrow[d, "q^k_h"] \\ J^hE \arrow[r, "J^hH"] \arrow[d, "p^h"'] & J^hF \arrow[d, "q^h"] \\ M \arrow[r, "h"] & N \end{tikzcd}$$

1.5 - Moreover, from the uniqueness of J^kH, it follows that J^k is a co-variant functor in the category of fibered spaces, namely

$$J^kH \circ J^kH' = J^k(H \circ H') \quad \text{and} \quad J^k id_E = id_{J^kE}.$$

1.6 - Sometimes, the following remark, which is a simple consequence of the local expression, will be useful. Let $N = M$ and let $s : M \to F$ be a section such that the following diagram commutes

$$\begin{tikzcd} E \arrow[rr, "H"] \arrow[dr, "p"'] & & F \\ & M \arrow[ur, "s"'] & \end{tikzcd}$$

Then the following diagram commutes

$$\begin{tikzcd} J^kE \arrow[rr, "J^kH"] \arrow[dr, "p^k"'] & & J^kF \\ & M \arrow[ur, "j^ks"'] & \end{tikzcd}$$

A further consequences is the following. Let $r : G \to F$ be a further fibered space, so that $q \circ r : G \to M$ is also a fibered space. Let $S : E \to G$ be a morphism over the section $s : M \to F$, so that the following diagram commutes

$$\begin{tikzcd} E \arrow[r, "S"] \arrow[d, "p"'] & G \arrow[d, "r"] \\ M \arrow[r, "s"] & F \end{tikzcd}$$

Then the following diagram commutes (putting $H \equiv r \circ S$ and taking all the J^k over M)

$$\begin{tikzcd} J^kE \arrow[r, "J^kS"] \arrow[d, "p^k"'] & J^kG \arrow[d, "J^kr"] \\ M \arrow[r, "j^ks"] & J^kF \end{tikzcd}$$

1.7 - If we consider the fibered space $p^h : J^hE \to M$, with $0 \leq h$, then we have the two bundle structures, for $0 \leq k$,

$$(p^h)^k_0 : J^kJ^hE \to J^hE \quad \text{and} \quad J^kp^h_0 : J^kJ^hE \to J^kE,$$

whose expressions are

$$(x^\lambda, y^i_\Lambda) \circ (p^h)^k_0 = (x^\lambda, y^i_{\Lambda 0}) \quad \text{and} \quad (x^\lambda, y^i_M) \circ J^kp^h_0 = (x^\lambda, y^i_{0\Lambda}) \, .$$

1.8 - There is a unique morphism $\sigma^{(k,h)} : J^{h+k}E \to J^kJ^hE$ over J^hE, such that the following diagram commutes, for each (local) section $s : M \to E$,

$$\begin{tikzcd}
J^{h+k}E \arrow[rr, "\sigma^{(k,h)}"] & & J^kJ^hE \\
& M \arrow[ul, "j^{h+k}s"] \arrow[ur, "j^kj^hs"'] &
\end{tikzcd}$$

Moreover $\sigma^{(k,h)}$ is an embedding. By identification of $J^{h+k}E$ with its image, we have

$$J^{h+k}E = \bigcap_{h'+k'=h+k} J^{k'}J^{h'}E$$

The expression of $\sigma^{(k,h)}$ is

$$(x^\lambda, y^i_{\Lambda M}) \circ \sigma^{(k,h)} = (x^\lambda, y^i_{\Lambda + M}).$$

The subbundle $J^{h+k}E \hookrightarrow J^kJ^hE$ is locally characterized by

$$y^i_{\Lambda M} = y^i_{\Lambda' M'} \quad \Longleftrightarrow \quad \Lambda + M = \Lambda' + M'.$$

Moreover the following diagram commutes

$$\begin{tikzcd}
& J^kE \\
J^{h+k}E \arrow[ur, "p^{h+k}_h"] \arrow[r, "\sigma^{(k,h)}"] \arrow[dr, "p^{h+k}_k"'] & J^kJ^hE \arrow[u, "J^kp^h_0"'] \arrow[d, "(p^h)^k_0"] \\
& J^hE
\end{tikzcd}$$

1.9 - We will often use the canonical isomorphism

$$J^k(E \underset{M}{\times} F) \to J^kE \underset{M}{\times} J^kF.$$

2. - THE TANGENT FUNCTOR AND JET SPACES

T denotes the covariant tangent functor in the category of manifolds and V denotes the covariant vertical functor in the category of fibered spaces.

2.1 - We will be concerned with the tangent bundles of M and of J^kE

$$\pi : TM \to M \qquad \text{and} \qquad \pi_k : TJ^kE \to J^kE$$

and with the bundles tangent to $p^k : J^kE \to M$ and to $p^k_h : J^kE \to J^hE$

$$Tp^k : TJ^kE \to TM \qquad \text{and} \qquad Tp^k_h : TJ^kE \to TJ^hE.$$

The standard charts of TM and of TJ^kE are, respectively

$$(x^\lambda, \dot{x}^\lambda) \qquad \text{and} \qquad (x^\lambda, y^i_\Lambda \,;\, \dot{x}^\lambda, \dot{y}^i_\Lambda)$$

and we have the expressions

$$(x^\lambda) \circ \pi \equiv x^\lambda, \qquad (x^\lambda, y^i_\Lambda) \circ \pi_k \equiv (x^\lambda, y^i_\Lambda),$$

$$(x^\lambda, \dot{x}^\lambda) \circ Tp^k \equiv (x^\lambda, \dot{x}^\lambda), \quad (x^\lambda, y^i_M \,;\, \dot{x}^\lambda, \dot{y}^i_M) \circ Tp^k_h \equiv (x^\lambda, y^i_M \,;\, \dot{x}^\lambda, \dot{y}^i_M),$$

with $o \leqslant |\Lambda| \leqslant k, \quad o \leqslant |M| \leqslant h.$

The following diagram commutes

$$\begin{tikzcd}
TJ^kE \arrow[r, "Tp^k_h"] \arrow[d, "\pi_k"'] \arrow[rr, bend left, "Tp^k"] & TJ^hE \arrow[r, "Tp^h"] \arrow[d, "\pi_h"] & TM \\
J^kE \arrow[r, "p^k_h"] \arrow[rr, bend right, "p^k"'] & J^hE \arrow[r, "p^h"] & M \arrow[u]
\end{tikzcd}$$

and Tp^k_h and Tp^k are linear morphisms.

