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ASYMPTOTIC RATES OF DECAY FOR SOME NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS AND VARIATIONAL PROBLEMS OF ARBITRARY ORDER

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Résumé : Nous étudions ici le problème de minimiser la fonctionnelle $(1/p) \int_0^\infty |u^{(m)}(x)|^p \, dx + \int_0^\infty \Gamma(u(x))dx$, $p > 1$, avec des conditions aux limites pour $x = 0$. $\Gamma(s)$ se comporte comme $|s|^r$ près de $s = 0$ $(r > 0)$. Nous obtenons des taux optimaux de décroissance à l'infini pour les solutions de ce problème. Si $0 < r < p$ les solutions ont un support compact. Presque aucune hypothèse de régularité n'est demandée à la fonction Γ, qui peut même être discontinue. Nous obtenons des résultats analogues pour les solutions tendant vers zéro à l'infini de l'équation différentielle d'Euler associée d'ordre 2m $(-1)^m d^m/dx^m (|u^{(m)}|^{p-1} \operatorname{sgn} u^{(m)}) + \gamma(u) = 0$, en supposant γ continue. Les solutions peuvent être oscillatoires et γ n'est pas supposée monotone. Nous considérons aussi des non-linéarités dans les dérivées intermédiaires. Les démonstrations sont basées sur les inégalités d'interpolation de Nirenberg et Gagliardo.

Summary : Consider the variational problem of minimizing the functional $(1/p) \int_0^\infty |u^{(m)}(x)|^p \, dx + \int_0^\infty \Gamma(u(x))dx$, $p > 1$, with boundary data at $x = 0$. $\Gamma(s)$ behaves

like $|s|^r$ near $s = 0$ $(r > 0)$. We obtain optimal rates of decay at infinity for the solutions of this problem. If $0 < r < p$ the solutions havec compact support. The function Γ needs to satisfy almost no regularity hypotheses, even it may be noncontinuous. We obtain analogous results for the solutions tending to zero at infinity of the associated 2m-order Euler differential equation $(-1)^m\, d^m/dx^m\, (\,|u^{(m)}|^{p-1} \operatorname{sgn} u^{(m)}) + \gamma(u) = 0$, assuming γ continuous. The solutions of the differential equation may not be solutions of the variational problem, since no monotony of γ is assumed. Nonlinearities in intermediate derivatives are also considered. The proofs are based on Nirenberg-Gagliardo interpolation inequalities.

1. - INTRODUCTION

1.1. - Statement of the problems and notations

Given the real number $p > 1$ and the everywhere defined function $\Gamma : R \to R \cup \{+\infty\}$, we consider the variational problem :

$$\text{(Problem p}\Gamma\text{)}\quad \begin{cases} \text{minimize } \; J(u) = \dfrac{1}{p}\displaystyle\int_0^\infty |u^{(m)}(x)|^p\, dx + \int_0^\infty \Gamma(u(x))dx \\ \text{in } W = \left\{ u : u \in L^1_{loc}(R_+),\ \ \Gamma(u(\,.\,)) \in L^1(R_+) \text{ and } u^{(m)} \in L^p(R_+) \right\} \\ \text{with the boundary conditions } u^{(j)}(0) = \alpha_j \;\;,\;\; 0 \leqslant j \leqslant m-1 \end{cases}$$

where u is real-valued, m, j are integers $(m \geqslant 1)$ and all α_j are real numbers. The derivatives are taken in the sense of distributions, so that $u^{(m)} \in L^1_{loc}(\overline{R}_+)$ implies that $u^{(m-1)}$ is locally absolutely continuous in $\overline{R}_+ = [0,\infty)$.

The Euler differential equations associated with Problem $p\Gamma$ and with the particular case $p = 2$ are respectively :

$$\text{(Equation p}\gamma\text{)}\qquad (-1)^m \frac{d^m}{dx^m}\left(|u^{(m)}|^{p-1} \operatorname{sgn} u^{(m)}\right) + \gamma(u) = 0\,,\qquad \gamma(s) = \Gamma'(s)$$

$$\text{(Equation 2}\gamma\text{)}\qquad (-1)^m u^{(2m)} + \gamma(u) = 0$$

For example, $\gamma(s) = |s|^{r-1} \operatorname{sgn} s$ for $\Gamma(s) = (1/r)\,|s|^r$. Setting

$$(1)\qquad w = |u^{(m)}|^{p-1} \operatorname{sgn} u^{(m)}$$

Equation $p\gamma$ can be written as the normal system :

(2) $$u^{(m)} = |w|^{p'-1} \operatorname{sgn} w \ ; \ w^{(m)} = (-1)^{m+1} \gamma(u)$$

where p' is the dual exponent of p : $(1/p) + (1/p') = 1$. Notice that

(3) $$|u^{(m)}|^p \operatorname{sgn} u^{(m)} = |w|^{p'} \operatorname{sgn} w$$

We shall say that Γ is *positive definite* if and only if

(4) $$\Gamma(0) = 0 \quad \text{and} \quad \inf_{|t| \geqslant |s|} \Gamma(t) > 0 \quad \text{for } s \neq 0$$

Given $\gamma \in C(R)$, the statement that u is a solution of Equation $p\gamma$ in $\overline{R}_+$ will always mean in this paper that (u,w) is a classical solution of the system (2), i.e. such that $u \in C^m(\overline{R}_+)$ and $w \in C^m(\overline{R}_+)$. For p = 2 $u \in C^{2m}(\overline{R}_+)$. (It is well-known that any distribution solution in R_+ of (2) is a classical solution in $[\epsilon,\infty)$ for all $\epsilon > 0$).

Functions and constants will be real-valued. C, K will be positive constants which may be different in different occurrences. «Compact support» will mean «compact support in $\overline{R}_+$».

1.2. - Main results

The question of existence is summarized in Section 2, although the following theorems are independent of existence theory.

THEOREM 1. *Let* $\gamma \in C(R)$ *and assume that near* $s = 0$

(5) $$B_1 |s|^r \leqslant s\gamma(s) \leqslant B_2 |s|^r \ , \quad B_1 > 0.$$

Let u *be a solution in* $\overline{R}_+$ *of Equation* $p\gamma$, $1 < p < \infty$, *such that* $u(x) \to 0$ *as* $x \to \infty$.

I. *If* $1 < r < p$ *then* u *has compact support.*

II. *If* $r = p$ *then as* $x \to \infty$ *and for* $0 \leqslant j \leqslant m$

$$u^{(j)}(x) = \mathbf{O}(e^{-Cx}), \ C > 0 \ ; \ w^{(j)}(x) = \mathbf{O}(e^{-C^*x}), \ C^* > 0,$$

where the constants C and C* depend only on m, j, p, B_1 and B_2 .

III. If $r > p$ *then as* $x \to \infty$ *and for* $0 \leqslant j \leqslant m$

$$u^{(j)}(x) = \mathbf{O}(x^{\sigma-j}) \quad ; \quad w^{(j)}(x) = \mathbf{O}(x^{\sigma^*-j})$$

(6) $$\sigma = -pm/(r-p) \quad ; \quad \sigma^* = -r'm/(p'-r')$$

This theorem is proved in Section 3. We note that (5) includes the sign condition $s\gamma(s) \geqslant 0$, but no hypothesis of monotony of γ is made. If (5) holds for all $s \in R$ and $1 < r < p$, a bound of the support is given by (18). The case $r = 1$ can be incorporated in Theorem 1 in the sense explained in Section 6.5. When $p = 2$ Point III simplifies to :

$$u^{(j)}(x) = \mathbf{O}(x^{\sigma-j}) \quad , \quad 0 \leqslant j \leqslant 2m.$$

If $1 < p \leqslant 2$ and $\gamma(s)$ is Lipschitz continuous near $s = 0$, a standard ODE uniqueness theorem applied to (2) implies that the support of u is noncompact unless $u \equiv 0$. The following theorem (proved in Section 4) gives much more precise results.

THEOREM 2. *The bounds of Points II and III of Theorem 1 are optimal in the sense that* u *is identically zero : a) in Point II if* $\exp(-x)$ *is replaced by* $\exp(-xg(x))$ *with* $g(x) \to +\infty$ *as* $x \to +\infty$*; and b) in Point III if any capital* $\mathbf{O}$ *is replaced by a small* o.

As by-product of the proofs of Theorems 1 and 2, we also obtain lower bounds for the support and properties of «blow up» and continuation of solutions to the negative real axis (Section 4).

THEOREM 3. *Let* u *be a solution of Problem* $p\Gamma$, $1 < p < \infty$, Γ *positive definite in the sense of (4). Assume that near* $s = 0$ Γ *is Borel measurable and*

(7) $$B_1 \, |s|^r \leqslant \Gamma(s) \leqslant B_2 \, |s|^r \quad , \quad B_1 > 0$$

I. If $0 < r < p$ *then* u *has compact support.*

II. If $r = p$ *then as* $x \to \infty$ *and for* $0 \leqslant j \leqslant m-1$ $u^{(j)}(x) = \mathbf{O}(e^{-Cx})$, *where the constant* $C > 0$ *depends only on* m, j, p, B_1 *and* B_2 .

