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BLOCH WAVES FOR A SOLID-FLUID MIXTURE

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Résumé : On étudie les vibrations d'un mélange infini, à structure périodique, constitué d'un solide élastique et d'un fluide visqueux, barotrope. La solution des équations du mouvement admet un développement en fonction d'ondes de Bloch. Cette représentation peut être utilisée pour trouver des approximations. Un exemple en est fourni par le lien avec la théorie de l'homogénéisation.

Summary : Vibrations of an unbounded, periodic, mixture of an elastic solid and a viscous compressible fluid are studied. The solution of the equations of motion admits an expansion in terms of Bloch waves. This representation can be used to find approximations. The connection with the homogenization theory is such an example.

I. - INTRODUCTION

Wave propagation in periodic structures is of interest by a lot of applications. Geophysical or engineering problems, for instance, led to several works on laminated media or fiber-reinforced composites [1]. Using Floquet's theory for ordinary linear differential equations with periodic coefficients, effects of dispersion were analysed, allowing comparison with approximate theories [6], [7], [3].

More generally, the study of periodic structures is related to an analysis of the properties of differential operators with periodic coefficients. The spectral resolution of these operators comes from the existence of generalized eigenfunctions, the Bloch waves. Then, it is easy to show that the solution of linear equations, that describe effects due to the periodicity of the structure, can be given in terms of Bloch waves. So, in particular, the solution of the equations of motion of a periodic, unbounded medium with elastic or viscoelastic behaviour, has an expansion depending on eigenvectors of operators associated with the elasticity problem. These operators act upon functions which are defined on the basic cell of the material. The components of these eigenvectors are determined by solving differential or integrodifferential systems [9], [10].

In this work, we study linearized vibrations of an unbounded, periodic mixture of an elastic solid and a compressible, barotropic, viscous fluid. The mechanical problem is stated in section 2 and then, we show, in section 3, that the solution admits a Bloch expansion, in terms of eigenfunctions of differential operators which are defined from the elasticity and viscosity coefficients. This expansion can lead us to find directly approximations of the solution. If the viscosity terms are small, the average method can be used. As another application, in section 4, we assume that the initial conditions are slowly varying functions of x, depending on a small parameter ϵ and we study the asymptotic behaviour of the solution. It is given by the function obtained from the homogenization method ([5], [4]) applied to the initial problem.

2. - STATEMENT OF THE PROBLEM

2.1. - Local equations

A mixture of an elastic solid and a compressible, barotropic, viscous fluid fills the whole space $\mathbb{R}^3$, the solid in the part Ω^s and the fluid in Ω^f. We consider a reference state, the rest, in which the pressure is a constant everywhere, and the density of mass ρ_o (ρ_o^f in Ω^f and ρ_o^s in Ω_o^s) is constant in each of regions Ω^f and Ω^s. We study the displacement field u, u = u(x,t), with respect to the rest reference.

Under the hypothesis of small perturbations, the stress perturbation tensor is given by :

$$(1) \qquad \sigma_{h\ell}(u) = \begin{cases} a_{h\ell mn}(x)\,\epsilon_{mn}(u) & \text{in } \Omega^s \\ c_o^2\,\rho_o^f\,\delta_{h\ell}\,\delta_{mn}\,\epsilon_{mn}(u) + \\ (\lambda\,\delta_{h\ell}\,\delta_{mn} + 2\mu\,\delta_{hm}\,\delta_{\ell n})\,\epsilon_{mn}\left(\dfrac{\partial u}{\partial t}\right) & \text{in } \Omega^f \end{cases}$$

with

(2) $$\epsilon_{mn}(u) = \frac{1}{2}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m}\right)$$

In expression (1), the perturbation of pressure, that depends on the density ρ_o^f (barotropic fluid), is related with the velocity of sound c_o , in the reference state, and with the linearized strain tensor $\epsilon_{mn}(u)$ by the continuity equation (see, for instance,[5]).

Let Y be the cube ( ]0,1[ )^3 $\subset$ IR^3. We assume that, in the rest reference, the mixture has a 2πY-periodic structure : geometric positions of fluid and solid parts are 2πY-periodic, $Y = Y^f \cup Y^s$, and the fluid and solid properties are 2πY-periodic. In the framework of homogenization theory, i.e. the limit case with small period, E. Sanchez-Palencia [5] and J. Sanchez-Hubert [4] studied the vibrations of such a mixture, in a bounded region of IR^3.

