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LAURENT VÉRON

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GLOBAL BEHAVIOUR AND SYMMETRY PROPERTIES OF SINGULAR SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS

Laurent Véron (1)

(1) Département de Mathématiques, Faculté des Sciences, Parc de Grandmont, 37200 Tours.

Résumé : Nous étudions le comportement des solutions de (E) : $\Delta u = g(u)$ dans un domaine extérieur Ω, lorsque g est une fonction croissante. Si g ne s'annulle qu'en 0 et $u(x) = o(|x|)$, $|x|^{N-2}\, u(x)$ admet une limite isotrope quand x tend vers l'infini. Quand g se comporte asymptotiquement comme une fonction puissance nous recherchons à quelle condition sur cette puissance toutes les solutions de (E) dans $\mathbb{R}^N - \{0\}$ sont à symétrie sphérique. Sous des hypothèses plus restrictives portant sur g nous montrons l'unicité d'une solution de (E) avec une singularité donnée en 0.

Summary : We investigate the behaviour of any solution of (E) : $\Delta u = g(u)$ in some exterior domain Ω, where g is a nondecreasing function. If g vanishes only at 0 and $u(x) = o(|x|)$, $|x|^{N-2}\, u(x)$ admits an isotropic limit when x tends to infinity. When g has a power-like growth we study under what condition on that power all the solutions of (E) in $\mathbb{R}^N - \{0\}$ are spherically symmetric. Under a more restrictive assumption on g we prove the uniqueness of a solution of (E) with a prescribed singularity at 0.

INTRODUCTION

This paper deals with the study of some local and global qualitative properties of any solution of the equation

$$\text{(E)} \qquad -\Delta u + g(u) = 0,$$

in some exterior domain Ω of IR^N, where g is a nondecreasing function defined on IR. More precisely we shall investigate the three following problems

(I) What is the asymptotic behaviour of u(x) when x tends to infinity ?

(II) If we suppose that $\Omega = IR^N - \{0\}$ and that u is possibly singular at 0, is u spherically symmetric ?

(III) When any possibly singular solution of (E) in $IR^N - \{0\}$ is uniquely determined ?

As that type of equation appeared in the modelisation of many physical phenomena, it has been intensively studied in supposing first that u is positive and radial and $g(u) = u^q$. For example the Thomas-Fermi theory of interaction among atoms leads, as a first approximation, to the following differential equation (see [12], and [9])

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} - u^{3/2} = 0. \tag{0.1}$$

The singularities and the asymptotic behaviour of any solution of (0.1) are now well known (see [12] and [9]). Recently some new results concerning the asymptotic behaviour and the description of the isolated singularities of non positive solutions of (E) when $g(u) = |u|^{q-1}u$ has been given in [14] and [16]. Those results where strongly linked to the existence of a very simple solutions of (E) in $IR^N - \{0\}$ if $1 < q < \frac{N}{N-2}$:

$$u_s(x) = \left[\left(\frac{2}{q-1}\right)\left(\frac{2q}{q-1} - N\right)\right]^{1/(q-1)} |x|^{-2/(q-1)}. \tag{0.2}$$

Moreover when $1 < q < \frac{N+1}{N-1}$ an infinite family of non-isotropic solutions of (E) was obtained under the following form

$$u(x) = |x|^{-2/(q-1)}\, v\left(\frac{x}{|x|}\right), \tag{0.3}$$

where v is any non constant solution of

$$-\Delta_{S^{N-1}} v + |v|^{q-1}v = \left(\frac{2}{q-1}\right)\left(\frac{2q}{q-1} - N\right) v \quad \text{on } S^{N-1}, \tag{0.4}$$

$\Delta_{S^{N-1}}$ being the Laplace-Beltrami operator on S^{N-1}.

However, as a physical law is just an approximation of a phenomena, it is natural to replace the exactitude of the definition of g by a less restrictive assumption if we want to take into account some secondary effects, for example $g(r) \underset{r \to +\infty}{\sim} c\, r^q$ (q = 3 in the Relativistic Thomas-Fermi Theory). So we no longer have explicit solutions of the equation (E), but in using some of the methods introduced in [5] and [16] we can give answers to the three problems

(I) *Suppose* g *vanishes only at* 0 *and* $u(x) = o(|x|)$ *or* g *vanishes at* 0 *and* $\lim_{|x|\to+\infty} u(x) = 0$, *then* $|x|^{N-2} u(x)$ *converges to some real number* γ *as* x *tends to infinity.*

(II) *Suppose* g *satisfies*

$$\liminf_{|r|\to+\infty} |g(r)|/|r|^{(N+1)/(N-1)} = +\infty, \tag{0.5}$$

or

$$(g(r) - g(s))(r-s) \geqslant c\,|r-s|^{2N/(N-1)} - d\,|r-s|^2, \quad \textit{for } c,d > 0, \tag{0.6}$$

or

$$\begin{cases} \lim_{r\to+\infty} (g(r) - c\,r^q)r^{-(q-1)(N+1)/2} = 0, \quad \textit{for some } 1 < q < \dfrac{N+1}{N-1}, \\ \\ \limsup_{r\to 0^+} g(r)/r < +\infty \quad \textit{and} \quad u \geqslant 0\,; \end{cases} \tag{0.7}$$

then any solution u *of* (E) *in* $\mathrm{IR}^N - \{0\}$ *is spherically symmetric.*

(III) *Suppose* g *vanishes only at* 0 *and*

$$\lim_{r\to+\infty} (g(r) - c\,r^q)r^{-(q-1)N/2} = 0, \quad \textit{for some } 1 < q < \frac{N}{N-2}, \tag{0.8}$$

then any solution u *of* (E) *in* $\mathrm{IR}^N - \{0\}$ *is uniquely determined by its isotropic singularity at* 0.

If we réplace Δ by $L = \sum_{i,j} a_{ij} \dfrac{\partial^2}{\partial x_i\, \partial x_j}$ a strongly elliptic operator with constant coefficient all our results remain true provided $|x|$ is replaced by some $(\sum_{i,j} \alpha_{ij} x_i x_j)^{1/2}$, the coefficients α_{ij} being obtained after the diagonalisation of the matrix $(\frac{1}{2}(a_{ij} + a_{ji}))$.

Results concerning symmetry and singularities of positive solutions of equations of type (E) when $g(r) = -r^q$ have been given in [7] and in [8]. For general g, symmetry of positive regular solutions vanishing for $|x| = R$ is also given in [8].

The contents of our work is the following :

1. Behaviour at infinity.
2. Spherically symmetric solutions.
3. Uniqueness of solutions.

1. - BEHAVIOUR AT INFINITY

In this paragraph Ω is an *exterior domain* (that is $\complement\,\Omega$ is compact) of $\mathbb{R}^N$, $N \geqslant 3$, and g is a *nondecreasing* function defined on IR and *vanishing* at 0. The equation we consider is the following

$$-\Delta u + g(u) = 0. \tag{1.1}$$

For the sake of simplicity we prefer to deal with C^2 solutions of (1.1) in Ω, so we shall suppose that g is Holder continuous although our results remain true when g is discontinuous and u is a C^1 solution of (1.1) in $D'(\Omega)$.

When $\displaystyle\int_{-1}^{1} (j(t))^{-1/2}dt < +\infty$, where $j(t) = \displaystyle\int_0^t g(s)ds$, any solution of (1.1) vanishing in some weak sense at infinity has a compact support (see [1]).

When $g(u) = |u|^{q-1} u$, $q \geqslant 1$, the behaviour of any solution u of (1.1) has been given by Veron in [14] :

(i) if $q = 1$, $|x|^{(N-1)/2} \exp(|x|)\, u(x)$ converges to some non isotropic limit,

(ii) if $1 < q < \dfrac{N}{N-2}$ and $u \geqslant 0$, $|x|^{2/(q-1)} u(x)$ converges to 0 or

$$\left(\left(\frac{2}{q-1}\right)\left(\frac{2q}{q-1} - N\right)\right)^{1/(q-1)} = \ell_{q,n}\,,$$

(iii) if $\dfrac{N+1}{N-1} \leqslant q < \dfrac{N}{N-2}$, $|x|^{2/(q-1)} u(x)$ converges to 0 or $\pm\, \ell_{q,N}$,

(iv) if $q = \dfrac{N}{N-2}$, $|x|^{N-2} (\mathrm{Log}\,|x|)^{(N-2)/2} u(x)$ converges to 0 or $\pm\left(\dfrac{N-2}{\sqrt{2}}\right)^{N-2}$

(v) if $q > \dfrac{N}{N-2}$, $|x|^{N-2} u(x)$ converges to some arbitrary real number.

Moreover, when $q > 1$ and when u vanishes at infinity, the hypothesis on g can be weakened and replaced by $\lim\limits_{r\to 0} g(r)/|r|^{q-1} r = c > 0$.

Our main result which generalises strongly the last one of [14] is

THEOREM 1.1. *Suppose* u *is a* C^2 *solution of* (1.1) *in* Ω *and*

(i) *either* $\lim\limits_{|x|\to+\infty} u(x)/|x| = 0$ *and* g *vanishes only at* 0,

(ii) *or* $\lim\limits_{|x|\to+\infty} u(x) = 0$.

Then $|x|^{N-2}u(x)$ *converges to some real number when* x *tends to infinity.*

We call (r,σ) the spherical coordinates in $\mathbb{R}^N = \mathbb{R}^+ \times S^{N-1}$ and $\bar{u}(r)$ the average of $u(r,\sigma)$ on S^{N-1} and we suppose that $\{x \mid |x|>R\} \subset \Omega$. The following estimate is fundamental.

PROPOSITION 1.1. *There exists a constant* C(N) *such that if the hypotheses of Theorem 1.1 are fulfilled the following estimate holds*

$$\| u(r,.) - \bar{u}(r) \|_{L^\infty(S^{N-1})} \leqslant \ldots \tag{1.2}$$

$$\leqslant C(N)\left(1 + \frac{1}{\log\frac{r}{\rho}}\right)^{(N-1)/2} \frac{r}{\rho}\left(\frac{r}{R}\right)^{1-N} \| u(R,.) - \bar{u}(R) \|_{L^2(S^{N-1})}$$

for any $R \leqslant \rho < r$.