III. If $r > p$ *then as* $x \to \infty$ *and for* $0 \leqslant j \leqslant m-1$ $u^{(j)}(x) = \mathbf{O}(x^{\sigma-j})$, *where* σ *is given by (6).*

If (7) holds for all $s \in R$ and $0 < r < p$, then (18) gives an explicit estimate of the support in terms of the data of Problem $p\Gamma$. No sign hypothesis for the derivative of Γ are needed.

In fact, the punctual derivative of Γ may fail to exist at every point. Even Γ may be noncontinuous at points different from the origin. (In this respect, Theorem 3 is new even for $2m = 4$, see below). The proof of Theorem 3 (Section 5) does not use at all the Euler differential equation. In return, it does not apply to the Euler equation solutions of Theorem 1 if $J(u)$ is not the absolute minimum of J.

Further developments are sketched in Sections 6 and 7. They include nonlinearities in intermediate derivatives, variable coefficents, $\Gamma(s)$ noncontinuous at $s = 0$, optimality of the bounds of Theorem 3, equations of odd order and other sign conditions for γ.

1.3. - Related references

Application of Nirenberg [20] and Gabliardo [16] interpolation inequalities for half-lines (see Appendix I) is the unifying feature of the proofs. A preliminary step (Section 2.2) is based on some inequalities of Redheffer [22] and Redheffer & Walter [24] . The book of Beckenbach & Bellman [2] has been a great help to us.

Compactness of the support results of Theorem 1 for $p = 2$ are included in our n-dimensional paper [8] , but we give here a much shorter one-dimensional proof. All other results of Theorems 1 and 3 on compactness of the support are new for $2m \geqslant 6$. Pioneering work on fourth order problems is due to Berkovitz & Pollard [5-I,II] , Redheffer [21] , Hestenes & Redheffer [18-I,II] . Other fourth order references are [9,10,6,7,12] . As far as we know, the results on power rates of decay of Theorems 1 to 3 are new for $2m \geqslant 4$, except some fourth order results of [6] . Fourth order models from optimal control in [5-I, 12] and from elasticity in [6,7].

For $m = 1$ Problem $p\Gamma$ can be explicitly solved (see below). The corresponding second order n-dimensional problem is included in the following works : I) Compactness of the support for $p = 2$ and $1 \leqslant r < 2$ in Benilan, Brezis & Crandall [4] and Redheffer [23]. First n-dimensional results on the subject in Brezis [11] . II) Compactness of the support for general p and $1 \leqslant r < p$ in Diaz & Herrero [29] and references therein. III) Asymptotic power bounds for $p = 2$ and $r > 2$ in Véron [25,26] . Most of second order papers use comparison principles to prove compactness of the support. Antoncev [1] and Diaz & Véron [13] already apply imbedding-interpolation inequalities for bounded domains.

There are many results on asymptotic rates of decay for more complicated second order ordinary differential equations : see Bellman [3] . Nonlinear higher order equations seem to be little studied from this point of view. Third order equations having u-power nonlinearities are considered, from another point of view, in Erbe [15] and references therein. Compact support results for nonoscillatory solutions of higher order equations are to be found in Kiguradze [27], see also the survey [28].

1.4. - Some comments

If $m = 1$, $u(0) = \alpha > 0$, Γ is positive definite and $\Gamma \in C(\overline{R}_+) \cap C^1(R_+)$, the solution of Problem pΓ is $u(x) = G^{-1}(x)$ for $0 \leqslant x < G(0)$; $u(x) = 0$ for $x \geqslant G(0)$, where

$$G(s) = \int_s^\alpha (p'\Gamma(t))^{-1/p}\, dt.$$

The support of u is compact if and only if G(0) is finite. For $\Gamma(s) = |s|^r$, the solution is a power if $r > p$, an exponential if $r = p$ and a power prolonged by zero if $0 < r < p$. The exponent of the powers is σ, given by (6).

For even m, nontrivial solutions of Theorem 1 are necessarily oscillatory. (This is implied by Lemma 2 and the sign condition $s\gamma(s) \geqslant 0$). If m is odd $\geqslant 3$, the solutions of some Problems pΓ, $\Gamma(s) = |s|^r$, are σ-powers (exponentials for $r = p$), but this happens only for very special boundary data. The factor $(-1)^m$ of Equation pγ causes these differences between even and odd m. Nonoscillatory solutions are easier to handle in many cases : see Section 7.3.

The proofs of this paper include long computations with exponents. In fact, if the equation (or the functional) has only two termes (as in Theorems 1 to 3), the exponents are determined by dimensional analysis. Then the key point of the proofs is to show that the parameters are within the allowed range of the inequalities to be applied. For example, in theorem 1 the parameter σ is associated to Equation pγ through the condition $(\sigma-m)(p-1) - m = \sigma(r-1)$. In Theorem 3 the functional J generates the equivalent condition $(\sigma-m)p = \sigma r$. Therefore, the dimensionality exponent of $\int |u^{(j)}|^q\, dx$ is $1 + q(\sigma-j)$. Nevertheless, the methods of this paper apply also to cases with more than two terms, e.g. in Section 6.1.

2. - PRELIMINARIES

2.1. - Existence, uniqueness and relation of Problem pΓ with its Euler equation

Problem pΓ, $1 < p < \infty$, has some solution if Γ is nonnegative and lower semicontinuous and the minimization set is nonempty. (Proof as in [6] , which in turn is inspired in [5-I]). This set is nonempty for any boundary data α_j if in addition $\Gamma(0) = 0$ and Γ is locally bounded. No hypotheses of the type $\Gamma(s) \geqslant |s|^r$ are needed. The solution is unique if Γ is convex. In particular, Problem pΓ, $\Gamma(s) = C|s|^r$, has a solution for all $r > 0$ and it is unique for $r \geqslant 1$.

Now we state two standard results of the calculus of variations. If $\Gamma \in C^1(R)$, then any solution of Problem pΓ is a classical solution in $\overline{R}_+$ of Equation pγ (or of the system (2)).

If in addition Γ is convex, then the unique solution of Problem pΓ is also the unique solution of Equation pγ belonging to W and satisfying the boundary conditions at $x = 0$.

The solutions of Equation pγ considered in Theorem 1 are also solutions of the corresponding Problem pΓ, $\Gamma(0) = 0$, if Γ is convex. The fact that they belong to W is not included in the hypotheses of Theorem 1, but it is easily implied by the conclusions of Theorem 1 itself.

2.2. Conditions in order that $u^{(j)}(x) \to 0$ as $x \to \infty$

We recall that $u^{(m)} \in L^1_{loc}(\bar{R}_+)$ implies $u^{(m-1)} \in AbsC_{loc}(\bar{R}_+)$.

LEMMA 1. *Let* $u \in L^1_{loc}(R_+)$. *Assume 1)* $u^{(m)} \in L^p(R_+)$, $1 \leqslant p \leqslant \infty$, *and 2)* $u(x) \to 0$ *as* $x \to \infty$ *or* $\Gamma(u(\,.\,)) \in L^1(R_+)$ *with* Γ *positive definite in the sense (4). Then* $u^{(j)}(x) \to 0$ *as* $x \to \infty$, $0 \leqslant j \leqslant m-1$.

This is a particular case of Theorem 1 of Redheffer & Walter [24]. The lemma applies to all functions of the set W of Problem pΓ if Γ is positive definite. Due to this result, the hypothesis (7) of Theorem 3 needs to hold only near $s = 0$. On the other hand, in Theorem 1 we assume that $u(x) \to 0$ as $x \to \infty$. Then the role of the above results is to give a sufficient condition for the existence of solutions satisfying the hypotheses of Theorem 1.

LEMMA 2. *Let* $u \in C^m(\bar{R}_+)$. *Let* w *be given by (1),* $m \geqslant 1$. *If* $u(x) \to 0$ *as* $x \to \infty$ *and* $w^{(m)} \in L^\infty(R_+)$, *then as* $x \to \infty$

$$u^{(j)}(x) \to 0, \quad 0 \leqslant j \leqslant m, \quad \textit{and}\; w^{(j)}(x) \to 0, \quad 0 \leqslant j \leqslant m-1$$

(For $p = 2$ *this lemma is included in the former). It implies that the solution* u *of Theorem 1 satisfies* $\lim_{x\to\infty} u^{(j)}(x) = \lim_{x\to\infty} w^{(j)}(x) = 0$, $0 \leqslant j \leqslant m$.