So, the elasticity coefficients a_{hlmn} in (1) are functions defined on $2\pi Y^s$ (notation of [2]), and they satisfy the usual symmetry (3) and positivity (4) conditions :

(3) $$a_{h\ell mn} = a_{\ell hmn} = a_{mnh\ell}$$

(4) $$\forall\ \epsilon_{ij},\ \epsilon_{ij} = \epsilon_{ji},\ \exists\ \alpha > 0\ ;\ a_{h\ell mn}(x)\ \epsilon_{mn}\ \overline{\epsilon}_{h\ell} \geqslant \alpha\ \epsilon_{ij}\ \overline{\epsilon}_{ij}$$

where α is a constant and $\overline{\epsilon}$ denotes the complex conjugate of ϵ. The viscosity coefficients λ and μ in (1) satisfy :

(5) $$\mu > 0,\quad \frac{\lambda}{\mu} \geqslant -\frac{2}{3}C,\quad 0 < C < 1.$$

We assume that there are no given body forces, so the equations of motion are given by :

(6) $$\frac{\partial^2 u_h}{\partial t^2} - \frac{1}{\rho_o}\frac{\partial \sigma_{h\ell}}{\partial x_\ell}(u) = 0$$

with $\sigma_{h\ell}(u)$ defined by (1).

Moreover, at the interface between the solid and the fluid, we must have the continuity of displacement and stress :

(7) $$[u_h] = 0 \qquad [\sigma_{h\ell}\, n_\ell] = 0$$

satisfied at points obtained by periodicity from points of $2\pi\Gamma$, Γ boundary between Y^s and Y^f. The brackets mean «jump of» and n denotes a unit vector, normal to $2\pi\Gamma$.

To these equations and boundary conditions, we must finally add initial data :

$$u(x,0) = f(x) \qquad \frac{\partial u}{\partial t}(x,0) = g(x). \tag{8}$$

2.2. - Variational formulation

Problem (6), (7), (8) can be expressed as an initial value problem in $(L^2(IR^3))^3$. From (6) and by means of (7), we show that the solution u satisfies :

$$\int_{IR^3} \rho_o \frac{\partial^2 u_h}{\partial t^2} \bar{v}_h \, dx + \int_{IR^3} \sigma_{h\ell}(u)\, \bar{\epsilon}_{h\ell}(v) dx = 0 \qquad \forall\, v \in (H^1(IR^3))^3.$$

Let us introduce the $2\pi Y$-periodic coefficients :

$$b_{h\ell mn}(x) = \begin{cases} a_{h\ell mn}(x) & x \in 2\pi Y^s \\ c_o^2\, \rho_o^f\, \delta_{h\ell}\, \delta_{mn} & x \in 2\pi Y^f \end{cases} \tag{9}$$

$$c_{h\ell mn}(x) = \begin{cases} 0 & x \in 2\pi Y^s \\ \lambda\, \delta_{h\ell}\, \delta_{mn} + 2\mu\, \delta_{hm}\, \delta_{\ell n} & x \in 2\pi Y^f \end{cases} \tag{10}$$

Then, we define two sesquilinear forms on $(H^1(IR^3))^3$:

$$v, w \in (H^1(IR^3))^3, \quad b(v,w) = \int_{IR^3} b_{h\ell mn}(x)\, \epsilon_{mn}(v)\, \bar{\epsilon}_{h\ell}(w) dx \tag{11}$$

$$v, w \in (H^1(IR^3))^3, \quad c(v,w) = \int_{IR^3} c_{h\ell mn}(x)\, \epsilon_{mn}(v)\, \bar{\epsilon}_{h\ell}(w) dx. \tag{12}$$

Introducing a weighted inner product, with weight ρ_o, on the Hilbert space $(L^2(IR^3))^3$, the problem has the following form :

. Find u, function of t with values in $(H^1(IR^3))^3$, such that :

$$\begin{cases} \left(\dfrac{d^2u}{dt^2}, v\right) + b(u,v) + c\left(\dfrac{du}{dt}, v\right) = 0 \qquad \forall\, v \in (H^1(IR^3))^3 & (13) \\ u(0) = f \qquad \dfrac{du}{dt}(0) = g. & (14) \end{cases}$$

In (14), f and g are data such that :

$$f \in (H^1(IR^3))^3 \qquad g \in (L^2(IR^3))^3. \tag{15}$$

Let B and C be the associated operators to the forms b and c. In an equivalent way, equation (13) can be written :

$$\frac{d^2u}{dt^2} + Bu + C\frac{du}{dt} = 0. \tag{16}$$

The problem seems to look like the one which is related to the vibrations of an unbounded, periodic medium with instantaneous memory [10]. But, the forms b and c are not coercive. Nevertheless, for any positive real number β, the form $a^\beta + \beta^2$ (with associated operator $A^\beta + \beta^2$, a^β defined by (17), is coercive on $(H^1(IR^3))^3$.

$$a^\beta = b + \beta c. \tag{17}$$

So, a change of unknown function is done in (13) or (15) :

$$u = e^{\beta t}\, v \quad , \quad \beta \text{ positive real number.}$$

And the new function v is solution of :

$$\frac{d^2v}{dt^2} + (A^\beta + \beta^2)v + (C + 2\beta)\frac{dv}{dt} = 0 \tag{18}$$

$$v(0) = f \qquad \frac{dv}{dt}(0) = g - \beta f. \tag{19}$$

Under the assumptions (3), (4), (5), (15), in the classical framework of semi-group theory, for instance (see J. Sanchez-Hubert [4]), the solutions of problems (16), (15) and consequently (18), (19) exist and are unique.

