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RÉSUMÉ. — Nous prouvons que la première valeur propre du laplacien-p est associée à une fonction propre qui est $C^{1,\alpha}(\overline{\Omega})$, strictement positive et unique à une constante multiplicative près si $\partial\Omega$ est connexe. De plus, nous montrons que c'est la seule valeur propre associée à une fonction propre positive.

ABSTRACT. — We prove that the first eigenvalue of the p-Laplacian is associated to a positive $C^{1,\alpha}(\overline{\Omega})$ eigenfunction, which is unique up to a multiplicative constant if $\partial\Omega$ is connected. Moreover, we show that it is the unique eigenvalue associated to a non negative eigenfunction.

Introduction

In this work, we are interested in the properties of the "first eigenvalue" of the p-laplacian. Let us recall that the typical eigenvalue problem for the p-laplacian is to find $\lambda \in \mathbf{R}$ and $u(\not\equiv 0)$ in $W_0^{1,p}(\Omega)$ weak solution of :

$$\begin{cases} -\text{div } \left(|\nabla u|^{p-2}\nabla u\right) = \lambda |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbf{R}^N$. In all the following, we will always assume that the boundary $\partial\,\Omega$ is of class $C^{2,\beta}$. The problem of the existence of such λ and u has been studied recently by J.P. GARCIA- AZORERO and I. PERAL-ALONZO [5] : They show the existence of an increasing sequence $(\lambda_k)_{k\geq 1}$ of eigenvalues such that $\lambda_k \to +\infty$.

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We prove here that, roughly speaking, the first eigenvalue λ_1, which is defined by

$$\lambda_1 = \inf\left(\int_\Omega |\nabla u|^p \,;\; u \in W_0^{1,p}(\Omega),\ \int_\Omega |u|^p = 1\right)$$

possesses all the properties of the first eigenvalue of a second -order non degenerate elliptic operator as soon as $\partial\Omega$ is connected. λ_1 is associated to a $C^{1,\alpha}(\overline{\Omega})$ eigenfunction which is positive in Ω and is unique (up to a multiplicative constant). Moreover, λ_1 is the unique eigenvalue associated to a nonnegative eigenfunction. This kind of properties has been studied for $N = 1$ by M. OTANI [13] and in a ball of $\mathbf{R}^N$ by F. de THELIN [4].

Of course, the main difficulty to prove these results relies on the – a priori – lack of regularity of the eigenfunctions. For example, in the unit ball, the existence of a radially symmetric eigenfunction, for which it is easy to show that it is smooth except perhaps in zero, simplifies all the arguments below. Therefore, the first part of this paper is devoted to the study of the regularity of a solution u of (1) for $\lambda = \lambda_1$. It would be too long to mention here all the works (that we know!) concerning the regularity for similar degenerate elliptic equations; we only give here those we have directly used and refer the reader to the references in these works. Essentially, these results are of two types: the first one concerns the minima (and quasi-minima) in the calculus of variations (cf. E. de GIORGI [3], M. GIAQUINTA and E. GIUSTI [6], E. Di BENEDETTO and N.S. TRUDINGER [2]).

These results give the local Hölder continuity and provide Hanack inequalities. Let us just recall that, in our case, the eigenfunctions for λ_1 are minima of

$$J(u) = \int_\Omega |\nabla u|^p - \lambda_1 \int_\Omega |u|^p. \tag{2}$$

The second ones concerns degenerate elliptic equations on divergence form (cf. E. Di BENEDETTO [1], D.A. LADYZENSKAYA and N.N. URALTSEVA [9], P. TOLKSDORF [14], K. UHLENBECK [16]). Very often, as in our case, it is the Euler equation of a variational problem. These results give the $C^{1,\alpha}_{\text{loc}}$ regularity of the weak solutions. Finally, in P. TOLKSDORF [15], an argument, which allows to use the Schwarz reflection principle, gives the $C^{1,\alpha}$ global regularity in certain cases. Using these results and these methods, we show that every eigenfunction for λ_1 is positive in Ω and belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ and is of class $C^{2,\alpha}$ in a neighbourhood of $\partial\Omega$. These results do not use the connectedness of $\partial\Omega$.