Proof of Lemma 2. By Lemma 1 we only need to establish that $w(x) \to 0$. In fact, we are going to prove by induction on j that $w^{(j)}(x) \to 0$, $0 \leqslant j \leqslant m-1$. The method of proof is taken from [24] and we use the following lemma of Redheffer [22]:

LEMMA 3. *Let* ϵ *and* δ *be positive constants and let* u *be a real-valued function which satisfies* $|u^{(k)}(x)| \geqslant \epsilon$ *on an interval of length* δ. *Then*

$$|u(x)| \geqslant \delta^k \epsilon/2^{k(k+1)} \quad \textit{on a subinterval of length}\; \delta/4^k.$$

We proceed with the proof. Firstly, we have by the continuity of $w^{(j)}$

$$\liminf_{x\to\infty} |w^{(j)}(x)| = 0 \quad , \quad 0 \leqslant j \leqslant m-1 \tag{8}$$

If not, u would be unbounded. By induction, it is enough to see that for $0 \leqslant j \leqslant m-1$ $u(x) \to 0$ and $w^{(j+1)} \in L^{\infty}(R_+)$ imply that $w^{(j)}(x) \to 0$. Set

$$\limsup |w^{(j)}(x)| = S \leqslant \infty \qquad \text{as } x \to \infty$$

We are going to see that $S > 0$ leads to contradiction. Take $0 < \epsilon < S$ and a sequence $\{t_n\}$ such that $t_n \to \infty$,

$$|w^{(j)}(t_n)| > \epsilon \quad \text{and} \quad \lim_{n\to\infty} |w^{(j)}(t_n)| = S. \tag{9}$$

Consider the closed interval $[a_n , b_n]$ «around» t_n such that

$$|w^{(j)}(a_n)| = |w^{(j)}(b_n)| = \epsilon \qquad \text{and} \qquad |w^{(j)}(x)| > \epsilon \quad \text{for } a_n < x < b_n .$$

In this construction we have used the continuity of $w^{(j)}$ and (8). Since $w^{(j+1)} \in L^{\infty}$ implies that $w^{(j)}$ is Lipschitz continuous, we have :

$$|w^{(j)}(t_n)| - \epsilon = |w^{(j)}(t_n) - w^{(j)}(a_n)| \leqslant (t_n - a_n) \, \| w^{(j+1)} \|_{L^{\infty}(a_n, b_n)} \leqslant$$

$$\leqslant (b_n - a_n) \, \| w^{(j+1)} \|_{L^{\infty}(R_+)}$$

From this relation, (9) and $S > \epsilon$ is derived that for some $\delta > 0$

$$b_n - a_n \geqslant \delta \quad \text{for all } n.$$

So we have an unbounded sequence of intervals whose lengths are bounded away from zero and $w^{(j)}$ also is bounded away from zero on them. By Lemma 3, the same thing holds for w, therefore for $u^{(m)}$ and for u, in contradiction to the hypothesis $u(x) \to 0$. Q.E.D.

3. - UPPER BOUNDS FOR SOLUTIONS OF THE DIFFERENTIAL EQUATION : THEOREM 1

Proof of Theorem 1. We shall repeatedly use the hypothesis (5). The values of u will be in the range of (5) on some half-line. By a translation, we can suppose that this half-line is $\bar{R}_+$. We define for $x \geqslant 0$

$$(10) \qquad I(x) = \int_x^\infty |u^{(m)}(t)|^p \, dt + \int_x^\infty u(t)\, \gamma(u(t))dt$$

We multiply by u the differential equation and integrate by parts between x and ∞. By (1), Lemma 2 and Lemma 9 (in Appendix II) we obtain for $x \geqslant 0$

$$(11) \qquad I(x) = (-1)^m \sum_{j=0}^{m-1} (-1)^j u^{(j)}(x)\, w^{(m-j-1)}(x)$$

(In particular, the integrals of (10) are finite). Now we raise to a power $\lambda > 0$; apply the power inequalities (63), integrate between x and ∞ and use Schwarz inequality :

$$(12) \quad \int_x^\infty I^\lambda dt \leqslant C \sum_{j=0}^{m-1} \left[\int_x^\infty |u^{(j)}|^{2\lambda} dt\right]^{1/2} \left[\int_x^\infty |w^{(m-j-1)}|^{2\lambda} dt\right]^{1/2}$$

By (10)-(11), $u^{(m)} \in L^p(R_+)$ and $u \in L^r(R_+)$. By (3), $w \in L^{p'}(R_+)$ and by Equation pγ $w^{(m)} \in L^{r'}(R_+)$, since $|\gamma(u)| \leqslant B_2 |u|^{r-1}$. Then Nirenberg inequalities (Lemma 7, in Appendix I) imply that for λ large enough the right-hand side of (12) is finite. For example, it is enough to take

$$(13) \qquad 2\lambda = \max \left\{ p, r, p', r' \right\}$$

In several important cases it is possible to take $\lambda = 1$ (see Remarks 2 and 3 below), but not for general $r > p$. The exponents of the final bounds will be independent of λ. Now we apply Nirenberg inequalities for the half-line (x,∞) in the following way :

$$(14) \quad \left[\int_x^\infty |u^{(j)}|^{2\lambda} dt\right]^{1/2\lambda} \leqslant C \left[\int_x^\infty |u^{(m)}|^p dt\right]^{a_j/p} \left[\int_x^\infty |u|^r dt\right]^{(1-a_j)/r}$$

$$\frac{1}{2\lambda} = j + a_j \left(\frac{1}{p} - m\right) + (1 - a_j)\frac{1}{r}$$

$$(15) \quad \left[\int_x^\infty |w^{(m-j-1)}|^{2\lambda} dt\right]^{1/2\lambda} \leqslant C \left[\int_x^\infty |w^{(m)}|^{r'} dt\right]^{b_j/r'} \left[\int_x^\infty |w|^{p'} dt\right]^{(1-b_j)/p'}$$

$$\frac{1}{2\lambda} = m-j-1 + b_j \left(\frac{1}{r'} - m\right) + (1 - b_j)\frac{1}{p'}.$$

In (15) we take into account that $|w^{(m)}|^{r'} \leqslant C\,|u|^{r}$ and $|w|^{p'} = |u^{(m)}|^{p}$. Then we insert (14) and (15) in (12) and apply the power inequality (64) of Appendix II :

$$\int_x^\infty I^\lambda\,dt \leqslant C \sum_{j=0}^{m-1} \left[\int_x^\infty |u^{(m)}|^p\,dt + \int_x^\infty |u|^r\,dt \right]^{\lambda/\beta_j}$$

$$\frac{1}{\beta_j} = \frac{a_j}{p} + \frac{1-b_j}{p'} + \frac{1-a_j}{r} + \frac{b_j}{r'} .$$

β_j turns out to be independent of j, as it was expected by dimensional analysis. So we finally obtain :

$$\int_x^\infty I^\lambda\,dt \leqslant C\,I(x)^{\lambda/\beta} \tag{16}$$

$$\frac{1}{\beta} = 1 + \frac{1}{\lambda}\,\frac{(1/r)-(1/p)}{m-(1/p)+(1/r)} = 1 + \frac{1}{\lambda}\,\frac{1}{1+\sigma r} \tag{17}$$

where σ is given by (6). Therefore, the function $f(x) = \int_x^\infty I^\lambda\,dt$ satisfies the first order differential inequality of Lemma 11 (in Appendix II).

Case $r < p$. Then $\sigma > 0$. From (17), $\beta < 1$ and by Lemma 11 f and u have compact support whose extreme a is bounded by (see formula (58)) :

$$a \leqslant C\,f(0)^{1-\beta} \leqslant \bar{C}\,I(0)^{\lambda(1-\beta)/\beta}$$

where we have used (16). Computing the exponent and taking into account that $I(0) \leqslant C\,J(u)$, we arrive at the final expression of the bound :

$$\left\{ \begin{array}{l} a \leqslant C\,J(u)^{1/(1+\sigma r)} \\[2ex] \text{C depends only on m, p, r, } B_1 \text{ and } B_2 \end{array} \right. \tag{18}$$

Remark 1. In (18) J is the functional of the associated Problem $p\Gamma$, $\Gamma(0) = 0$, but (for Theorem 1) it is not required that u be a solution of Problem $p\Gamma$. If this also holds (e.g. if Γ is convex), then $J(u)$ is the minimum of J and the bound (18) becomes more useful. We also note that the proof gives a constructive method to compute an admissible value of the constant C.

Case $r > p$. Then $\sigma < 0$ and $1 + \sigma r < 0$. From (17) $\beta > 1$ and from (16) and Lemma 11, formula (62) :

$$f(x) = \int_x^\infty I^\lambda \, dt = \mathbf{O}\,(x^{-1/(\beta-1)}) \tag{19}$$

Since I is nonnegative and nonincreasing :

$$f(x/2) = \int_{x/2}^\infty I^\lambda \, dt \geqslant \int_{x/2}^x I^\lambda \, dt \geqslant (x/2)\; I(x)^\lambda \tag{20}$$

From (19), (20) and (17) :

$$I(x) = \mathbf{O}\,(x^{-(1/\lambda)(1+1/(\beta-1))}) = \mathbf{O}\,(x^{1+\sigma r}) \tag{21}$$

We bound $|u(x)|$ in terms of $I(x)$ using again Nirenberg inequalities (now with $j = 0$ and $q = \infty$) and (64) :

$$(22)\quad |u(x)| \leqslant C \left[\int_x^\infty |u^{(m)}|^p \, dt\right]^{c/p} \left[\int_x^\infty |u|^r \, dt\right]^{(1-c)/r} \leqslant C\; I(x)^{(c/p)+(1-c)(1/r)}$$

$$0 = c\,(\frac{1}{p} - m) + (1-c)\frac{1}{r}.$$

Computing the exponent and inserting in (21), we obtain finally that $u(x) = \mathbf{O}\,(x^\sigma)$. The asymptotic bound for $w^{(m)}$ is obtained now from the differential equation, i.e. from $|w^{(m)}| \leqslant B_2\, |u|^{r-1}$. Bounds for $w^{(j)}$ and $u^{(j)}$ follow then by integration, taking into account that $\sigma^* < 0$, $\sigma < 0$, $w^{(j)}(x) \to 0$ and $u^{(j)}(x) \to 0$.

