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On the composition of nondegenerate quadratic forms with an arbitrary index

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RÉSUMÉ. — On considère deux formes bilinéaires non dégénérées avec indices et signatures quelconques : $(a,b)_S$ – symétrique et $(f,g)_V$ – symétrique ou antisymétrique qui satisfont la condition $(a,a)_S\ (f,g)_V = (af,ag)_V$. Dans le cas où les deux indices sont zéro et la forme $(f,g)_V$ est symétrique, le problème a été résolu par A. HURWITZ (1923). On montre que la solution générale est liée aux algèbres de CLIFFORD ainsi qu'à des structures complexes et hermitiennes convenables.

ABSTRACT. — Two non-degenerate bilinear forms of arbitrary indices and signatures are considered : $(a,b)_S$ – symmetric and $(f,g)_V$ – symmetric or antisymmetric. The problem of determining all such forms which satisfy the condition $(a,a)_S\ (f,g)_V = (af,ag)_V$ is solved. In the case where the both indices are zero and $(f,g)_V$ is symmetric, the problem was solved by A. HURWITZ (1923). The general solution is shown to be connected with Clifford algebras as well as with suitable complex and hermitian structures.

Introduction

In 1923 there appeared a famous, posthumous paper by A. HURWITZ [11] solving the problem of determining all pairs of positive integers (n,p) and all systems of real numbers :

$$c^k_{j\alpha} \in \mathbf{R},\ j,k = 1,\dots,n; \qquad \alpha = 1,\dots,p, \quad p \le n, \tag{1}$$

such that the collection of bilinear forms $F_j = a_\alpha c^k_{j\alpha} f_k$ satisfies the condition

$$\sum_j F_j^2 = \sum_\alpha a_\alpha^2 \sum_k f_k^2 \qquad \textit{(the Hurwitz condition)}. \tag{2}$$

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In other words, he solved the problem of determining all pairs of $n-$ and $p-$dimensional positive-definite symmetric bilinear forms $(f,g)_n$ and $(a,b)_p$, satisfying the condition $(a,a)_p\ (f,f)_n = (af,af)_n$. It is obvious that the solution has to rely upon a suitable choice of the multiplication $(a,f)=af$, and so it determines the real structures constants (1) in connection with the classification problem for real Clifford algebras (cf. e.g. [22], pp. 272–273).

Following several earlier attempts (cf. the papers by Adem [1–3] and the list of references given there), including our own studies [14–17] on geometrical realizations of possible multiplication schemes, we are going to consider two real vector spaces S and V equipped with non-degenerate pseudo-euclidean real scalar products $(\ \ ,\ \ )_S$ and $(\ \ ,\ \ )_V$. Namely, for $f,g,h\in V$; $a,b,c\in S$, and $\alpha,\beta\in\mathbf{R}$ we suppose that :

$$\begin{array}{ll} (a,b)_S\in\mathbf{R}, & (f,g)_V\in\mathbf{R},\\ (b,a)_S=(a,b)_S, & (g,f)_V=\delta(f,g)_V, \delta=1 \text{ or } -1,\\ (\alpha a,b)_S=\alpha(a,b)_S, & (\alpha f,g)_V=\alpha(f,g)_V,\\ (a,b+c)_S=(a,b)_S+(a,c)_S; & (f,g+h)_V=(f,g)_V+(f,h)_V. \end{array} \tag{3}$$

In S and V we choose some bases (ϵ_α and (e_j), respectively, with $\alpha = 1,\ldots,\ \dim S=p$; $k=1,\ldots \dim V=n$. We assume that $p\le n$. For the metrics :

$$n\equiv[n_{\alpha\beta}]:=[(\epsilon_\alpha,\epsilon_\beta)_S];\quad \kappa\equiv[\kappa_{jk}]:=[(e_j,e_k)_V], \tag{4}$$

by the postulates (3), we get :

$$\begin{array}{ll} n^{-1}\equiv[n^{\alpha\beta}],\ n^T=n, & \kappa^{-1}\equiv[\kappa^{jk}],\ \kappa^T=\delta\kappa,\\ \det\ n\neq 0; & \det\kappa\neq 0 \end{array} \tag{5}$$

Without any loss of generality we can chose the basis (ϵ_α) so that

$$n=\operatorname{diag}\ \underbrace{(1,\ldots,1,-1,\ldots,-1)}_{\text{p times}},\text{and hence}\quad n^{-1}=n \tag{6}$$

The *multiplication* of elements of S by elements of V is defined as a mapping $S\times V\to V$ with the properties.

(i) $(a+b)f=af+bf$ and $a(f+g)=af+ag$ for $f,g\in V$ and $a,b\in S$;

(ii) $(a,a)_S(f,g)_V=(af,ag)_V$ *(the generalized Hurwitz condition)*;

(iii) there exists the unit element ϵ_0 in S with respect to the multiplication : $\epsilon_0 f=f$ for $f\in V$.

By (i), the multiplication is an $\mathbf{R}$-linear operation on V; by (iii), the multiplication by $\alpha \in \mathbf{R}$ is identified with the multiplication by $\alpha\epsilon_0$.

The product af is uniquely determined by the multiplication scheme for base vectors :

$$\epsilon_\alpha e_j = c^k_{j\alpha} e_k, \qquad \alpha = 1, \ldots, p; \qquad j = 1, \ldots, n. \tag{7}$$

The scheme, together with the postulates (3), yields in particular the following formulae for the real structure constants (1) :

$$c^k_{j\alpha} = (e^k, \epsilon_\alpha e_j)_V, \tag{8}$$

i.e. they are simply the matrix elements for ϵ_α treated as an $\mathbf{R}$-linear endomorphism of V. If the multiplication $S \times V \to V$ does not leave invariant proper subspaces of V, the corresponding pair (V, S) is said to be *irreducible*. In such a case we call (V, S) a *pseudo-euclidean Hurwitz pair.*

If the scalar products $(\quad,\quad)_S$ and $(\quad,\quad)_V$ are euclidean, it is sufficient to consider the corresponding euclidean norms $\|\quad\|_S$ and $\|\quad\|_V$, and to replace the generalized Hurwitz condition (ii) by $\|a\|_S \, \|f\|_V = \|af\|_V$ which is just the original Hurwitz condition (2). In this case the corresponding *euclidean* Hurwitz pair is simply called a *Hurwitz pair* [14, 15].

Now the programme of our paper may be described so that we aim at solving the following.

Problem. Determine all the pseudo-euclidean Hurwitz pairs effectively, i.e. find all the admissible scalar products $(\quad,\quad)_S$ and $(\quad,\quad)_V$ so that they correspond to a pseudo-euclidean Hurwitz pair (V, S).

Denote by ind S the index of S, that is, the number of $n_{\alpha\alpha} = -1$ in (6). Set :

$$r = p - s - 1, \quad s = \text{ ind } S, \quad \text{where } p = \text{ dim } S. \tag{9}$$

Now we may say we have to determine all the admissible systems

$$(n, r, s, \kappa_{jk} : j, k = 1, \ldots, n), \quad \text{where } n = \text{ dim } V, \tag{10}$$

κ_{jk} being determined in (4), what gives rise to the calculation of the structure constants (1) according to the formulae (8). All the results of Hurwitz [11] are included in our results obtained in the case $s = 0$ and $[\kappa_{jk}]$ symmetric and positively defined.

Let us describe briefly the earlier approaches to the problem. Chevalley [5] and Lee [21] used already Clifford algebras in a systematic way for studying composition of quadratic forms. The study of which quadratic forms admit such compositions was done over arbitrary fields (of characteristic not 2), independently of ADEM [1–3], by SHAPIRO [23–27]. The duality of the quadratic structure on V compatible with $C^{(r,s)}$– action, where $C^{(r,s)}$ is the associated Clifford algebra, appears as a consequence of the general theory due to FRÖLICH and MC EVETT [8]. The monograph [9] on orthogonal designs points out additional combinatorial aspects. Finally, more general types of composition for sums of squares with their relation to algebraic topology have recently been discussed in [27].

Thus, our rask may be described as a specification of some results given in [24, 25, 9], namely of [9], pp. 220–227, in the sense of giving the complete and effective determination and classification of all the admissible metrics (4) corresponding to $(\ ,\ )_S$ and $(\ ,\ )_V$. Yet this statement shows that our approach goes outside the consideration of pseudo-euclidean bilinear forms $(\ ,\ )_S$ and symmetric or anti-symmetric (skew-symmetric) bilinear forms $(\ ,\ )_V$. The geometrical aspect of the problem, completely abandoned in [24, 25, 9] gave rise to discussions of J. ŁAWRYNOWICZ with the unforgetable Professor A. ANDREOTTI, yielding a series of papers [14–17] with a geometrical approach enabling an original, the simplest foundation of the regular mapping theory within CLIFFORD analysis, and also physical models connected with particle physics [16, 17], including solitons (solitary waves) [12, 13, 28, 29]. As noticed by HESTENES [10], p. 9, Clifford algebras "become vastly richer when given geometrical and/or physical interpretations". Another geometrical approach has been proposed in [6, 7].

Whithin our approach, from Lemmas 1 and 2 in Sec. 1 it follows that the metric κ in (4) can be expressed in terms of a function $(r,s) \to a'_{rs}$, which is double periodic, exactly $(8,8)$ – periodic. For the sake of convenience we will use the notation $(r,s) \to a_{r+1,s}$. The rest of that section is devoted in each case to the characterization of the representation space, the calculation of its dimension and the dimension of V, as well as the description of the possibility of constructing the real and imaginary Majorana representations of Clifford algebras (Theorem 1). The real (resp. imaginary) Majorana representation of a Clifford algebra $C^{(r,s)}$ is defined by the choice of its generators as real (resp. purely imaginary) matrices (cf. [23], p. 699). Thus the section is of a preparatory character and is merely a repetition of a fragment of our previous paper [16].