In the second part, we are interested in the uniqueness results. Let us first mention that similar uinqueness problems for nonlinear "first eigenvalue " have been considered in P.L. LIONS [11,12] and that we use here the same general ideas as in [11,12]. The main point is to have a strong comparison principle which allows to use a method due to T. LAETSCH [10]. Therefore, we first prove a strong comparison property between a weak subsolution u and a weak supersolution v, both in $W_0^{1,p}(\Omega) \cap C(\Omega)$ of

$$\begin{cases} -\mathrm{div}\left(|\nabla w|^{p-2}\nabla w\right) = f & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

where $f \in L^q(\Omega)$ $\left(\dfrac{1}{p} + \dfrac{1}{q} = 1\right)$. We show that if we assume that there exists a neighbourhood of $\partial\Omega$ in which u is C^2, v is C^1 and $|\nabla u| > 0$ then either u and v coincide in this neighbourhood or $u < v$ in Ω. Moreover, if we assume that u, v are positive in Ω, then either $u \equiv v$ in a neighbourhood of $\partial\Omega$ or there exists $\theta > 1$ such that $\theta u \leq v$. To do so, we need the connectedness of $\partial\Omega$. This results complements the strong comparison principle of P. TOLKSFORF [15]. Then, we prove the uniqueness results announced above by the Laetsch's method and we conclude this paper by mentionning a bifurcation results which can be treated by the same methods.

N.B.. — When this manuscript was typed, we have heard from J.I. DIAZ and J.E. SAA that they have obtained very general uniqueness results for equations with the p-Laplacian. (See [17,18]). We have also learned that S. SAKAGUCHI has obtained independently some results similar to ours.

I. On the eigenfunction for λ_1

In this section, we investigate the properties of a non-zero solution u in $W_0^{1,p}(\Omega)$ of

$$\forall \varphi \in W_0^{1,p}(\Omega), \quad \int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi = \lambda_1 \int_\Omega |u|^{p-2} u \varphi. \tag{4}$$

Let us remark that every solution u of (4) satisfies

$$J(u) = \mathrm{Inf}\, \left\{ J(v)\,;\, v \in W_0^{1,p}(\Omega) \right\} = 0 \tag{5}$$

and therefore, the variational problem (5) is equivalent to (4), which is its associated Euler Equation. This property will allow us to use results concerning both minima in the calculus of variations and degenerate elliptic equations on divergence form. Our results are the following.

PROPOSITION I.1.— *Let u be a weak solution of (4) different from 0, then $u \in C^{0,\alpha}_{loc}(\Omega)$ for some $0 < \alpha < 1$ and satisfies*

$$u > 0 \quad in \quad \Omega \qquad or \qquad -u > 0 \quad in \ \Omega.$$

PROPOSITION I.2.— *Let u be a weak solution of (4), then*

$$u \in L^q(\Omega) \qquad \forall q \in [1, +\infty[.$$

THEOREM 1.3.— *Let u be a weak solution of (4), then there exists $\alpha \in (0,1)$ and $\varepsilon > 0$ such that*

$$u \in C^{1,\alpha}(\overline{\Omega}) \cap C^{2,\beta}(\overline{\Omega}_\varepsilon)$$

where

$$\Omega_\varepsilon = \{x \in \Omega / d(x, \partial\Omega) < \varepsilon\}.$$

Before proving these results, let us precise that the local $C^{0,\alpha}$ regularity is a consequence of the results of E. de GIORGI [3], O.A. LADYZENSKAYA and N.N. URALT'SEVA [9] or M. GIAQUINTA and E. GIUSTI [6]. Moreover, the local bounds in L^∞_{loc} depend only on the L^p norm of u and the distance to $\partial\Omega$ (cf. E. Di BENEDETTO and N.S. TRUDINGER [2], via Harnack inequalities). Then, one gets easily $C^{1,\alpha}_{\text{loc}}$ estimates by the results of E. Di BENEDETTO [1] or P. TOLKSDORF [14]. So, in theorem I.3, only the boundary estimates have to be shown. To do so, we adapt a method of P. TOLKSDORF [15] which replaces the boundary estimates by an interior estimates via the Schwarz reflection principle.