We first need the L^2 version of (1.2)

LEMMA 1.1. *Suppose* $u \in C^2(\Omega)$ *is a solution of (1.1) such that* $\lim\limits_{|x| \to +\infty} u(x)/|x| = 0$, *then*

$$\| u(r,.) - \bar{u}(r) \|_{L^2(S^{N-1})} \leqslant \left(\frac{r}{R}\right)^{1-N} \| u(R,.) - \bar{u}(R) \|_{L^2(S^{N-1})} \tag{1.3}$$

for any $R \leqslant r$.

Proof. If $\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on S^{N-1}, the function u satisfies

$$\frac{\partial^2 u}{\partial r^2} + \frac{N-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Delta_{S^{N-1}} u = g(u), \tag{1.4}$$

in $[R,+\infty) \times S^{N-1}$. In averaging (1.4) we obtain

$$\int_{S^{N-1}} \frac{\partial}{\partial r^2}(u-u)(u-u)d\sigma + \frac{N-1}{r}\int_{S^{N-1}} \frac{\partial}{\partial r}(u-u)(u-u)d\sigma - \frac{1}{r^2}\int_{S^{N-1}} -\Delta(u-u)(u-u)d\sigma \geqslant 0, \tag{1.5}$$

as

$$\int_{S^{N-1}} (g(u) - \overline{g(u)})(u-\bar{u})d\sigma = \int_{S^{N-1}} (g(\bar{u}) - g(\bar{u}))(u-u)d\sigma + \int_{S^{N-1}} (g(\bar{u}) - \overline{g(u)})(u-\bar{u})d\sigma \ldots$$

$$\ldots = \int (g(u) - g(\bar{u}))\,(u-\bar{u})d\sigma \geqslant 0.$$

Moreover $\displaystyle\int_{S^{N-1}} -\Delta_{S^{N-1}}(u-\bar{u})(u-\bar{u})d\sigma \geqslant (N-1)\int_{S^{N-1}} (u-\bar{u})^2\, d\sigma$ as $\bar{u}$ is the projection of u on the first eigenspace of $-\Delta_{S^{N-1}}$ and N−1 is the second eigenvalue of $-\Delta_{S^{N-1}}$ (see [3]), so we deduce

(1.6) $$\int_{S^{N-1}} \frac{\partial^2}{\partial r^2}(u-\bar{u})(u-\bar{u})d\sigma + \frac{N-1}{r}\int_{S^{N-1}} \frac{\partial}{\partial r}(u-\bar{u})(u-\bar{u})d\sigma - \frac{N-1}{r^2}\int_{S^{N-1}} (u-\bar{u})^2 d\sigma \geqslant 0.$$

We set $w(r) = \left(\int_{S^{N-1}} (u-\bar{u})^2(r,\sigma)d\sigma\right)^{1/2}$ and we have when $w \neq 0$:

$$w\frac{dw}{dr} = \int_{S^{N-1}} \frac{\partial}{\partial r}(u-\bar{u})(u-\bar{u})d\sigma, \quad \left|\frac{dw}{dr}\right| \leqslant \left(\int_{S^{N-1}} (\frac{\partial}{\partial r}(u-\bar{u}))^2 d\sigma\right)^{1/2} \text{ and}$$

$\int_{S^{N-1}} \frac{\partial}{\partial r^2}(u-\bar{u})(u-\bar{u})d\sigma \leqslant w\frac{d^2w}{dr^2}$. If we set $\Gamma = \{r > R : w(r) > 0\}$, we get

(1.7) $$\frac{d^2w}{dr^2} + \frac{N-1}{r}\frac{dw}{dr} - \frac{N-1}{r^2}w \geqslant 0,$$

on Γ. By the maximum principle w cannot assume a strictly positive maximum value, so the set Γ can only be of two types

(i) $\Gamma = (R,T)$, T finite and $w(r) = 0$ on $(T,+\infty)$,

(ii) $\Gamma = (R,T) \cup (T',+\infty)$, and $w(T) = w(T') = 0$ if T and T' are finite.

Let us consider now the following differential equation

(1.8) $$\frac{d^2y}{dr^2} + \frac{N-1}{r}\frac{dy}{dr} - \frac{N-1}{r^2}y = 0.$$

That equation admits two linearly independant solutions

(1.9) $$\phi_1(r) = r \qquad \text{and} \qquad \phi_2(r) = r^{1-N}.$$

Now we set $\psi_\epsilon(r) = \epsilon r + \| u(R,.) - \bar{u}(R) \|_{L^2(S^{N-1})} (\frac{r}{R})^{1-N}$, $\epsilon \geqslant 0$. As ψ_ϵ satisfies (1.8), we have

(1.10) $$\frac{d^2}{dr^2}(w-\psi_\epsilon) + \frac{N-1}{r}\frac{d}{dr}(w-\psi_\epsilon) - \frac{N-1}{r^2}(w-\psi_\epsilon) \geqslant 0,$$

on Γ. If we are in the first case or in the second when $T < +\infty$, we take $\epsilon = 0$ and we deduce by the maximum principle that $0 \leqslant w(r) \leqslant \psi_0(r)$ on (R,T), which is (1.3). In the second case with

$T = +\infty$, or on $(T', +\infty)$, we take $\epsilon > 0$. As $\lim_{r \to +\infty} w(r)/r = 0$, $w - \psi_\epsilon$ is non positive at the end points of the interval, so $w - \psi_\epsilon$ remains non positive. Making $\epsilon \to 0$ we deduce $w \leqslant \psi_0$ which ends the proof.

Remark 1.1. In Lemma 1.1 we need not assume $g(0) = 0$ (see Theorem 2.1 for an application of this method).

We set $u^+ = \mathrm{Max}(u,0)$, $u^- = \mathrm{Max}(-u,0)$ and we have

LEMMA 1.2. *Under the assumptions of Theorem 1.1 we have*

$$u^+(x) \leqslant \left(\frac{|x|}{R}\right)^{2-N} \| u^+(R,.) \|_{L^\infty(S^{N-1})}, \tag{1.11}$$

$$u^-(x) \leqslant \left(\frac{|x|}{R}\right)^{2-N} \| u^-(R,.) \|_{L^\infty(S^{N-1})}, \tag{1.12}$$

for any x *such that* $|x| \geqslant R$.

Proof. Multiplying (1.4) by u and integrating over S^{N-1} yields

$$\frac{d^2}{dr^2} \int_{S^{N-1}} u^2 d\sigma + \frac{N-1}{r} \frac{d}{dr} \int_{S^{N-1}} u^2 d\sigma \geqslant 0. \tag{1.13}$$

By the maximum principle $r \mapsto \int_{S^{N-1}} u^2(r,\sigma) d\sigma$ is asymptotically monotone so there exists $\gamma \in \mathbb{R}^+ \cup \{+\infty\}$ such that $\lim_{r \to +\infty} \| u(r,.) \|^2_{L^2(S^{N-1})} = \gamma^2 \, | S^{N-1} |$. From the estimate (1.3) and the continuity of $r \mapsto u(r)$, either $\lim_{r \to +\infty} \bar{u}(r) = \gamma$ or $\lim_{r \to +\infty} \bar{u}(r) = -\gamma$ and $\lim_{r \to +\infty} u(r,.) = \lim_{r \to +\infty} \bar{u}(r)$ in $L^2(S^{N-1})$.

We first suppose that $\gamma = 0$ (which is an hypothesis if $\lim_{|x| \to +\infty} u(x) = 0$) and set p a convex function vanishing on $(-\infty, 0)$, increasing on $(0, +\infty)$ and such that $0 \leqslant p' \leqslant 1$. We set $\theta^+(x) = \left(\frac{|x|}{R}\right)^{2-N} \| u^+(R,.) \|_{L^\infty(S^{N-1})}$. θ^+ is a positive harmonic function and we have

$$\frac{d^2\theta^+}{dr^2} + \frac{N-1}{r} \frac{d\theta^+}{dr} + \frac{1}{r^2} \Delta_{S^{N-1}} \theta^+ \leqslant g(\theta^+). \tag{1.14}$$

As we have

$$-\int_{S^{N-1}} \Delta_{S^{N-1}}(u-\theta^+) \, p'(u-\theta^+) d\sigma \geqslant 0, \tag{1.15}$$

and

$$\frac{\partial^2}{\partial r^2}\,p(u-\theta^+) \geqslant p'(u-\theta^+)\,\frac{\partial^2}{\partial r^2}\,(u-\theta^+), \tag{1.16}$$

we deduce from the monotonicity of g that

$$\frac{d^2}{dr^2}\int_{S^{N-1}} p(u-\theta^+)d\sigma + \frac{N-1}{r}\,\frac{d}{dr}\int_{S^{N-1}} p(u-\theta^+)d\sigma \geqslant 0. \tag{1.17}$$

The function $r \mapsto \int_{S^{N-1}} p(u(r,\sigma) - \theta^+(r))d\sigma$ vanishes at R and as $p(u-\Theta^+) \leqslant (u-\theta^+)^+$, we have : $\lim_{r\to+\infty} \int_{S^{N-1}} p(u(r,\sigma) - \theta^+(r))d\sigma = 0$. By the maximum principle $\int_{S^{N-1}} p(u-\theta^+)d\sigma \leqslant 0$, which is (1.11). In considering $\theta^-(x) = -\left(\frac{x}{R}\right)^{2-N} \| u^-(R,.) \|_{L^\infty(S^{N-1})}$, we obtain (1.12) in the same way.

We suppose now that $\gamma > 0$ (so g vanishes only at 0) and, for example, $\lim_{r\to+\infty} \bar{u}(r) = \gamma$. The function $\int_{S^{N-1}} p(\theta^- - u)d\sigma$ satisfies

$$\frac{d^2}{dr^2}\int_{S^{N-1}} p(\theta^- - u)d\sigma + \frac{N-1}{r}\,\frac{d}{dr}\int_{S^{N-1}} p(\theta^- - u)d\sigma \geqslant 0\ ; \tag{1.18}$$

it vanishes at R and as $p(\theta^- - u) \leqslant p(\theta^- - \bar{u}) + |\bar{u} - u|$, we deduce from (1.3) that $\lim_{r\to+\infty} \int_{S^{N-1}} p(\theta^-(r) - u(r,\sigma))d\sigma = 0$. By the maximum principle we get (1.12) which implies that u(x) is bounded below on $\{x \mid |x| \geqslant R\}$.