After these preliminaries we can concentrate, in Section 2, on determining avoid introducing many notions, unnecessary for final results, and simplify essentially the formulae obtained, e.g. (23) and (30) below. Thirdly, the results seem to be of some interest to theoretical physicists (cf. [17]) what motivates additionally the use of the matrix language.

Let us pass to the matrix notation for the real structure constants (1) :

$$C_\alpha := [c^k_{j\alpha}], \qquad \overline{C}_\alpha := \kappa C^T_\alpha \kappa^{-1} \tag{13}$$

Then set :

$$C_\alpha = i\,\gamma_\alpha\, C_t, \quad t \quad \text{fixed}, \;\; \alpha = 1,\ldots,p; \quad \alpha \neq t, \tag{14}$$

LEMMA 1.— *Given a pseudo-eucliden Hurwitz pair (V,S), the matrices γ_α, introduced in (14), are uniquely determined, up to an orthogonal transformation $0 \in 0(n)$, by the conditions (14) and :*

$$C_t\overline{C}_t = n_{tt}\, I_n, \quad t \quad \text{fixed}, \; t \in \{1,\ldots,p\}, \tag{15}$$
$$\overline{\gamma}_\alpha = -\gamma_\alpha, \quad \operatorname{re}\gamma_\alpha = 0, \; \alpha = 1,\ldots,p; \;\; \alpha \neq t, \tag{16}$$
$$\{\gamma_\alpha, \gamma_\beta\} \equiv \gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 2\widehat{n}_{\alpha\beta} I_n, \quad \alpha,\beta = 1,\ldots,p; \;\; \alpha,\beta \neq t, \tag{17}$$
$$\widehat{n}_{\alpha\beta} := n_{\alpha\beta}/n_{tt}, \quad n_{tt} = 1 \quad \text{or} \quad -1, \tag{18}$$

where I_n stands for the identity $n \times n$-matrix.

Proof.— We rewrite the generalized Hurwitz condition (ii) in the coordinate form.

We have :

$$(a,a)_S\,(f,f)_V = a^\alpha a^\beta f^j\, f^k\, \eta_{\alpha\beta}\kappa_{jk},$$
$$(af,ag)_V = \tfrac{1}{2} a^\alpha a^\beta f^j f^k[(\epsilon_\alpha e_j, \epsilon_\beta e_k)_V + (\epsilon_\beta e_j, \epsilon_j, \epsilon_\alpha e_k)_V]$$
$$= \tfrac{1}{2} a^\alpha a^\beta f^j f^k[(e^r, \epsilon_\alpha e_j)_V \kappa_{rs}(e^s, \epsilon_\beta e_k)_V + (e^r, \epsilon_\beta e_j)_V \kappa_{rs}(e^s, \epsilon_\alpha e_k)_V$$

Hence, by (8), the property (ii) becomes :

$$a^\alpha b^\beta f^j\, f^k(c^r_{j\alpha}\kappa_{rs}c^s_{k\beta} + c^r_{j\beta}\kappa_{rs}c^s_{k\alpha}) = a^\alpha\, a^\beta\, f^j\, f^k\, \eta_{\alpha\beta}\,\kappa_{jk}$$

or, equivalently,

$$c^r_{j\alpha}\kappa_{rs}c^s_{k\beta} + c^r_{j\beta}\,\kappa_{rs}\,c^s_{k\alpha} = 2\,\eta_{\alpha\beta}\,\kappa!_{jk}.$$

In the matrix notation (13) the latter relation reads :

$$C_\alpha\overline{C}_\beta + C_\beta\overline{C}_\alpha = 2\,\eta_{\alpha\beta}\, I_\eta, \quad \alpha,\beta = 1,\ldots,\; \dim S. \tag{19}$$

Now we observe that the **R**-linearity of ϵ_α as an endomorphism of V, together with the relations (7) and (19), is equivalent to the conditions (i) and (ii) which are required for the chosen multiplication. Besides, (19) yields the invertibility of C_α. Let us fix an arbitrary integer $t \in \{1, \ldots, p\}$. Introducing the matrices γ_α, $\alpha \neq t$, determined by (14), we arrive at the system (15) – (17), where $\widehat{\eta}_{\alpha\beta} = \eta_{\alpha\beta}/\eta_{tt}$; $\eta_{\alpha\beta}$ being chosen diagonal as in (6). Since $\eta_{tt} = 1$ or -1, we get the system (14) – (18), equivalent to the original system of the equations (7), (13), (14), and (19). Since the Hurwitz pair (V, S) is given, the real structure constants (8) are uniquely defined, up to an orthogonal transformation $0 \in 0(n)$, and this, by (13) and (14), yields the uniqueness (in the same sense) of the matrices γ_α, $\alpha = 1, \ldots, p$; $\alpha \neq t$. Thus the proof is completed.

We see et once that γ_α are generators of a real CLIFFORD algebra. The precise result reads as follows :

LEMMA 2.— *Given a pseudo-euclidean Hurwitz pair (V, S), the matrices γ_α satisfying the conditions (14) – (18) are generators of a real Clifford algebra $C^{(r,s)}$ with (r, s) determined by the signature of $\widehat{n} := [\widehat{\eta}_{\alpha\beta}]$ and by (9). These generators are chosen in the (imaginary) Majorana representation. Conversely, any pseudo-euclidean Hurwitz pair (V, S) is a geometrical realization of a real Clifford algebra $C^{(r,s)}$, and the relationship is given by the conditions (14) – (18); (r, s) being determined by the signature of $\widehat{\eta}$ and by (9).*

Proof.— The first conclusion follows from Lemma 1, especially from the conditions (17) and (18), if we take into account (9). The second conclusion is a consequence of (15) and (16). The third one is established due to the uniqueness, up to an orthogonal transformation $0 \in 0(n)$, of γ_α, $\alpha = 1, \ldots, p$; $\alpha \neq t$, for any fixed $t \in \{1, \ldots, p\}$; the uniqueness being also asserted in the same lemma.

By Lemmas 1 and 2 it is natural to make the following

Assumption (A). Suppose that (V, S) is a pseudo-euclidean Hurwitz pair, for which we admit the notation (9) and $n = \dim V$. Let γ_α, $\alpha = 1, \ldots, p-1$, be the associated generators of the corresponding Clifford algebra $C^{(r,s)}$, whereas (ϵ_α) and (e_j) – some bases of S and V, respectively, restricted by the condition (6), the metrics η and κ being defined by (4).

Denote by $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, and $\mathbf{H}$ the real, complex and quaternion number fields, and let $\mathbf{M}(N, \mathbf{F})$ be the algebra of $N \times N$-matrices over $\mathbf{F}$. Let further :

$$\ell := \left[\frac{1}{2}m\right], \qquad m := r + s = p - 1, \tag{20}$$

where [ ] stands for the function "entier". We have

THEOREM 1.— *Let us take the assumption (A) and the notation (20). Then the following assertions hold.*

(I) For each pair (r,s) of non-negative integers r and s the algebra $C^{(r,s)}$ is isomorphic to :

$$\begin{array}{ll} \mathbf{M}(2^{\frac{1}{2}m}, \mathbf{R}) & \text{for } r+1-s \equiv 1,3 \pmod 8, \\ \mathbf{M}(2^{\left[\frac{1}{2}m\right]}, \mathbf{R}) + \mathbf{M}(2^{\left[\frac{1}{2}m\right]}, \mathbf{R}) & \qquad\qquad\quad 2 \pmod 8, \\ \mathbf{M}(2^{\left[\frac{1}{2}m\right]}, \mathbf{C}) & \qquad\qquad\quad 0,4 \pmod 8, \\ \mathbf{M}(2^{\frac{1}{2}m-1}, \mathbf{H}) & \qquad\qquad\quad 5,7 \pmod 8, \\ \mathbf{M}(2^{\left[\frac{1}{2}m-1\right]}, \mathbf{H}) + \mathbf{M}(2^{\left[\frac{1}{2}m-1\right]}, \mathbf{H}) & \qquad\qquad\quad 6 \pmod 8. \end{array} \tag{21}$$

(II) *The dimension of the representation space (21) is :*

$$2^{\ell} \qquad \text{for} \quad r+1-s \equiv 3,4,5,7,0,1 \pmod 8$$

and

$$2^{\ell+1} \qquad \text{for} \quad r+1-s \equiv 2,6 \pmod 8.$$

(III) *The dimension n of V equals :*

$$2^{\ell} \qquad \text{for} \quad r+1-s \equiv 7,0,1 \pmod 8$$

and

$$2^{\ell+1} \qquad \text{for} \quad r+1-s \equiv 2,3,4,5,6 \pmod 8.$$

(IV) *If $r - s \equiv 0,1$ (mod 8), one can construct both the real and the imaginary Majorana representation (shortly : RMR and IMR). If $r - s \equiv 2$ (mod 8), one can construct the RMR; its imaginary analogue IRM can only be constructed after doubling the dimension of the representation space (21). If $r - s \equiv 5,6,7$ (mod 8), one can construct the IMR; the RMR can only be constructed after doubling the dimension of (21). Finally, if $r-s \equiv 3,4$ (mod 8), the RMR and IMR can only be constructed after doubling the dimension of (21).*