Proof of Proposition I.1 .— Let us $\varphi = u^+$ in (4). We obtain:

$$J(u^+) = 0 = \inf\left\{J(v)\,;\; v \in W^{1,p}_0(\Omega)\right\},$$

so u^+ is a solution of (4) and (5). But, by a result of E. Di BENEDETTO and N.S. TRUDINGER [2], the nonnegative solutions of (5) satisfy an Harnack inequality. Then either $u^+ > 0$ and so $u > 0$, or $u_+ \equiv 0$. In this last case, the same argument for $-u$ yields the result.

Proof of Proposition I.2. — Of course, this result is interesting only in the case when $p < N$ since if $p > N$, $W_0^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ and if $p = N$, $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [1, +\infty[$. So, we assume that $p < N$. Let u be a solution of (4). Changing u in $-u$, we may assume that $u > 0$ in Ω, by Proposition I.1. Now, we define v by

$$v = \inf(u; M),$$

and we take $\varphi = v^{kp+1}$ ($k \geq 0$) in (4); we obtain

$$(kp+1)\int_\Omega v^{kp} \cdot |\nabla v|^p = \lambda_1 \int_\Omega u^{p-1} v^{kp+1}.$$

The left-hand side is equal to

$$\frac{(kp+1)}{(k+1)^p}\int_\Omega \left|\nabla\left(v^{k+1}\right)\right|^p$$

and $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ with $q = Np \cdot (N-p)^{-1}$. Therefore

$$\left(\int_\Omega v^{(k+1)q}\right)^{p/q} \leq \frac{C(k+1)^p}{(kp+1)} \cdot \lambda_1 \int_\Omega u^{p-1} v^{kp+1}.$$

Finally, if $u \in L^r(\Omega)$, we choose k such that

$$(k+1)p = r$$

and the right-hand side is estimated by

$$\frac{C(k+1)^p}{(kp+1)} \cdot \lambda_1 \int_\Omega |u|^r.$$

We conclude that $u \in L^{r'}(\Omega)$, where $r' = (k+1)q = \dfrac{qr}{p}$. But $q > p$, so an easy bootstrap argument gives the result.

Proof of Theorem I.3 . — The proof consists essentially in obtaining boundary estimates in $C^{1,\alpha}$ by using the method of P. Tolksdorf [15] and the estimates of E. Di Benedetto [1]. Since the arguments are routine adaptations of those of [15] and [1], we only sketch the proof. Let $x_0 \in \partial\Omega$; since $\partial\Omega$ is a $C^{2,\beta}$- surface, there exists a $C^{2,\beta}$ diffeomorphism φ from a neighbourhood of 0 in $\mathbf{R}^{N-1}$ into a neighbourhood of x_0 in $\partial\Omega$. We extend it to a $C^{1,\beta}$ diffeomorphism ψ from a neighbourhood of 0 in $H^+ = \left\{(y', y_n) \in \mathbf{R}^{N-1}\mathbf{R}^+\right\}$ in a neighbourhood of x_0 in $\overline{\Omega}$ by setting

$$\psi(y', y_n) = \varphi(y') - y_n n(\varphi(y'))$$

where $n(x)$ is the outward unit vector at $x \in \partial\Omega$. This transformation has the following property: if we define v by

$$v(y) = u(\psi(y))$$

then

$$|\nabla u(\psi(y))|^2 = |A(y)\nabla'_y v|^2 + \left|\frac{\partial v}{\partial y_n}\right|^2$$

where

$$\nabla'_y v = \left(\frac{\partial v}{\partial y_1}, \ldots, \frac{\partial v}{\partial y_{N-1}}\right)$$

and $A(y)$ is, for all y, an invertible $(N-1) \times (N-1)$ matrix. So, this transformation decouples the $(N-1)$ first derivatives, and the y_n derivative. A similar idea was used by R. Jensen [8]. Finally, if u is solution of (5), v is solution of