3. - BLOCH EXPANSION OF THE SOLUTION

3.1. - Bloch waves

By means of operatof A^β, we use the technics of Bloch expansion, described by A. Bensoussan, J.L. Lions and G. Papanicolaou [2] in the scalar case, and by N. Turbé [8] for the equations of elasticity.

The periodicity of the coefficients in operator A^{β} leads to the definition of a set of sesquilinear forms (or a set of operators) which act upon functions defined on the basic cell $2\pi Y$. Let $(H^1_p(2\pi Y))^3$ (p for periodic) be the space of the vectors of $(H^1(2\pi Y))^3$ which take equal values on two opposite points of two opposite sides of the cell $2\pi Y$. With the help of coefficients (9) (resp. (10)), we define the set of forms b(k) (resp. c(k)) :

$$(20)\quad \begin{cases} k \in Y & u, v \in (H^1_p(2\pi Y))^3 \\ b(k)(u,v) = \displaystyle\int_{2\pi Y} b_{h\ell mn}(x) \left(\frac{\partial u_m}{\partial x_n} + ik_n u_m\right) \overline{\left(\frac{\partial v_h}{\partial x_\ell} + ik_\ell v_h\right)} dx \end{cases}$$

The space $(L^2(2\pi Y))^3$ is equipped with the weighted inner product, defined by ρ_o, and we denote by B(k) (resp. C(k)) the operators which are associated with the forms b(k) (resp. c(k)) :

$$(21)\qquad (B(k)u)_h(x) = -\frac{1}{\rho_o}\left(\frac{\partial}{\partial x_\ell} + ik_\ell\right)\left[b_{h\ell mn}(x)\left(\frac{\partial u_m}{\partial x_n} + ik_n um\right)\right]$$

And we introduce :

$$(22)\qquad a^{\beta}(k) = b(k) + \beta c(k) \qquad\qquad A^{\beta}(k) = B(k) + \beta C(k)$$

The operators $A^{\beta}(k)$, $k \in Y$, act upon the vectors of $(H^1_p(2\pi Y))^3$ for which the stress belong to $(H^1_p(2\pi Y))^3$ and that satisfy the transmission conditions (7).

As it is done in the study of the elastic, periodic medium (see N. Turbé [8]), we prove that, for any $k \in Y$, $A^{\beta}(k) + \beta^2$ is a positive, selfadjoint operator with compact resolvent. As a consequence :

> for each $k \in Y$, there exist eigenvalues $0 \leqslant \omega_0^2(k) \leqslant \omega_1^2(k) \ldots \to +\infty$ with corresponding eigenfunctions $\varphi^0(x;k)$, $\varphi^1(x;k)$... of the operator $A^{\beta}(k)$ which form an orthonormal basis in $(L^2(2\pi Y))^3$.

Any function of $(L^2(IR^3))^3$, with complex values, can be then expanded by means of the eigenvectors of the operators $A^{\beta}(k)$ (N. Turbé [8]) :

$$(23)\qquad u(x) = \int_Y e^{ik.x}\, dk \sum_{m=0}^{+\infty} \widehat{u}_m(k)\, \varphi^m(x;k) \qquad (k.x = k_\ell x_\ell)$$

with

$$(24) \qquad \hat{u}_m(k) = \int_{\mathbb{R}^3} e^{-ik.x}\, u_h(x)\, \overline{\varphi_h^m}(x;k)\, \rho_o(x)dx.$$

3.2. - Representation of the solution

The set of basis $\{\varphi^m(.\,;k)\}$ is used in equation (13). Let v and w be two vectors of $(H^1(\mathbb{R}^3))^3$ which components, in the Bloch expansion, are denoted by $\hat{v}_m(k)$ and $\hat{w}_m(k)$. As we have it for an elastic, periodic medium :

$$(25) \qquad a^\beta(v,w) = \int_Y dk \sum_{m=0}^{+\infty} \omega_m^2(k)\hat{v}_m(k)\overline{\hat{w}_m}(k).$$

In the same way, we can represent c(v,w) with the help of $\hat{v}_m(k)$ and $\hat{w}_m(k)$. First, let us suppose that v and w belong to $(\mathscr{D}(\mathbb{R}^3))^3$. We construct the auxiliary functions $\tilde{v}(x;k)$ and $\tilde{w}(x;k)$ (used in the proof of the Bloch expansion theorem [9]) :

$$(26) \qquad k \in Y,\, x \in 2\pi Y,\ \tilde{v}(x;k) = \sum_{\gamma \in 2\pi\mathbb{Z}^3} v(x+\gamma)e^{-ik.(x+\gamma)}$$