Case $r = p$. From (17) $\beta = 1$. The proof is like in the case $r > p$. Note that $I(x)^\lambda = (1/x)\,\mathbf{O}(e^{-Cx})$ implies $I(x) = \mathbf{O}\,(e^{-\bar{C}x})$ for some other constant $\bar{C}$.

Remark 2. If $1 \leqslant r \leqslant p$ ($r = 1$ in the sense of Section 6.5) it is possible to take $\lambda = 1$. Then from (11) to (12) we apply Hölder's inequality (with appropriate exponents) rather than Schwarz inequality. We explain the question a little more : The optimal interpolation exponent q_j in Nirenberg inequalities is obtained for $a = j/m$; therefore $u^{(j)} \in L^s(R_+)$ for all $s \geqslant q_j$ and $w^{(j)} \in L^s(R_+)$ for all $s \geqslant \bar{q}_j$, where

$$\frac{1}{q_j} = \frac{j}{m}\,\frac{1}{p} + \frac{m-j}{m}\,\frac{1}{r}\quad , \quad \frac{1}{\bar{q}_j} = \frac{j}{m}\,\frac{1}{r'} + \frac{m-j}{m}\,\frac{1}{p'}$$

If $r = p$ then $q_j = p$ and $\bar{q}_j = p'$. If $r < p$ then q_j and $\bar{q}_j$ increase with j. Noting that $(1/q_j) + (1/\bar{q}_{m-j}) = 1$, we obtain

$$u^{(j)}\,.\,w^{(k)} \in L^1(R_+) \quad \text{if} \quad 0 \leqslant j+k \leqslant m.$$

This shows how to apply, firstly, Hölder's inequality and, secondly, Nirenberg inequalities to

$$\int_x^\infty I\,dt \leqslant C \sum_{j=0}^{m-1} \int_x^\infty |u^{(j)} w^{(m-j-1)}|\,dt.$$

Remark 3. For $p = 2$ (and any $r \geqslant 1$, $r = 1$ again in the sense of Section 6.5) a primitive of I can be written without integrals. Indeed, using Lemma 10

$$I_1(x) \equiv \int_x^\infty I\,dt = \sum_{j=0}^{m-1} c_j\, u^{(j)}(x)\, u^{(2m-j-2)}(x)$$

Here we apply Nirenberg inequalities in the form :

$$\| u^{(j)} \|_\infty \leqslant C\, \| u^{(2m)} \|_{r'}^{a_j}\, \| u \|_r^{1-a_j}\ ,\quad a_j = (j + \frac{1}{r}) \,/\, (2m - \frac{1}{r'} + \frac{1}{r})$$

and by the differential equation $\| u^{(2m)} \|_{r'} \leqslant C \,\| u \|_r^{r-1}$. Now (16) and (17) become :

$$I_1(x) \leqslant C \left[\int_x^\infty |u|^r\,dt \right]^{1/\beta} \leqslant C\, I(x)^{1/\beta}\ ,\quad \frac{1}{\beta} = 1 + \frac{1}{1+\sigma r}.$$

4. - LOWER BOUNDS, OPTIMALITY OF UPPER BOUNDS, «BLOW UP» AND CONTINUATION OF SOLUTIONS

We begin with a general lemma.

LEMMA 4. *Let* $u^{(m-1)}$ *and* $w^{(m-1)}$ *be locally absolutely continuous in the real open interval* $(\bar{a},\infty)$, $\bar{a} \geqslant -\infty$, m *integer* $\geqslant 1$. *Assume that for all* $x > \bar{a}$ $u \in L^r((x,\infty))$, $1 \leqslant r < \infty$, $w \in L^{p'}((x,\infty))$, $1 < p \leqslant \infty$ *and that*

(23) $$|u^{(m)}(x)| \leqslant K\, |w(x)|^{p'-1} \quad and \quad |w^{(m)}| \leqslant K^*\, |u(x)|^{r-1} \quad \text{a.e. } in\ (\bar{a},\infty)$$

Then Lemma 12 (see Appendix II) is satisfied on $(\bar{a},a)$ *by the function*

(24) $$f(x) = \int_x^\infty |u|^r\,dt + \int_x^\infty |w|^{p'}\,dt \quad with\quad \beta = m \,/\, (m - \frac{1}{p} + \frac{1}{r})$$

$(\bar{a},a)$ *being the interval where* f *is positive.*

If $p' = 1$ or $r = 1$, (23) is to be understood as $|u^{(m)}| \leqslant K$ or $|w^{(m)}| \leqslant K^*$. Let us see the proof. Applying Nirenberg inequalities (Lemma 7) for the half-line (x,∞) with $q = \infty$, $j = 0$:

$$|u(x)| \leqslant C \, \|u^{(m)}\|_p^b \, \|u\|_r^{1-b} \leqslant C \, \|w\|_{p'}^{b(p'-1)} \, \|u\|_r^{1-b}$$

$$|w(x)| \leqslant C \, \|w^{(m)}\|_{r'}^c \, \|w\|_{p'}^{1-c} \leqslant C \, \|u\|_r^{c(r-1)} \, \|w\|_{p'}^{1-c}$$

$$0 = b\left(\frac{1}{p} - m\right) + (1-b)\frac{1}{r} \quad , \quad 0 = c\left(\frac{1}{r'} - m\right) + (1-c)\frac{1}{p'}$$

where (23) has been used. Therefore, by the power inequality (64) :

$$|f'(x)| = |u(x)|^r + |w(x)|^{p'} \leqslant C \, f(x)^{\beta_1} + C \, f(x)^{\beta_2}$$

$$\beta_1 = r\left[b \, \frac{p'-1}{p} + (1-b)\frac{1}{r}\right], \; \beta_2 = p'\left[c \, \frac{r-1}{r} + (1-c)\frac{1}{p'}\right]$$

Some computations show that $\beta_1 = \beta_2 = \beta$ of (24). So the lemma is proved.

Now we assume again the hypotheses of Theorem 1 with (5) holding for all $s \in R$. (It will be clear which questions require (5) only near $s = 0$). We consider a continuation of $u(x)$ to the left of $x = 0$ as solution of the differential system (2). So we have u defined in $(\bar{a},\infty)$, $-\infty \leqslant \bar{a} < 0$. Continuation may not be unique, but the following arguments apply to any continuation. Since u may be oscillatory, lower bounds will not apply directly to u.

Lemma 4 is a reversed form of the differential inequality (16). Now I (defined by (10)) plays the role of $\int I^\lambda \, dt$ in (16). Of course, β in (17) is not the same as β in (24). The relation between Lemma 4 and Theorem 1 results from recalling (2) and noting that by (3) and (5) :

$$C \, I(x) \leqslant \int_x^\infty |u|^r dt + \int_x^\infty |w|^{p'} dt \leqslant \bar{C} \, I(x) \tag{25}$$

The power exponent corresponding to I by (24)-(25) is $1/(1-\beta)$, i.e. $1 + \sigma r$. So for $r > p$ we have $\beta > 1$, $1 + \sigma r < 0$ and writting down Lemma 12 we obtain that if $I(0) \neq 0$:

$$I(x) \geqslant [C x + I(0)^{1/(1+\sigma r)}]^{1+\sigma r} \quad , \quad x \geqslant 0 \tag{26}$$

The crucial point is that the exponent $1 + \sigma r$ is the same as in (21). Therefore, we have upper and lower power bounds both «to the left» and «to the right». (Exponential bounds if $r = p$). These considerations and Lemmas 11 and 12 imply easily the following consequences :

Case $r < p$ (then $\sigma > 0$) :

1. The solution u is bounded «to the left» by a σ-power and therefore is continued to the whole R ; i.e. we obtain a result of global existence on R. (Argue as in (22)).

2. The following lower bound for the support holds :

$$a \geqslant \overline{C} \quad J(u)^{1/(1+\sigma r)}$$

$\overline{C}$ depending only on m, p, r, B_1 and B_2 . Compare with (18). Remark 1 is also pertinent.

3. Point 1 can be applied to the extreme of the support, a, to obtain :

$$u(x) = \mathbf{O}\,((a-x)^{\sigma}) \qquad \text{ax} \qquad x \to a^-,$$

which may be regarded as a regularity result at $x = a$.

4. The σ-power bound of Point 1 is optimal in the sense explained in Theorem 2 (see Point 6 below). In particular, u is unbounded at $-\infty$ unless $u \equiv 0$.

Case $r > p$ (then $\sigma < 0$ and $1 + \sigma r < 0$) :

5. The solution u «blows up» at a finite $\bar{a}$ unless $u \equiv 0$. Upper and lower bounds for $\bar{a}$ are obtained.

6. *Proof of Theorem 2 for* $r > p$. A small o for some $u^{(j)}$ or some $w^{(j)}$ implies a small o for all $u^{(j)}$ and $w^{(j)}$ (by integration and the differential equation). This implies $I(x) = o(x^{1+\sigma r})$, which contradicts (26).