2.2 - The vertical spaces of $p^k : J^kE \to M$ and of $p^k_h : J^kE \to J^hE$ are the linear sub-bundles of $\pi_k : TJ^kE \to J^kE$

$$VJ^kE \equiv \ker Tp^k \qquad \text{and} \qquad V_hJ^kE \equiv \ker Tp^k_h,$$

which are locally characterized by

$$\dot{x}^\lambda = o \qquad \text{and} \qquad \dot{x}^\lambda = \dot{y}^i_M = 0, \text{ with } 0 \leqslant |M| \leqslant h.$$

And so naturally we have the following sequence of inclusions

$$V_{k-1}J^kE \hookrightarrow \dots \hookrightarrow V_oJ^kE \hookrightarrow VJ^kE \hookrightarrow TJ^kE.$$

The transverse spaces of $p^k : J^kE \to M$ and of $p^k_h : J^kE \to J^hE$ are the pull-back bundles of $\pi : TM \to M$ and of $\pi_h : TJ^hE \to J^hE$ with respect to $p^k : J^kE \to M$ and to $p^k_h : J^kE \to J^hE$, respectively,

$$HJ^kE \equiv J^kE \underset{M}{\times} TM \qquad \text{and} \qquad H_hJ^kE \equiv J^kE \underset{J^hE}{\times} TJ^hE,$$

whose standard charts are

$$(x^\lambda, y^i_\Lambda ; \dot{x}^\lambda) \qquad \text{and} \qquad (x^\lambda, y^i_\Lambda ; \dot{x}^\lambda, \dot{y}^i_M)$$

with $o \leqslant |\Lambda| \leqslant k, \quad o \leqslant |M| \leqslant h.$

And so naturally we have the following sequence of projections

$$TJ^kE \to H_{k-1}J^kE \to \dots \to H_oJ^kE \to HJ^kE.$$

Moreover one has the following exact sequences

$$0 \to VJ^kE \to TJ^kE \to HJ^kE \to 0$$

$$0 \to V_hJ^kE \to TJ^kE \to H_hJ^kE \to 0.$$

2.3 - Moreover we will be concerned with the k-jet spaces of the fibered spaces $\pi : TM \to M$ and $p^h \circ \pi_h : TJ^hE \to M$

$$\pi^k : J^kTM \to M \qquad \text{and} \qquad (p^h \circ \pi_h)^k : J^kTJ^hE \to M,$$

with $o \leqslant h,k$, whose standard charts are

$$(x^\lambda, \dot{x}^\lambda_\Lambda) \qquad \text{and} \qquad (x^\lambda, y^i_{M\Lambda} ; \dot{x}^\lambda_\Lambda, \dot{y}^i_{M\Lambda})$$

with $\quad o \leqslant |\Lambda| \leqslant k, \quad o \leqslant |M| \leqslant h.$

In (5.2) we will use the following result.

2.4 - LEMMA. *Let* $s : M \to E$ *be a (local) section. Then the following diagrams commute*

$$\begin{tikzcd} J^kTE \arrow[r, "J^k\pi_0"] & J^kE \\ J^kTM \arrow[u, "J^kTs"] \arrow[r, "\pi^k"] & M \arrow[u, "J^ks"'] \end{tikzcd} \qquad \begin{tikzcd} J^kTE \arrow[r, "J^kTp"] & J^kTM \\ J^kTM \arrow[u, "J^kTs"] \arrow[ur, "id"'] & J^kTM \end{tikzcd}$$

Proof. The first one follows from (1.6). The second one follows from the functorial properties of J^k and T.

3. - AFFINE STRUCTURES ON JET SPACES

3.0 - Let $a : A \to M$ be an affine bundle (see also [3]). Then we denote by $\bar{a} : \bar{A} \to M$ its vector bundle. We note that $VA = A \underset{M}{\times} \bar{A}$.

Let $b : B \to N$ be a further affine bundle and $F : A \to B$ be an affine morphism over $f : M \to N$. Then we denote by $DF : A \to \mathrm{Hom}(\bar{A},\bar{B})$ the «fibre derivative» of F.

3.1 - In order to exploit further affine structures coming from a given one, we need the following lemmas.

a) LEMMA. *Let* $a : A \to M$ *be an affine bundle. Then* $a^k : J^kA \to M$ *is naturally an affine bundle, whose vector bundle is* $\bar{a}^k : J^k\bar{A} \to M$.

Proof. The covariant functor J^k prolongs naturally the translation $+ : A \underset{M}{\times} \bar{A} \to A$ and its properties to $\overset{k}{+} \equiv J^k+ : J^k\bar{A} \underset{M}{\times} J^kA \to J^kA$, so that, for each (local) section $s : M \to A$ and $v : M \to \bar{A}$, we have

$$j^ks \overset{k}{+} j^kv = j^k(s+v) \quad \text{and} \quad \overset{k}{o} = j^ko.$$

LEMMA. *Let* $f : F \to E$ *be a linear (affine) bundle (whose vector bundle is* $\bar{f} : \bar{F} \to E$), *so that* $q \equiv f \circ p : F \to M$ $(\bar{q} \equiv \bar{f} \circ p : \bar{F} \to M)$ *is a fibered space,* f *and* $J^kf : J^kF \to J^kE$ ($\bar{f}$ *and* $J^k\bar{f}$: $J^k\bar{F} \to J^kE$) *are morphisms over* M.

b) Then $J^kf : J^kF \to J^kE$ is a linear (affine) bundle (whose vector bundle is $J^k\bar{f} : J^k\bar{F} \to J^kE$).

c) Then $q^k_h : J^kF \to J^hF$ is a linear (affine) morphism over $p^k : J^kE \to J^hE$ (whose fibre derivative is $\bar{q}^k_h : J^k\bar{F} \to J^h\bar{F}$), so that the following diagrams commute

$$\begin{tikzcd} J^kF \arrow[r, "q^k_h"] \arrow[d, "J^kf"'] & J^hF \arrow[d, "J^hf"] \\ J^kE \arrow[r, "p^k_h"] & J^hE \end{tikzcd} \qquad \begin{tikzcd} J^k\bar{F} \arrow[r, "\bar{q}^k_h"] \arrow[d, "J^k\bar{f}"'] & J^h\bar{F} \arrow[d, "J^h\bar{f}"] \\ J^kE \arrow[r, "p^k_h"] & J^hE \end{tikzcd}$$

d) Then $(J^kf, q^k_h) : J^kF \to J^kE \underset{J^hE}{\times} J^hF$ is an affine bundle.