Proof. — The reasoning, based on Lemmas 1 and 2, is completely analogous to that given in [6] in the euclidean case $s = 0$. The only important change is that we have to take into account the recurrence relations.

$$C^{(r,s)} \otimes C^{(1,1)} = C^{(r+1,s+1)}, \quad C^{(r,s)} \hat{\otimes} C^{(2,0)} = C^{(r+2,s)}, \quad C^{(r,s)} \hat{\otimes} C^{(0,2)} = C^{(r,s+2)}.$$

At the end of this section we illustrate the assertion (III) of Theorem 1 giving the table of $\log_2 n$ in terms of $r+1$ and S for $1 \leq r+1 \leq 9, 0 \leq s \leq 10$:

2. Classification according to the admissible systems (n, r, s)

Consider the sequence of matrices :

$$\check{\gamma}_\alpha = \gamma_\alpha, \quad \alpha = 1, \ldots, r; \quad \widehat{\gamma}_\beta = \gamma_{r+\beta}, \quad \beta = 1, \ldots, s \tag{22}$$

with γ_α, $\gamma_{r+\beta}$ as in the assumption (A) and, further, the matrices :

$$A = (-1)^r \check{\gamma}_1 \check{\gamma}_2 \ldots \check{\gamma}_r, \qquad B = (-1)^s \widehat{\gamma}_1 \widehat{\gamma}_2 \ldots \widehat{\gamma}_s; \tag{23}$$

if $s = 0$ we set $B = I_n$. One verifies directly their properties :

LEMMA 3.— *In contrast to the matrices (22) which are imaginary, the matrices (23) are real. Besides,*

$$A^T = (-1)^{\frac{1}{2}r(r+1)} A, \qquad B^T = (-1)^{\frac{1}{2}s(s-1)} B$$

and

$$\begin{aligned} A\check{\gamma}_\alpha &= (-1)^{r-1} \check{\gamma}_\alpha A, & B\check{\gamma}_\alpha &= (-1)^s \check{\gamma}_\alpha B, \\ A\widehat{\gamma}_\beta &= (-1)^r \widehat{\gamma}_\beta B; & B\widehat{\gamma}_\beta &= (-1)^{s-1} \widehat{\gamma}_\beta B. \end{aligned}$$

Assumption (B). Consider the particular cases of (22), where each irreducible representation of the Clifford algebra $C^{(r,s)}$ can be embedded in an irreducible representation of either

$C^{(r+1,s)}$ with generators $\gamma_1, \ldots, \gamma_{r+1+s}$

or

$C^{(s,r+1)}$ $\gamma_1, \ldots, \gamma_{r+s+1}$

or

$C^{(r+1,s)}$ and then of $C^{(r+2,s)}$ $\gamma_1, \ldots, \gamma_{r+2+s}$

or

$C^{(r,s+1)}$ and then of $C^{(r,s+2)}$ $\gamma_1, \ldots, \gamma_{r+s+2}$

Then the corresponding sequence of matrices (22) can naturally be modified as follows : either

$$\check{\gamma}_\alpha = \gamma_\alpha, \quad \alpha = 1, \ldots, r+1; \quad \widehat{\gamma}_\beta = \gamma_{r+1+\beta}, \ \beta = 1, \ldots, s$$

or

$$\check{\gamma}_\alpha = \gamma_\alpha, \quad \alpha = 1, \ldots, r; \quad \widehat{\gamma}_\beta = \gamma_{r+\beta}, \ \beta = 1, \ldots, s+1$$

or

$$\check{\gamma}_\alpha = \gamma_\alpha, \quad \alpha = 1, \ldots, r+2; \quad \widehat{\gamma}_\beta = \gamma_{r+2+\beta}, \ \beta = 1, \ldots, s$$

or

$$\check{\gamma}_\alpha = \gamma_\alpha, \quad \alpha = 1, \ldots, r; \quad \widehat{\gamma}_\beta = \gamma_{r+\beta}, \beta = 1, \ldots, s+2,$$

respectively.

Now we return to the general situation which includes the one covered by the assumption (B). We consider the finite sequence of matrix functions of r and s :

$$\begin{array}{ll} K(\delta) = A, & K(\delta i) = B, \\ K(2\delta) = iA\check{\gamma}_{r+1}, & K(2\delta i) = i\,B\,\widehat{\gamma}_{s+1}, \\ K(3\delta) = iA\widehat{\gamma}_{s+1}, & K(3\delta i) = iB\check{\gamma}_{r+1}, \\ K(4\delta) = A\check{\gamma}_{r+1}\,\check{\gamma}_{r+2}, & K(4\delta i) = B\widehat{\gamma}_{s+1}\,\widehat{\gamma}_{s+2} \end{array} \tag{24}$$

for $\delta = 1$ and -1. We may treat (24) as a function $z \to K(z)$ of a complex variable, defined for $z = m\delta$ and $z = m\delta i$. For the sake of convenience we take into consideration also the point $z = 0$, assuming that in this case the value $K(z)$ is undefined. We are interested in investigating the composition $K(a_{\rho\sigma})$ with (ρ, σ) as in (11), where $A = [a_{\rho\sigma}]$ is a complex 8×8-matrix, defined by the formulae :

$$\begin{array}{rll} a_{\rho\sigma} = & \delta \ \text{ for } & \rho = 3+\delta,\, 7+\delta,\, 1 \le \sigma \le 8,\, \delta = 1 \text{ or } -1; \\ & 2 \ \text{ for } & (\rho,\sigma) = (3,5),\, (3,6),\, (7,1),\, (7,2); \\ & -2 & \qquad\qquad (1,3),\, (1,4),\, (5,7),\, (5,8); \\ & 3 & \qquad\qquad (1,7),\, (5,3); \\ & -3 & \qquad\qquad (3,1),\, (7,5); \\ & 4\,i & \qquad\qquad (1,6),\, (5,2); \\ & -4\,i & \qquad\qquad (3,8),\, (7,4); \\ & 0 \ \text{ for } & (\rho,\sigma) = (3,2),\, (1,8) \\ & \text{and for } & \rho = 2+\delta,\, \sigma = 2+\delta,\ 3+\delta,\ 6+\delta,\ \delta = 1 \text{ or } -1 \\ & & \rho = 6+\delta,\ \sigma = 2+\delta,\, 5+\delta,\ 6+\delta,\ 7+\delta,\ \delta = 1 \text{ or } -1. \end{array} \tag{25}$$

One verifies directly the properties of A :

LEMMA 4.— *$a_{j+4,\,k+4} = a_{jk}$ and $a_{j,\,k+4} = a_{j+4,\,k}$ for $1 \le j,\ k < 4$. The above construction leads us to :*

THEOREM 2.— *Let us take the assumption (A). Then for each pair (r,s) there are two possible metrics :*

$$\kappa = K_1 \qquad or \qquad \kappa = K_2 \qquad at\ most \tag{26}$$

(zero, one or two possibilities). The functions K_1 and K_2 can be chosen to be expressible in terms of the (8, 8)-periodic function $(r,s) \to a_{r+1,s}$, defined

for $1 \leq r+1, s < 8$ *by (25) and having the properties listed in Lemma 4. Explictly,*

$$K_1(r,s) = K(a_{r+1,s})\,(r,s), \quad K_2(r,s) = K(i\,\overline{a}_{s,r+1})\,(r,s)$$

or, equivalently,

$$K_1(r,s) = K(a_{\rho\sigma})\,(r,s), \qquad K_2(r,s) = K(i\,\overline{a}_{\sigma\rho})\,(r,s), \tag{27}$$

where k is given by (24), $\overline{a}_{s,r+1}$ *denotes the complex conjugate of* $a_{s,r+1}$*, the indices* ρ *and* σ *are related to r and s by (11), k and* k_0 *appearing in (11) are integers, and* $k > 0$*. In particular,*

$$\begin{aligned} K_1(r,s) \text{ and } K_2(r,s) \text{ are defined whenever } a_{r+1,s} \neq 0, \\ \text{are undefined whenever } a_{r+1,s} = 0, \end{aligned} \tag{28}$$

and $\kappa^T = \delta\kappa$.

Proof. — Let us take the assumption (A) and consider a corresponding system (10) with the notation (9). It is interesting to notice that the pair (r,s) is not determined uniquely, yet this observation is of no importance to us now. By Lemma 2 the metric $\kappa = [\kappa_{jk}]$ in (10) has to be an element of the Clifford algebra $C^{(r,s)}$ with generators γ_α, $\alpha = 1,\ldots,p-1$, namely :

$$\kappa = a\,I_n + i\,b^\alpha\,\gamma_\alpha + c^{\alpha\beta}\,\gamma_\alpha\,\gamma_\beta + i\,d^{\alpha\beta\delta}\,\gamma_\alpha\,\gamma_\beta\,\gamma_\delta + \cdots, \tag{29}$$

where the coefficients a, b, $c^{\alpha\beta}$, $d^{\alpha\beta\delta}$, ... are real and antisymmetric with respect to the transposition of the indices α, β, δ, ...

By Lemma 1 the metric κ has to satisfy all the contraints (14) – (18) given in that lemma, in particular :

$$\kappa\gamma_\alpha^T\,\kappa^{-1} = -\gamma_\alpha, \qquad \alpha = 1,\ldots,p-1,$$

or, equivalently,

$$\kappa\check{\gamma}_\alpha = \check{\gamma}_\alpha\,\kappa, \quad \alpha = 1,\ldots,r; \quad \kappa\check{\gamma}_\beta = -\widehat{\gamma}_\beta\,\kappa, \quad \beta = 1,\ldots,s. \tag{30}$$

Now, we are going to consider, separately, eight cases

$$\rho - \sigma \equiv q \pmod 8, \qquad q = 0,1,\ldots,7,$$

ρ and σ being given in (11), where k and k_0 are integers, and $k \geq 0$. In each of them we have to derive all the admissible possibilities, combining (29) with (30).