$\tilde{v}$ is an element of $(L^2(2\pi Y))^3$ we expand in the basis of the eigenfunctions $\{\varphi^m(.\,;k)\}$:

$$(27) \qquad \tilde{v}(x;k) = \sum_{m=0}^{+\infty} \hat{v}_m(k)\, \varphi^m(x;k).$$

Let us consider then the scalar $c(k)\,(\tilde{v}(.\,;k), \tilde{w}(.\,;k))$. From definitions (20) and (26), it comes :

$$c(k)(\tilde{v}(.\,;k), \tilde{w}(.\,;k)) = \int_{2\pi Y} c_{h\ell mn}(x) \sum_{\gamma,\delta \in 2\pi\mathbb{Z}^3} \frac{\partial v_m}{\partial x_n}(x+\gamma)\, \frac{\partial \overline{w}_h}{\partial x_\ell}(x+\delta)\, e^{-ik.(\gamma-\delta)}\, dx$$

We integrate this expression over k, $k \in Y$. The functions v and w have compact supports and the coefficients $c_{h\ell mn}$ are periodic, so we have :

$$\int_Y c(k)\,(\tilde{v}(.\,;k), \tilde{w}(.\,;k))\, dk = c(v,w)$$

But if decomposition (27) is used, it results :

$$c(k)\,(\tilde{v}(.\,;k), \tilde{w}(.\,;k)) = \hat{v}_m(k)\overline{\hat{w}_n}(k)\, c(k)\, (\varphi^m(.\,;k), \varphi^n(.\,;k))$$

Let us define :

$$(28) \qquad c^{mn}(k) = c(k)\, (\varphi^m(.\,;k), \varphi^n(.\,;k)).$$

Finally, it comes :

$$c(v,w) = \int_Y c^{mn}(k)\widehat{v}_m(k)\overline{\widehat{w}_n}(k)dk. \tag{29}$$

Expression (29) holds true for v and w in $(\mathscr{D}(\mathbb{R}^3))^3$. But $(\mathscr{D}(\mathbb{R}^3))^3$ is densely embedded into $(H^1(\mathbb{R}^3))^3$, so (29) holds true for any v and w in $(H^1(\mathbb{R}^3))^3$. Let us note that expression (25) can be obtained in the same way, using the properties of the functions φ^m. Also, we have :

$$v, w \in (L^2(\mathbb{R}^3))^3 \quad , \quad (v,w) = \int_Y \widehat{v}_m(k)\overline{\widehat{w}_m}(k)dk. \tag{30}$$

The solution u of (13) has a Bloch expansion (23), the components of which depend on the time parameter t. Expressions (25), (29) and (30) are introduced into (13) and it results that the functions $\widehat{u}_m(k,t)$, for fixed k, are determined from the differntial system (31)-(32) :

$$\frac{d^2\widehat{u}_m}{dt^2} + \omega_m^2(k)\widehat{u}_m - \beta \sum_{n=0}^{+\infty} c^{nm}(k)\widehat{u}_n + \sum_{n=0}^{+\infty} c^{nm}(k)\frac{d\widehat{u}_n}{dt} = 0 \tag{31}$$

$$\left\{ \begin{aligned} &\widehat{u}_m(k,0) = \int_{\mathbb{R}^3} e^{-ik.x}\, f_\ell(x)\, \overline{\varphi}_\ell^m(x;k)\, \rho_o(x)dx \\ &\frac{d\widehat{u}_m}{dt}(k,0) = \int_{\mathbb{R}^3} e^{-ik.x}\, g_\ell(x)\, \overline{\varphi}_\ell^m(x;k)\, \rho_o(x)dx \end{aligned} \right. \tag{32}$$

This coupled system, with an infinite number of unknown functions, is not easy to solve. But it can be used to find approximations of the solution. Let us point out two examples : from the average method if the viscosity terms are small [9], from an expansion of the solution if the data are slowly varying functions.

The parameter β appears in the expression of u, not only from the coefficients $\widehat{u}_m(k,t)$, by (31), but also from the choice of the basis φ^m. But this fact is only a technics in the writing of the solution u. An example of this fact is given in the next section.

4. - CONNECTION WITH THE HOMOGENIZATION THEORY

4.1. - Setting of the problem

Let ϵ be a small positive parameter. We suppose that the initial data f and g are slowly varying functions of x by means of ϵ :

$$(33) \qquad u(x,0) = f^{\epsilon}(\epsilon x) \qquad \frac{\partial u}{\partial t}(x,0) = \epsilon g^{\epsilon}(\epsilon x)$$

f^{ϵ} and g^{ϵ} have asymptotic power series expansions :

$$(34) \qquad f^{\epsilon}(y) = f^{0}(y) + \epsilon f^{1}(y) + \ldots \qquad g^{\epsilon}(y) = g^{0}(y) + \epsilon g^{1}(y) + \ldots$$

The period of the medium seems to be small compared to the data scale. So the assumptions of the approximate homogenization theory are satisfied. We prove that an expansion of the solution u, in powers of ϵ, has for first term the function given by the homogenization theory, which is written down in J. Sanchez-Hubert[4].