$$\widetilde{J}(v) = \inf\left\{\widetilde{J}(\omega), \omega \in W\right\} \tag{5'}$$

where

$$\widetilde{J}(v) = \int_{U+}\left(|A(y)\nabla_{y'}v|^2 + \left|\frac{\partial v}{\partial y_n}\right|^2\right)^{p/2} |\det \nabla\psi| - \lambda_1 \int_{U+} |v|^p |\det \nabla\psi|$$

$$U^+ = \{(y', y_n) = y/|y'| < R,\ 0 < y_n < \eta\}$$

for R, η small enough, and

$$W = \left\{\omega \in W^{1,p}(U^+)/\omega = \quad \text{on } y_n = 0 \ \text{ and } \omega(y) = u(\psi(y)) \ \text{ for } |y'| = R\right\}.$$

Now, we use the Schwarz reflection principle i.e., we extend v to $U = \{(y', y_n)/|y'| \leq R,\ |y_n| < \eta\}$ by setting

$$v(y', y_n) = -v(y', -y_n) \quad \text{for } y_n \leq 0$$

and the other functions are extended by parity on y_n. The particular form of $\widetilde{J}$ implies that $\widetilde{v}$ is solution of an analogous variational problem in

$$U^- = \{(y', y_n)/|y'| \leq R,\ -\eta < y_n < 0\}.$$

Then, if we denote by

$$G(y, p', p_n) = \left(|A(y)p'|^2 + |p_n|^2\right)^{p/2} |\det(\nabla\psi)|$$

for $y \in U$, $p' \in \mathbf{R}^{N-1}$, $p_n \in \mathbf{R}$, v is a weak solution of

$$-\operatorname{div}\left(\nabla_p G\left(y,\, \nabla_{y'} v,\, \frac{\partial v}{\partial y_n}\right)\right) = \lambda_1 |v|^{p-2} v |\det \nabla\psi| \tag{6}$$

in U^+ and U^-. But, since v and its derivatives are continuous on $y_n = 0$, we have (6) in U and $v \in W^{1,p}(U)$.

Now, we use the $C^{1,\alpha}_{\text{loc}}$-estimates of E. Di BENEDETTO [1]. Let us just recall that these estimates depend only on N, p, ψ, λ_1 and the L^q-norm of $|v|^{p-2}v$ for a $q > Np'$, $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$. Finally, let u be a solution of (4) and (5) different from zero; we consider the approximated problems

$$\forall \varphi \in W_0^{1,p}(\Omega) \int_\Omega (\varepsilon + |\nabla u^\varepsilon|^2)^{(p-2)/2} \nabla u^\varepsilon \nabla\varphi = \lambda_1 \int_\Omega |u|^{p-2} u\varphi. \tag{4}$$

By standard arguments, u^ε exists and belongs to $W^{2,q}(\Omega)$ for all $q \in [1, +\infty[$ since $|u|^{p-2}u \in L^q(\Omega)$ for all $q \in [1, +\infty[$. Moreover, u^ε satisfies the a priori estimates above which depend only on the constants of the problem and the L^q norm of u for a $q > Np'$. So, by Ascoli theorem, there exists $\alpha \in]0, 1[$ such that u_ε converges in $C^{1,\alpha}(\overline{\Omega})$ and the classical uniqueness result for the Dirichlet problem (3) implies that the limit is necessarely u. Hence, $u \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. Moreover, since $u > 0$ in Ω or $-u > 0$ in Ω, by the strong maximum principle (cf. P. TOLKSDORF [15]), we have

$$\left|\frac{\partial u}{\partial n}\right| > 0 \quad \text{on } \partial\Omega;$$

therefore, since $|\nabla u| \in C^{0,\alpha}(\overline{\Omega})$, there exists $\eta > 0$ such that

$$|\nabla u| \geq \delta > 0 \quad \text{in } \Omega_n = \{x \in \Omega / d(x, \partial\Omega) < \eta\}.$$

So, in Ω_n, (4) becomes strictly elliptic and, by classical results, (see D. GILBARG and N.S. TRUDINGER [7]), $u \in C^{2,\beta}(\overline{\Omega}_\varepsilon)$, for all $\varepsilon < \eta$.