It seems that the easiest case is when $q = 1$. We find, by a direct verification, that the only possible pseudo-euclidean Hurwitz pairs are those satisfying one of the following four sets of conditions :

$$\begin{array}{llccl} (\rho,\sigma) = & (1,8), & \kappa = B & \Rightarrow \delta = & 1; \\ \text{or} & (4,3), & " \quad A & " & 1; \\ \text{or} & (3,2), & " \quad B & " & -1; \\ \text{or} & (2,1), & " \quad A & " & -1, \end{array}$$

where $\Rightarrow$ abbreviates "what implies" and δ is defined by $\kappa^T = \delta\kappa$. In the calculations we utilize the formulae given in Lemma 3. By (11), within each set of the conditions we have still one additional point (ρ, δ), namely, in our sets we have the points $(1 + 4,\ 8 - 4) = (5,\ 4)$, $(8,\ 7)$, $(7,\ 6)$, and $(6,\ 5)$, respectively. Hence, by (24) and (25), we arrive at (26) with :

$$\begin{array}{lll} K_1 \text{ given in } (27); & (\rho,\sigma) = (4,\ 3),\ (8,\ 7),\ (2,\ 1),\ (6,\ 5); \\ K_2 \text{ given in } (27); & (\rho,\sigma) = (1,\ 8),\ (5,\ 4),\ (3,\ 2),\ (7,\ 6). \end{array} \tag{31}$$

Now we return our attention to the case $q = 7$, which is the most similar to the case $q = 1$. In analogy to that case we find :

$$\begin{array}{llccl} (\rho,\sigma) = & (7,\ 8), & \kappa = B & \Rightarrow \delta = & 1; \\ \text{or} & (8,\ 1), & " \quad A & " & 1; \\ \text{or} & (1,\ 2), & " \quad B & " & -1; \\ \text{or} & (2,\ 3), & " \quad A & " & -1. \end{array}$$

By (24) and (25) we arrive at (26) with :

$$\begin{array}{lll} K_1 \text{ given in } (27); & (\rho,\sigma) = (8,\ 1),\ (4,\ 5),\ (2,\ 3),\ (6,\ 7); \\ K_2 \text{ given in } (27); & (\rho,\sigma) = (7,\ 8),\ (3,\ 4),\ (1,\ 2),\ (5,\ 6). \end{array} \tag{32}$$

The case $q = 2$ is quite different. By the assertion (I) in Theorem 1 we have to take into account that each irreducible representation of $C^{(r,s)}$ can be in our case, owing to the congruence $r - (s + 1) \equiv 0 \pmod 8$, embedded in an irreducible representation of the Clifford algebra $C^{(r,s+1)}$ which is isomorphic to the corresponding matrix ring. Consequently, κ has to belong to $C^{(r,s+1)}$ and this is why we are led to the possibilities :

$$\begin{array}{llllllll} (\rho,\sigma) = & (2,8), & \kappa = A, & \delta = -1 & \text{or } \kappa = B & \Rightarrow \delta = & 1; \\ \text{or} & (4,\ 2), & A, & " \quad 1 & " \quad B & " & -1; \\ \text{or} & (5,\ 3), & iA\check{\gamma}_{s+1} & " \quad 1 & " \quad iB\check{\gamma}_{s+1} & " & 1; \\ \text{or} & (3,\ 1), & iA\check{\gamma}_{s+1} & " \quad -1 & " \quad iB\check{\gamma}_{s+1} & " & -1. \end{array}$$

We arrive at (26) with :

$$K_1,\ K_2 \text{ given by (27)};\quad (\rho,\sigma) = \begin{array}{l}(2,\ 8),\ (6,\ 4),\ (4,\ 2),\ (8,\ 6);\\ (5,\ 3),\ (1,\ 7),\ (3,\ 1),\ (7,\ 5).\end{array} \tag{33}$$

If $q = 6$, then, as in the preceding case, we observe, by the assertion (I) in Theorem 1, that each irreducible representation of $C^{(r,s)}$ can be embedded in an irreducible representation of $C^{(r+1,s)}$ wich is isomorphic to the corresponding matrix ring.

Consequently, κ has to belong to $C^{(r+1,s)}$, so we find

$$\begin{array}{llllllll}
(\rho,\sigma) = & (8,\ 2), & \kappa = A \Rightarrow \delta = & 1 & \text{or} & \kappa = B \Rightarrow & \delta = & -1;\\
\text{or} & (6,\ 8), & \text{"}\ A\ \text{"} & -1 & & \text{"}\ B\ \text{"} & & 1;\\
\text{or} & (7,1), & \text{"}\ iA\check{\gamma}_{r+1}\ \text{"} & 1 & & \text{"}\ iB\check{\gamma}_{r+1}\ \text{"} & & 1;\\
\text{or} & (1,\ 3), & \text{"}\ iA\check{\gamma}_{r+1}\ \text{"} & -1 & & \text{"}\ iB\check{\gamma}_{r+1}\ \text{"} & & -1.
\end{array}$$

We arrive at (26) with :

$$K_1,\ K_2 \text{ given by (27)};\quad (\rho,\sigma) = \begin{array}{l}(8,\ 2),\ (4,\ 6),\ (6,\ 8),\ (2,\ 4);\\ (7,\ 1),\ (3,\ 5),\ (1,\ 3),\ (5,\ 7)\end{array} \tag{34}$$

In the case $q = 3$ each irreducible representation of $C^{(r,s)}$ can be embedded in an irreducible representation of $C^{(r,s+1)}$, and then of $C^{(r,s+2)}$. We get :

$$\begin{array}{llllllll}
(\rho,\sigma) = & (6,\ 3), & \kappa\ = A \Rightarrow\ \ \delta = & -1 & \text{or} & \kappa = iB\check{\gamma}_{s+1} \Rightarrow \delta = & & 1;\\
\text{or} & (3,\ 8), & \text{"}\ B\widehat{\gamma}_{s+1}\widehat{\gamma}_{s+2}\ \text{"} & -1 & & \text{"}\ B\ \text{"} & & 1;\\
\text{or} & (4,\ 1), & \text{"}\ A\ \text{"} & 1 & & \text{"}\ iB\widehat{\gamma}_{s+1}\ \text{"} & & -1;\\
\text{or} & (5,\ 2), & \text{"}\ B\widehat{\gamma}_{s+1}\widehat{\gamma}_{s+2}\ \text{"} & 1 & & \text{"}\ B\ \text{"} & & -1.
\end{array}$$

We arrive at (26) with :

$$K_1,\ K_2 \text{ given by (27)};\quad (\rho,\sigma) = \begin{array}{l}(6,\ 3),\ (2,\ 7),\ (3,\ 8),\ (7,\ 4);\\ (4,\ 1),\ (8,\ 5),\ (5,\ 2),\ (1,\ 6).\end{array} \tag{35}$$

If $q = 5$, then, as in the preceding case, we observe that each irreducible representation of $C^{(r,s)}$ can be embedded in an irreducible representation of $C^{(r+1,s)}$, and then of $C^{(r+2,s)}$, so we find

$$\begin{array}{llllllll}
(\rho,\sigma) = & (7,2), & \kappa = iA\check{\gamma}_{r+1} \Rightarrow \delta = & 1 & \text{or} & \kappa\ = B \Rightarrow & \delta = & -1;\\
\text{or} & (8,3), & \text{"}\ A\ \text{"} & 1 & & \text{"}\ A\check{\gamma}_{r+1}\check{\gamma}_{r+2}\ \text{"} & & -1;\\
\text{or} & (5,8), & \text{"}\ iA\check{\gamma}_{r+1}\ \text{"} & -1 & & \text{"}\ B\ \text{"} & & 1;\\
\text{or} & (6,1), & \text{"}\ A\ \text{"} & -1 & & \text{"}\ A\check{\gamma}_{r+1}\check{\gamma}_{r+2}\ \text{"} & & 1.
\end{array}$$

We arrive at (26) with :

$$K_1,\ K_2 \text{ given by (27)};\quad \begin{aligned}(\rho,\sigma) = {} & (7, 2), (3, 6), (8, 3), (4, 7);\\ & (5, 8), (1, 4), (6, 1), (2, 5).\end{aligned} \tag{36}$$

In the case $q = 4$ we choose $C^{(r,s)}$ irreducible and, by the assertion (I) in Theorem 1, it is isomorphic to the matrix algebra $\mathbf{M}(2^{[\frac{1}{2}m]}, \mathbf{C})$, $m = r + s$, so the only possible pseudo-euclidean Hurwitz pairs are those satisfying one of the following two sets of conditions :

$$\begin{aligned} (\rho,\sigma) = (4, 8), &\quad \kappa = A \text{ or } \kappa = B \Rightarrow \delta = 1;\\ \text{or} \qquad (6, 2), &\quad \kappa = A \text{ or } \kappa = B \Rightarrow \delta = -1.\end{aligned}$$

We arrive at (26) with :

$$K_1,\ K_2 \text{ given by (27)};\quad (\rho,\sigma) = (4, 8),\ (8, 4);\ (6, 2),\ (2, 6). \tag{37}$$