Case $r = p$ (then $\beta = 1$) :

7. The solution u is bounded «to the left» by an exponential (tending to $+\infty$ as $x \to -\infty$). Therefore, u is continued to the whole R. This exponential bound is optimal in the sense of Theorem 2. In particular, u is bounded by no power near $-\infty$ unless $u \equiv 0$.

8. *Proof of Theorem 2 for* $r = p$. It is an easy modification of Point 6.

Remark 4. For even m the «blow up» of Point 5 is necessarily oscillatory. For odd $m \geqslant 3$ the «blow up» is oscillatory unless all $u^{(j)}$ and $w^{(j)}$, $0 \leqslant j \leqslant m$, are monotone. Heidel [17] finds an oscillatory «blow up» in a third order nonlinear equation. See Kiguradze [28] for higher order.

Remark 5. For $p = 2$ this section can be approached in a very different way throught the integral representation :

$$u(x) = \frac{1}{(2m-1)!} \int_x^\infty (t-x)^{2m-1}\, u^{(2m)}(t)dt$$

which holds (in the usual Legesgue sense) due to the asymptotic bounds of Theorem 1. By (5) $B_1 \,|u|^{r-1} \leqslant |u^{(2m)}| \leqslant B_2 \,|u|^{r-1}$. The method consists in considering iterations and using the monotony of the function s^{r-1}.

5. - UPPER BOUNDS FOR SOLUTIONS OF THE VARIATIONAL PROBLEM : THEOREM 3

LEMMA 5. *Let* u *be a solution of Problem* $p\Gamma$, $1 < p < \infty$, Γ *Borel measurable in* R. *Assume that for all* $s \in R$

$$0 \leqslant \Gamma(s) \leqslant B_2 \,|s|^r \quad , \quad r > 0 \tag{27}$$

Then for all $x \geqslant 0$ *and for any positive function* $z : \overline{R}_+ \to R_+$

$$\begin{aligned} &\frac{1}{p}\int_x^\infty |u^{(m)}|^p\, dt + \int_x^\infty \Gamma(u)\, dt \leqslant \\ &\leqslant K \sum_{j=o}^{m-1} \left[\frac{|u^{(j)}(x)|^p}{z(x)^{(m-j)p-1}} + z(x)^{1+jr}\, |u^{(j)}(x)|^r \right] \end{aligned} \tag{28}$$

the constant K *depending only on* m, p, r *and* B_2 .

Proof. From the definition of Problem $p\Gamma$, it readily follows that for every $x \geqslant 0$

$$u \text{ minimizes } \frac{1}{p}\int_x^\infty |v^{(m)}|^p\, dt + \int_x^\infty \Gamma(v)\, dt$$

with the boundary conditions $v^{(j)}(x) = u^{(j)}(x)$, $0 \leqslant j \leqslant m-1$.

Therefore, it is enough to obtain (28) for $x = 0$, proving that $z > 0$ can be arbitrarily chosen. We recall that $u^{(j)}(0) = \alpha_j$.

Let P be the polynomial of degree $\leqslant 2m-1$ such that :

$$P^{(j)}(0) = \alpha_j \quad , \quad P^{(j)}(z) = 0 \quad , \quad 0 \leqslant j \leqslant m-1$$

The dependence on z becomes more transparent considering a second polynomial Q defined by $P(x) = Q(x/z)$. Then

$$Q^{(j)}(0) = z^j \alpha_j \quad , \quad Q^{(j)}(1) = 0 \quad , \quad 0 \leqslant j \leqslant m-1 \tag{29}$$

Some elementary computations give :

$$Q(t) = \sum_{i=0}^{m-1} A_i (1-t)^{m+i} \quad , \quad A_i = \sum_{i,j=0}^{m-1} a_{ij} z^j \alpha_j \tag{30}$$

where *each constant* a_{ij} *depends only on m,* by (29). Therefore for all i

$$|A_i| \leqslant C \sum_{j=0}^{m-1} z^j |\alpha_j| \tag{31}$$

Consider now the function $v(x) = P(x)$ for $0 \leqslant x \leqslant z$; $v(x) = 0$ for $x > z$. Note that $\Gamma(v(x))$ is Lebesgue measurable because Γ is Borel measurable. Since v belong to the minimization set of Problem pΓ, from the definition of minimum and (27) we obtain :

$$C\,J(u) \leqslant C\,J(v) \leqslant \int_0^z |P^{(m)}(x)|^p\,dx + \int_0^z |P(x)|^r\,dx =$$

$$= (1/z)^{mp-1} \int_0^1 |Q^{(m)}(t)|^p\,dt + z \int_0^1 |Q(t)|^r\,dt$$

where we have used $P(x) = Q(x/z)$, $P^{(m)}(x) = (1/z)^m Q^{(m)}(x/z)$.

(The cas $B_2 = 0$ is trivial). Using now (30), (31) and the power inequality (63) :

$$C\,J(u) \leqslant (1/z)^{mp-1} \sum_{j=0}^{m-1} z^{jp} |\alpha_j|^p + z \sum_{j=0}^{m-1} z^{jr} |\alpha_j|^r$$

which proves the lemma.

We shall also need the following simple lemma, whose idea is already in [5-I].

LEMMA 6. *Let* u *be a solution of Problem* pΓ, $\Gamma \geqslant 0$, $\Gamma(0) = 0$. *If* $u^{(j)}(x) = 0$, $0 \leqslant j \leqslant m-1$, *then* $u(y) = 0$ *for all* $y \geqslant x$.

Because any other admissible prolongation of u would make the functional strictly greater.

Proof of Theorem 3. If u is a solution of Problem pΓ we call a (finite or not) the extreme of its support and we set the notation :

$$J(x) = \frac{1}{p}\int_x^\infty |u^{(m)}|^p\,dt + \int_x^\infty \Gamma(u)dt$$

By Lemma 1 $u(x) \to 0$ as $x \to \infty$, so that $u(x)$ lies within the range of (7) for x close enough to a. This is assumed in the sequel. Power inequalities (63) will be repeatedly applied without more notice.

Case $r < p$. Then $\sigma > 0$. Furthermore, $\sigma > m$ since $(\sigma - m)p = \sigma r$. For $x < a$ we set :

$$z(x) = \sum_{i=0}^{m-1} |u^{(i)}(x)|^{q_i} \quad , \qquad q_i = 1/(\sigma - i) > 0 \tag{32}$$

Then $z(x) > 0$ by Lemma 6. The choice of q_i is guided by dimensional analysis : in this way all terms preserve the same dimensionality. Other choices of $z(x)$ (e.g. $z(x) \equiv 1$) also give some bounds, but not the optimal bounds. Taking into account that

$$\frac{|u^{(j)}(x)|^p}{z(x)^{(m-j)p-1}} \leqslant |u^{(j)}(x)|^\mu \quad , \quad \mu = p - q_j((m-j)p-1) = q_j(1+\sigma r) > 0$$

and applying Lemma 5 we obtain :

$$C\,J(x) \leqslant \sum_{j=0}^{m-1} |u^{(j)}(x)|^{q_j(1+\sigma r)} + \sum_{i,j=0}^{m-1} |u^{(i)}(x)|^{q_i(1+jr)}\,|u^{(j)}(x)|^r$$

Now we raise to a power $\lambda > 0$ (see comments in Section 3), integrate between x and ∞ and apply Schwarz inequality :

$$\begin{aligned} C\int_x^\infty J^\lambda\,dt \leqslant & \sum_{j=0}^{m-1} \int_x^\infty |u^{(j)}|^{\lambda q_j(1+\sigma r)}dt + \\ & + \sum_{i,j=0}^{m-1} \left[\int_x^\infty |u^{(i)}|^{2\lambda q_i(1+jr)}dt\right]^{1/2} \left[\int_x^\infty |u^{(j)}|^{2\lambda r}dt\right]^{1/2} \end{aligned} \tag{33}$$

We choose λ large enough to assure the finiteness of the integrals. Since $u \in L^r(R_+)$ and $u^{(m)} \in L^p(R_+)$, Nirenberg inequalities and $0 < r < p$ imply (Lemma 8) that $u^{(j)} \in L^p(R_+)$, $0 \leqslant j \leqslant m$. So we can choose

$$\lambda = \max_{i,j} \left\{ \frac{p}{q_j(1+\sigma r)} , \frac{p}{2q_i(1+jr)} , \frac{p}{2r} \right\}.$$

We finish the proof in a way similar to Section 3. We apply Nirenberg inequalities (Lemma 8) to the termes and factors of (33). Afterwards, we pass from products to sums through the power inequality (64). So we obtain, taking into account (7) :

$$\int_x^\infty |u^{(j)}|^{\lambda q_j(1+\sigma r)}\,dt \leqslant C\,J(x)^{\lambda/\beta_j}$$

$$\left[\int_x^\infty |u^{(i)}|^{2\lambda q_i(1+jr)}\,dt\right]^{1/2}\ \left[\int_x^\infty |u^{(j)}|^{2\lambda r}\,dt\right]^{1/2} \leqslant C\,J(x)^{\lambda/\beta_{ij}}$$

The crucial point is that all β_j and all β_{ij} turn out to be equal. (We call β their common value). This was to be expected by dimensional nalysis and is checked through long computations with exponents. Therefore :

$$\int_x^\infty J^\lambda\,dt \leqslant C\,J(x)^{\lambda/\beta}\ ,\quad \frac{1}{\beta} = 1 + \frac{1}{\lambda}\,\frac{1}{1+\sigma r} \tag{34}$$

This relation is like (16)-(17) in Section 3 and the proof is completed as there.