Proof. b) It follows, in the same way as (a), by taking into account the commutative diagram

$$\begin{tikzcd} J^kF \underset{J^kE}{\times} J^k\bar{F} \arrow[rr, "J^k+"] \arrow[dr, "J^kf"] \arrow[ddr, "q^k"'] & & J^kF \arrow[ddl, "q^k"] \\ & J^kE \arrow[ur, "J^kf"] \arrow[d, "p^k"] & \\ & M & \end{tikzcd}$$

and sections $(s,v) : M \to F \underset{M}{\times} F$ such that $f \circ s = \bar{f} \circ v : M \to E$.

c) It follows from $q^k_o \circ (j^ks \overset{k}{+} j^kv) = q^k_o \circ j^k(s+v) = s + v$.

d) The fibre of J^kF over $(\sigma,y) \in J^kE \underset{J^hE}{\times} J^hF$ is the affine subspace of the fibre of J^kF over $\sigma \in J^kE$ which is projected onto $y \in J^hF$.

3.2 - We will be concerned with different structures of the space TJ^kE. Besides the vector bundle

$$\pi^k : TJ^kE \to J^kE$$

we will consider the affine bundle

$$\varkappa_k \equiv (\pi_k, Tp^k) : TJ^kE \to HJ^kE \equiv J^kE \underset{M}{\times} TM,$$

whose vector bundle is (up to an obvious pull-back) VJ^kE and the affine bundle

$$\mu_{(k,h)} \equiv (\pi_k, Tp^k_h) : TJ^kE \to H_hJ^kE \equiv J^kE \underset{J^hE}{\times} TJ^hE,$$

whose vector bundle is (up to an obvious pull-back) V_hJ^kE.

3.3 - We have a basic affine structure on jet spaces. Namely

$$p^k_{k-1} : J^kE \to J^{k-1}E$$

is an affine bundle, whose vector bundle is (up to an obvious pull-back)

$$\overline{J^kE} = \underset{k}{\vee} T^*M \otimes VE$$

where $\underset{k}{\vee}$ is the symmetrized tensor product.

Then (see 3.0) we have

$$V_{k-1}J^kE = J^kE \underset{E}{\times} (\underset{k}{\vee} T^*M \otimes VE).$$

3.4 - Moreover, the previous lemmas allow us to discover further linear and affine structures, which will be useful in the following.

PROPOSITION. *a)* $J^k\pi_0 : J^kTE \to J^kE$ *is a vector bundle.*

b) $J^k\pi_0 : J^kVE \to J^kE$ *is a linear sub-bundle of the previous bundle.*

c) $(J^k\pi_0, (p \circ \pi_0)^k_0) : J^kTE \to J^kE \underset{E}{\times} TE$ *is an affine bundle.*

d) $(J^k\pi_0, (\pi_0)^k_{k-1}) : J^kTE \to J^kE \underset{J^{k-1}E}{\times} J^{k-1}TE$ *is an affine bundle, whose vector bundle is (up to an obvious pull-back)* $\underset{k}{\vee} T^*M \to TE$.

e) $(J^k\pi_0, J^kTp) : J^kTE \to J^kE \underset{M}{\times} J^kTM$ *is an affine bundle, whose vector bundle is (up to an obvious pull-back)* J^kVE.

Proof. a) It follows from (3.1b), taking into account the fact that $\pi_0 : TE \to E$ is a vector bundle.
b) It follows from (3.1b) and (3.4a), by taking into account the fact that VE is a subbundle of TE over E.
c) It follows from (3.1d), by taking into account the fact that $\pi_0 : TE \to E$ is a vector bundle.
d) It follows from (3.1d), by taking into account the fact that $\pi_0 : TE \to E$ is a vector bundle, and using (3.3).

e) It follows from (3.1b), by taking into account the fact that (see 3.2) $\mu_0 : TE \to E \underset{M}{\times} TM$ is an affine bundle.

4. - THE FIRST FUNDAMENTAL STRUCTURE ON JET SPACES

4.1 - The exact sequence (2.2)

$$0 \to VJ^kE \to TJ^kE \to HJ^kE \to 0$$

has not in general a canonical splitting, but its pull-back over $J^{k+1}E$ splits.

PROPOSITION. *There is a unique morphism*

$$c^{k+1} : p^{k+1} * TM \to TJ^kE$$

over $p_k^{k+1} : J^{k+1} E \to J^kE$, *such that, for each (local) section* $s : M \to E$, *the following diagram commutes*

$$\begin{tikzcd} J^{k+1}E \underset{M}{\times} TM \arrow[rr, "c^{k+1}"] & & TJ^kE \\ & TM \arrow[ul, "(j^{k+1}s \circ \pi, id_{TM})"] \arrow[ur, "Tj^ks"'] & \end{tikzcd}$$

Moreover c^{k+1} *is a linear morphism, with respect to the fibres, which induces a splitting of the previous sequence over* $J^{k+1}E$.

Its expression is

$$(x^\lambda, y^i_M ; \dot{x}^\lambda, \dot{y}^i_M) \circ c^{k+1} = (x^\lambda, y^i_M ; \dot{x}^\lambda, y^i_{M+\mu}\dot{x}^\mu) \ , \ o \leqslant |M| \leqslant k.$$

Proof. *Uniqueness.* Let $(\sigma,u) \in J^{k+1}E \underset{M}{\times} TM$ be a point over $x \in M$. Let $s : M \to E$ be one of the (local) sections such that $j^{k+1}s(x) = \sigma$. Then we have $c^{k+1}(\sigma,u) = (x^\lambda, y^i_M ; \dot{x}^\lambda, \dot{y}^i_M) \circ Tj^ks(u) = (x^\lambda, \partial_M s^i ; \dot{x}^\lambda, \partial_{M+\mu} s^i \, \dot{x}^\lambda)(u)$.

Existence. The previous formula does not depend on the choice of s.

The complementary linear epimorphism is denoted by

$$\bar{\vartheta}^{k+1} : p_k^{k+1} * TJ^kE \to VJ^kE.$$

Its expression is

$$(x^\lambda, y^i_M ; \dot{y}^i_M) \circ \bar{\vartheta}^{k+1} = (x^\lambda, y^i_M ; \dot{y}^i_M - y^i_{M+\mu} \dot{x}^\mu), \quad o \leqslant |M| \leqslant k.$$

Moreover, taking into account the projection $(\pi_{k+1}, Tp^{k+1}_k) : TJ^{k+1}E \to p^{k+1}_k * TJ^kE$, we can also characterize $\bar{\vartheta}^{k+1}$ by means of the vector valued form

$$\vartheta^{k+1} : J^{k+1}E \to T^*J^{k+1}E \otimes VJ^kE.$$

Its expression is

$$\vartheta^{k+1} = (dy^i_M - y^i_{M+\mu}\, dx^\mu) \otimes \partial_i, \quad o \leqslant |M| \leqslant k.$$

We call c^{k+1} (or, equivalently, $\bar{\vartheta}^{k+1}$, or ϑ^{k+1}) the FIRST FUNDAMENTAL STRUCTURE of order k+1 on jet spaces.