As $g(r) = g^+(r) - g^-(r)$, we set $g_N^+(r) = \min(N, g^+)$, $N > 0$ and we have in averaging (1.4)

$$\frac{d^2\bar{u}}{dr^2} + \frac{N-1}{r}\,\frac{d\bar{u}}{dr} \geqslant \overline{g_N^+(u)} - \overline{g^-(u)}. \tag{1.19}$$

But $g^-(u) = g^-(u^-)$ and $\lim_{r\to+\infty} u^-(r,.) = 0$ in $L^2(S^{N-1})$. As u^- is bounded below, $\lim_{r\to+\infty} \overline{g^-(u^-)} = 0$. On the other hand, by Lebesgue's Theorem, $\lim_{r\to+\infty} \overline{g_N^+(u)} = \overline{g_N^+(\gamma)} = |S^{N-1}| \min(N, g(\gamma)) = \alpha > 0$. There exists $R' > R$ such that

$$\frac{d^2\bar{u}}{dr^2} + \frac{N-1}{r}\,\frac{d\bar{u}}{dr} \geqslant \frac{\alpha}{2}, \tag{1.20}$$

on $(R', +\infty)$. Integrating (1.20) twice yields

$$\bar{u}(r) \geqslant \frac{\alpha}{2N}\,r^2 + c\,r^{2-N} + c', \tag{1.21}$$

on $(R',+\infty)$ which contradicts the fact that $\lim_{|x|\to+\infty} u(x)/|x| = 0$. So $\gamma = 0$ which ends the proof.

LEMMA 1.3. *Suppose* g *is a continuous nondecreasing function vanishing at* 0 ; *then for any* $\rho > 0$ *and any real* a *there exists a unique function* v *twice continuously differentiable satisfying*

$$(1.22)\qquad \begin{cases} \dfrac{d^2v}{dr^2} + \dfrac{N-1}{r}\dfrac{dv}{dr} = g(v) & \text{on } (\rho,+\infty), \\ v(\rho) = a, & \limsup_{r\to+\infty} r^{N-2}|v(r)| < +\infty. \end{cases}$$

Proof. *Uniqueness* : Consider the following change of variable and unknown

$$(1.23)\qquad s = \frac{r^{N-2}}{N-2}, \qquad v(r) = r^{2-N} w(s).$$

The function w satisfies

$$(1.24)\qquad s^2 \frac{d^2w}{ds^2} = (N-2)^{\frac{4-N}{N-2}} s^{\frac{N}{N-2}} g\left(\frac{w}{s(N-2)}\right).$$

Suppose $\tilde{w}$ is another solution of (1.24) with the same initial data, then

$$(1.25)\qquad s^2 \frac{d^2}{ds^2} |w - \tilde{w}| \geqslant 0,$$

so the function $s \mapsto |w - \tilde{w}|(s)$ is nonnegative, convex, vanishes at $\dfrac{\rho^{N-2}}{N-2}$ and $\lim_{s\to+\infty} \dfrac{1}{s}|w(s) - \tilde{w}(s)| = 0$, so it is identically zero.

Existence : For any $T > \rho$ set v_T the solution of the following two points problem

$$(1.26)\qquad \begin{cases} \dfrac{d^2v_T}{dr^2} + \dfrac{N-1}{r}\dfrac{dv_T}{dr} = g(v_T) & \text{on } (\rho,T), \\ v_T(\rho) = a, \quad v_T(T) = 0. \end{cases}$$

The function v_T exists and is unique ; moreover $|v_T|$ decreases. Thanks to the uniqueness of the solution of (1.26), the function $T \mapsto |v_T(r)|$ is nondecreasing for any $r > \rho$. As $|v_T(r)| \leqslant |a|$ and g is continuous, we deduce in integrating (1.26) that $\dfrac{dv_T}{dr}$ and $\dfrac{d^2v_T}{dr^2}$ remain bounded on every compact interval of $[\rho,T)$; so $v_T(r)$ converges uniformly on every compact interval to some C^2 function v, as T tends to $+\infty$. Moreover $|v_T|$ is majorized by the function ψ defined on $[\rho,+\infty)$ by $\psi(r) = (\frac{\rho}{r})^{N-2}|a|$ (which satisfies (1.22) with $g \equiv 0$). So $r^{N-2}v(r)$ remains bounded on $[\rho,+\infty)$ and (1.22) is satisfied.

LEMMA 1.4. *For any* $\rho > 0$ *and* $\alpha \in L^2(S^{N-1})$ *there exists a unique function* $\omega \in L^\infty((\rho,+\infty)\,;\,L^2(S^{N-1})) \cap C^0([\rho,+\infty)\,;\,L^2(S^{N-1})) \cap C^2((\rho,+\infty) \times S^{N-1})$ *satisfying*

$$(1.27)\qquad \begin{cases} s^2 \dfrac{\partial^2\omega}{\partial s^2} + \dfrac{1}{(N-2)^2}\Delta_{S^{N-1}}\,\omega = 0 & \text{on } (\rho,+\infty)\times S^{N-1}, \\ \omega(\rho,.) = \alpha(.) \quad \text{on} \quad S^{N-1}. \end{cases}$$

Moreover there exists a constant $C = C(N)$ *such that the following estimate holds*

$$(1.28)\qquad \|\omega(s,.)\|_{L^\infty(S^{N-1})} \leqslant C\left(1 + \frac{1}{\log\frac{s}{\rho}}\right)^{(N-1)/2} \|\alpha(.)\|_{L^2(S^{N-1})}$$

for any $s > \rho$.

Proof. For the uniqueness set $\tilde{\omega}$ a solution of (1.27) taking the value $\tilde{\alpha}$ for $s = \rho$. We have : $s^2 \displaystyle\int_{S^{N-1}} \frac{\partial^2}{\partial s^2}(\omega - \tilde{\omega})(\omega - \tilde{\omega})d\sigma \geqslant 0$. Hence $s \mapsto \displaystyle\int_{S^{N-1}} (\omega - \tilde{\omega})^2(s,\sigma)d\sigma$ is a convex function. As it is bounded it is nonincreasing.

For the existence we set $t = \log s$ and $\phi(t,\sigma) = \omega(s,\sigma)$. The function ϕ satisfies on $(\log\rho,+\infty)$

$$(1.29)\qquad \frac{\partial^2\phi}{\partial t^2} - \frac{\partial\phi}{\partial t} + \frac{1}{(N-2)^2}\Delta_{S^{N-1}}\,\phi = 0.$$

If $(T(t))_{t\geqslant 0}$ is the semigroup of contractions of $L^2(S^{N-1})$ generated by $-\left(-\dfrac{1}{(N-2)^2}\Delta_{S^{N-1}} + \dfrac{1}{4}I\right)^{1/2}$, it is easy to check that $\exp((t-\log\rho)/2)\,T(t-\log\rho)\alpha$ satisfies the equation (1.29) with initial data α and is bounded ; so it is ϕ.

Set H_0 the subspace of $L^2(S^{N-1})$ of constant functions and $H' = (H_0)^\perp$. We have the following hilbertian direct sum : $L^2(S^{N-1}) = H' \oplus H_0$, and both H_0 and H' are invariant under $(T(t))_{t\geqslant 0}$.

As $\displaystyle\int_{S^{N-1}} u\left(\frac{u}{4} - \frac{1}{(N-2)^2}\Delta_{S^{N-1}} u\right) d\sigma \geqslant \frac{N^2}{4(N-2)^2}\int_{S^N} u^2 d\sigma$, for any $u \in H'$,

the restriction $T'(t)$ of $T(t)$ to H' satisfies (see [4])

$$(1.30) \qquad \| T'(t)u \|_{L^2(S^{N-1})} \leqslant \exp(-tN/(2N-4)) \| u \|_{L^2(S^{N-1})},$$

for any $u \in H'$: Moreover we have the following regularizing effect (see [15])

$$(1.31) \qquad \| T'(t)u \|_{L^\infty(S^{N-1})} \leqslant C(1+\frac{1}{t})^{(N-1)/2} \| u \|_{L^2(S^{N-1})}$$

for any $u \in L^2(S^{N-1})$ and any $t > 0$. In combining (1.30) and (1.31), and using the semigroup property, we have for any $u \in H'$, any $t > 0$ and any $\epsilon > 0$:

$$(1.32) \quad \| T'(t)u \|_{L^\infty(S^{N-1})} \leqslant C(1+\frac{1}{\epsilon t})^{(N-1)/2} \exp(-t(1-\epsilon)N/(2N-4)) \| u \|_{L^2(S^{N-1})}.$$

Now we write $\alpha = \alpha_0 + \alpha'$ with $\alpha_0 \in H_0$ and $\alpha' \in H'$ (and in fact $\alpha_0 = \frac{1}{|S^{N-1}|}\int \alpha(\sigma)d\sigma$). We have

$$(1.33) \qquad T(t)\alpha = T(t)\alpha_0 + T(t)\alpha' ;$$

but $T(t)\alpha_0 = \exp(-t/2)\alpha_0$. In taking $\epsilon = \frac{2}{N}$ in (1.32) we get

$$(1.34) \qquad \| T(t)\alpha \|_{L^\infty(S^{N-1})} \leqslant C(1+\frac{1}{t})^{(N-1)/2} \exp(-t/2) \| \alpha \|_{L^2(S^{N-1})}.$$

In replacing t by $\log s - \log \rho$, we obtain (1.28).