Case $r > p$. Now $\sigma < 0$ and $1 + \sigma r < 0$. The proof is analogous setting

$$\frac{1}{z(x)} = \sum_{i=0}^{m-1} |u^{(i)}(x)|^{\bar{q}_i}\ ,\quad \bar{q}_i = 1/(i-\sigma) > 0 \tag{35}$$

and taking into account that

$$z(x)^{1+jr}\,|u^{(j)}(x)\,|^r \leqslant |u^{(j)}(x)|^{\bar{\mu}}\ ,\quad \bar{\mu} = r - \bar{q}_j(1+jr) = -\bar{q}_j(1+\sigma r) > 0$$

In the step analogous to (22) we bound directly the supremum norm of all derivatives $u^{(j)}$, $0 \leqslant j \leqslant m-1$.

Case $r = p$. It is much easier : set $z(x) \equiv 1$ and $\lambda = 1$.

6. - FURTHER DEVELOPMENTS FOR GENERAL p

6.1. - Nonlinearities in intermediate derivatives

We consider the more general functional :

$$J(u) = \sum_{k=0}^{m} \int_0^\infty \Gamma_k(u^{(k)}(x))\,dx \tag{36}$$

$$\Gamma_m(s) = B_m\,|s|^{r_m}\ ,\quad B_m > 0 \tag{37}$$

$$(38)\qquad \left.\begin{array}{l} \Gamma_k \text{ Borel measurable} \\ B_k\,|s|^{r_k} \leqslant \Gamma_k(s) \leqslant C_k\,|s|^{r_k}, \quad B_k \geqslant 0 \end{array}\right\}\quad 0 \leqslant k \leqslant m-1$$

We set $r_m = p$, $r_0 = r$ and consider the interpolation exponents p_k given by :

$$(39)\qquad \frac{1}{p_k} = \frac{k}{m}\,\frac{1}{p} + \frac{m-k}{m}\,\frac{1}{r}$$

Note that $p_m = p$, $p_0 = r$. Some computations show that

$$p_k = \frac{\sigma r}{\sigma - k} = \frac{(\sigma - m)p}{\sigma - k}$$

where $\sigma = pm/(p-r)$ is the same parameter as in the former sections. Therefore, *if* $r_k = p_k$ *all terms of (36) have the same dimensionality.*

THEOREM 4. *Replace the functional of Theorem 3 by (36)-37), where* $r_m = p > 1$, $r_0 = r > 0$. *Assume that* Γ_0 *is positive definite, (38) holds near* $s = 0$, $B_0 > 0$ *and*

$$r_k \geqslant \max\{1, p_k\}, \qquad 1 \leqslant k \leqslant m-1,$$

where p_k *is given by (39). Then conclusions I, II and III of Theorem 3 hold.*

We sketch how to reduce the proof to that of Section 5. The analogous of Lemma 5 is :

$$J(x) \leqslant C \sum_{j=0}^{m-1} \sum_{k=0}^{m} z(x)^{1+(j-k)r_k}\,|u^{(j)}(x)|^{r_k}$$

We divide these terms between two classes according to the sign of $1 + (j-k)r_k$. (Here the hypotheses $p > 1$ and $r_k \geqslant 1$ for $1 \leqslant k \leqslant m-1$ are used. Note that for r_0 it is sufficient to have $r_0 > 0$).

$$(40)\qquad 1 + (j-k)r_k > 0 \quad \text{if} \quad 0 \leqslant k \leqslant j \leqslant m-1$$

$$(41)\qquad 1 + (j-k)r_k \leqslant 0 \quad \text{if} \quad 0 \leqslant j < k \leqslant m$$

In the *case* $r < p$ we choose $z(x)$ as in (32). Then for the terms corresponding to (41) we obtain :

$$(42)\qquad \frac{|u^{(j)}(x)|^{r_k}}{z(x)^{(k-j)r_k-1}} \leqslant |u^{(j)}(x)|^{\mu}, \quad \mu = r_k - q_j((k-j)r_k - 1) = q_j(1+(\sigma-k)r_k)$$

where $\sigma-k>0$ because $\sigma>m$. Since $u^{(j)}(x)\to 0$, for x large enough we obtain greater quantities if we replace r_k by p_k in all exponents. Here we use $r_k \geqslant p_k$, $j-k\geqslant 0$ in (40) and $\sigma-k>0$ in (42). After this replacement, *all terms have the same dimensionality.* Then the proof is completed as in Section 5 : all terms generate the *same* exponent β given by (34).

In the *case* $r>p$ we choose z(x) as in (35) and the roles of the classes (40) and (41) are interchanged. In the *case* $r=p$ we take $z(x)\equiv 1$.

Remark 6. When all $\Gamma_k \in C^1(R)$ the solutions considered in Theorem 4 satisfy in R_+

$$\sum_{k=0}^{m} (-1)^k \frac{d^k}{dx^k} \gamma_k(u^{(k)})=0 \quad , \quad \gamma_k(s)=\Gamma'_k(s)$$

where the derivatives and the equality are in the sense of distributions. (This is shown through the usual way in calculus of variations).

6.2. - Variable coefficients

The proof of Theorem 1 extends at once to the equation

$$(-1)^m \frac{d^m}{dx^m} (a(x)\,|u^{(m)}(x)|^{p-1} \operatorname{sgn} u^{(m)}(x)) + c(x)\,\gamma(u(x))=0$$

if the functions a, 1/a, c and 1/c belong to $L_+^\infty(R_+)$. (Here u is a solution in Carathéodory's sense of the system corresponding to (2)).

The proof of Theorems 3 and 4 extends to the functional

$$\sum_{k=0}^{m} \int_0^\infty a_k(x)\,\Gamma_k(u^{(k)}(x))dx$$

if a_k, $0\leqslant k\leqslant m$, $1/a_m$ and $1/a_0$ belong to $L_+^\infty(R_+)$.

Remark 7. For $p=2$, $\gamma(s)=\pm\,|s|^{r-1}\operatorname{sgn} s$, $a(x)=x^\lambda$, $c(x)=x^\mu$, the above differential equation becomes a 2m-order Emden-Fowler equation. Application of the above methods requires a greater apparatus of weighted interpolation inequalities not to be considered here. A comprehensive survey with references on second order Emden-Fowler equations is to be found in [3] . These second order methods rely upon the nonoscillatory character of the solutions and therefore do not apply to higher order equations. We also recall the survey of Kiguradze [28] for some higher order nonoscillatory results.

6.3. - Compactness of the support if $\Gamma(s) \geqslant B > 0$ for $s \neq 0$

THEOREM 5. *Lest* u *be a solution of Problem* $p\Gamma$, $1 < p < \infty$. *Assume that* $\Gamma(0) = 0$ *and* $\Gamma(s) \geqslant B > 0$ *for all* $s \neq 0$, *where* B *is a constant. Then* u *has compact support.*

The proof is very short. The finiteness of the functional implies that the set $\{u \neq 0\}$ has finite measure. On the other hand, Lemma 6 (in Section 5) implies that the support of u is an interval and the set $\{u = 0\} \cap \operatorname{supp} u$ is at most countable. (This is already shown in [5-I]). Therefore the measure of the set $\{u \neq 0\}$ is equal to the measure of supp u.

The simplest example is $\Gamma(0) = 0$, $\Gamma(s) = 1$ for $s \neq 0$. Then (see [7]) any solution of Problem 2Γ is in its support equal to a polynomial of degree 2m–1. The solution is *not* unique for some boundary data.

6.4. - Optimality of the bounds of Theorem 3

The optimality of the bounds of Theorem 3 is implied by the reversed inequality of (34) and Lemma 12 (argue as in Section 4). So we are going to sketch the proof of the reversed inequality of (34) for an appropriate λ. Take the case $r > p$. Inserting (35) in Lemma 5 we obtain :

$$(43) \qquad C\,J(x) \leqslant \sum_{i,j=0}^{m-1} |u^{(i)}(x)|^{\bar{q}_i(mp-jp-1)}\,|u^{(j)}(x)|^p + \sum_{j=0}^{m-1} |u^{(j)}(x)|^{-\bar{q}_j(1+\sigma r)}$$

Recall that $1+\sigma r < 0$. We are going to bound the supremum norm of $u^{(j)}$ on (x,∞) by $\displaystyle\int_x^\infty J^\lambda\,dt$. To this purpose the following interpolation inequalities are needed :

$$\|u^{(j)}\|_\infty \leqslant C\|u^{(m)}\|^{a}_{p,\mu}\,\|u\|^{1-a}_{r,\nu}\ ,\quad 0 = j + a_j\left(-\frac{1}{p}+\frac{1}{\mu}-m\right) + (1-a_j)\left(\frac{1}{r}+\frac{1}{\nu}\right).$$

$\|u\|_{r,\nu}$ means $\Big\| \|u\|_r \Big\|_{L^\nu(R_+)}$, $\|u\|_r$ being the function $\|u\|_{L^r(t,\infty)}$.