If $q = 0$, arguing exactly as in the preceding case, we find

$$\begin{aligned} (\rho,\sigma) = (2, 2), &\quad \kappa = A \text{ or } \kappa = B \Rightarrow \delta = -1;\\ \text{or} \qquad (8, 8), &\quad \kappa = A \text{ or } \kappa = B \Rightarrow \delta = 1.\end{aligned}$$

We arrive at (26) with :

$$K_1,\ K_2 \text{ given by (27)};\quad (\rho,\sigma) = (2, 2),\ (6, 6),\ (8, 8),\ (4, 4). \tag{38}$$

Combining the formulae (31) – (38) we obtain (27) for $1 \le \rho, \sigma \le 8$, except of those (ρ, σ) for which $a_{\rho\sigma} = 0$. This means that :

$$\begin{aligned} K_1(\rho,\sigma) \text{ is undefined for } (\rho,\sigma) = {} & (1, 1), (1, 2), (1, 5), (1, 8),\\ & (3, 2), (3, 3), (3, 4), (3, 7),\\ & (5, 1), (5, 4), (5, 5), (5, 6),\\ & (7, 3), (7, 6), (7, 7), (7, 8);\\ K_2(\rho,\sigma) \text{ is undefined for } (\rho,\sigma) = {} & (1, 1), (1, 5), (2, 1), (2, 3),\\ & (3, 3), (3, 7), (4, 3), (4, 5),\\ & (5, 1), (5, 5), (6, 5), (6, 7),\\ & (7, 3), (7, 7), (8, 1), (8, 7).\end{aligned}$$

Therefore, if $(\rho, \sigma) = (1, 1), (3, 3), (5, 5), (7, 7), (5, 1), (1, 5), (7, 3), (3, 7)$, the metric $\kappa(\rho + 1, \sigma)$ does not exist. For the other listed pairs (ρ, σ) we have only one solution : $\kappa = K_1$ or $\kappa = K_2$. Otherwise we have two solutions : $\kappa = K_1$ and $\kappa = K_2$. Thus, by the $(8, 8)$-periodicity of (r, s)

$a_{r+1,s}$, we have also (28), what completes the proof since the conclusion $\kappa^T = \delta\kappa$ is obvious.

3. Classification according to the admissible systems $(n,\, r,\, s,\, \kappa)$

The formulae (26) for κ in an arbitrary basis (e_j) of V appear to be pretty complicated because of an involved character of (27), (24), and (23). Therefore it is natural to simplify these formulae by choosing a suitable basis.

In order to shorten the list of cases we replace the fundamental square (11) of indices $(n,\, \rho,\, \sigma) = (n(\rho,\, \sigma),\, \rho,\, \sigma)$ by

$$\begin{array}{ll} & (n,\, \mu, \nu),\ \mu = r - 4(k + k_0) + 1, \quad \nu = s - 4(k - k_0), \quad 1 \le \mu, \nu \le 8, \\ & \qquad (\mu,\, \nu),\ (\nu,\, \mu) \neq (1,\, 6),\, (1,\, 7),\, (1,\, 8),\, (2,\, 6),\, (4,\, 8) \\ \text{and} & \qquad (\mu,\, \nu),\ (\nu,\, \mu) = (9,\, 6),\, (9,\, 7),\, (1,\, 0),\, (10,\, 6),\, (4,\, 0). \end{array} \tag{39}$$

Then we replace (25) by

$$\begin{array}{rll} b_{\mu\nu} = & 1 \text{ for } (\mu,\nu) = & (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (4,7), \\ & & (0,1), (8,2), (8,3), (0,4), (8,5), (8,6), (8,7); \\ & -1 & (2,2), (6,10); \\ & 2 & (3,6), (7,2); \\ & -2 & (1,3), (5,7); \\ & 3 & (9,7), (5,3); \\ & 4 & (2,5), (6,9); \\ & i & (1,4), (1,0), (2,4), (2,8), (3,4), (3,8), (4,0) \\ & & (5,4), (5,8), (6,4), (6,8), (7,4), (7,8), (8,8) \\ & -i & (10,6), (6,6); \\ b_{\mu\nu} = & 2\,i \text{ for } (\mu,\nu) = & (2,7), (6,3); \\ & -2\,i & (3,1), (7,5); \\ & 3\,i & (3,5), (7,9); \\ & 4\,i & (9,6), (5,2); \\ & 0 & (1,1), (1,2), (1,5), (2,1), (2,3), (3,2), (3,3); \\ & & (3,7), (5,1), (5,5), (5,6), (6,5), (6,7), (7,3); \\ & & (7,6), (7,7); \end{array}$$

and denote the system of all $b_{\mu\nu}$ by $B := (b_{\mu\nu})$.

Finally, we consider the system $-B := (-b_{\mu-2,\nu-2})$ with

$$\begin{array}{llll} b_{\mu-2,\nu-2} := b_{\mu+6,\nu+6} & \text{if} & 0 \leq \mu \leq 2,\ 0 \leq \nu \leq 2; & \\ \quad " \qquad\quad b_{\mu+6,\nu-2} & " & 1 \leq \mu \leq 2,\ 3 \leq \nu \leq 8,\ (\mu,\nu) \neq (2,3) & \\ & & \text{and } (\mu,\nu) = (0,4); & (41) \\ \quad " \qquad\quad b_{\mu-2,\nu+6} & " & 3 \leq \mu \leq 8,\ 1 \leq \nu \leq 2,\ (\mu,\nu) \neq (3,2) & \\ & & \text{and } (\mu,\nu) = (4,0). & \end{array}$$

One verifies directly the properties of B and $-B = ((-b)_{\mu\nu})$:

LEMMA 5.— *$b_{j+4,k+4} = b_{jk}$ for $1 \leq j,\ k \leq 4$ and $(j,k) = (0,1), (1,0), (2,5), (3,5), (4,0), (5,2), (5,3)$.*

$b_{j,k+4} = b_{j+4,k}$ for $(j,k) = (1,1), (2,3), (2,4), (3,2), (3,3), (3,4), (4,2), (4,3), (6,6)$.

and analogous formulae hold for $(-b)_{\mu\nu}$, where μ,ν are as in (39).

The above construction leads us to :

THEOREM 3.— *Let us take the assumption (A). Then the following assertions hold.*

(V) For each pair (r,s) the functions K_1 and K_2 in (26) can be chosen to be expressible in terms of the (8, 8)-period function $(r,s) \to b_{r+1,s}$ defined for

$$\begin{array}{r} 1 \leq r+1, s \leq 8,\ (r+1,s) \neq (1,6), (1,7), (1,8), (4,8), (6,1), (8,1) \\ \text{and } (r+1,s) = (9,6), (9,7), (9,8), (4,0), (6,9), (8,9) \end{array}$$

by (40) and having the properties listed in Lemma 5. Explicitly,

$$K_1(r,s) = K(b_{\mu\nu})(r,s),\ K_2(r,s) = K(-b_{\mu-2,\nu-2})(r,s), \tag{42}$$

where K is given by (24), the indices μ and ν are related to r and s by (39), $-b_{\mu-2,\nu-2}$ outside the fundamental set of indices specified in (39) has to be understood as in (41), k and k_0 appearing in (39) are indices, and $k \geq 0$. In particular,

$$\begin{array}{llll} K_1(r,s) \text{ is defined whenever} & b_{r+1,s} \neq 0, & \\ \quad " \qquad \text{undefined} \qquad " & b_{r+1,s} = 0, & \\ K_2(r,s) \text{ is defined} \qquad " & b_{r-1,s-2} \neq 0, & (43) \\ \quad " \qquad \text{undefined} \qquad " & b_{r-1,s-2} = 0, & \end{array}$$

and $\kappa^T = \delta\kappa$.

(VI) The basis (e_j) of V can be chosen so that we have one of three possibilities listed in (12) exclusively, where I_n and $I_{\frac{1}{2}n}$ stand for the identity $n \times n-$ and $\frac{1}{2}n \times \frac{1}{2}n$-matrices, respectively. The second possibility occurs when

$$s = 0 \qquad \textit{and, simultaneously,} \qquad \kappa = K_1. \tag{44}$$

The first possibility occurs when

$$\begin{aligned} s \neq 0, \quad (\mu, \nu) \neq (1, 3), (2, 2), (3, 1), (10, 6), & \\ (5, 7), (6, 6), (7, 5), (6, 10), & \qquad \kappa = K_1 \end{aligned} \tag{45}$$

and when

$$\begin{aligned} (\mu, \nu) = (7, 1), (8, 8), (9, 7), (8, 4), & \\ (3, 5), (4, 4), (5, 3), (4, 0), & \qquad \kappa = K_2 \end{aligned} \tag{46}$$

The third possibility occurs when

$$\begin{aligned} (\mu, \nu) = (1, 3), (2, 2), (3, 1), (10, 6), & \\ (5, 7), (6, 6), (7, 5), (6, 10), & \qquad \kappa = K_1 \end{aligned} \tag{47}$$

and when

$$\begin{aligned} (\mu, \nu) \neq (7, 1), (8, 8), (9, 7), (8, 4), & \\ (3, 5), (4, 4), (5, 3), (4, 0), & \qquad \kappa = K_2 \end{aligned} \tag{48}$$

Remark 1.— It is clear that the assertion (VI) can be reformulated in terms of $a_{\rho\sigma}$, so that the introduction of $b_{\mu\nu}$ is not absolutely necessary. However, it simplifies the classification (44) – (48) considerably. In a subsequent paper [11] we are going to rearrange the proofs of Theorems 2 and 3 so that one can see that the choice of $[a_{\rho\sigma}]$ and $(b_{\mu\nu})$ si really optimal in a suitable sense. This is necessary for discovering several symmetries of the admissible systems (n, r, s, κ), but of course then the corresponding proofs are longer.