In the expression, of the form (23), of the solution u, we achieve the change : $k = \epsilon K$, and we study the function $\epsilon^3 \tilde{u}, \tilde{u}$ defined by (35), when ϵ tends to zero.

$$(35) \qquad \tilde{u} = \tilde{u}\,(\epsilon K;x,t,\epsilon) = \sum_{m=0}^{+\infty} \hat{u}_m(\epsilon K,t,\epsilon)\,\varphi^m(x;\epsilon K).$$

This function $\tilde{u}(t)$ is a solution to (36)-(37) :

$$(36) \qquad \frac{d^2\tilde{u}}{dt^2} + [A^{\beta}(\epsilon K) - \beta C(\epsilon K)]\,\tilde{u} + C(\epsilon K)\,\frac{d\tilde{u}}{dt} = 0$$

$$(37) \qquad \left\{ \begin{array}{l} \tilde{u}(0) = \displaystyle\sum_{m=0}^{+\infty} \varphi^m(x;\epsilon K) \int_{IR^3} e^{-i\epsilon K.y}\, f^{\epsilon}_{\ell}(\epsilon y)\, \bar{\varphi}^m_{\ell}(y;\epsilon K)\, \rho_0(y)dy \\ \\ \dfrac{d\tilde{u}}{dt}(0) = \displaystyle\sum_{m=0}^{+\infty} \varphi^m(x;\epsilon K)\epsilon \int_{IR^3} e^{-i\epsilon K.y}\, g^{\epsilon}_{\ell}(\epsilon y)\, \bar{\varphi}^m_{\ell}(y;\epsilon K)\, \rho_0(y)dy \end{array} \right.$$

4.2. - Study of the operators $A^{\beta}(\epsilon K)$ and $C(\epsilon K)$

Let us investigate the behaviour of the eigenvalues and eigenfunctions of the operator $A^{\beta}(\epsilon K)$. The study which is done for the elastic, periodic material (N. Turbé[9]) is used here.

For $m \neq 0$, the eigenvalue $\omega^2_m(0)$ being simple or multiple, it is possible to find a vector $\varphi^m(x;0)$ such that the eigenvalue and eigenvector of $A^{\beta}(\epsilon K)$ are written :

$$\omega^2_m(\epsilon K) = \omega^2_m(0) + \ldots \qquad \varphi^m(x;\epsilon K) = \varphi^m(x;0) + \ldots$$

And it results that :

(38) $$A^{\beta}(\epsilon K)\,\varphi^{m}(x;\epsilon K)=\omega_m^2(0)\,\varphi^{m}(x;0)+\ldots$$

For m = 0, $\omega_o^2(0)=0$ is an eigenvalue of $A^{\beta}(0)$ associated with any arbitrary constant vector. This eigenvalue produces three holomorphic branches of eigenvalues. Then $\omega_o^2(\epsilon K)$ and $\varphi^{o}(x;\epsilon K)$ admit the expansions (([8]) :

$$\omega_o^2(\epsilon K)=\epsilon^2\,\Omega^2(K)+\ldots$$

$$\varphi^{o}(x;\epsilon K)=\varphi^{o}(x;0)+\epsilon\varphi^{o}_{,p}(x)\,K_p+\epsilon^2\varphi^{o}_{,pq}(x)\,K_p\,K_q+\ldots$$

where the constant vector $\varphi^{o}(x;0)$, which depends on K, is one of the three eigenvectors $\psi^{(r)}(K)$ (r = 1,2,3), orthonormed vectors in $(L^2(2\pi Y))^3$, associated with the eigenvalues $\Omega^2=\Omega^2{}_{(r)}(K)$, obtained from :

(39) $$\Omega^2(K)\,\varphi^{o}(x;0)=K_p\,K_q\left[A^{\beta}(0)\,\varphi^{o}_{,pq}(x)+\frac{1}{2}\,A^{\beta}_{,p}\,\varphi^{o}_{,q}(x)+\frac{1}{2}\,A^{\beta}_{,q}\,\varphi^{o}_{,p}(x)+A^{\beta}_{,pq}\,\varphi^{o}(x;0)\right].$$

In (39), the operators are respectively defined by (40) and (41).