The ranges of the parameters are $0 \leqslant j \leqslant m-1$, $1 \leqslant p \leqslant \infty$, $0 < r \leqslant \infty$, $0 < \mu,\nu \leqslant \infty$ and $(1/p) + (1/\mu) < m-j$. These inequalities are a consequence of Nirenberg-Gagliardo inequalities and Taylor's formula.

Firstly, we apply these inequalities for the half-line (x,∞) with $\mu = p\lambda$ and $\nu = r\lambda$. Secondly, we apply the power inequalities (63) and (64) and the hypothesis (7). So we obtain :

$$|u^{(j)}(x)| \leqslant C\left[\int_x^\infty J^\lambda\,dt\right]^{\frac{a_j}{p\lambda}+\frac{1-a_j}{r\lambda}}$$

The parameter range of the inequalities requires $(1/p) + (1/p\lambda) < m-j$. This is satisfied for $0 \leqslant j \leqslant m-1$ taking $\lambda > 1/(p-1)$. Inserting in (43) raised to the power λ, the exponents of all terms turn out to be equal and we obtain the reversed inequality of (34).

Remark 8. The above interpolation inequalities also apply to obtain global bounds on $\overline{R}_+$ in Theorems 1, 3 and 4 (rather than asymptotic bounds). For $r > p$ these global bounds have the form :

$$|u^{(j)}(x)| \leqslant \left[C\,x + K\,J(u)^{1/(1+\sigma r)}\right]^{\sigma-j} \quad , \qquad \forall\, x \geqslant 0$$

where C and K depend only on m, j, p, r and the constants of the hypotheses (5), (7) or (37)-(38). (Here these hypotheses must hold for all $s \in R$).

6.5. - On the case r = 1

Consider Problem $p\Gamma$, $\Gamma(s) = |s|$. It is proved as for $p = m = 2$ in [5-II] and [18-I,II] that the unique solution u satisfies

$$\text{(44)} \qquad w^{(m)} \in L^{\infty}(R_+)$$

$$\text{(45)} \qquad \| w^{(m)} \|_{\infty} \leqslant 1 \ \text{ and } \ (-1)^m\, w^{(m)}(x) = -\operatorname{sgn} u(x) \ \text{ where } u \neq 0 \text{ in } \overline{R}_+$$

We recall that w is given by (1) and $w^{(m)} \in L^{\infty}(R_+)$ implies that $w^{(m-1)}$ is Lipschitz continuous in $\overline{R}_+$. Therefore :

$$(-1)^m\, u(x)\, w^{(m)}(x) = -\,|u(x)| \quad \text{in } \overline{R}_+$$

These results allow us to include this case in the proof of Theorem 1 (not only in that of Theorem 3). Now the choice (13) gives $\lambda = \infty$, but it is possible to choose a smaller λ in the way explained in Remark 2.

Remark 9. Conversely, if u satisfies (44)-(45) and belongs to the minimization set of Problem $p\Gamma$, then u is the solution of Problem $p\Gamma$. This can be proved e.g. as in [14, p. 95]. On the other hand, (45) is equivalent to the multivalued equation $(-1)^m\, w^{(m)} \in -\operatorname{sgn} u$, provided that (44) holds.

7. - FURTHER DEVELOPMENTS FOR $p = 2$

7.1. - «One-sided» hypotheses

THEOREM 6. *Let* $\gamma \in C(R)$ *and assume that near* $s = 0$

(46) $$B\,|s|^r \leqslant s\gamma(s) \quad , \qquad B > 0.$$

Let u be a solution of Equation 2γ *such that* $u(x) \to 0$ *as* $x \to \infty$. *Then u has compact support if* $1 < r < 2$.

The main feature of this theorem is the «one-sided» hypothesis (46) rather than the «two-sided» (5) of Theorem 1. Note that $\gamma(0) = 0$ by continuity. By Equation 2γ $u^{(2m)} \in L^\infty(R_+)$ and then by Lemma 1 :

(47) $$u^{(j)}(x) \to 0 \quad \text{as} \quad x \to \infty, \quad 0 \leqslant j \leqslant 2m$$

After these preliminary remarks, the proof of Theorem 1 of [8] applies. (This proof consists in multiplying by u the equation and integrating by parts 2m + 1 times). For $r = 2$ exponential decay is obtained.

Following [8, Theorem 5] , we can replace in Theorem 6 Equation 2γ by $A\,u + \gamma(u) = 0$, where

$$A\,u = \sum_{\nu=0}^{m} (-1)^\nu\, c_\nu\, u^{(2\nu)} \quad , \quad c_\nu \geqslant 0 \ \text{ constants}, \ \ c_m > 0$$

Now (47) is deduced from the following theorem of Esclangon-Landau (see Landau [19]) :

$$A\,u \in L^\infty(R_+) \ \text{ and } \ u \in L^\infty(R_+) \ \text{ imply } \ u^{(j)} \in L^\infty(R_+) \ , \quad 0 \leqslant j \leqslant m,$$

where A is a normal m-order linear ordinary differential operator with L^∞ coefficients. The order m may be even or odd. (Far-reaching generalizations of Esclangon-Landau theorem can be found e.g. in [2] and [22]).

7.2. - Other sign hypotheses for γ and equations of odd order

The first inequality (5) of Theorem 1 for $p = 2$ and the inequality (46) of Theorem 6 can be replaced by the following statement :

There exist two real constants λ, μ such that

$$(48) \qquad \lambda s\gamma(s) - \mu\Gamma(s) \geqslant |s|^r \quad \text{and} \quad \lambda + \mu(m - \frac{1}{2}) > 0$$

Then the second inequality (5) of Theorem 1 can be replaced by

$$|\gamma(s)| \leqslant C\,|s|^{r-1}$$

Let us sketch how to obtain (48). We call Γ the primitive of γ such that $\Gamma(0) = 0$. Multiplying Equation 2γ by u', integrating and using (47) we arrive at :

$$(49) \qquad (-1)^m \sum_{j=1}^{m-1} (-1)^{j+1}\, u^{(j)}\, u^{(2m-j)} = \frac{1}{2}\,|u^{(m)}|^2 - \Gamma(u)$$

We are led to (48) performing a linear combination of (49) and

$$(-1)^m\, u\, u^{(2m)} + u\,\gamma(u) = 0$$

before integrating by parts.

Take now an odd order equation of the form :

$$(-1)^m\, u^{(2m-1)} + \gamma(u) = 0$$

In this case (48) is replaced by

$$(50) \qquad \mu\Gamma(s) + \lambda s\gamma(s) \geqslant |s|^r \quad \text{and} \quad \mu > 0.$$

The functions $\gamma(s) = |s|^{r-1}\,\mathrm{sgn}\, s$ and $\gamma(s) = -|s|^{r-1}\,\mathrm{sgn}\, s$ satisfy (48) and (50).

7.3. - Nonoscillatory solutions

THEOREM 7. *Let* $u^{(m-1)}$ *be* monotone *and locally absolutely continuous in* R_+, $m \geqslant 2$. *Assume that* $u \in L^\infty(R_+)$ *and*

$$(51) \qquad B\,|u(x)|^{r-1} \leqslant |u^{(m)}(x)| \quad \text{a.e. in } R_+.$$

I. *If* $1 < r < 2$ *then* u *has compact support.*

II. *If* $r = 2$ *then* u *decays exponentially at* ∞.

III. *If* $r > 2$ *then as* $x \to \infty$

$$u^{(j)}(x) = O\,(x^{\sigma-j}) \quad , \quad 0 \leqslant j \leqslant m-1 \quad , \quad \sigma = -m/(r-2)$$

(Point I overlaps some results of Kiguradze [27,28]. The method of proof is different).

Note that m may be even or odd. The proof is rather brief thanks to the relation :

$$\| u^{(m)} \|_{L^1(x,\infty)} = |u^{(m-1)}(x)| \tag{52}$$

because $u^{(m-1)}$ is monotone. Note that for all $x > 0$ $u^{(m)} \in L^1(x,\infty)$: if not, u would be unbounded. Then by (51) $u \in L^{r-1}(x,\infty)$ and by Lemma 1 $u^{(j)}(x) \to 0$ as $x \to \infty$, $0 \leqslant j \leqslant m-1$. (This and the monotony of $u^{(m-1)}$ imply that $u^{(j)}$, $0 \leqslant j \leqslant m-1$, is monotone and has constant sign).

Now we apply the interpolation inequalities of Lemma 8 for the half-line (x,∞) with $q = \infty$, $p = 1$:

$$|u^{(m-2)}(x)| \leqslant \| u^{(m-2)} \|_\infty \leqslant C \, \| u^{(m)} \|_1^a \, \| u \|_{r-1}^{1-a} \leqslant C \, \| u^{(m)} \|_1^{a+\frac{1-a}{r-1}} =$$

$$= C \, |u^{(m-1)}(x)|^{a+\frac{1-a}{r-1}}$$

where we have used (51) and (52) and a is given by :

$$0 = m-2 + a(1-m) + (1-a)\,/\,(r-1).$$

The proof concludes applying Lemma 11 (in Appendix II). If $r > 2$ we obtain first the asymptotic bound for $u^{(m-2)}$. The bounds for the $u^{(j)}$, $0 \leqslant j \leqslant m-3$, are deduced by integration taking into account that $\sigma < 0$ and $u^{(j)}(x) \to 0$ as $x \to \infty$. For $u^{(m-1)}$ use an integral representation like that of Remark 5 (valid by positivity and Fubini's theorem) and, finally, use the monotony of $u^{(m-1)}$.