Proof of Proposition 1.1. Consider the change of variable and unknown

$$(1.35) \qquad s = \frac{r^{N-2}}{N-2}, \quad u(r,\sigma) = r^{2-N} v(s,\sigma).$$

The function v satisfies

$$(1.36) \qquad s^2 \frac{\partial^2 v}{\partial s^2} + \frac{1}{(N-2)^2}\Delta_{S^{N-1}} v = (N-2)^{\frac{4-N}{N-2}} s^{\frac{N}{N-2}} g(\frac{v}{s(N-2)}),$$

in $[\frac{R^{N-2}}{N-2}, +\infty) \times S^{N-1}$. Let y be the solution (from Lemma 1.3) of

$$(1.37) \qquad \begin{cases} s^2 \dfrac{d^2 y}{ds^2} = (N-2)^{\frac{4-N}{N-2}} s^{\frac{N}{N-2}} g(\dfrac{y}{s(N-2)}) \quad \text{on } (\rho',+\infty), \ \rho' > R, \\[2ex] y(\rho') = a, \quad y \text{ bounded.} \end{cases}$$

We set $w = v - y$ and we have for $s \geqslant \rho'$

(1.38) $$s^2\frac{\partial^2 w}{\partial s^2} + \frac{1}{(N-2)^2}\Delta_{S^{N-1}} w = (N-2)^{\frac{4-N}{N-2}} s^{\frac{N}{N-2}} hw,$$

where
$$h = \begin{cases} \left(g\left(\frac{v}{s(N-2)}\right) - g\left(\frac{y}{s(N-2)}\right)\right) / (v-y) & \text{if } v \neq y, \\ 0 & \text{if } v = y. \end{cases}$$

The function h is nonnegative as g is nondecreasing. If ω^+ is the solution of (1.27) taking the value $(v(\rho',.) - a)^+$ for $s = \rho'$, ω^+ is nonnegative and satisfies

(1.39) $$s^2\frac{\partial^2 \omega^+}{\partial s^2} + \frac{1}{(N-2)^2}\Delta_{S^{N-1}} \omega^+ \leqslant (N-2)^{\frac{4-N}{N-2}} s^{\frac{N}{N-2}} h\omega^+.$$

Introducing the nondecreasing convex function p as we have done it in Lemma 1.2, we get $s^2\frac{d^2}{ds^2}\int_{S^{N-1}} p(w-\omega^+)(s,\sigma)d\sigma \geqslant 0$; hence $w \leqslant \omega^+$. In the same way w is minorized on $(\rho', +\infty)$ by the solution ω^- of (1.27) taking the value $-(v(\rho,.) - a)^-$ for $s = \rho'$. Combining those estimates with (1.28) we get

(1.40) $$\| v(s,.) - y(s) \|_{L^\infty(S^{N-1})} \leqslant C\left(1 + \frac{1}{\log\frac{s}{\rho'}}\right)^{(N-1)/2} \| v(\rho',.) - a \|_{L^2(S^{N-1})}.$$

In averaging (1.40) we deduce

(1.41) $$\| v(s,.) - \bar{v}(s) \|_{L^\infty(S^{N-1})} \leqslant 2C\left(1 + \frac{1}{\log\frac{s}{\rho'}}\right)^{(N-1)/2} \| v(\rho',.) - a \|_{L^2(S^{N-1})}.$$

We take now $a = \bar{v}(\rho')$, $s = \frac{r^{N\ 2}}{N-2}$, $\rho' = \frac{\rho^{N-2}}{N-2}$ and apply (1.3) between R and ρ, we get (1.2).

Remark 1.2. We can deduce from Lemma 1.1 a first property of symmetry of the solutions of (1.1) : *suppose* g *is a monotone nondecreasing function and* u *is a* C^2 *solution of (1.1) satisfying* $\lim_{|x| \to +\infty} u(x)/|x| = 0$. *If* u *is spherically symmetric on* $\{x \mid |x| = R\}$ *then it remains spherically symmetric on* $\{x \mid |x| > R\}$.

Proof of Theorem 1.1. In Proposition 1.1 we take $r = 2\rho$ and make $r \to +\infty$. In taking the notations of the transformation (1.35) we get

$$(1.42) \qquad \lim_{s \to +\infty} \| v(s,.) - \bar{v}(s) \|_{L^\infty(S^{N-1})} = 0.$$

As $\{ \bar{v}(s) \}$ is bounded, there exists a sequence $s_n \to +\infty$ such that $v(s_n)$ converges to some number c when $n \to +\infty$.

<u>If $C > 0$</u> (or $C < 0$ in the same way) there exists some n_0 such that $v(s_n,.) > \frac{C}{2}$ for $n \geqslant n_0$ (it is a consequence of (1.42)). If we apply the maximum principle to the function v in the spherical shell $(s_{n_0}, s_n) \times S^{N-1}$, we deduce that $v(s,\sigma) \geqslant 0$ in that shell and therefore in $(s_{n_0}, +\infty) \times S^{N-1}$. In averaging (1.36) on S^{N-1} we deduce $s^2 \dfrac{d^2\bar{v}}{ds^2} \geqslant 0$ for $s \geqslant s_{n_0}$. Hence v is convex and, as it is bounded, it converges when s goes to $+\infty$. The only admissible limit is C and finally $\lim_{s \to +\infty} v(s,.) = C$ in $L^\infty(S^{N-1})$.

<u>If $C = 0$</u> then $\lim_{s \to +\infty} \| v(s,.) \|_{L^\infty(S^{N-1})} = 0$, otherwise there would exist a sequence $s'_n \to +\infty$ and $\epsilon > 0$ such that $\| v(s'_n,.) \|_{L^\infty(S^{N-1})} > \epsilon$ for $s'_n \geqslant s'_{n_0}$ and there would exist a sequence s''_n extracted from s'_n and a number λ, $|\lambda| > \frac{\epsilon}{2}$, such that $\lim_{s''_n \to +\infty} \bar{v}(s''_n) = \lambda$. Applying what have been done when $C \neq 0$, we would have $\lim_{s \to +\infty} v(s,.) = \lambda$ in $L^\infty(S^{N-1})$, which contradicts $\lim_{s_n \to +\infty} \bar{v}(s_n) = 0$.

2. - SPHERICALLY SYMMETRIC SOLUTIONS

In this paragraph g is a continuous nondecreasing function defined on IR (not necessarily vanishing at 0) and we still consider the equation

$$(2.1) \qquad -\Delta u + g(u) = 0\ ;$$

but the equation is taken in $D'(\mathrm{IR}^N - \{0\})$ and u may have a singularity at 0. The following result is fundamental and its proof is very similar to the one of Lemma 1.1 (comparison of w with $\epsilon\phi_1 + \epsilon'\phi_2$, $\epsilon, \epsilon' > 0$).

THEOREM 2.1. *Suppose* $u \in C^2(\mathrm{IR}^N - \{0\})$ *is a solution of* (2.1) *in* $D'(\mathrm{IR}^N - \{0\})$ *such that*

i) $\lim_{r \to +\infty} r^{-1} \| u(r,.) - \bar{u}(r) \|_{L^2(S^{N-1})} = 0$,

ii) $\lim_{r \to 0} r^{N-1} \| u(r,.) - \bar{u}(r) \|_{L^2(S^{N-1})} = 0$,

where $(r,\sigma) \in \mathbb{R}^+ \times S^{N-1}$ *are the spherical coordinates in* $\mathbb{R}^N$ *and* $\bar{u}(r) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} u(r,\sigma) d\sigma$, *then* u *is spherically symmetric.*

The following «universal» estimate on u when g has an asymptotic growth corresponding to a power greater than 1 is originated in [5].

LEMMA 2.1. *Suppose* g *satisfies, for some* $q > 1$,

$$\left\{ \begin{array}{l} \liminf_{r \to +\infty} g(r)/r^q > 0 , \\ \\ \limsup_{r \to +\infty} g(r)/ |r|^q < 0 , \end{array} \right. \tag{2.2}$$

and $u \in C^2(\mathbb{R}^N - \{0\})$ *is a solution of* (2.1) *in* $D'(\mathbb{R}^N - \{0\})$ *; then*

$$|u(x)| \leqslant C |x|^{-2/(q-1)} + D , \tag{2.3}$$

for $x \neq 0$, *where* C *and* D *depend on* g *and* N.

Proof. From the hypothesis (2.2) there exist two constants A and $B > 0$ such that

$$\left\{ \begin{array}{ll} g(r) \geqslant Ar^q - B & \text{on } r > 0 , \\ \\ g(r) \leqslant -A |r|^q + B & \text{on } r < 0 , \end{array} \right. \tag{2.4}$$

which yields

$$-\Delta u + Au^q \leqslant B \qquad \text{a.e. on } \{x \mid u(x) > 0\}. \tag{2.5}$$

For $x_0 \neq 0$ set $G = \{x \in \mathbb{R}^N,\ |x - x_0| < \frac{1}{2} |x_0|\}$ and consider the function

$$v(x) = \lambda \left(\frac{1}{4} |x_0|^2 - |x - x_0|^2\right)^{-2/(q-1)} + \mu ,$$

where λ and μ are to be determined in order that

$$-\Delta v + Av^q \geqslant B, \tag{2.6}$$

in G. For simplification set $v(r) = \lambda(R^2 - r^2)^{-2/(q-1)} + \mu$. We have in G

$$-\Delta v + Av^q \geqslant \lambda(R^2 - r^2)^{-2q/(q-1)}(A\lambda^{q-1} - \frac{2NR^2}{q-1} + \frac{2}{q-1}(N - 2\frac{q+1}{q-1})\, r^2) + A\mu^q .$$

Set $\beta = \max\left(\frac{2N}{q-1}, 4\frac{q+1}{(q-1)^2}\right)$ and we take $\lambda = (\frac{\beta}{A})^{1/(q-1)} R^{2/(q-1)}$ and $\mu = (\frac{B}{A})^{1/q}$, so we get (2.6).

By Kato's inequality (see [10]) we have as in [5]

$$\Delta(u - v)^+ \geqslant \operatorname{sign}^+(u - v)\, \Delta(u - v) \geqslant 0 \qquad \text{in } \mathcal{D}'(G), \tag{2.7}$$

in $\mathcal{D}'(G)$. Moreover $(u - v)^+$ vanishes in some neighbourhood of ∂G, so $(u - v)^+ \equiv 0$ in G and

$$u(x_0) \leqslant v(x_0) = (\frac{16\beta}{A})^{1/(q-1)} |x_0|^{-2/(q-1)} + (\frac{B}{A})^{1/q} : \tag{2.8}$$

In the same way $u(x_0) \geqslant - v(x_0)$.