(40) $$(A^{\beta}_{,p}\,u)_j=-\frac{i}{\rho_o}\left\{(b_{jpmn}+\beta c_{jpmn})\,\frac{\partial u_m}{\partial x_n}+\frac{\partial}{\partial x_\ell}[(b_{j\ell mp}+\beta c_{j\ell mp})u_m]\right\}$$

(41) $$(A^{\beta}_{,pq}\,u)_j=\frac{1}{\rho_o}\,(b_{jpmq}+\beta c_{jpmq})u_m$$

The term $\varphi^{o}_{,p}(x)$ is function of $\varphi^{o}(x;0)$:

(42) $$\varphi^{o}_{,p}(x)=i\,\varphi^{o}_{\ell}\,(x;0)\,\chi^{\ell}_{p}(x)$$

where the vector $\chi^{\ell}_{p}(x)$, classically introduced in the homogenization of elasticity problems ([5]) are solutions of :

(43) $$\chi^{\ell}_{p}\in(H^1_p(2\pi Y))^3\quad,\qquad A^{\beta}(0)\chi^{\ell}_{p}=i\,A^{\beta}_{,p}\,e^{\ell}$$

($e^{\ell}=\ell^{\text{th}}$ vector of the IR^3 natural basis).

So, equation (39) produces a linear, homogeneous system, with variables $\varphi^{o}_{j}\,(x;0)$, the solution of which furnishes $\psi^{(r)}(K)$ and $\Omega^2_{(r)}\,(K)$ (r = 1,2,3).

From these properties, it follows that :

(44) $$A^{\beta}(\epsilon K)\,\varphi^{o}(x;\epsilon K)=\epsilon^2\,\Omega^2_{(r)}(K)\,\psi^{(r)}(K)+\ldots$$

(without an add on r, r = 1,2,3).

In the same way, it is possible to obtain an estimate of $C(\epsilon K)\,\varphi^m(x;\epsilon K)$. The operator $C(\epsilon K)$ is expanded in powers of ϵ :

$$C(\epsilon K) = C(0) + \epsilon\, C_{,p}\, K_p + \epsilon^2\, C_{,pq}\, K_p\, K_q$$

where $C_{,p}$ and $C_{,pq}$ are defined by similar expressions to (40) and (41), with the coefficients $c_{h\ell mn}$ instead of $b_{h\ell mn} + \beta c_{h\ell mn}$. Then it comes :

$$C(\epsilon K)\,\varphi^m(x;\epsilon K) = C(0)\,\varphi^m(x;0) + \ldots \qquad (m \neq 0)$$

$$C(\epsilon K)\,\varphi^o(x;\epsilon K) = \epsilon\, K_p\,[C_{,p}\,\varphi^o(x;0) + C(0)\,\varphi^o_{,p}(x)] +$$

$$\epsilon^2\, K_p\, K_q \left[C(0)\,\varphi^o_{,pq}(x) + \frac{1}{2} C_{,p}\,\varphi^o_{,q}(x) + \frac{1}{2}\; C_{,q}\,\varphi^o_{,p}(x) + C_{,pq}\,\varphi^o(x;0)\right]$$

4.3. - Approximate solution

In the initial conditions (37), we adopt the data scale and we change y into y / ϵ. Then, we use the following property : if $h(x,y)$ is smooth on $IR^3 \times 2\pi Y$ and has compact support in x

$$\int_{IR^3} h(x,\frac{x}{\epsilon})\,dx \;\rightarrow\; \int_{IR^3} dx\, \frac{1}{(2\pi)^3} \int_{2\pi Y} h(x,y)dy \quad \text{as} \quad \epsilon \rightarrow 0.$$

The basis $\{\varphi^m(x;0)\}$ is orthonormal and the vectors $\varphi^o(x;0)$ are constant, so we deduce :

(46)
$$\begin{cases} \epsilon^3\, \tilde{u}(0) = (2\pi)^{\frac{3}{2}}\, \tilde{\rho}_o\, \hat{f}^o_\ell(K)\, \overline{\psi}^{(r)}_\ell(K)\, \psi^{(r)}(K) + \ldots \\ \epsilon^3\, \dfrac{d\tilde{u}}{dt}(0) = \epsilon(2\pi)^{\frac{3}{2}}\, \tilde{\rho}_o\, \hat{g}^o_\ell(K)\, \overline{\psi}^{(r)}_\ell(K)\, \psi^{(r)}(K) + \ldots \end{cases}$$

where $\tilde{\rho}_o$ is the mean value of the density

(47)
$$(2\pi)^3\, \tilde{\rho}_o = \int_{2\pi Y} \rho_o(x)dx$$

and $\hat{f}^o(K)$ and $\hat{g}^o(K)$ denote the Fourier transforms of the vectors $f^o(x)$ and $g^o(x)$.