4.3 - c^{k+1} and c^{h+1}, with $o \leqslant h < k$, are related by the following commutative diagram, which follows immediately from the local expressions,

$$\begin{tikzcd} p^{k+1} * TM \arrow[r, "c^{k+1}"] \arrow[d, "p^{k+1}_{h+1} \times id_{TM}"'] & TJ^kE \arrow[d, "Tp^k_h"] \\ p^{h+1} * TM \arrow[r, "c^{h+1}"] & TJ^hE \end{tikzcd}$$

4.4 - The first fundamental structure is functorially invariant

PROPOSITION. *Let* $H : E \to F$ *be a morphism over the diffeomorphism* $h : M \to N$. *Then the following diagram commutes*

$$\begin{tikzcd} p^{k+1} * TM \arrow[r, "c^{k+1}"] \arrow[d, "J^{k+1}H \times Th"'] & TJ^kE \arrow[d, "TJ^kH"] \\ q^{k+1} * TN \arrow[r, "c^{k+1}"] & TJ^kF \end{tikzcd}$$

Proof. It follows by taking into account the diagram

$$\begin{tikzcd}[column sep=huge]
TM \arrow[r, "(j^{k+1}s \circ \pi, id_{TM})"] \arrow[d, "Th"'] \arrow[rr, bend left, "Tj^ks"] & J^{k+1}E \underset{M}{\times} TM \arrow[d, "J^{k+1}H \times Th"] & TJ^kE \arrow[d, "TJ^kH"] \\
TN \arrow[r, "(j^{k+1}H^*s \circ \pi, id_{TN})"] \arrow[rr, bend right, "Tj^kH^*s"'] & J^{k+1}F \underset{N}{\times} TN & TJ^kF
\end{tikzcd}$$

which commutes by the properties of the covariant functors J^k and T, and the diagrams

$$\begin{tikzcd}[column sep=huge]
TM \arrow[r, "(j^{k+1}s \circ \pi, id_{TM})"] \arrow[rr, bend left, "Tj^ks"] & J^{k+1}E \underset{M}{\times} TM \arrow[r, "c^{k+1}"] & TJ^kE \\
TN \arrow[r, "(j^{k+1}H^*s \circ \pi, id_{TN})"] \arrow[rr, bend right, "TjkH^*s"'] & J^{k+1}F \underset{N}{\times} TN \arrow[r, "c^{k+1}"] & TJ^kF
\end{tikzcd}$$

which commute by the definition of c^{k+1}, where $s : M \to E$ is a (local) section and $H^*s \equiv H \circ s \circ h^{-1} : N \to F$.

4.5 - The form ϑ^{k+1} determines a differential system $\Delta^{k+1} \equiv \ker \vartheta^{k+1} \hookrightarrow TJ^{k+1}E$ or, equivalently,

$$(\Delta^{k+1})^{\perp} \equiv (\ker \vartheta^{k+1})^{\perp} \hookrightarrow T^*J^{k+1}E.$$

Locally, Δ^{k+1} is generated by the vector fields $\partial_\mu + y^i_{M+\mu}\, \partial^M_i$ and ∂^Λ_i and $(\Delta^{k+1})^{\perp}$ is generated by the forms

$$\alpha^i_M \equiv dy^i_M - y^i_{M+\mu}\, dx^\mu, \quad 0 \leqslant |M| \leqslant k, \quad |\Lambda| = k+1.$$

Δ^{k+1} has constant rank and is not involutive. In fact we can easily prove that the exterior product of $d\alpha^i_M$ times all the α is different from 0.

Moreover we note that $(d\alpha^i_M)^m \wedge \alpha^i_M = 0$. Hence α^i_M can be viewed as a «contact form». More precisely, in the particular case of $E \equiv M \times IR$, we have $JE = IR \times T^*M$ and ϑ^1 is the veritable contact form induced by the Liouville form. For this reason, we call Δ^{k+1} the CONTACT SYSTEM of degree k+1.

4.6 - The fundamental form characterizes the sections of $p^{k+1} : J^{k+1}E \to M$ which are the lifts of their projections on E.

PROPOSITION. *Let* $S : M \to J^{k+1}E$ *be a section. Then the following conditions are equivalent*

a) $S = j^{k+1}(p^{k+1} \circ S)$

b) $TS : TM \to \Delta^{k+1}$.

Proof. It follows from the local expression by induction on $|M|$ such that $o \leqslant |M| \leqslant k+1$.

4.7 - PROBLEM. Is there a way to get a splitting of the sequence

$$0 \to V_h J^k E \to TJ^k E \to H_h J^k E \to 0$$

up to a suitable pull-back ?

5. - THE SECOND FUNDAMENTAL STRUCTURE ON JET SPACES

We can introduce a new fundamental map, which accounts for the exchange between J^k, j^k and T, in terms of a morphism between the spaces $J^k TE$ and $TJ^k E$.

5.1 - First we recall the following map (see also [3]).

LEMMA. *There is a unique morphism*

$$i^k : J^k VE \to VJ^k E$$

over $J^k E$, such that, for each 1-parameter family of (local) sections $\sigma : IR \times M \to E$, the following diagram commutes

$$\begin{tikzcd}
J^k VE \arrow[rr, "i^k"] & & VJ^k E \\
& M \arrow[ul, "j^k \partial \sigma"] \arrow[ur, "\partial j^k \sigma"'] &
\end{tikzcd}$$

where ∂ *is the «variational derivative», i.e. the derivative with respect to the parameter evaluated at* $o \in \mathrm{IR}$. *Moreover* i^k *is linear over* J^kE *(see 3.4b) and is an isomorphism. Its expression is*

$$(x^\lambda, y^i_\Lambda ; \dot{y}^i_\Lambda) \circ i^k = (x^\lambda, y^i_\Lambda ; \dot{y}^i_\Lambda).$$

Proof. We have

$$(x^\lambda, y^i_\Lambda ; y^i_\Lambda) \circ j^k \partial\sigma = (x^\lambda, \partial_\Lambda \sigma^i{}_{|o} ; \partial_\Lambda \partial\sigma^i) = (x^\lambda, \partial_\Lambda \sigma^i{}_{|o} ; \partial\partial_\Lambda \sigma^i) = (x^\lambda, y^i_\Lambda ; \dot{y}^i_\Lambda) \circ \partial j^k \sigma.$$