From that result we get

THEOREM 2.2. *Suppose* g *satisfies*

$$\begin{cases} \liminf\limits_{r \to +\infty} g(r)/r^{(N+1)/(N-1)} = +\infty, \\ \\ \limsup\limits_{r \to -\infty} g(r)/|r|^{(N+1)/(N-1)} = -\infty, \end{cases} \tag{2.9}$$

and $u \in C^2(\mathbb{R}^N - \{0\})$ *satisfies* (2.1) *in* $\mathcal{D}'(\mathbb{R}^N - \{0\})$ *; then* u *is spherically symmetric.*

Proof. From (2.9), for any $n > 0$, there exists $B_n \geqslant 0$ such that

$$\begin{cases} g(r) \geqslant nr^{(N+1)/(N-1)} - B_n & \text{for } r \geqslant 0, \\ \\ g(r) \leqslant -n\,|r|^{(N+1)/(N-1)} + B_n & \text{for } r \leqslant 0. \end{cases} \tag{2.10}$$

From (2.8) we get $|u(x)| \leqslant \left(\frac{16\beta}{n}\right)^{1/(q-1)} |x|^{1-N} + \left(\frac{B_n}{n}\right)^{(N-1)/(N+1)}$, for $x \neq 0$, which implies $\limsup_{r \to 0} r^{N-1} \| u(r,.) - \bar{u}(r) \|_{L^2(S^{N-1})} \leqslant 2\left(\frac{16\beta}{n}\right)^{1/(q-1)}$. Letting $n \to +\infty$ we obtain the condition ii) of Theorem 2.1 ; as for the condition i) it is an immediate consequence of (2.3).

When the rate of growth of g at infinity is of order $\frac{N+1}{N-1}$, it is not enough to make a hypothesis on g but we have to make it on g' and we get :

THEOREM 2.3. *Suppose* g *satisfies*

$$(g(r) - g(s))(r - s) \geqslant C\, |r - s|^{2N/(N-1)} - D(r - s)^2, \tag{2.11}$$

for some $C > 0$, $D \geqslant 0$ *and all* r *and* s *real. If* $u \in C^2(\mathbb{R}^N - \{0\})$ *is a solution of* (2.1) *in* $D'(\mathbb{R}^N - \{0\})$, *then* u *is spherically symmetric.*

We first need the following result

LEMMA 2.2. *Under the hypotheses of Theorem 2.3, we have*

$$\| u(r,.) - \bar{u}(r) \|_{L^2(S^{N-1})} \leqslant \frac{r}{R} \| u(R,.) - \bar{u}(R) \|_{L^2(S^{N-1})} \tag{2.12}$$

for $0 < r \leqslant R$.

Proof. The function u satisfies

$$\frac{\partial^2 u}{\partial r^2} + \frac{N-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Delta_{S^{N-1}} u = g(u), \tag{2.13}$$

in $(0,+\infty) \times S^{N-1}$. We set $y(r,\sigma) = r^{N-1} u(r,\sigma)$. From Lemma 2.1 y is bounded on every compact or $[0,+\infty) \times S^{N-1}$ and it satisfies

$$\frac{\partial^2 y}{\partial r^2} + \frac{1-N}{r^2}\frac{\partial y}{\partial r} + \frac{N-1}{r^2} y + \frac{1}{r^2}\Delta_{S^{N-1}} y = r^{N-1} g(r^{1-N} y). \tag{2.14}$$

Now we set

$$s = \frac{r^N}{N}, \quad v(s,\sigma) = y(r,\sigma). \tag{2.15}$$

The function v satisfies

$$(2.16)\qquad (Ns)^2 \frac{\partial^2 v}{\partial s^2} + (N-1)v + \Delta_{S^{N-1}} v = (Ns)^{(N+1)/N} g((Ns)^{(1-N)/N} v),$$

in $(0,+\infty) \times S^{N-1}$. If $\bar{v}$ is the average of v on S^{N-1} we get, as in Lemma 1.1,

$$(2.17)\qquad (Ns)^2 \int_{S^{N-1}} \frac{\partial^2}{\partial s^2}(v-\bar{v})(v-\bar{v})d\sigma \geqslant 0\ ;$$

hence $s \mapsto \| v(s,.) - \bar{v}(s) \|^2_{L^2(S^{N-1})}$ is convex. As it is bounded, it admits a limit when $s \to 0$. From (2.11) we get

$$(2.18)\quad (Ns)^2 \frac{d^2}{ds^2} \int_{S^{N-1}} (v-\bar{v})^2\, d\sigma \geqslant C \int_{S^{N-1}} (v-\bar{v})^{2N/(N-1)} d\sigma - D(Ns)^{2/N} \int_{S^{N-1}} (v-\bar{v})^2 d\sigma.$$

As $\int_{S^{N-1}} | v-\bar{v}|^{2N/(N-1)} d\sigma \geqslant C' \left(\int_{S^{N-1}} (v-\bar{v})^2\, d\sigma \right)^{N/(N-1)}$, we see in integrating (2.18) twice that the only admissible limit for $\| v - \bar{v} \|^2_{L^2(S^{N-1})}$ is 0. From (2.17) we also deduce that the function $s \mapsto \| (v-\bar{v})(s,.) \|_{L^2(S^{N-1})}$ is convex (see the proof of Lemma 1.1). As it vanishes at 0 we get, for $0 < s < \sigma$:

$$(2.19)\qquad \| v(s,.) - \bar{v}(s) \|_{L^2(S^{N-1})} \leqslant \frac{s}{\sigma} \| v(\sigma,.) - \bar{v}(\sigma) \|_{L^2(S^{N-1})},$$

which is (2.12).

Remark 2.1. The assumption of monotonicity on g can be avoided for obtaining estimates of the type (2.12) : if we suppose that g satisfies

$$(2.20)\qquad (g(r)-g(s))\,(r-s) \geqslant C\, |r-s|^{q+1} - D(r-s)^2,$$

for some C and $D > 0$, $q \geqslant \frac{N+1}{N-1}$ and all r and s real, we first deduce from Lemma 2.1 the boundedness of $|x|^{2/(q-1)}\, u(x)$ on every compact of IR^N. With the change of variable of Lemma 2.2 of [16] we obtain the following estimate

$$(2.21)\quad \| u(r,.) - \bar{u}(r) \|_{L^2(S^{N-1})} + dr^{\alpha} \leqslant \left(\frac{r}{R}\right)^{2q/(q-1)-N} \left(\| u(R,.) - \bar{u}(R) \|_{L^2(S^{N-1})} + dR^{\alpha}\right)$$

$$\text{for } r < R,$$

where d depends on D and $\alpha > 0$. If we suppose moreover that g is differentiable and satisfies

$$(2.22)\qquad |g'(r)| \leqslant C' |r|^{q-1} + D',$$

for some C' and D' > 0 and all r, then we can obtain as in the Appendix of [16]

$$\lim_{r\to 0} \| u(r,.) - \bar{u}(r) \|_{L^\infty(S^{N-1})} = 0. \tag{2.23}$$

Such a relation can be used for proving that the isolated singularities of the solutions of (2.1) are radial.

Proof of the Theorem 2.3. From Lemma 1.1, we have for any $\rho < r$,

$$\| u(r,.) - \bar{u}(r) \|_{L^2(S^{N-1})} \leqslant \left(\frac{r}{\rho}\right)^{1-N} \| u(\rho,.) - \bar{u}(\rho) \|_{L^2(S^{N-1})} ; \tag{2.24}$$

and from the Lemma 2.2, $\lim_{\rho\to 0} \| u(\rho,.) - \bar{u}(\rho) \|_{L^2(S^{N-1})} = 0$, which implies $\| u(r,.) - \bar{u}(r) \|_{L^2(S^{N-1})} = 0$ for all $r > 0$ and ends the proof.

When $1 < q < \frac{N+1}{N-1}$ there exist non spherically symmetric solutions of

$$-\Delta u + |u|^{q-1} u = 0, \tag{2.25}$$

in $IR^N - \{0\}$. For example if v is a non constant solution of the equation

$$-\Delta_{S^{N-1}} v + |v|^{q-1} v = \left(\frac{2}{q-1}\right)\left(\frac{2q}{q-1} - N\right) v \qquad \text{on } S^{N-1}, \tag{2.26}$$

(such a solution exists as $\left(\frac{2}{q-1}\right)\left(\frac{2q}{q-1} - N\right) > N-1$ which is the second eigenvalue of $-\Delta_{S^{N-1}}$) then $x \mapsto |x|^{-2/(q-1)} v\left(\frac{x}{|x|}\right)$ is a non isotropic solution of (2.25). However such a solution cannot keep a constant sign, so we shall restrict ourself to positive solutions of (2.1). Our first result is an extension of Theorem 1.1 of [16].

PROPOSITION 2.1. *Suppose* g *satisfies*

$$\begin{cases} \text{i)} \quad \lim_{r\to+\infty} g(r)/r^q = c, \\ \\ \text{ii)} \quad \limsup_{r\to 0^+} g(r)/r < +\infty, \end{cases} \tag{2.27}$$

for some $c > 0$ *and* $1 < q < \frac{N}{N-2}$ *and* Ω *is an open subset of* IR^N *containing* 0. *If* $u \in C^2(\Omega - 0)$ *is a non negative solution of* (2.1) *in* $D'(\Omega - 0)$ *then we have the following alternative* *i)* *either* $\lim_{x\to 0} |x|^{2/(q-1)}\, u(x) = \left((\frac{2}{c(q-1)})(\frac{2q}{q-1} - N) \right)^{1/(q-1)}$,

ii) *or* $\lim_{x\to 0} |x|^{N-2}\, u(x) = \gamma$ *for some* $\gamma \geqslant 0$.

Proof. We shall just sketch it as it is not far from the proof of Theorem 1.1 of [16] (at least in its first part). Moreover we need not suppose that g is nondecreasing. The two assertions are distinct according $|x|^{N-2}\, u(x)$ is bounded or not near 0.

Part 1 : $|x|^{N-2}\, u(x)$ is bounded in some neighbourhood of 0 (and we can even suppose that u has not a constant sign if $|g(r)|/|r|^q$ is bounded when $r \to -\infty$). We make the change of variable (1.35) of Proposition 1.1 and we deduce from Lemma 6.4 of [16] that $\lim_{r\to 0} r^{N-2} \| u(r,.) - \bar{u}(r) \|_{L^\infty(S^{N-1})} = 0$. We end the proof as in Theorem 1.1 of [16].

Part 2 : $|x|^{N-2}\, u(x)$ is unbounded near 0. If we write (2.1) as follows

$$-\Delta u + \frac{g(u)}{u} u = 0, \tag{2.28}$$

we deduce from (2.27) and Lemma 2.1 that $\frac{g(u)}{u} \leqslant C|x|^{-2} + D$. Using Trudinger's estimates in Harnack inequalities as in the Lemma 1.4 of [16], we deduce that $\lim_{x\to 0} |x|^{N-2}\, u(x) = +\infty$.