Let us now find the first term of the expansion of $\epsilon^3\, \tilde{u}(t)$. We replace expression

(35) of $\widetilde{u}(t)$ into (36) and we use (38), (44) and (45). Then, by taking the scalar product with $\varphi^m(x;0)$, we have the equations satisfied by the coefficients $\widehat{u^o_m}(t)$

$$\epsilon^3 \widehat{u_m}(t) = \widehat{u^o_m}(t) + \ldots$$

According to the initial conditions (46), we single out two cases. For m = 1,2,... it comes :

$$\left\{ \begin{aligned} & \frac{d^2 \widehat{u^o_m}}{dt^2} + \omega^2_m(0)\, \widehat{u^o_m} - \beta \sum_{n=1}^{+\infty} c^{nm}(0)\, \widehat{u^o_n} + \sum_{n=1}^{+\infty} c^{nm}(0)\, \frac{d\widehat{u^o_n}}{dt} = 0 \\ & \widehat{u^o_m}(0) = 0 \qquad \frac{d\widehat{u^o_m}}{dt}(0) = 0 \end{aligned} \right.$$

which has the solution 0. Therefore, as for as the elastic, periodic material ([8]), the first term of the solution comes from the contribution m = 0.

For m = 0, the components $\widehat{u^o_{(r)}}(t)$ are solutions of (48)-(49) :

$$\text{(48)} \qquad \frac{d^2 \widehat{u^o_{(r)}}}{dt^2} + \epsilon^2\, \Omega^2_{(r)}(K)\, \widehat{u^o_{(r)}} - \beta\, \epsilon^2 \sum_{s=1}^{3} \gamma^{sr}(K)\, \widehat{u^o_{(s)}} + \epsilon^2 \sum_{s=1}^{3} \gamma^{sr}(K)\, \frac{d\widehat{u^o_{(s)}}}{dt} = 0$$

$$\text{(49)} \qquad \widehat{u^o_{(r)}}(0) = (2\pi)^{\frac{3}{2}}\, \widetilde{\rho}_o\, \hat{f}^o_\ell(K)\, \overline{\psi}^{(r)}_\ell(K) \qquad \frac{d\widehat{u^o_{(r)}}}{dt}(0) = \epsilon (2\pi)^{\frac{3}{2}}\, \widetilde{\rho}_o\, \hat{g}^o_\ell(K)\, \overline{\psi}^{(r)}_\ell(K)$$

where the coefficients $\gamma^{sr}(K)$ are given by :

$$\text{(50)} \qquad \gamma^{sr}(K) = \left[\frac{1}{2}\,(C_{,p}\, \psi^{(s)}_{,q}, \psi^{(r)}) + \frac{1}{2}\,(C_{,q}\, \psi^{(s)}_{,p}, \psi^{(r)}) + (C_{,pq}\, \psi^{(s)}, \psi^{(r)}) \right] K_p\, K_q$$

The system (48) is a linear differential system, with the following characteristic equation :

$$\text{(51)} \qquad \det\,(p^2\, \delta_{sr} + \epsilon^2\, \Omega^2_{(s)}\, \delta_{sr} - \epsilon^2\, \beta\, \gamma^{sr} + \epsilon^2\, p\, \gamma^{sr}) = 0$$

that has six roots $p_\nu = p_\nu(K,\epsilon)$ (ν = 1,6). Therefore, the solutions of (48) are of the form $A^{(r)}_\nu \exp(p_\nu\, t)$, where the coefficients $A^{(r)}_\nu$ satisfy :

$$\text{(52)} \qquad p^2\, A^{(r)} + \epsilon^2\, \Omega^2_{(r)}\, A^{(r)} - \beta\, \epsilon^2 \sum_{s=1}^{3} \gamma^{sr}\, A^{(s)} + \epsilon^2\, p \sum_{s=1}^{3} \gamma^{sr}\, A^{(s)} = 0$$

$$(p = p_\nu \quad , \qquad \nu = 1 \ldots 6).$$

The solution of the linear, homogeneous system (52) depends on six constants, obtained from the initial conditions (49). So, the first term of the expansion of u is :

$$(53)\qquad u^o = \sum_{\substack{\nu=1,6\\ r=1,3}} \int_{IR^3} e^{iK.\epsilon x}\, A_\nu^{(r)}(K,\epsilon)\, e^{p_\nu(K,\epsilon)t}\, \psi^{(r)}(K)\, dK$$

the coefficients p_ν satisfy (51) and $A_\nu^{(r)}$ (52), (54).

$$(54)\qquad \sum_{\nu=1}^{6} A_\nu^{(r)} = (2\pi)^{\frac{3}{2}}\, \widetilde{\rho}_o\, \hat{f}_\ell^o(K)\, \overline{\psi}_\ell^{(r)}(K) \qquad \sum_{\nu=1}^{6} A_\nu^{(r)}\, p_\nu = \epsilon(2\pi)^{\frac{3}{2}}\, \widetilde{\rho}_o\, \hat{g}_\ell^o(K)\, \overline{\psi}_\ell^{(r)}(K)$$

By using (54), we easily show that :

$$u^o(\epsilon x,0) = f^o(\epsilon x) \qquad \frac{\partial u^o}{\partial t}(\epsilon x,0) = \epsilon\, g^o(\epsilon x).$$