5.2 - PROPOSITION. *There is a unique affine morphism (see 3.4d and 3.2)*

$$r^k : J^kTE \to TJ^kE$$

over

$$\mathrm{id}_{J^kE} \times \pi^k : J^kE \underset{M}{\times} J^kTM \to J^kE \underset{M}{\times} TM$$

such that

a) *for each (local) section* $s : M \to E$, *the following diagram commutes*

$$\begin{tikzcd}
J^kTE \arrow[r, "r^k"] & TJ^kE \\
J^kTM \arrow[u, "J^kTs"] \arrow[r, "(\pi)^k_o"] & TM \arrow[u, "Tj^ks"']
\end{tikzcd}$$

b) $Dr^k = i^k$ *(see 3.0).*

Moreover r^k *is a surjective map and its local expression is*

$$(x^\lambda, y^i_\Lambda ; \dot{x}^\lambda, \dot{y}^i_\Lambda) \circ r^k = (x^\lambda, y^i_\Lambda ; \dot{x}^\lambda, \dot{y}^i_\Lambda - y^i_{\phi+\mu} \dot{x}^\mu_\psi),$$

with $o \leqslant |\Lambda| \leqslant k$ *and where the summation is extended to all the multi-indices such that* $\phi + \psi = \Lambda$, $|\psi| > 0$.

Proof. *Uniqueness.* Let $(\sigma, u^k) \in J^kE \underset{M}{\times} J^kTM$ be a point which is projected on $(\sigma, u) \in J^kE \underset{M}{\times} TM$ by the map $\mathrm{id}_{J^kE} \times \pi^k$ and let $x \in M$ be the common basis of σ, u^k, u. Let $s : M \to E$ be one of the (local) sections such that $j^ks(x) = \sigma$. Then (see 2.4) $a \equiv J^kTs(u^k) \in J^kTE$ and $b \equiv Tj^ks(u) \in TJ^kE$ are, respectively, points of the affine fibres of

$$(J^k\pi_o, J^kTp) : J^kTE \to J^kE \underset{M}{\times} J^kTM \text{ over } (\sigma, u^k)$$

and of

$$(\pi_k, Tp^k) : TJ^kE \to J^kE \underset{M}{\times} TM \text{ over } (\sigma, u).$$

Hence there is a unique affine morphism r^k between these affine fibres, such that $r^k(a) = b$ and $Dr^k = i^k$.

Existence. The local expression shows that this map r^k does not depend on the choice of s.

We call r^k the SECOND FUNDAMENTAL STRUCTURE of order k on jet spaces. The local expression of r^k shows the further properties.

5.3 - PROPOSITION.

a) r^k *is a linear morphism (see 3.4a) over* J^kE.

b) r^k *is an affine morphism (see 3.4c) over* $J^kE \underset{E}{\times} TE$.

5.4 - We can view i^k as the restriction of r^k.

PROPOSITION. *The following diagram commutes (see 5.1 and 5.4b)*

$$\begin{tikzcd} J^kTE \arrow[r, "r^k"] & TJ^kE \\ J^kVE \arrow[u, hook] \arrow[r, "i^k"] & VJ^kE \arrow[u, hook] \end{tikzcd}$$

5.5 - r^k and r^h, with $o \leqslant h < k$, are related by the following commutative diagram

$$\begin{tikzcd} J^kTE \arrow[r, "r^k"] \arrow[d, "(p\circ\pi_o)^k_h"'] & TJ^kE \arrow[d, "Tp^k_h"] \\ J^hTE \arrow[r, "r^h"] & TJ^hE \end{tikzcd}$$

5.6 - We have a relation between r^k and ϑ^1.

PROPOSITION. r^k *is an affine morphism (see 3.4d) over*

$$id_{J^kE} \times r^{k-1} : J^kE \underset{J^{k-1}E}{\times} J^{k-1}TE \to J^kE \underset{J^{k-1}E}{\times} TJ^{k-1}E$$

whose fibre derivative is (up to an obvious pull-back)

$$\mathrm{id}_{\underset{k}{\vee} T^*M} \otimes \bar{\vartheta}' : \underset{k}{\vee} T^*M \otimes TE \to \underset{k}{\vee} T^*M \otimes VE.$$

5.7 - The second fundamental structure is functiorially invariant.

PROPOSITION. *Let* H : E → F *be a morphism over the diffeomorphism* h : M → N. *Then the following diagrams commute*

$$a)\quad \begin{tikzcd} J^kVE \arrow[r, "i^k"] \arrow[d, "J^kVH"'] & VJ^kE \arrow[d, "VJ^kH"] \\ J^kVF \arrow[r, "i^k"] & VJ^kF \end{tikzcd} \qquad b)\quad \begin{tikzcd} J^kTE \arrow[r, "r^k"] \arrow[d, "J^kTH"'] & TJ^kE \arrow[d] \\ J^kTF \arrow[r, "r^k"] & TJ^kF \end{tikzcd}$$

Proof. a) Let $a \in J^kVE$ be a point which is projected on $x \in M$. Then there is a 1-parameter family of (local) sections $\sigma : \mathrm{IR} \times M \to E$, *such that* $j^k\partial\sigma(x) = a$. Then the proof follows from the commutative diagrams

$$\begin{tikzcd} & M \arrow[dl, "j^k\partial\sigma"'] \arrow[dr, "\partial j^k\sigma"] & \\ J^kVE \arrow[rr, "i^k"] & & VJ^kE \end{tikzcd} \qquad \begin{tikzcd} J^kVF \arrow[rr, "i^k"] & & VJ^kF \\ & N \arrow[ul, "j^k\partial H^*\sigma"] \arrow[ur, "\partial j^kH^*\sigma"'] & \end{tikzcd}$$

$$\begin{tikzcd} & M \arrow[dl, "j^k\partial\sigma"'] \arrow[dr, "\partial j^k\sigma"] \arrow[ddd, "h"] & \\ J^kVE \arrow[d, "J^kVH"'] & & VJ^kE \arrow[d, "VJ^kH"] \\ J^kVF & & VJ^kF \\ & N \arrow[ul, "j^k\partial H^*\sigma"] \arrow[ur, "\partial j^kH^*\sigma"'] & \end{tikzcd}$$

b) Let $(\sigma,u) \in J^kE \underset{M}{\times} J^kTM$ be a point which is projected on $x \in M$ and let $s : M \to E$ be a (local) section, such that $j^ks(x) = \sigma$.