II. Uniqueness results

The aim of the section is to prove that the positive eigenfunction associated to λ_1 is unique (up to a multiplicative constant) and that λ_1 is the unique eigenvalue associated to a nonnegative eigenfunction,in the case when $\partial\Omega$ is connected. We will assume, in all this part, that $\partial\Omega$ *is connected.* Before giving this result, we need the following strong comparison result.

THEOREM II.1.— *Let u and v be respectively weak sub and supersolution of $(\mathcal{S})$ in $W_0^{1,p}(\Omega)\cap C(\Omega)$. We assume in addition that there exists $\varepsilon>0$ such that $u\in C^2(\overline{\Omega}_\varepsilon)$ with $|\nabla u|>0$ in $\overline{\Omega}_\varepsilon$ and $v\in C^1(\Omega_\varepsilon)$. Then, either $u\equiv v$ in $\overline{\Omega}$ or $u<v$ in Ω. Moreover, if u and v are positive in Ω, then either $u\equiv v$ in $\overline{\Omega}_\varepsilon$ or there exists $\theta>1$ such that $\theta u\leq v$.*

Remark 1.— Of course, the same result holds if $v\in C^2(\overline{\Omega}_\varepsilon)$ with $|\nabla v|>0$ in $\overline{\Omega}_\varepsilon$ and $u\in C^1(\Omega_\varepsilon)$.

Remark 2.— In order to compare the result of Theorem II. 1 with the strong comparison result of [15], we remark that we only need u and v to be smooth in a neighbourhood of $\partial\Omega$; but with this weaker assumption, our result is weaker than the result of [15] since if $u(x)=v(x)$ for some $x\in\Omega$, we can only conclude that $u\equiv v$ in $\overline{\Omega}_\varepsilon$. Nevertheless, we are more interested – for application to the eigenvalue problem – in the case when $u,v>0$ and this type of result was not considered in [15].

Now, we can state the

THEOREM II.2.— *Let $v\in W_0^{1,p}(\Omega)$ be a weak solution of*

$$-\ div\left(|\nabla v|^{p-2}\nabla v\right)=\lambda|v|^{p-2}v\quad in\ \Omega \tag{7}$$

with $v\geq 0$ in Ω, then u a positive eigenfunction for λ_1; then

$$\begin{aligned}&(i)\quad \lambda=\lambda_1\\ &(ii)\quad \exists\mu\in\mathbf{R}\quad such\ that\ v=\mu u.\end{aligned}$$

Remark 3.— Let us just remark that weak solutions of (7) in $W_0^{1,p}(\Omega)$ are Hölder continuous: this is a consequence of the fact that such solutions are in the De GIORGI classes (cf. [2], [3], [6]...). Therefore, the Harnack inequality of [2] concludes that $v>0$ in Ω.

We first prove Theorem II.2, using Theorem II.1.

Proof of Theorem II.2.— Let $x_0\in\Omega_\varepsilon$, Ω_ε being defined in Theorem I.3. Let $\mu=\dfrac{v(x_0)}{u(x_0)}$ and let $\omega=\mu u$. We are going to prove that $\omega\equiv v$. In fact, first, we want to prove that $\omega \leq 0$. To do so, we use a method due to Th. LAETSCH [10]. We define t_0 by

$$t_0 = \sup\{t \in (0,1)/t\omega \leq v\}.$$

If $t_0 = 1$, the result is proved. If $t_0 < 1$, $t_0\omega$ is subsolution of

$$-\operatorname{div}\left(|\nabla(t_0\omega)|^{p-2}\nabla(t_0\omega)\right) \leq \lambda|v|^{p-2}v.$$

because $\lambda \geq \lambda_1$ and $t_0\omega \leq v$. By Theorem II. 1, either $t_0\omega \equiv v$ in Ω_ε or there exists $\theta > 1$ such that $\theta t_0\omega \leq v$. But, at $x_0 \in \Omega_\varepsilon$, $t_0\omega(x_0) < v(x_0)$; so, we are in the second case and the inequality $\theta t_0\omega \leq v$ with $\theta > 1$ contradicts the definition of t_0. Hence, $t_0 = 1$ and the result is proved. Now arguing exactly as before with $t_0 = 1$, we find that $\omega \equiv v$ in Ω_ε, therefore $\lambda = \lambda_1$ and the same argument as above proves the opposite equality $\omega \geq v$.