Proof of Theorem 3 – The first step. Let us take the assumption (A) and consider the corresponding system (10) with the notation (9). The assertion (V) is a direct consequence of Theorem 3. In order to prove the assertion (VI) we observe that the formula (24) can be written as

$$K(z) = A(z)\, B(z), \quad z = m\delta,\, m\delta i; \quad m = 1,2,3,4; \quad \delta = 1, -1 \tag{49}$$

with $A(z)$ and $B(z)$ defined as follows.

Given $z = m\delta$ or $-m\delta$, consider the sequence of matrices

$$\check{\gamma}_\alpha = \gamma_\alpha, \quad \alpha = 1, \ldots, r(z); \quad \widehat{\gamma}_\beta = \gamma_{r(z)+\beta}, \quad \beta = 1, \ldots, s(z)$$

with γ_α, $\gamma_{r(z)+\beta}$ as in the assumption (A) or (B) according to the case. This means that,

$$\begin{array}{llll} r(\delta) = r, & s(\delta) = 0, & r(\delta i) = 0, & s(\delta i) = s, \\ r(2\delta) = r+1, & s(2\delta) = 0, & r(2\delta i) = 0, & s(2\delta i) = s = 1, \\ r(3\delta) = r, & s(3\delta) = 1, & r(3\delta i) = 1, & s(3\delta i) = s, \\ r(4\delta) = 0, & s(4\delta) = s+2, & r(4\delta i) = r+2, & s(4\delta i) = 0 \end{array} \tag{50}$$

for $\delta = 1$ and -1. Then we can define the matrix functions

$$A(z) = (-1)^{r(z)} \check{\gamma}_1 \check{\gamma}_2 \ldots \check{\gamma}_{r(z)}, \quad B(z) = (-1)^{s(z)} \widehat{\gamma}_1 \widehat{\gamma}_2 \ldots \widehat{\gamma}_{s(z)}; \tag{51}$$

if $s(z) = 0$, we set $B(z) = I_n$. By Lemma 3 these matrix functions have the following properties :

LEMMA 4.— *The matrix functions (51) are real. Besides,*

$$A^T(z) = (-1)^{\frac{1}{2}r(z)[r(z)+1]} A(z), \quad B^T(z) = (-1)^{\frac{1}{2}s(z)[s(z)-1]} B(z)$$

and

$$\begin{array}{lll} A(z)\check{\gamma}_\alpha & = & (-1)^{r(z)-1} \check{\gamma}_\alpha A(z), \quad B(z)\check{\gamma}_\alpha = (-1)^{s(z)} \check{\gamma}_\alpha B(z), \\ A(z)\widehat{\gamma}_\beta & = & (-1)^{r(z)} \widehat{\gamma}_\beta B(z), \quad B(z)\widehat{\gamma}_\beta = (-1)^{s(z)-1} \widehat{\gamma}_\beta B(z). \end{array}$$

Proof of Theorem 3 - The second step By Lemma 4, the formulae (26), (42), and (49) yield

$$\kappa^T = \kappa \Leftrightarrow \kappa^2 = I_n, \qquad \kappa^T = -\kappa \Leftrightarrow \kappa^2 = -I_n,$$

that is,

$$\delta = 1 \Leftrightarrow \kappa^2 = I_n \qquad \delta = -1 \Leftrightarrow \kappa^2 = -I_n,$$

being determined by (24) and (25).

Suppose first that $\delta = -1$. Then $\kappa^2 = -I_n$ implies that the basis (e_j) of V can be chosen so that we have the third possibility in (12) : $\kappa = J_n$. Since in the case concerned, by (26), we have $\kappa = K(m)$ or $K(m\,i)$, $1 \leq m \leq 4$, then when applying Theorem 2 we obtain (47) and (48).

Then, suppose that $\delta = 1$. If $r(z) + s(z) = 0$, we get $r = s = 0$, and hence $\kappa = I_n$, that is, the second possibility in (12). If $r(z) + s(z)$ is even and positive, we obtain $Tr\kappa = -Tr\kappa = 0$; therefore the basis (e_j) of V can be chosen so that we have the first possibility in (12) : $\kappa = H_n$.

Suppose, in turn, that $\delta = 1$, $r(z) + s(z)$ is odd, and in the basis sector of an irreducible representation of the Clifford algebra involved, i.e.

$$C^{(r,s)},\ C^{(r+1,s)},\ C^{(r,s+1)},\ C^{(r+2,s)},\quad \text{or}\quad C^{(r,s+2)},$$

there is a generator $\gamma_q \neq \gamma_\alpha$ being a factor in the product on the right-hand side of (49). As a consequence, we get

$$Tr\kappa = \eta_{qq}\ Tr\kappa\gamma_q^2 = \eta_{qq}\gamma_q\kappa\gamma_q = -\eta_{qq}\ Tr\kappa\gamma_q^2 = -Tr\kappa = 0,$$

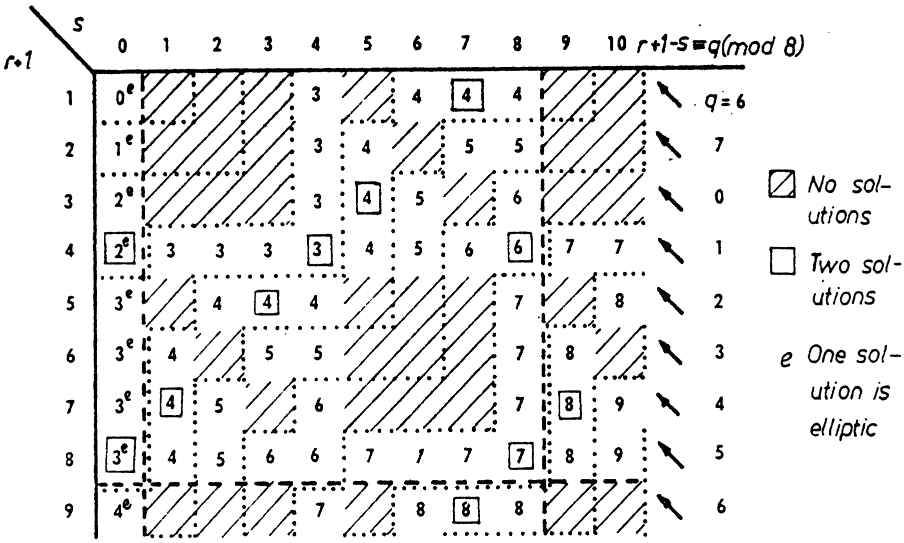
and hence still the first possibility in (12) : $\kappa = H_n$. When applying Theorem 2 again we obtain (45) and (46).

The only cases left are when

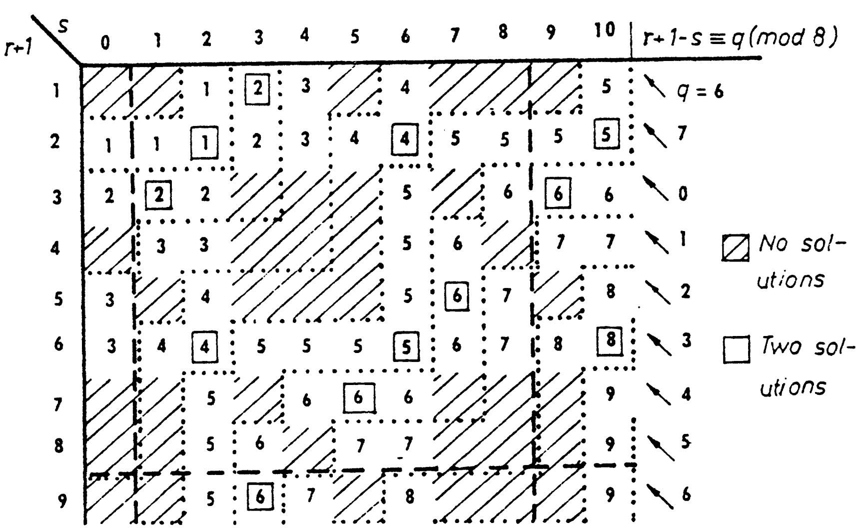
$$r > 0, \qquad s = 0 \qquad \text{and, simultaneously,} \qquad \kappa = K_1.$$

They are completely analogous to the already discussed case $r - s = 0$, when we have the sole solution $\kappa = K_1$. In all these cases the basis (e_j) of V can be chosen so that we have $\kappa = I_n$, that is, the second possibility in (12) occurs when (44) holds. Thus we have also proved the assertion (VI), as desired.

At the end of this section we illustrate the assertion (VI) of Theorem 3, rearranging Tab. 1 so that the cases (44) (elliptic), (45) – (46) (hyperbolic), and (47) – (48) (symplectic) are indicated separately.