Then, from the commutative diagrams

$$\begin{tikzcd} J^kTM \arrow[r, "(\pi)^k_o"] \arrow[d, "J^kTs"'] & TM \arrow[d, "Tj^ks"] \\ J^kTE \arrow[r, "r^k"] & TJ^kE \end{tikzcd}$$

$$\begin{tikzcd} J^kTM \arrow[r, "J^kTs"] \arrow[d, "J^kTh"'] & J^kTE \arrow[d, "J^kTH"] \\ J^kTN \arrow[r, "J^kTH*s"] & J^kTF \end{tikzcd} \qquad \begin{tikzcd} TJ^kE \arrow[r, "Tj^ks"] \arrow[d, "TJ^kH"'] & TM \arrow[d, "Th"] \\ TJ^kF \arrow[r, "Tj^kH*s"] & TN \end{tikzcd}$$

$$\begin{tikzcd} J^kTF \arrow[r, "r^k"] \arrow[d, "J^kTH*s"'] & TJ^kF \arrow[d, "Tj^kH*s"] \\ J^kTN \arrow[r, "(\pi)^k_o"] & TN \end{tikzcd}$$

it follows that the following diagram commutes with respect to the point $J^kTs(u) \in J^kTE$ which is on the fibre $(\sigma,u) \in J^kE \underset{M}{\times} J^kTM$

$$\begin{tikzcd} J^kTE \arrow[r, "r^k"] \arrow[d, "J^kTH"'] & TJ^kE \arrow[d, "TJ^kH"] \\ J^kTF \arrow[r, "r^k"] & TJ^kF \end{tikzcd}$$

Moreover, the diagram (a) can be viewed as the fibre derivative of (b). Then (b) commutes for each point of the fibre (σ,u) $J^kE \underset{M}{\times} J^kTM$.

5.8 - r^k allows the exchange between ∂ and J^k. This result will be utilized in (7.5).

PROPOSITION. *Let* $H : \mathbb{R} \times E \to E$ *be a* 1*-parameter family of morphisms over the* 1*-parameter family of diffeomorphisms* $h : \mathbb{R} \times M \to M$, *such that* $H_o = id_E$ (*hence* $h_o = id_M$). *Then*

$$\partial J^k H = r^k \circ J^k H : J^k E \to TJ^k E.$$

Proof. We have (see 1.2)

a) $$x^\alpha \circ J^k H = h_o^\alpha = x^\alpha$$

b) $$y_A^i \circ \partial J^k H = \partial_j H_o^i \, y_M^j \, (\partial_1 h_o^1)^{b(1)} \dots (\partial_m h_o^m)^{b(m)} = y_A^i$$

for

$$H_o^i = y^i \,, \quad h_o^\alpha = x^\alpha,$$

$$B^{(\alpha,1)} = \dots = B^{(\alpha,M_\alpha)} = 0 + \alpha,$$

$$B^{(1,1)} + \dots + B^{(m,b(m))} = A, \quad \text{hence} \quad M = A \,;$$

c) $$\dot{x}^\alpha \circ \partial J^k H = \partial h^\alpha \equiv u^\alpha \,;$$

d) $\dot{y}_A^i \circ \partial J^k H =$

$$= \partial_\Lambda \, \partial_I \, \partial H^i \, y^1_{M(1,1)} \dots y^l_{M(l,I_l)} \, (\partial_1 h_o^1)^{b(1)} \dots (\partial_m h_o^m)^{b(m)} -$$

$$- \partial_j H_o^i \, y_m^j (\partial_1 h_o^1)^{b(1)} \dots \partial_{B(\mu,1)} \partial h^\mu (\partial_\mu h_o^\mu)^{b(\mu)-1} \dots (\partial_m h_o^m)^{b(m)}$$

$$= \partial_\Lambda \, \partial_I u^i \, y^1_{M(1,1)} \dots y^l_{M(l,I_l)} - y^i_{N+\mu} \, \partial_B u^\mu$$

with

$$\Lambda + M^{(1,1)} + \dots + M^{(l,I_l)} = A,$$

$$I_h > 0 \Rightarrow |M^{(h,1)}|, \dots, |M^{(h,I_h)}| > 0,$$

for

$$b(\mu) = \Lambda_\mu + M_\mu^{(1,1)} + \dots + M_\mu^{(l,I_l)},$$

$$B^{(\mu,1)} + \dots + B^{(\mu,b(\mu))} = 0 + b(\mu),$$

hence

$$\Lambda + M^{(1,1)} + \dots + M^{(l,I_l)} = A,$$

and with $\quad N + B = A,$

for $b(\lambda) = M_\lambda$,

$$A_1 = M_1 + B_1 ,\dots, A_\mu = B_\mu + M_\mu - 1 ,\dots, A_m = M_m + B_\mu ,$$

hence $N + B = A$, where $M = N + \mu$.

5.9 - r^k also allows the exchange between ∂ and j^k.

PROPOSITION. *Let* $h : \mathbb{R} \times M \to M$ *be a* 1*-parameter family of maps such that* $h_o = id_M$. *Then, for each (local) section* $s : M \to E$, *the following diagram commutes*

$$\begin{tikzcd}
J^kTE \arrow[rr, "r^k"] & & TJ^kE \\
& M \arrow[ul, "j^k\partial(s\circ h)"] \arrow[ur, "\partial(j^ks\circ h)"'] &
\end{tikzcd}$$

Proof. It follows from the following commutatives diagrams

$$\begin{tikzcd}
& J^kTE \\
M \arrow[ur, "j^k(s\circ h)"] \arrow[r, "j^k\partial h"'] & J^kTM \arrow[u, "J^kTs"']
\end{tikzcd}
\qquad
\begin{tikzcd}
J^kTE \arrow[r, "r^k"] & TJ^kE \\
J^kTM \arrow[u, "J^kTs"] \arrow[r, "(\pi)^k_o"'] & TM \arrow[u, "Tj^ks"']
\end{tikzcd}$$

$$\begin{tikzcd}
& & TJ^kE \\
M \arrow[urr, "\partial j^k(s\circ h)"] \arrow[r, "j^k\partial h"] \arrow[rr, bend right, "\partial h"'] & J^kTM \arrow[r, "(\pi)^k_o"] & TM \arrow[u, "Tj^ks"']
\end{tikzcd}$$