4.4. - Homogenized constitutive law

We prove that the function u^o in (53) satisfies :

$$\frac{\partial^2 u_j^o}{\partial t^2} - \frac{1}{\widetilde{\rho}_o} \frac{\partial \sigma_{j\ell}^h}{\partial x_\ell} = 0$$

where the homogenized stress tensor $\sigma_{j\ell}^h$ has the expression obtained by J. Sanchez-Hubert [4].

$$\frac{\partial^2 u_j^o}{\partial t^2} = -\epsilon^2 \int_{IR^3} \sum_{s,r,\nu} e^{iK.\epsilon x}\, dK\, (\Omega_{(r)}^2\, \psi_j^{(r)} - \beta\gamma^{rs}\, \psi_j^{(s)} + p_\nu \gamma^{rs}\, \psi_j^{(s)})\, A_\nu^{(r)}\, e^{p_\nu t}$$

The term $\Omega_{(r)}^2\, \psi_j^{(r)}$ is deduced from (39) with the scalar product of (39) by e^j, IR^3 natural basis vector. From definition (50), it comes $\gamma^{rs}\, \psi_j^{(s)}$. With an evident notation, we obtain :

$$(55)\qquad \frac{\partial^2 u_j^o}{\partial t^2} = -\frac{\epsilon^2 K_p K_q}{\widetilde{\rho}_o (2\pi)^3} \sum_{\nu,r} \int_{IR^3} e^{iK.\epsilon x}\, dK\, A_\nu^{(r)}\, e^{p_\nu t} \Big[((B_{,pq} + p_\nu C_{,pq})\, \psi^{(r)}, e^j)$$

$$+ \frac{1}{2}\, ((B_{,p} + p_\nu C_{,p})\, \psi_{,q}^{(r)}, e^j) + \frac{1}{2}\, ((B_{,q} + p_\nu C_{,q})\, \psi_{,p}^{(r)}, e^j) \Big]$$

Let us note again that, in (55), $\psi^{(r)}$ is an eigenvector of the operator $B(0) + \beta C(0)$, β any positive real number. The term $-\epsilon^2 K_p K_q$, in (55), is associated with a x_p, x_q – derivative. As for the time dependance, it is given by the expression of the Laplace transform, for any point β.

With the definition of the classical homogenized coefficients (see E. Sanchez-Palencia [5]), we obtain :

$$\mathcal{L}\left(\frac{\partial^2 u_j^o}{\partial t^2}\right)(\beta) = \frac{1}{\tilde{\rho}_o} \frac{\partial}{\partial x_p} (b_{jpmq} + \beta c_{jpmq})^h \mathcal{L}\left(\frac{\partial u_m^o}{\partial x_q}\right)(\beta)$$

where $\mathcal{L}(u)$ denotes the Laplace transform of the function u.

Therefore, the homogenized constitutive law is given by a convolution product, associated with a non-instantaneous memory.

REFERENCES

[1] J.D. ACHENBACH. *«Waves and vibrations in directionally reinforced composites»*. Composite materials, vol. 2, Mechanics of composite materials L.J. Broutman, R.M. Kroch (ed.). Academic Press, New-York (1975), pp. 309-351.

[2] A. BENSOUSSAN, J.L. LIONS, G. PAPANICOLAOU. *«Asymptotic analysis for periodic structures»*. North-Holland, Amsterdam (1978).

[3] T.J. DELPH, G. HERRMANN, R.K. KAUL. *«Harmonic wave propagation in a periodically layered, infinite elastic body, antiplane strain»*. J. Appl. Mech., 45 (1978), pp. 343-349.

[4] J. SANCHEZ-HUBERT. *«Asymptotic study of the macroscopic behaviour of a solid-fluid mixture»*. Math. Meth. Appl. Sci., 2 (1980) pp. 1-11.

[5] E. SANCHEZ-PALENCIA. *«Non-homogeneous media and vibration theory»*. Springer, Berlin (1980).

[6] C.T. SUN, J.D. ACHENBACH, G. HERRMANN. *«Continuum theory for a laminated medium»*. *J.* Appl. Mech. 35 (1968), pp. 467-473.

[7] C. SVE. *«Time harmonic waves traveling obliquely in a periodically laminated medium»*. J. Appl. Mech. 38 (1971), pp. 477-482.

[8] N. TURBE. *«Applications of Bloch expansion to periodic elastic and viscoelastic media»*. Math. Meth. Appl. Sci. 4 (1982) pp. 433-449.

[9] N. TURBE. Thèse de Doctorat, Université Pierre et Marie Curie (1982).

[10] N. TURBE. Communication colloque du Groupe Français de Rhéologie (1982).

[11] C.H. WILCOX. *«Theory of Bloch waves»*. J. Anal. Math. 33 (1978) pp. 146-167.

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