Tab.2. $(r+1,s) \mapsto \log_2 n$.
The elliptic and hyperbolic cases



Tab. 3. $(r+1,s)$ — $\log_2 n$.
The sympletic case

4. The duality pairing involving complex and hermitian structures

Finally, when considering the metric κ of V, we observe that the symplectic case (47) – (48) gives rise to the standard complex structure J_n in (12), and – in a suitable basis (e_j) of V – to an arbitrary preassumed complex structure $J = RJ_nR^T$ obtained from J_n by transformations belonging to the subspace $0(n)/U\left(\frac{1}{2}n\right)$ of $0(n)$. Moreover, the totality of solutions for κ determines the hermitian structures $A = [a_{\rho\sigma}]$ and $A^+ = [i\, a_{\sigma\rho}]$, defined by (25), which – via the transformations $A \to B$ and $A^+ \to -B$, defined by (29) – (31) – give rise to a natural duality pairing of two *types of the metric* κ :

symmetric (elliptic or hyperbolic) – the cases (44) – (46)

and

antisymmetric (symplectic) – the cases (47) – (48).

Given a set E of 64 pairs (μ, ν) of indices $\mu, \nu = 0, 1, 2, \ldots$, we consider, for a fixed $(\mu, \nu) \in E$, *five elementary transformations* (symmetry with respect to $\mu = \nu$ and four translations) :

$$\begin{array}{ll} S(\mu, \nu) = (\nu, \mu); & \\ T_-^+(\mu, \nu) = (\mu + 1, \nu - 1), & T_+^-(\mu, \nu) = (\mu - 1, \nu + 1); \\ T_+^+(\mu, \nu) = (\mu + 2, \nu + 2), & T_-^-(\mu, \nu) = (\mu - 2, \nu - 2). \end{array} \tag{52}$$

We confine ourselves to the following *admissible* image pairs :

$$\begin{array}{lll} S(\mu, \nu) \in E & \text{for } S, & T_+^+(\mu,\nu) \in E \\ & \text{for} & T_+^+, \quad T_-^-(\mu, \nu) \in E \text{ for } T_-^-, \\ T_-^+(\mu, \nu) \in E & \text{and} & n[T_-^+(\mu,\nu)] = n(\mu, \nu) \text{ for } T_-^+, \\ T_+^-(\mu, \nu) \in E & \text{and} & n[T_+^-(\mu, \nu)] = n(\mu, \nu)] \text{ for } T_+^-. \end{array}$$

By Theorem 1 (cf. Tab. 1) we get

LEMMA 5.— $n[S(\mu, \nu)] = n(\mu, \nu)$, $n[T_+^+(\mu, \nu)] = n(\mu, \nu) + 2$, $n[T_-^-(\mu, \nu)] = n(\mu, \nu) - 2$, *provided that the corresponding image pair is admissible.*

Because of the dependence on the conditions involving $n = \dim V$, the above transformations are strictly related to matrix transformations of the form

$$K_1(r,s) = K(C_{\mu\nu})(r,s), \quad K_2(r,s) = K(d_{\mu\nu})(r,s),$$

where K_1 and K_2 have to be as in (26), and K is given by (24). Precisely, each of the transformations $(\mu, \nu) \to (\mu', \nu')$, given by (52), generates the transformation

$$K(C_{\mu\nu})(\mu, \nu) \mapsto K(d_{\mu\nu})(\mu, \nu)$$

which may provide, in principle, the following types of pairings :

$$s\,a = \text{ symmetric } \mapsto \text{antisymmetric } a\,s,\ s\,s,\ a\,a.$$

Leaving a detailed study of these duality pairings to a subsequent paper [11], here we confine ourselves to a specific choice only.

Namely, we take for E the set of pairs (μ, ν) of indices as in (39), $c_{\mu\nu} = b_{\mu\nu}$, $d_{\mu\nu} = -b_{\mu-2,\nu-2}$ with the convention (41), and define a global transformation $(\mu, \nu) \mapsto (\mu', \nu')$, the choice of which is motivated by a *principle of triality*, formulated below in Lemma 6 :

$$\text{1 st triad : } \begin{matrix}(4,1)\\(4,2)\\(5,2)\end{matrix} \xrightarrow{S} \begin{matrix}(1,4),\\(2,4),\\(2,5);\end{matrix} \qquad \text{2nd triad : } \begin{matrix}(1,4)\\(2,4)\\(2,5)\end{matrix} \xrightarrow{S} \begin{matrix}(4,1)\\(4,2),\\(5,2);\end{matrix}$$

$$\text{3nd triad : } \begin{matrix}(6,3)\\(6,4)\\(7,4)\end{matrix} \xrightarrow{S} \begin{matrix}(3,6),\\(4,6),\\(4,7);\end{matrix} \qquad \text{4 th triad : } \begin{matrix}(3,6)\\(4,6)\\(4,7)\end{matrix} \xrightarrow{S} \begin{matrix}(6,3);\\(6,4),\\(7,4);\end{matrix}$$

$$\text{5th triad : } \begin{matrix}(8,5)\\(8,6)\\(9,6)\end{matrix} \xrightarrow{S} \begin{matrix}(5,8),\\(6,8),\\(6,9);\end{matrix} \qquad \text{6th triad : } \begin{matrix}(5,8)\\(6,8)\\(6,9)\end{matrix} \xrightarrow{S} \begin{matrix}(8,5);\\(8,6)\\(9,6);\end{matrix}$$

$$\text{7th triad : } \begin{matrix}(2,7)\\(2,8)\\(3,8)\end{matrix} \xrightarrow{S} \begin{matrix}(7,2),\\(8,2),\\(8,3);\end{matrix} \qquad \text{8th triad : } \begin{matrix}(7,2)\\(8,2)\\(8,3)\end{matrix} \xrightarrow{S} \begin{matrix}(2,7),\\(2,8),\\(3,8);\end{matrix}$$

$$\begin{array}{lcl}
 & (3,1) \xrightarrow{T^-_+S} (0,4), & \qquad (0,4) \xrightarrow{ST^+_-} (3,1), \\
\text{9 th triad}: & (2,2) \xrightarrow{T^+_+S} (4,4), & \text{10th triad}: (4,4) \xrightarrow{ST^-_-} (2,2), \\
 & (1,3) \xrightarrow{T^+_-S} (4,0); & \qquad (4,0) \xrightarrow{ST^-_+} (1,3);
\end{array}$$

$$\begin{array}{lcl}
 & (9,7) \xrightarrow{ST^+_-} (6,10), & \qquad (6,10) \xrightarrow{T^-_+S} (9,7), \\
\text{11th triad}: & (8,8) \xrightarrow{ST^-_-} (6,6), & \text{12th triad}: (6,6) \xrightarrow{T^+_+S} (8,8), \\
 & (7,9) \xrightarrow{ST^-_+} (10,6); & \qquad (10,6) \xrightarrow{T^+_-S} (7,9);
\end{array}$$

$$\begin{array}{lcccl}
 & (4,3) & & (1,2), & \\
\text{13th triad}: & (7,5) & \xrightarrow{T^-_-S} & (3,5), & \\
 & (8,7) & & (5,6); &
\end{array}
\qquad
\begin{array}{lccc}
 & (3,4) & & (2,1), \\
\text{14th triad}: & (5,7) & \xrightarrow{ST^-_-} & (5,3), \\
 & (7,8) & & (6,5);
\end{array}$$

$$\begin{array}{lccc}
 & (5,4) & & (6,7), \\
\text{15th triad}: & (5,3) & \xrightarrow{ST^+_+} & (5,3), \\
 & (1,0) & & (2,3);
\end{array}
\qquad
\begin{array}{lccc}
 & (4,5) & & (7,6); \\
\text{16th triad}: & (3,5) & \xrightarrow{T^+_+S} & (7,5); \\
 & (0,1) & & (3,2).
\end{array}$$

The above triads correspond to the triples of pairs $(\mu,\,\nu)=(\mu_j,\,\nu_j)$, $j=1,2,3$, at which $b_{\mu\nu}\neq 0$; we will call them *regular triads*. Then we extend the mapping in a natural way to those $(\mu,\,\nu)$ at which $b_{\mu\nu}=0$, that is, by (42) and (43), $K(b_{\mu\nu})$ is undefined : to *singular triads* and *singular monads*.

1) Four singular triads :

$$\begin{array}{lccc}
 & (2,1) & & (3,4), \\
\text{17th triad}: & (1,1) & \xrightarrow{T^+_+S} & (3,3), \\
 & (1,2) & & (4,3);
\end{array}
\qquad
\begin{array}{lccc}
 & (2,3) & & (1,0); \\
\text{18th triad}: & (3,3) & \xrightarrow{ST^-_-} & (1,1); \\
 & (3,2) & & (0,1);
\end{array}$$

$$\begin{array}{lccc}
 & (6,5) & & (7,8), \\
\text{19th triad}: & (5,5) & \xrightarrow{ST^+_+} & (7,7), \\
 & (5,6) & & (8,7);
\end{array}
\qquad
\begin{array}{lccc}
 & (6,7) & & (5,4); \\
\text{20th triad}: & (7,7) & \xrightarrow{T^-_-S} & (5,5); \\
 & (7,6) & & (4,5).
\end{array}$$

2) Four singular monads (Nos 21 – 24) :

$$(5,1)\xrightarrow{S}(1,5);\quad (1,5)\xrightarrow{S}(5,1);\quad (7,3)\xrightarrow{S}(3,7);\quad (3,7)\xrightarrow{S}(7,3).$$

A regular triad

$$((\mu j,\,\nu j)\mapsto(\mu' j,\,\nu' j): j=1,2,3),\quad (\mu,\,\nu)\quad \text{as in (39)} \tag{53}$$

is said to be : 1° *dimension-preserving* if

$$n(\mu_j', \nu_j') - n(\mu_j, \nu_j) = 0 \qquad \text{for} \qquad j = 1,2,3;$$

2° *dimension-changing of order* $(\delta_1, \delta_2, \delta_3)$ if

$$n(\mu_j', \nu_j') - n(\mu_j, \nu_j) = \delta_j \qquad \text{for} \qquad j = 1,2,3.$$

A dimension-preserving regular triad (53) is said to be of *reduced dimension* (n_1, n_2, n_3) if

$$n_j = n(\mu_j, \nu_j) - \min\{n(\mu_k, \nu_k) : k = 1,2,3\}.$$

A straightforward calculation, based on Theorem 1 (cf. Tab. 1) and Lemma 5, yields.