Now, we turn to the proof of Theorem II.1.

Proof of Theorem II.1.— We only treat the case when u and v are positive in Ω, the other result being obtained in the same way. Since $v \in C^1(\overline{U}^\eta)$, where $U^\eta = \{x \in \Omega/\eta < d(x,\partial\Omega) < \varepsilon\}$, we can use the strong comparison priciple of P. TOLKSDORF [15] in U_η since we already know by the weak maximum principle that $u \leq v$ in Ω. Hence, if u is not equal to v in Ω_ε then for η small enough U_η is connected and this result gives

$$u < v \quad \text{in } U^\eta.$$

In particular, on $\{x \in \Omega/d(x,\partial\Omega) = \varepsilon/2\}$, $u < v$. Let b be defined on $\Omega_{\varepsilon/2}$ by

$$b(x) = C\left(\exp\left(\alpha d(x,\partial\Omega)\right) - 1\right)$$

$d(\cdot,\partial\Omega)$ is $C^{2,\beta}$ on $\Omega_{\varepsilon/2}$ and for C and α small enough, it is easy to check that $u + b$ is still a subsolution of

$$-\operatorname{div}\left(|\nabla\omega|^{p-2}\nabla\omega\right) = f \quad \text{in } \Omega_{\varepsilon/2}$$

and that $u + b \leq v$ on $\partial\Omega_{\varepsilon/2}$. So, by the weak maximum principle

$$u + b \leq v \quad \text{in } \Omega_{\varepsilon/2}.$$

Now, we can make two remarks. First, since ∇u is bounded in $\Omega_{\varepsilon/2}$ and since $\nabla b(x) \cdot \nabla d(x) \geq \alpha C > 0$ in $\Omega_{\varepsilon/2}$, there exists $\eta_1 > 0$ such that

$$\eta_1 u \leq b \quad \text{in } \Omega_{\varepsilon/2}.$$

The second remark is that, on $\partial(\Omega - \overline{\Omega}_{\varepsilon/2})$

$$u + \delta \leq v$$

where $\delta = C\left(\exp\left(\frac{\alpha\varepsilon}{2}\right) - 1\right)$. So, by the weak maximum principle in $\Omega - \overline{\Omega}_{\varepsilon/2}$, we have

$$u + \delta \leq v \quad \text{in } \Omega - \overline{\Omega}_{\varepsilon/2}.$$

Denoting by $\eta_2 = \inf\left\{\delta(u(x))^{-1},\, x \in \Omega - \overline{\Omega}_{\varepsilon/2}\right\}$, we have

$$\eta_2 u \leq \delta \quad \text{in } \Omega - \overline{\Omega}_{\varepsilon/2}.$$

If we set $\eta = \inf(\eta_1, \eta_2)$, we get

$$(1+\eta)u \leq v \quad \text{in } \Omega$$

and the proof is complete.

Let us conclude by giving a bifurcation result which can be proved by the same methods as above.

THEOREM II.3.— *Let $f: \mathbf{R} \to \mathbf{R}$ be a nondecreasing function satisfying $f(0) = 0$. We assume that $t \to f(t)t^{1-p}$ is decreasing for $t > 0$ and that $\lim\limits_{+\infty} f(t)t^{1-p} < \lambda_1$. Then*

(i) I

f $\lim\limits_{0} f(t)t^{1-p} \leq \lambda_1$, 0 is the unique nonnegative solution of

$$\begin{cases} -\, div\ \left(|\nabla u|^{p-2}\nabla u\right) = f(u) & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases} \tag{8}$$

(ii) I

f $\lim\limits_{0} f(t)t^{1-p} > \lambda_1$, there exists exactly two nonnegative solutions of (8) : 0 and a solution which is positive in Ω.

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