For any $c' > c$ there exists $\rho > 0$ such that $g(u(x)) \leqslant c'(u(x))^q$ on $\{x \mid |x| < \rho\}$, so $-\Delta u + c'u^q \geqslant 0$ on such a shell. For any $\alpha > 0$ set v_α the solution of

$$\begin{cases} -\Delta v_\alpha + c' v_\alpha^q = 0 & \text{for } 0 < |x| < \rho\,, \\ \\ \lim_{x\to 0} |x|^{N-2}\, v_\alpha(x) = \alpha, \quad v_\alpha(x) = \min_{|x|=\rho} u(x) & \text{for } |x| = \rho\,. \end{cases} \tag{2.29}$$

Such a solution exists (see Lemma 1.6 of [16]). Moreover, from the maximum principle, $v_\alpha(x) \leqslant u(x)$ for any x with $0 < |x| < \rho$. When α goes to $+\infty$, $v_\alpha(x)$ increases and converges to $v_\infty(x)$ and

$$\lim_{x\to 0} |x|^{2/(q-1)} v_\infty(x) = \left((\frac{2}{c'(q-1)})(\frac{2q}{q-1} - N) \right)^{1/(q-1)}, \tag{2.30}$$

from [16]. If we set

$$\ell = \left((\frac{2}{c(q-1)})(\frac{2q}{q-1} - N\,) \right)^{1/(q-1)}, \tag{2.31}$$

and make $c' \searrow c$, we deduce $\liminf_{x\to 0} |x|^{2/(q-1)}u(x) \geqslant \ell$. Now suppose $\limsup_{x\to 0} |x|^{2/(q-1)}u(x) > \ell$. There exist a sequence $x_n \to 0$ and $\ell' > \ell$ such that $\lim_{n\to+\infty} |x_n|^{2/(q-1)}u(x_n) = \ell'$. Set $v_n(x) = |x_n|^{2/(q-1)}u(|x_n| x)$; v_n satisfies

$$- \Delta v_n(x) + |x_n|^{2q/(q-1)} g(|x_n|^{-2/(q-1)} v_n(x)) = 0 \quad \text{in } \mathbb{R}^N - \{0\}. \tag{2.32}$$

By compactness there exists a subsequence n_k and a function v such that $v_{n_k}(x)$ converges to $v(x)$ uniformly on every compact of $\mathbb{R}^N - \{0\}$, and v satisfies

$$- \Delta v + cv^q = 0 \qquad \text{in } \mathbb{R}^N - \{0\}. \tag{2.33}$$

From Lemma 1.4 of [16] there exist two constants $K > 0$ and $\tau > 0$ such that the following inequality holds for any $R > 0$ and any $0 < |x| < R/$

$$v(x) \leqslant \ell\, |x|^{-2/(q-1)} \left(1 + K\left(\frac{|x|}{R}\right)^{\tau}\right). \tag{2.34}$$

Making $R \to +\infty$ we deduce $v(x) \leqslant \ell\, |x|^{-2(q-1)}$ for $x \neq 0$. For any $\epsilon > 0$ there exists n_{k_0} such that for $n_k \geqslant n_{k_0}$ and $|x| = 1$

$$|x_{n_k}|^{2/(q-1)}\, u(|x_{n_k}| x) - v(x) < \epsilon. \tag{2.35}$$

If we take $x = \dfrac{x_n}{|x_n|}$ and make $n_k \to +\infty$ we deduce $\ell' - \ell < \epsilon$ which contradicts $\ell' > \ell$; so $\ell = \ell' = \lim_{x\to 0} |x|^{2/(q-1)}u(x)$.

THEOREM 2.4. *Suppose* g *is defined on* $\mathbb{R}^+$ and satisfies for some $c > 0$ *and some* $1 < q < \dfrac{N+1}{N-1}$

$$\begin{cases} \text{i)} & \lim_{r\to+\infty} (g(r) - cr^q)r^{-(N+1)(q-1)/2} = 0, \\ \\ \text{ii)} & \limsup_{r\to 0^+} g(r)/r < +\infty. \end{cases} \tag{2.36}$$

If $u \in C^2(\mathbb{R}^N - \{0\})$ *is a non negative solution of (2.1) in* $D'(\mathbb{R}^N - \{0\})$, *it is spherically symmetric.*

Before proving that result we introduce the generalised Sommerfeld exponant τ (see

[12] and [16]) which is the positive root of the equation

$$X^2 - \left(2\frac{q+1}{q-1} - N\right)X - 2\left(\frac{2q}{q-1} - N\right) = 0. \tag{2.37}$$

We have the following result which will also be used in Section 3,

PROPOSITION 2.2. *Suppose* q *and* p *are two real numbers such that* $1 < q < \frac{N}{N-2}$, $0 < p < \frac{q-1}{2}\tau$ *and* g *is defined on* $\mathbb{R}^+$ *and satisfies for some* $c > 0$

$$\lim_{r \to +\infty} (g(r) - cr^q)r^{p-q} = 0. \tag{2.38}$$

If $u \in C^2(\mathbb{R}^N - \{0\})$ *is a positive solution of (2.1) in* $D'(\mathbb{R}^N - \{0\})$ *satisfying* $\lim_{x \to 0} |x|^{2/(q-1)} u(x) = \ell$ *(defined in (2.31), then for any* $\epsilon > 0$ *there exist* $\rho > 0$ *and* $k \geqslant 0$ *such that*

$$|\ell - |x|^{2/(q-1)} u(x)| \leqslant \epsilon |x|^{2p/(q-1)} + k |x|^{2/(q-1)}, \tag{2.39}$$

for any $0 < |x| < \rho$.

Proof. First we shall prove that for any $\epsilon > 0$, there exist $\rho > 0$ and $k \geqslant 0$ such that the following inequality holds for any $0 < |x| < \rho$:

$$u(x) \leqslant \ell |x|^{-2/(q-1)}(1+\epsilon/\ell |x|^{2p/(q-1)}) + k. \tag{2.40}$$

Step 1. We set $\psi(x) = \ell |x|^{-2/(q-1)}$ and we define as in the Proposition A.4 of [5] $\phi(x) = \mathrm{Max}(\psi(x), u(x))$. From Kato's inequality we get

$$\Delta\phi = \Delta\frac{1}{2}(\psi+u+|\psi-u|) \geqslant \frac{1}{2}(\Delta\psi+\Delta u) + \frac{1}{2}\mathrm{sign}(\psi-u)\Delta(\psi-u).$$

As $\Delta\psi = c\psi^q$, we get $\Delta\phi \geqslant \frac{1}{2}(c\psi^q+g(u)+\mathrm{sign}(\psi-u)(c\psi^q-g(u))$, or

$$\Delta\phi \geqslant \mathrm{Min}(c\phi^q, g(\phi)). \tag{2.41}$$

Moreover there exists $D > 0$ such that

$$g(\phi(x)) \geqslant c(\phi(x))^q - D(\phi(x))^{q-p}, \tag{2.42}$$

for $0 < |x| < 1$.

Step 2. Set $w(r,\sigma) = r^{2/(q-1)}\phi(r,\sigma)$ and $\overline{w}(r)$ its average on S^{N-1}. We have $w(r,\sigma) \geqslant \ell$ and

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\left(N - \frac{q+3}{q-1}\right)\frac{\partial w}{\partial r} + \frac{1}{r^2}\Delta_{S^{N-1}} w \geqslant \frac{c}{r^2}(w^{q-1} - \ell^{q-1}) - Dr^{2p/(q-1)-2}w^{q-p}\,. \tag{2.43}$$

As w is bounded on $\{x \mid 0 < |x| < 1\}$ we deduce in averaging (2.43) that

$$\frac{d^2\overline{w}}{dr^2} + \frac{1}{r}\left(N - \frac{q+3}{q-1}\right)\frac{dw}{dr} \geqslant -D_1\, r^{2p/(q-1)-2}, \tag{2.44}$$

for $0 < r < 1$, D_1 being a constant. Now we set $s = \dfrac{r^{2(q+1)/(q-1)-N}}{2(q+1)/(q-1)\ N}$ and $\overline{v}(s) = \overline{w}(r)$. We have

$$\frac{d^2 v}{ds^2} + D_2 s^{\theta-2} \geqslant 0, \tag{2.45}$$

on $\left\{s \mid 0 < s < \dfrac{q-1}{2(q+1)-N(q-1)}\right\}$ where D_2 is non negative and $\theta = \dfrac{2p}{2(q+1)-N(q-1)}$. Hence the function $s \mapsto \overline{v}(s) + \dfrac{D_2}{\theta(\theta-1)}\, s^\theta$ is convex (or $s \mapsto v(s) + D_2(s \log s - s)$ if $\theta = 1$) which implies that $\overline{v}(s) \leqslant \overline{v}(0) + s\, D_3$, where D_3 depends on θ, q, N and $\overline{u}(1)$; so there exists a constant A such that

$$\overline{u}(r) \leqslant \ell\, r^{-2/(q-1)} + A\, r^{2q/(q-1)-N}, \tag{2.46}$$

for $0 < r < 1$. Moreover that relation is true for any $0 < p$.