5.10 - By replacing $p : E \to M$ with $p^h : J^hE \to M$ in the definition of r^k, we obtain the map

$$r^{(k,h)} : J^kTJ^hE \to TJ^kJ^hE \qquad\qquad 0 \leqslant h,k.$$

Its expression is

$$(x^\lambda, y^i_{\Lambda M}\, ; \dot{x}^\lambda, \dot{y}^i_{\Lambda M}) \circ r^{(k,h)} = (x^\lambda, y^i_{\Lambda M}\, ; \dot{x}^\lambda, \dot{y}^i_{\Lambda M} - y^i_{\Lambda\ \phi+\mu}\, \dot{x}^\mu_\psi\,),$$

with

$$0 \leqslant |\Lambda| \leqslant h\,, \quad 0 \leqslant |M| \leqslant k\,, \quad \phi + \psi = M\,, \qquad |\psi| > 0\,.$$

In particular we have $r^{(k,o)} = r^k$. We may also note that the image of $r^{(k,h)}$ is larger than the subspace

$$TJ^{h+k}E \hookrightarrow TJ^kJ^hE.$$

6. - INFINITESIMAL CONTACT TRANSFORMATIONS ON JET SPACES

We can introduce the infinitesimal contact transformations of any order, by using both the first and the second fundamental structures.

6.1 - DEFINITION. *An INFINITESIMAL CONTACT TRANSFORMATION is a vector field*

$$u : J^{k+1}E \to TJ^{k+1}E$$

such that

$$L_u\, \Delta^{k+1} \subset \Delta^{k+1}.$$

Of course, if u, v are i.c.t., then [u,v] is an i.c.t.

PROPOSITION. *Let* $u : J^{k+1}E \to TJ^{k+1}E$ *be a vector field, whose expression is*

$$u = u^\lambda \partial_\lambda + u^i_\Lambda\, \partial^\Lambda_i + u^i_\theta\, \partial^\theta_i\,, \quad \textit{with} \quad 0 \leqslant |\Lambda| \leqslant k, \quad |\theta| = k+1.$$

Then the following conditions are equivalent :

a) u *is an i.c.t.*

b) $u^i_{\Lambda+\lambda} = (\partial_\lambda u^i_\Lambda + \partial^M_j\, u^i_\Lambda\, y^j_{M+\lambda}) - y^i_{\Lambda+\mu}\, (\partial_\lambda u^\mu + \partial^M_j\, u^\mu\, y^j_{M+\lambda}\,)$

$$\partial_j^\theta u_\Lambda^i = y_{\Lambda+\lambda}^i\, \partial_j^\theta u^\lambda\,, \qquad \textit{with} \qquad 0 \leqslant |M| \leqslant k.$$

Proof. It follows from the expression of $L_u v$, with $v : J^{k+1}E \to \Delta^{k+1}$.

By extension we may naturally consider each vector field $u : E \to TE$ as an i.c.t. of order 0.

6.2 - We can characterize the i.c.t. by means of r^k, $r^{(1,k-1)}$ and $r^{(1,k)}$.

LEMMA. *Let* $u : J^kE \to TJ^kE$ *be a vector field. Then*

$$r^{(1,k-1)} \circ J^1(Tp_{k-1}^k \circ u) = TJ^1 p_{k-1}^k \circ r^{(1,k)} \circ J^1 u : J^{k+1}E \to TJ^1 J^{k-1}E.$$

Proof. It follows from the local expression.

PROPOSITION. *Let* $u : J^kE \to TJ^kE$ *be a vector field. Then the following conditions are equivalent :*

a) u *is an i.c.t.*

b) The following diagram commutes

$$\begin{tikzcd} J^{2k}E \arrow[r, "J^k(Tp_0^k \circ u)"] \arrow[d, "p_k^{2k}"'] & J^kTE \arrow[d, "r^k"] \\ J^kE \arrow[r, "u"] & TJ^kE \end{tikzcd}$$

c) The following diagram commutes

$$\begin{tikzcd} J^{k+1}E \arrow[r, "r^{(1,k-1)} \circ J^1(Tp_{k-1}^k \circ u)"] & TJ^1J^{k-1}E \\ J^kE \arrow[u, "p_k^{k+1}"] \arrow[r, "u"] & TJ^kE \arrow[u, "T\sigma^{(1,k-1)}"'] \end{tikzcd}$$

d) The following diagram commutes

$$\begin{array}{ccc} J^{k+1}E & \xrightarrow{\quad TJ^1 p^k_{k-1} \circ r^{(1,k)} \circ J^1 u \quad} & TJ^1 J^{k-1}E \\ \Big\uparrow{\scriptstyle p^{k+1}_k} & & \Big\uparrow{\scriptstyle T\sigma^{(1,k-1)}} \\ J^k E & \xrightarrow{\qquad\qquad u \qquad\qquad} & TJ^k E \end{array}$$

Proof. a) $\Longleftrightarrow$ b). It follows, after a long calculation, by induction on $|\Lambda|$ such that $0 \leqslant |\Lambda| \leqslant k$, by the formula

$$\dot{y}^i_\Lambda \circ r^k \circ J^k(Tp^k_o \circ u) = \partial_A\, \partial_I u^i \ldots y^h_{\theta 1+\psi(h,1;1)} \cdots y^h_{\theta l_k+\psi(h,l_k;\mathcal{I}(h,l_k))} \cdots -$$

$$- y^i_{B+\mu}\, \partial_C \partial_J u^\mu \ldots y^h_{\theta 1+\phi(h,1;1)} \cdots y^h_{\theta l_k+\phi(h,l_k;J(h,l_k))}$$

with

$$0 \leqslant |\Lambda| \leqslant k\ ,\quad 0 \leqslant |\theta^a| \leqslant k\,,\quad 1 \leqslant h \leqslant l$$

$$A + \psi^{(1,1;1)} + \ldots + {}^{(l,l_k;\mathcal{I}(l,l_k))} = B + C + \phi^{(1,1;1)} + \ldots + \phi^{(l,l_k;\mathcal{I}(l,l_k))} = \Lambda$$

$$l(a,b) > 0 \;\Rightarrow\; |\psi(a,b;1)|, \ldots, |\psi(a,b;l(a,b))| > 0$$

$$J(a,b) > 0 \;\Rightarrow\; |\phi(a,b;1)|, \ldots, |\phi(a,b;J(a,b))| > 0$$

$$|C + \phi^{(h,1;1)} + \ldots + \phi^{(h,l_k;J(h,l_k))}| > 0$$

a) $\Longleftrightarrow$ c). It follows from the formula

$$(\dot{y}^i_\Sigma, \dot{y}^i_{\Sigma\mu}) \circ r^{(1,k-1)} \circ J^1(Tp^k_{k-1} \circ u) = (u^i_\Sigma, \partial_\mu u^i_\Sigma + \partial^M_j u^i_\Sigma\, y^j_{M+\mu} - y^i_{\Sigma+\lambda}(\partial_\mu u^\lambda + \partial^M_j u^\lambda y^j_{M+\mu}))$$

with

$$0 \leqslant |\Sigma| \leqslant k-1.$$

c) $\Longleftrightarrow$ d). It follows from the previous lemma.