Remark 10. a) If in addition $u^{(m)}$ is assumed to be monotone, we can add $j = m$ in Point III. It is enough to take into account that then

$$\| u^{(m)} \|_{L^\infty(x,\infty)} = |u^{(m)}(x)| \quad \text{a.e.}$$

and to use in the proof the L^∞ norm rather than the L^1 norm.

b) If the reversed inequality of (51) holds, then lower bounds are easily obtained through Lemma 12. A lower bound for u requires only the monotony of u (rather than the monotony of $u^{(m-1)}$).

c) Remark 5 (in section 4) is also pertinent here.

APPENDIX I. Nirenberg-Gagliardo interpolation inequalities

LEMMA 7. *Let* $u \in L^r(R_+)$ *and* $u^{(m)} \in L^p(R_+)$, $1 \leqslant r,p \leqslant \infty$. *Then for the derivatives* $u^{(j)}$, $0 \leqslant j \leqslant m-1$, *the following inequalities hold :*

$$\| u^{(j)} \|_q \leqslant C \| u^{(m)} \|_p^a \| u \|_r^{1-a} \tag{53}$$

where

$$\frac{1}{q} = j + a\left(\frac{1}{p} - m\right) + (1-a)\frac{1}{r} \tag{54}$$

for all a *in the interval* $j/m \leqslant a \leqslant 1$, *the constant* C *depending only on* m, j, p, r, a.

This is the one-dimensional form of Nirenberg interpolation inequalities [20] . (In dimension 1 there are no exceptional cases). Negative values of q mean Hölder norms and will not be used here. Gagliardo inequalities [16] are very close. A survey of previous results is to be found in [2].

We have $0 < r < 1$ in several places of the present paper. The following lemma covers these needs.

LEMMA 8. *Assume that* $0 < r \leqslant p \leqslant q \leqslant \infty$ *and* $1 \leqslant p < \infty$. *If* $u \in L^1_{loc}(R_+) \cap L^r(R_+)$ *and* $u^{(m)} \in L^p(R_+)$, *then for* $0 \leqslant j \leqslant m-1$ *(53) holds with a given by (54), the constant* C *depending only on* m, j, p, q, r.

Proof. In [8, Lemma 9] it is proved that if $u \in L^p(R_+) \cap L^r(R_+)$ and $u^{(m)} \in L^p(R_+)$, $1 \leqslant p \leqslant \infty$, $0 < r \leqslant p$, then

$$\| u \|_p \leqslant C \| u^{(m)} \|_p^b \| u \|_r^{1-b}, \quad \frac{1}{p} = b\left(\frac{1}{p} - m\right) + (1-b)\frac{1}{r}. \tag{55}$$

Since $| s |^r$ is positive definite, Lemma 1 implies that $u(x) \to 0$ as $x \to \infty$. Therefore $u \in L^\infty(R_+)$ and by standard L^p interpolation $u \in L^p(R_+)$ and (55) holds. Applying Lemma 7 with $r = p$ and $p \leqslant q \leqslant \infty$:

$$\| u^{(j)} \|_q \leqslant C \| u^{(m)} \|_p^c \| u \|_p^{1-c}, \quad \frac{1}{p} = j + c\left(\frac{1}{p} - m\right) + (1-c)\frac{1}{p}$$

Inserting (55) in these relations we finish the proof of Lemma 8.

APPENDIX II. Auxiliary lemmas

In this appendix we collect for expository reasons some simple results repeatedly used in this paper.

Formulae of integration by parts

LEMMA 9. *Assume that 1)* $u^{(m-1)}$ *and* $w^{(m-1)}$, $m \geqslant 1$, *belong to* $\mathrm{AbsC}_{loc}(\bar{R}_+) \cap L^\infty(R_+)$, *2)* $u(x)$ *and* $w(x)$ *tend to zero as* $x \to \infty$ *and 3)* $u^{(m)}\, w \geqslant 0$ *and* $(-1)^m\, u\, w^{(m)} \leqslant 0$ a.e. *in* R_+. *Then for all* $x \geqslant 0$

$$\int_x^\infty u^{(m)}(t)\, w(t)\, dt - (-1)^m \int_x^\infty u(t)\, w^{(m)}(t)\, dt = (-1)^m \sum_{j=0}^{m-1} (-1)^j\, u^{(j)}(x)\, w^{(m-j-1)}(x)$$

and both integrands belong to $L^1(R_+)$.

Proof. The corresponding formula in a bounded interval $[x,y]$ is straightforward. Letting $y \to \infty$ the lemma is obtained taking into account that

$$u^{(j)}(x) \to 0 \text{ and } w^{(j)}(x) \to 0 \text{ as } x \to \infty,\ 0 \leqslant j \leqslant m-2, \tag{56}$$

because of Lemma 1.

If we set $w = u^{(m)}$ a primitive of the above right-hand side can be written without integrals. So using again (56) we obtain the following Lemma.

LEMMA 10. *Assume that 1)* $u^{(2m-1)} \in \mathrm{AbsC}_{loc}(\bar{R}_+) \cap L^\infty(R_+)$, $m \geqslant 1$, *2)* $u(x) \to 0$ *as* $x \to \infty$ *and 3)* $(-1)^m\, u\, u^{(2m)} \leqslant 0$ a.e. *in* R_+. *Set*

$$I(x) = (-1)^m \sum_{j=0}^{m-1} (-1)^j\, u^{(j)}(x)\, u^{(2m-j-1)}(x).$$

Then $I \in L^1(R_+)$ *and for all* $x \geqslant 0$

$$\int_x^\infty I(t)\, dt = (-1)^m \sum_{j=0}^{m-2} (-1)^{j+1}\, (j+1)\, u^{(j)}(x)\, u^{(2m-j-2)}(x) + \frac{m}{2}\, (u^{(m-1)}(x)\,)^2$$

A first order differential inequality

The proof of the following two lemmas is obtained easily by explicit integration.

LEMMA 11. *Let* $f : (\bar{a},a) \to R$ *be locally absolutely continuous, nonincreasing and positive in the open interval* $(\bar{a},a)$, $-\infty \leqslant \bar{a} < a \leqslant \infty$. *Let* $\beta \in R$. *Assume that*

$$K\, f(x)^{\beta} \leqslant |f'(x)| \quad \text{a.e. } in\ (\bar{a},a), \quad K>0 \tag{57}$$

Then for $\bar{a} < x_1 \leqslant x \leqslant x_2 < a$:

I. If $\beta < 1$ *a is finite and*

$$a-x \leqslant \frac{1}{K(1-\beta)}\, f(x)^{1-\beta} \tag{58}$$

$$\left[f(x_2)^{1-\beta} + K(1-\beta)(x_2-x)\right]^{1/(1-\beta)} \leqslant f(x) \leqslant \left[f(x_1)^{1-\beta} - K(1-\beta)(x-x_1)\right]^{1/(1-\beta)} \tag{59}$$

II. If $\beta = 1$

$$f(x) \leqslant f(x_1)\, e^{-K(x-x_1)} \tag{60}$$

III. If $\beta > 1$ $\bar{a}$ *is finite and*

$$x-\bar{a} \leqslant \frac{1}{K(\beta-1)}\, f(x)^{-(\beta-1)} \tag{61}$$

$$\left[f(x_2)^{-(\beta-1)} - K(\beta-1)(x_2-x)\right]^{-1/(\beta-1)} \leqslant f(x) \leqslant \left[f(x_1)^{-(\beta-1)} + K(\beta-1)(x-x_1)\right]^{-1/(\beta-1)} \tag{62}$$

LEMMA 12. *If in the former lemma the inequality (57) is reversed, then also (59), (60) and (62) are reversed, provided that the quantities between brackets are positive. Inequalities (58) and (61) are also reversed if* $f(a^-) = 0$ *for* $\beta < 1$ *and* $f(\bar{a}^+) = \infty$ *for* $\beta > 1$.

Power inequalities

If $s>0$ and all $a_i \geqslant 0$ then

$$C_{k,s}\,(a_1^s + \dots + a_k^s) \leqslant (a_1 + \dots + a_k)^s \leqslant \bar{C}_{k,s}\,(a_1^s + \dots + a_k^s), \quad C_{k,s} > 0 \tag{63}$$

If $s>0$, $t>0$, $A \geqslant 0$ and $B \geqslant 0$ then

$$A^s B^t \leqslant \frac{s}{s+t}\, A^{s+t} + \frac{t}{s+t}\, B^{s+t} \leqslant K_{s,t}\,(A+B)^{s+t} \tag{64}$$

(63) is obtained either from Jensen's inequality or from Hölder's inequality. (64) is a form of Young's inequality obtained setting $p = 1 + (t/s)$, $p' = 1 + (s/t)$ in

$$A^s B^t \leqslant \frac{1}{p} A^{sp} + \frac{1}{p'} B^{tp'} \quad , \quad p > 1.$$

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