Step 3. We set $\omega(r) = \ell\, r^{-2/(q-1)}(1+\epsilon/\ell\, r^{2p/(q-1)})$ and we claim that we can find σ such that

$$-\Delta\omega + g(\omega) \geqslant 0, \tag{2.47}$$

on $\{x \mid 0 < |x| < \sigma\}$. For a given $\delta > 0$, there exists $\sigma' > 0$ such that $g(\omega(r)) \geqslant c(\omega(r))^q - \delta(\omega(r))^{q-p}$ for $0 < r < \sigma'$. We get

$$\frac{d\omega}{dr} = -\frac{2}{q-1}\ell\, r^{-(q+1)/(q-1)} + \epsilon\, 2\frac{p-1}{q-1}\, r^{(2p-q-1)/(q-1)},$$

$$\frac{d^2\omega}{dr^2} = \frac{2(q+1)}{(q-1)^2}\,\ell\, r^{-2q/(q-1)} + \epsilon\,\frac{2(p-1)(2p-q-1)}{(q-1)^2}\, r^{2(p-q)/(q-1)},$$

$$c\omega^q = c\ell^q\, r^{-2q/(q-1)}(1+\epsilon/\ell\, r^{2p/(q-1)})^q \geqslant c\ell^q\, r^{-2q/(q-1)}(1+q\,\epsilon/\ell\, r^{2p/(q-1)}),$$

$$\delta\omega^{q-p} = \delta\,\ell^{q-p}(1 + \epsilon/\ell\, r^{2p/(q-1)})^{q-p}\, r^{2(p-q)/(q-1)}.$$

So we get

$$-\Delta\omega + c\omega^{q} - \delta\omega^{q-p} \geqslant -\left(\epsilon\,\frac{2(p-1)(2p-q-1)}{(q-1)^2} + 2(N-1)\,\epsilon\frac{p-1}{q-1}\right) r^{2(p-q)/(q-1)} \ldots$$

$$+\, c\,\epsilon\,\ell^{q-1} q\, r^{2(p-q)/(q-1)} - \delta\,\ell^{q-p}(1 + \epsilon/\ell\, r^{2p/(q-1)})^{q-p}\, r^{2(p-q)/(q-1)}.$$

And the right hand side of that inequality can be written as

$$\left\{\epsilon\left[2\,(\frac{2q}{q-1} - N) + (2\frac{q+1}{q-1} - N)\,(\frac{2p}{q-1}) - (\frac{2p}{q-1})^2\right] - \delta\ell^{q-p}\,(1 + \epsilon/\ell\, r^{2p/(q-1)})^{q-p}\right\} r^{2(p-q/(q-1)},$$

and, as $0 < \dfrac{2p}{q-1} < \tau$, the coefficient of ϵ is positive. So we first choose σ_1 such that $(1 + \epsilon/\ell\, r^{2p/(q-1)})^{q-p} \leqslant 2$ for $0 \leqslant r \leqslant \sigma_1$. We then choose δ such that

$$2\,\delta\,\ell^{q-p} < \epsilon\left[2\,(\frac{2q}{q-1} - N) + (2\frac{q+1}{q-1} - N)\,(\frac{2p}{q-1}) - (\frac{2p}{q-1})^2\right],$$

and then we take $\sigma = \min(\sigma_1\,, \sigma')$, which implies (2.47).

Step 4. We follow now the end of the proof of Proposition A.4 of [5] . Set $k = \underset{|x|=\sigma}{\mathrm{Max}}\ \phi(x)$. As g is nondecreasing we have

$$-\Delta(\omega+k) + g(\omega+k) \geqslant 0, \tag{2.48}$$

on $\{x \mid 0 < |x| < \sigma\}$. Let ξ_n be a sequence of smooth functions such that

$$\xi_n(x) = \begin{cases} 1 & \text{for } |x| \geqslant \dfrac{1}{n} \\ \\ 0 & \text{for } |x| \leqslant \dfrac{1}{2n} \end{cases}, \quad 0 \leqslant \xi_n \leqslant 1,\ |\Delta\xi_n| \leqslant Kn^2.$$

Let θ be a smooth nondecreasing function vanishing on $(-\infty,0]$, strictly positive on $(0,+\infty)$ and such that $\theta = 1$ on $[1, +\infty)$ and set $j(t) = \displaystyle\int_0^t \theta(s)ds$. We have from Steps 1 and 3, in setting $\Omega = \{x \mid 0 < |x| < \sigma\}$,

$$(2.49)\qquad \int_\Omega \nabla(u-\omega-k)\,.\,\nabla\xi_n\theta(u-\omega-k)dx+\int_\Omega |\nabla(u-\omega-k)|^2\,\xi_n\theta'(u-\omega-k)dx\,\ldots$$

$$\ldots+\int_\Omega (g(u)-g(\omega+k))\xi_n\,\theta(u-\omega-k)dx\leqslant 0.$$

As $\nabla(u-\omega-k)\,\theta(u-\omega-k)=\nabla j(u-\omega-k)$, so we get

$$\int_\Omega |\nabla(u-\omega-k)|^2\,\xi_n\theta'(u-\omega-k)dx+\int_\Omega (g(u)-g(\omega+k)\xi_n\theta'(u-\omega-k)dx\leqslant\ldots$$

$$\ldots\int_\Omega j(u-\omega-k)\Delta\xi_n dx\leqslant Kn^2\int_{\frac{1}{2n}\leqslant|x|\leqslant\frac{1}{n}} j(u-\omega-k)dx.$$

But $j(u-\omega-k)\leqslant j(u-\omega)\leqslant j(\phi-\ell\,|x|^{-2/(q-1)})\leqslant\phi-\ell\,|x|^{-2/(q-1)}$ and from Step 2, $0\leqslant\phi(r)-\ell\,r^{-2/(q-1)}\leqslant A\,r^{2q/(q-1)-N}$ for $0<r<1$.

So we get : $Kn^2\int_{\frac{1}{2n}\leqslant|x|\leqslant\frac{1}{n}} j(u-\omega-k)dx\leqslant KA\,\frac{q+1}{2q}\,n^{-2/(q-1)}$. As $n\to+\infty$ we get by Fatou's Lemma

$$(2.50)\qquad \int_\Omega |\nabla(u-\omega-k)|^2\,\theta'(u-\omega-k)dx+\int_\Omega (g(u)-g(\omega+k))\theta'(u-\omega-k)dx\leqslant 0,$$

which implies that both terms are 0. If we make $\theta(r)\to r^+$ we deduce that $\nabla(u-\omega-k)^+=0$ a.e. But $(u-\omega-k)^+$ vanishes on $\partial\Omega$ so it is identically 0 and we have

$$(2.51)\qquad u(x)\leqslant\ell\,|x|^{-2/(q-1)}\,(1+\epsilon/\ell\,|x|^{2p/(q-1)})+k,$$

for $0<|x|<\sigma$, which is (2.40).

For próving the reverse inequality

$$(2.52)\qquad u(x)\geqslant\ell\,|x|^{-2/(q-1)}(1-\epsilon/\ell\,|x|^{2p/(q-1)})-k,$$

we do the same in introducing $\phi_1(x)=\mathrm{Min}(\psi(x),u(x))$ which satisfies

$$(2.53)\qquad \Delta\phi_1\leqslant \mathrm{Mas}(c\,\phi_1^q\,,g(\phi_1)).$$

With the same change of variable we obtain by concavity

$$(2.54)\qquad \bar u(r)\geqslant\ell\,r^{-2/(q-1)}-A\,r^{2q/(q-1)-N},$$

for $0 < r < 1$. We then construct a subsolution $\omega_1(r) = \ell\, r^{-2/(q-1)}(1 - \epsilon/\ell\, r^{2p(q\ 1)})$ for the equation (2.1) (the only slight change being in the estimation of $(\omega_1(r))^q$ where we have : $(\omega_1(r))^q \leqslant \ell^q\, r^{-2q/(q-1)}(1 - q'\epsilon/\ell\, r^{2p/(q-1)})$ where $1 < q' < q$ but $q - q'$ can be as small as we want in restricting r). We end the proof as in the Step 4.

Proof of the Theorem 2.4. From (2.36) and Lemma 2.1 any solution of (2.1) is bounded at infinity. So, if $\lim_{x \to 0} |x|^{N-2} u(x) = \gamma$, we deduce from Theorem 2.1 that u is spherically symmetric.

Now suppose that $\lim_{x \to 0} |x|^{2/(q-1)} u(x) = \ell$. We have $\frac{2p}{q-1} = \frac{2q}{q-1} - (N+1) > 0$, and

$\left(\frac{q+1}{q-1} - N\right)^2 - \left(2\frac{q+1}{q-1} - N\right)\left(\frac{q+1}{q-1} - N\right) - 2\left(\frac{2q}{q-1} - N\right) < 0$, so we have (2.39) and

$r^{2/(q-1)} \| u(r,.) - \bar{u}(r) \|_{L^\infty(S^{N-1})} \leqslant 2\epsilon^{(q+1)/(q-1)-N} + 2k\, r^{2/(q-1)}$, for $0 < r < \rho$. So we deduce

$$\limsup_{r \to 0} r^{N-1} \| u(r,.) - \bar{u}(r) \|_{L^\infty(S^{N-1})} \leqslant 2\epsilon. \tag{2.55}$$

Making $\epsilon \to 0$ we obtain $\lim_{r \to 0} r^{N-1} \| u(r,.) - \bar{u}(r) \|_{L^\infty(S^{N-1})} = 0$ and then we conclude with Theorem 2.1.

Remark 2.2. The following nonlinear Liouville-Hadamard type result is a consequence of Theorem 2.1 : *a* C^2 *solution* u *of (2.1) in* $\mathbb{R}^N$ *such that* $u(x) = o(|x|)$ $(|x| \to \infty)$ *is a constant.*

3. - UNIQUENESS OF SOLUTIONS

In that part we shall still suppose that g is a continuous nondecreasing function defined on IR (Holder continuous as we want to deal with strong solutions) and we consider the equation

$$-\Delta u + g(u) = 0, \tag{3.1}$$

taken into $D'(\mathbb{R}^N - \{0\})$ and we investigate under what assumption on g is a (possibly singular) solution of (3.1) uniquely determined. If u is a solution of (3.1) and $\theta \in O(n)$, $u \circ \theta$ is also a solution of (3.1) ; so if u is uniquely determined, u must be spherically symmetric. The following easy-to-prove result is the key-stone of this section.

THEOREM 3.1. *Suppose* u_1 *and* u_2 *belonging to* $C^2(\mathbb{R}^N - \{0\})$ *are two solutions of (3.1) in* $D'(\mathbb{R}^N - \{0\})$. *If they satisfy*

$$(3.2)\qquad \begin{cases} \text{i)} & \lim\limits_{r\to 0} r^{N-2} \| u_1(r,.) - u_2(r,.) \|_{L^2(S^{N-1})} = 0, \\ \text{ii)} & \lim\limits_{r\to +\infty} \| u_1(r,.) - u_2(r,.) \|_{L^2(S^{N-1})} = 0, \end{cases}$$

then $u_1 = u_2$.