7. - PROLONGATION OF VECTOR FIELDS ON JET SPACES

7.1 - We recall the following

DEFINITION. *Let* $u^k : J^kE \to TJ^kE$ *be a vector field. Then we say that* u^k *is projectable on* $u : M \to TM$, *or on* $u^h : J^hE \to TJ^hE$, *with* $0 \leqslant h < k$, *if the following diagrams commute, respectively,*

$$\begin{tikzcd} J^kE \arrow[r, "u^k"] \arrow[d, "p^k"'] & TJ^kE \arrow[d, "Tp^k"] \\ M \arrow[r, "u"] & TM \end{tikzcd} \qquad \begin{tikzcd} J^kE \arrow[r, "u^k"] \arrow[d, "p^k_h"'] & TJ^kE \arrow[d, "Tp^k_h"] \\ J^hE \arrow[r, "u^h"] & TJ^hE \end{tikzcd}$$

7.2 - THEOREM. *Let* $u^o : E \to TE$ *be a vector field. Then there is a unique i.c.t.* $u^k : J^kE \to TJ^kE$, *which is projectable on* u^o. *Namely, we have*

$$u^k = r^k \circ J^k u^o.$$

The expression of

$$u^k = u^\lambda \partial_\lambda + u^i_\Lambda \partial^\Lambda_i, \qquad 0 \leqslant |\Lambda| \leqslant k,$$

is given by

$$u^i_\Lambda = \partial_\phi \partial_I u^i u^1_{\psi(1,1)} \cdots y^1_{\psi(1,I_1)} \cdots y^l_{\psi(l,1)} \cdots y^l_{\psi(l,I_l)} -$$

$$- y^i_{M+\mu} \partial_\theta \partial_J u^\mu y^1_{\Sigma(1,1)} \cdots y^1_{\Sigma(1,J_1)} \cdots y^l_{\Sigma(l,1)} \cdots y^l_{\Sigma(l,J_l)}$$

with

$$\phi + \psi^{(1,1)} + \ldots + \psi^{(l,I_l)} = M + \theta + \Sigma^{(1,1)} + \ldots + \Sigma^{(l,J_l)} = \Lambda$$

$$I_h > 0 \Rightarrow |\psi^{(h,1)}|, \ldots, |\psi^{(h,I_h)}| > 0$$

$$J_h > 0 \Rightarrow |\Sigma^{(h,1)}|, \ldots, |\Sigma^{(h,J_h)}| > 0$$

$$|M| < |\Lambda|.$$

Proof. *Uniqueness.* If u^k is an i.c.t. which is projectable on u^o, then it is determined by u^o, taking into account its inductive expression (or more precisely 6.2b).

Existence. $r^k \circ J^k u^o$ is an i.c.t. for (6.2b).

We call

$$u^k \equiv r^k \, . \, u^o \equiv r^k \circ J^k u^o$$

the PROLONGATION of order k of u^o.

7.3 - There is a natural relation among the prolongations of u^o at different orders.

PROPOSITION. *Let* $u^o : E \to TE$ *be a vector field. Then* u^k *is projectable on* u^h, *for* $0 \leqslant h < k$.

Proof. It will result from the local expressions of u^k and u^h.

7.4 - PROPOSITION. *The map* $u^o \mapsto u^k$ *is a homomorphism of Lie algebras, i.e.*

$$r^k \, . \, [u^o, v^o] = [r^k \, . \, u^o, r^k v^o] \, .$$

Proof. It will result, after a long calculation, by induction on $|\Lambda|$ such that $0 \leqslant |\Lambda| \leqslant k$, taking into account the local expression.

7.5 - We find an equivalent prolongation in the particular case of projectable vector fields. First we recall the following obvious

LEMMA. *Let* $u^o : E \to TE$ *be a vector field. Then the following conditions are equivalent.*

a) The flow of u^o $H : \mathrm{IR} \times E \to E$ *is a* 1*-parameter group of (local) isomorphisms over the* 1*-parameter group of (local) diffeomorphisms* $h : \mathrm{IR} \times M \to M$.

b) u^o *is projectable on* $u : M \to TM$.

Moreover, in such a case, we have

$$u^o = \partial H \qquad\qquad u = \partial h.$$

In particular, the following conditions are equivalent.

a') The flow of u^o $H : \mathrm{IR} \times E \to E$ *is a* 1*-parameter group of (local) isomorphisms over* M.

b') u^o *is vertical.*

PROPOSITION. *Let* $u^0 : E \to TE$ *be a vector field projectable on* $u : M \to TM$ *and let* $H : \mathbb{R} \times E \to E$ *be its flow over* $h : \mathbb{R} \times M \to M$. *Then we have*

$$u^k = \partial J^k H.$$

Proof. It results from (5.8).

7.6 - We could try to prolong any vector field $u^h : J^hE \to TJ^hE$ to $u^k : J^kE \to TJ^kE$, with $0 \leqslant h < k$, by means of $r^{(k-h,h)}$, but we find that

$$u^k \equiv r^{(k-h,h)} \circ J^{k-h}u^h : J^kE \to TJ^{k-h}J^hE$$

generally does not take its values in

$$TJ^kE \hookrightarrow TJ^{k-h}J^hE.$$

We can find the local necessary and sufficient condition for it to hold. In particular we have the following condition.

PROPOSITION. *Let* $u^h : J^h E \to TJ^hE$ *be a vector field. If* $r^{(k-h,h)} \circ J^{k-h}u^h$ *is factorizable through a vector field* $u^k : J^kE \to TJ^kE$, *i.e. if the following diagram commutes*

$$\begin{tikzcd}
J^{k-h}TJ^hE \arrow[r, "r^{(k-h,h)}"] & TJ^{k-h}J^hE \\
J^kE \arrow[u, "J^{k-h}u^h"] \arrow[r, "u^k"] & TJ^kE \arrow[u, "T\sigma^{(k-h,h)}"']
\end{tikzcd}$$

then u^h *is an i.c.t.*

Proof. It results from the local expression.

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