LEMMA 6.— *(the principle of triality). Each regular triad is type-changing. The triads 1-8 are dimension-preserving of reduced dimension (0, 0, 1). The triads 9-16 are dimension-changing of order* $\rho(k)$, $k = 9, \ldots, 16$, *with*

$$\begin{array}{llll} \rho(k) = (0,2,0) \quad \text{for} & k = 9, & \rho(k) = (0,-2,0) \quad \text{for} & k = 1, \\ \phantom{\rho(k) = {}}(0,-2,0) & \phantom{k = {}}11, & \phantom{\rho(k) = {}}(0,2,0) & \phantom{k = {}}12, \\ \phantom{\rho(k) = {}}(-2,-2,-2) & \phantom{k = {}}13, & \phantom{\rho(k) = {}}(2,2,2) & \phantom{k = {}}14, \\ \phantom{\rho(k) = {}}(2,2,2) & \phantom{k = {}}15, & \phantom{\rho(k) = {}}(-2,-2,-2) & \phantom{k = {}}16. \end{array} \tag{54}$$

The above triality is somehow similar to E. Cartan's triality (cf. [4], pp. 119-120, and [22], pp. 435-462) : we have 16 triples of objects and each member of a fixed triple plays almost the same role. This is still seen better from the following direct consequence of Lemme 6 and Theorem 3, which is a counterpart of the principle of triality for the *matrix triads*

$$(K(b_{\mu_j \nu_j})(\mu_j, \nu_j) \mapsto K(-b_{\mu_j'-2, \nu_j'-2})(\mu_j', \nu_j') : j = 1, 2, 3) \tag{55}$$

generated by (53) :

THEOREM 4.— *Let us take the assumption* (A) *and consider the matrix functions (42), where* K *is given by (24),* $b_{\mu\nu}$ *are defined by (40),* $-b_{\mu-2,\nu-2}$ *satisfy the convention (41),* (μ, ν) *are as in (39),* k *and* k_0 *appearing in (39) are integers, and* $k \geq 0$. *Besides, consider 16 regular matrix triads (55) generated by the corresponding 16 regular triads (53). Then each of them* *is type-changing. Moreover, the matrix triads 1-8 are dimension-preserving of reduced dimension (0, 0, 1). Contrary to those triads, the matrix 9-16 are dimension-changing of order $\rho(k)$, $k = 9, \ldots, 16$, with $\rho(k)$ given by (54).*

Needless to say that the definitions of dimension-preserving and dimension-changing regular matrix triads of order $(\delta_1, \delta_2, \delta_3)$, and of the reduced dimension of the first ones, are exactly the same as for the initial, usual triads.

Remark 2.— Our procedure of duality pairing can be extended in a natural way to all pairs $(r+1, s) = 0, 1, 2, \ldots$, and even to the pairs with $r = 0$. It is easily seen that our procedure is unextendable for

$$r + 1 - s = r + 1 + s = 8k - 1 \qquad \text{or} \qquad 8k,\ k = 1, 2, \ldots$$

Theorem 4 gives an example of the required type-changing duality pairing of the symmetric and antisymmetric pseudo-euclidean real vector spaces being higher-dimensional memebers of some pseudo-euclidean Hurwitz pairs. The pairing involves in an explicit way the complex and hermitian structures. A further study, including the full motivation for this choice and a description of other possible choices will be given in [11].

Références

[1] ADEM (J.). — Construction of some normed maps, *Bol. Soc. Mat. Mexicana (2)*, t. **20**, 1975, p. 59-75.

[2] . — On maximal sets of anticommuting matrices, *Ibid. (2)*, t. **23**, 1978, p. 61-67.

[3] ADEM (J.). — On the Hurwitz problem over an arbitrary field I-II Ibid. (2), *25* 1980, *29-51 and (2)* , t. **26**, 1986, p. 29-41.

[4] CARTAN (E.). — The theory of spinors, *transl. from French Paris : Hermann* 1966.

[5] CHEVALLEY (C.). — The algebraic theory of spinors. *Columbia : Columbia Univ. Press* 1954.

[6] CRUMEYROLLE (A.). — Bilinéarité et géométrie affine attachées aux espaces de spineurs complexes, minkowskiens ou autres, *Ann. Inst. Henri Poincaré*, t. **34**, 1981, p. 351-372.

[7] Construction d'algèbres de Lie graduées orthosymplectiques et conformosymplectiques minkowskiennes. — *in : Seminar on deformations, Proceeding, Łódź-Warsaw* 1982◁84, *Lecture Notes in Math. 1165, Berlin-Heidelberg-New York-Tokyo : Springer* 1985, p. 52-83.

[8] FROHLICH (A.), MCEVETT (A.). — Forms over rings with involution, *J. Algebra*, t. **12**, 1969, p. 79-104.

[9] GERAMITA (A.V.), SEBERRY (J.). — Orthogonal designs. *Lecture Notes in Pure and Applied Math. 45, New York-Basel : Marcel Dekker* 1979.

[10] HESTENES (D.). — A unified laguage for mathematics and physics, *in : Clifford algebras and their applications in mathematical physics, Proceeding, Canterbury* 1985 *Dordrecht : Reidel* 1986 p. 1-23.

[11] HURWITZ (A.). — Uber die Komposition der quadratischen Formen. *Math. Ann. 88* 1923, p. 1-25; *reprinted in : A. Hurwitz, Mathematische Werke II. Basel Birkhäuser* 1933 p. 641-666.

[12] KALINA (J.), ŁAWRYNOWICZ (J.), SUZUKI (O.). — A field equation defined by a Hurwitz pair, *in Proc. of the 13th Winter School on Abstract Analysis, Srni* 1985. *Suppl. Rend. Circ. Mat. Palermo (2) 9* 1985 p. 117-128.

[13] . — Partial differential equations connected with some Clifford structures and the related quasiconformal mappings, *in Proc. Conf. "Metodi di Analisi Reale nelle Equaziono alle Derivate Parziali" Cagliari* 1985, to appear.

[14] ŁAWRYNOWICZ (J.), REMBIELINSKI (J.). — Hurwitz pairs equipped with complex structures, *in : Seminar on deformations, Proceedings, Łódź-Warsaw—1982—84— Lecture Notes in Math. 1165, Berlin-Heidelberg-New-York-Tokyo : Springer* 1985, p. 184-195.

[15] . — Supercomplex vector spaces and spontaneous symmetric breaking, *in : Seminari di Geometria* 1984, *Bologna : Università di Bologna* 1985, p. 131-154.

[16] . — Pseudo-euclidean Hurwitz pairs and generalized Fueter Equations. *(a) Inst. Math. Polish Acad. Sci. Preprint* p. 355, II + 10 1985 (b) In : Clifford algebras and their applications in mathematical physics, *Proceedings, Canterbury* 1985, *Dordrecht : Reidel* 1986 p.39-48.

[17] . — Pseudo-euclidean Hurwitz pairs and the Kaluza-Klein theories.

J. Phys. A. Math. Gen., to appear.

[18] . — Complete classification for pseudo-euclidean Hurwitz pairs including the symetry operations. *Bull. Soc. Sci. Lettres Łódź 36, no. 29 (Série : Recherches sur les déformations 3, no. 39)* 1986, p. 15.

[19] RAMIREZ DE ARELLANO (E.) WENE (G.P.). — *The correspondance between type-changing transformations of pseudo-euclidean Hurwitz pairs and Clifford algebras* in preparation.

[20] LAM, Kee Yuen. — The algebraic theory of quadratic forms, *Reading, MA : W.A. Benjamin, Inc.* 1973..

[21] LEE (H.C.). — Sur le théorème de Hurwitz-Radon pour la composition des formes quadratique, *Comment. Math. Helv.*, t. **21**, 1948, p. 261-269.

[22] PORTEOUS (I.R.). — Topological geometry, *2nd ed. Combridge : Cambridge Univ. Press* 1981.

[23] ROMAN (P.). — Advanced quantum theory, *Reading, MA : Addison-Wesley Publ. Co.* 1965.

[24] SHAPIRO (D.B). — Spaces of similarities I-II, *J. Algebra 46* 1977, p. 148-164 and p. 165-181.

[25] . — Spaces of similarities IV, *Pac. J. Math*, t. **69**, 1977, p. 223-224.

[26] . — Hasse principle for the Hurwitz problem, *J. reine angew. Math.*, t. **301**, 1978, p. 179-190.

[27] . — Products of sums of squares, *Expo. Math*, t. **2**, 1984, p. 235-261.

[28] SUZUKI (O.), ŁAWRYNOWICZ (J.), KALINA (J.), KANEMAKI (S.). — A geometric approach to the Kadomtsev-Petviasvili system I, *Proc. Inst. Nat. Sci. Coll. Hum. Sci. Nihon Univ*, t. **21**, 1986, p. 11-34.

[29] . — A geometric approach to the Kadomtsev-Petviasvili system II. Ibid., to appear.

[30] WADSWORTH (A.), SHAPIRO (D.B.). — Spaces of similarities III. *J. Algebra* t. **46** 1977 p. 182-188.