Proof. We make the change of variable

$$(3.3)\qquad s = \frac{r^{N-2}}{N-2}, \quad u_i(r,\sigma) = r^{2-N} v_i(r,\sigma), \quad i = 1,2.$$

The function v_i satisfies

$$(3.4)\qquad s^2 \frac{\partial^2 v_i}{\partial x^2} + \frac{1}{(N-2)^2} \Delta_{S^{N-1}} v_i = (N-2)^{(4-N)/(N-2)} s^{N/(N-2)} g(\frac{v_i}{s(N-2)}),$$

in $(0, +\infty) \times S^{N-1}$. If we set $w = v_1 - v_2$, then we get : $s^2 \int_{S^{N-1}} \left(\frac{\partial^2}{\partial s} \omega\right) \omega \, d\sigma \geqslant 0$ which implies that he function $s \mapsto \| w(s,.) \|_{L^2(S^{N-1})}$ is convex. As it vanishes at 0 and satisfies $\lim\limits_{s\to +\infty} \frac{1}{s} \| w(s,.) \|_{L^2(S^{N-1})} = 0$, it is identically 0.

As a consequence we have the following

COROLLARY 3.1. *Suppose* g *vanishes only at* 0 *and satisfies*

$$(3.5)\qquad \begin{cases} \text{i)} & \liminf\limits_{r\to +\infty} g(r)/r^{N/(N-2)} > 0, \\ \text{ii)} & \limsup\limits_{r\to -\infty} g(r)/|r|^{N/(N-2)} < 0. \end{cases}$$

Then the only $u \in C^2(\mathbb{R}^N - \{0\})$ *satisfying (3.1) in* $D'(\mathbb{R}^N - \{0\})$ *is the zero function.*

Proof. From a result of Brezis and Veron [6] the function u can be extended to whole $\mathbb{R}^N$ into a C^2 function. Moreover from Lemma 2.1 and Theorem 1.1, $|x|^{N-2} u(x)$ admits a limit when $|x|$ goes to $+\infty$. Applying Theorem 3.1 to u and 0, we get $u = 0$.

Remark 3.1. The assumption $g^{-1}(0) = 0$ can be cancelled if we consider the solutions of (3.1) vanishing in some sense at infinity, for example such that $\lim\limits_{r\to+\infty} \|u(r,.)\|_{L^2(S^{N-1})} = 0$. Some other conditions are discussed in [1].

When the growth of g at infinity is comparable to some power q with $1 < q < \frac{N}{N-2}$, there exist two types of isotropic singularities at 0. We deduce from Proposition 2.1 and Theorems 1.1 and 3.1.

COROLLARY 3.2. *Suppose* g *vanishes only at* 0 *and satisfies*

$$\lim_{|r|\to+\infty} |g(r)|/|r|^q = c, \tag{3.6}$$

for some $c > 0$ *and* $1 < q < \frac{N}{N-2}$. *If* $u \in C^2(\mathbb{R}^N - \{0\})$ *is a solution of (3.1) in* $D'(\mathbb{R}^N - \{0\})$ *such that* $|x|^{N-2}u(x)$ *remains bounded in some neighbourhood of* 0, *then* u *is uniquely determined by the value of* $\gamma = \lim\limits_{x\to 0} |x|^{N-2}u(x)$.

In fact in Corollary 3.2, we have not only the uniqueness with respect to the singularity at 0, but also the existence, as a consequence of

LEMMA 3.1. *Suppose* g *vanishes at* 0 *and satisfies (3.6) for some* $c > 0$ *and some* $1 < q < \frac{N}{N-2}$. *Then for any* γ *there exists a unique* $u \in C^2(0, +\infty)$ *satisfying*

$$\begin{cases} \dfrac{d^2u}{dr^2} + \dfrac{N-1}{r}\dfrac{du}{dr} - g(u) = 0 & \text{on } (0, +\infty), \\[2ex] \lim\limits_{r\to 0} r^{N-2}u(r) = \gamma, \quad \lim\limits_{r\to+\infty} u(r) = 0. \end{cases} \tag{3.7}$$

Proof. If we set $s = \frac{r^{N-2}}{N-2}$ and $u(r) = r^{2-N}v(s)$, then (3.7) is equivalent to

$$\begin{cases} s^2\dfrac{d^2v}{ds^2} - (N-2)^{(4-N)/(N-2)}\, s^{N/(N-2)}\, g\left(\dfrac{v}{(N-2)s}\right) = 0 \quad \text{on } (0, +\infty), \\[2ex] \lim\limits_{s\to 0} v(s) = \gamma, \quad \lim\limits_{s\to+\infty} \dfrac{1}{s}v(s) = 0. \end{cases} \tag{3.8}$$

The uniqueness comes from the same argument of convexity as the one of Lemma 1.3. For the existence, we consider for any $\epsilon > 0$ the solution v_ϵ (coming also from the Lemma 1.3) of the equation

$$(3.9)\quad \begin{cases} (s+\epsilon)^2 \dfrac{d^2v_\epsilon}{ds^2} - (N-2)^{(4-N)/(N-2)}(s+\epsilon)^{N/(N-2)}\, g\left(\dfrac{v}{(N-2)(s+\epsilon)}\right) = 0 \text{ on } (0,+\infty), \\ \\ v_\epsilon(0) = \gamma, \quad \lim\limits_{s\to+\infty} \dfrac{1}{s} v_\epsilon(s) = 0 . \end{cases}$$

As the function $s \mapsto |v_\epsilon(s)|$ is convex it is nonincreasing. From (3.6) we have

$$(3.10)\qquad |g(r)| \leqslant c\,|r|^q + d,$$

for any r and some $c, d > 0$; so we have for any $0 < s < T$

$$(3.11)\quad \left|\frac{dv_\epsilon}{ds}(s)\right| < \left|\frac{dv_\epsilon}{ds}(T)\right| + K\int_s^T ((\sigma+\epsilon)^{N/(N-2)-q-2}\,|v_\epsilon|^q + (\sigma+\epsilon)^{N/(N-2)-2})d\sigma .$$

But as $|v_\epsilon| \leqslant \gamma$, $|g(v_\epsilon)|$ is bounded and it is the same with $\dfrac{d^2v_\epsilon}{ds^2}$ and $\dfrac{dv_\epsilon}{ds}$ on any interval $(\alpha,+\infty)$, $\alpha > 0$. Integrating again (3.11) yields

$$(3.12)\quad |v_\epsilon(t) - v_\epsilon(s)| \leqslant A_1(t-s) + A_2((t+\epsilon)^{N/(N-2)-q} - (s+\epsilon)^{N/(N-2)-q}) + \ldots$$

$$\ldots\ A_3((t+\epsilon)^{N/(N-2)} - (s+\epsilon)^{N/(N-2)}),$$

for $0 < s < t < T$. As the functions $t \mapsto t^{N/(N-2)-q}$ and $t \mapsto t^{N/(N-2)}$ are uniformly continuous on $[0,T+1]$, the set of functions $(v_\epsilon \mid \epsilon \in (0,1])$ is equicontinuous on $[0,T]$. Using Arzela Ascoli theorem and the diagonal process, there exists a continuous function v on $[0,+\infty)$ and a sequence $\epsilon_n \to 0$ such that v_{ϵ_n} converges to v on $[0,T]$, for any $T > 0$. The function v satisfies the equation (3.8), is nonincreasing and $v(o) = \gamma$.

Remark 3.1. If we define $\tilde{u}$ on $\mathbb{R}^N - \{0\}$ by $\tilde{u}(x) = u(|x|)$, where u satisfies (3.7), one can see that $\tilde{u}$ is a solution of

$$(3.13)\qquad -\Delta u + g(u) = (N-2)\,|S^{N-1}|\,\gamma\,\delta_0 ,$$

in $D'(\mathbb{R}^N)$, unique if g vanishes only at 0.

THEOREM 3.2. *Suppose* g *vanishes only at* 0 *and satisfies for some* $c > 0$ *and some* $1 < q < \dfrac{N}{N-2}$

(3.14) $$\lim_{r \to +\infty} (g(r) - c\, r^q) r^{-N(q-1)/2} = 0.$$

Then there exists only one $u \in C^2(\mathbb{R}^N - \{0\})$ *solution of (3.1) such that* $\lim_{x \to 0} |x|^{2/(q-1)} u(x) = \ell$.

Proof. Existence : For any $\gamma > 0$ set u_γ the solution of (3.7) on $(0, +\infty)$. From the Lemma 2.1, there exist A and $B > 0$ such that

(3.15) $$0 \leqslant u_\gamma(r) \leqslant \frac{A}{r^{2/(q-1)}} + B,$$

for any $r > 0$ and $\gamma > 0$. Setting $s = \frac{r^{N-2}}{N-2}$ and $u_\gamma(r) = r^{2-N}\, v_\gamma(s)$, the function v_γ satisfies the equation (3.8) with initial data γ and vanishes at $+\infty$. From the uniqueness, for any $s > 0$, the function $\gamma \mapsto v_\gamma(s)$ is nondecreasind and as

(3.16) $$0 \leqslant v_\gamma(s) \leqslant \frac{A}{((N-2)s)^{2/(q-1)(N-2)}} + B,$$

it converges as $\gamma \to +\infty$ to some function v_∞ satisfying (3.8). Setting $u_\infty(r) = r^{2-N}\, v_\infty(s)$ the function u_∞ satisfies (3.7) and $\lim_{r \to 0} r^{N-2} u_\infty(r) = +\infty$. If $u(x) = u_\infty(|x|)$, u satisfies (3.1) and, from the Proposition 2.1, $\lim_{x \to 0} |x|^{2/(q-1)} u(x) = \ell$.

Uniqueness : Set u_1 and u_2 two solutions of (3.1) such that $\lim_{x \to 0} |x|^{2/(q-1)} u_i(x) = \ell$ for $i = 1,2$. We apply the Proposition 2.2 with $p = q - \frac{N}{2}(q-1)$ and we get from (2.39)

(3.17) $$|x|^{2/(q-1)} (u_1(x) - u_2(x)) \leqslant 2\epsilon\, |x|^{2/(q-1)+2-N} + k\, |x|^{2/(q-1)}$$

which implies $\lim_{x \to 0} |x|^{N-2} |u_1(x) - u_2(x)| = 0$. As u_1 and u_2 vanishes at infinity we deduce $u_1 = u_2$ from the Theorem 3.1.

Remark 3.2. When $g(r) = c\, |r|^{q-1}\, r$ the solution u of Theorem 3.2 is

(3.18) $$u(x) = \ell\, |x|^{-2/(q-1)}$$

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