

BANG-YEN CHEN

YOSHIHIKO TAZAWA

**Slant surfaces of codimension two**

*Annales de la faculté des sciences de Toulouse 5<sup>e</sup> série*, tome 11,  
n° 3 (1990), p. 29-43

<[http://www.numdam.org/item?id=AFST\\_1990\\_5\\_11\\_3\\_29\\_0](http://www.numdam.org/item?id=AFST_1990_5_11_3_29_0)>

© Université Paul Sabatier, 1990, tous droits réservés.

L'accès aux archives de la revue « Annales de la faculté des sciences de Toulouse » (<http://picard.ups-tlse.fr/~annales/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques  
<http://www.numdam.org/>

## Slant surfaces of codimension two

BANG-YEN CHEN<sup>(1)</sup> and YOSHIHIKO TAZAWA<sup>(2)</sup>

---

**RÉSUMÉ.** — Une immersion isométrique d'une variété riemannienne dans une variété presque hermitienne est dite oblique si l'angle de Wirtinger est constant [1]. Le but de cet article est d'étudier et de caractériser les surfaces obliques dans le plan complexe  $\mathbb{C}^2$  utilisant l'application de Gauss. Nos démontrons que chaque surface n'ayant aucun point tangent complexe dans une variété presque hermitienne  $\tilde{M}$  de dimension 4 est oblique par rapport à une structure presque complexe bien choisie sur  $\tilde{M}$ .

**ABSTRACT.** — A slant immersion is an isometric immersion from a Riemannian manifold into an almost Hermitian manifold with constant Wirtinger angle [1]. In this article we study and characterize slant surfaces in the complex 2-plane  $\mathbb{C}^2$  via the Gauss map. We also prove that every surface without complex tangent points in a 4-dimensional almost Hermitian manifold  $\tilde{M}$  is slant with respect to a suitably chosen almost complex structure on  $\tilde{M}$ .

---

### 1. Introduction

Let  $x : M \rightarrow \tilde{M}$  be an isometric immersion of a Riemannian manifold  $M$  with Riemannian metric  $g$  into an almost Hermitian manifold  $\tilde{M}$  with an almost complex structure  $\tilde{J}$  and an almost hermitian metric  $\tilde{g}$ . For each nonzero vector  $X$  tangent to  $M$  at  $p \in M$ , the angle  $\theta(X)$  between  $\tilde{J}X$  and the tangent space  $T_pM$  of  $M$  at  $p$  is called the Wirtinger angle of  $X$ . The immersion  $x$  is said to be *general slant* if the Wirtinger angle  $\theta(X)$  is constant (which is independent of the choice of  $p \in M$  and  $X \in T_pM$ ). In this case the angle  $\theta$  is called the *slant angle* of the slant immersion.

---

(1) Department of Mathematics, Michigan State University, East Lansing, Michigan 48824 (U.S.A.)

(2) Department of Mathematics, Tokyo Denki University, Kanda, Tokyo 101 (Japan)

If  $x$  is a totally real (or Lagrangian) immersion, then  $T_p^\perp M \supseteq \tilde{J}(T_p M)$  for any  $p \in M$  where  $T_p^\perp M$  is the normal space of  $M$  in  $\tilde{M}$  at  $p$ . Thus a totally real immersion is a general slant immersion with Wirtinger angle  $\theta \equiv \pi/2$ . If  $M$  is also an almost Hermitian manifold with almost complex structure  $J$ , then the immersion  $x : M \rightarrow \tilde{M}$  is called *holomorphic* (respectively, *anti-holomorphic*) if we have

$$x_*(JX) = \tilde{J}(x_*X) \quad (\text{respectively, } x_*(JX) = -\tilde{J}(x_*X)) \quad (1.1)$$

for any  $X \in T_p M$ . It is clear that holomorphic and anti-holomorphic immersions are general slant immersions with  $\theta \equiv 0$ . A general slant immersion which is neither holomorphic nor anti-holomorphic is simply called a *slant immersion* [1]. In paragraph 2 we review the geometry of the Grassmannian  $G(2, 4)$  for later use. In paragraph 3 we investigate the relationship between 2-planes in the Euclidean 4-space  $E^4$  and the complex structures on  $E^4$ . By applying the relationship we obtain a pointwise observation concerning slant surfaces. In paragraph 4 we study slant surfaces via their Gauss map. In particular we obtain in this section a new characterization of slant surfaces and we also prove that a non-minimal surface in  $E^4$  can be slant with respect to at most four compatible complex structures on  $E^4$ . In paragraph 5 we prove that every Hermitian manifold is a proper slant surface with any prescribed slant angle with respect to a suitable almost complex structure. In the last section we define doubly slant surfaces and show that their Gaussian and the normal curvatures vanish identically.

## 2. Geometry of $G(2, 4)$

In the section we recall some results concerning the geometry of the Grassmannian  $G(2, 4)$  of oriented 2-planes in  $E^4$  (for details, see [2, 3, 7, 8]).

Let  $E^m = (\mathbb{R}^m, \langle \cdot, \cdot \rangle)$  be the Euclidean  $m$ -space with the canonical inner product  $\langle \cdot, \cdot \rangle$ . Let  $\{\epsilon_1, \dots, \epsilon_m\}$  be the canonical basis of  $E^m$ . Then  $\Psi := \epsilon_1 \wedge \dots \wedge \epsilon_m$  gives the canonical orientation of  $E^m$ . For each  $n \in \{1, \dots, m\}$ , the space  $\wedge^n E^m$  is an  $\binom{m}{n}$ -dimensional real vector space with the inner product, also denote by  $\langle \cdot, \cdot \rangle$ , defined by

$$\langle X_1 \wedge \dots \wedge X_n, Y_1 \wedge \dots \wedge Y_n \rangle = \det(\langle X_i, Y_j \rangle) \quad (2.1)$$

and be extended linearly. The two vector spaces  $\bigwedge^n(E^m)^*$  and  $(\bigwedge^n E^m)^*$  are identified in a natural way by

$$\Phi(X_1 \wedge \cdots \wedge X_n) = \Phi(X_1, \dots, X_n) \quad (2.2)$$

for any  $\Phi \in \bigwedge^n(E^m)^*$  and any  $X_1, \dots, X_n \in E^m$ . The Grassmannian  $G(n, m)$  of oriented  $n$ -planes in  $E^m$  was identified with the set  $D_1(n, m)$  of unit decomposable  $n$ -vectors in  $\bigwedge^n E^m$ . The identification  $\varphi : G(n, m) \rightarrow D_1(n, m)$  is given by  $\varphi(V) = X_1 \wedge \cdots \wedge X_n$  for any positive orthonormal basis  $\{X_i\}$  of  $V \in G(n, m)$ .

The star operator  $*$  :  $\bigwedge^2 E^4 \rightarrow \bigwedge^2 E^4$  is defined by

$$(*\xi, \eta)\Psi = \xi \wedge \eta \quad \text{for } \xi, \eta \in \bigwedge^2 E^4. \quad (2.3)$$

If we regard a  $V \in G(2, 4)$  as an element in  $D_1(2, 4)$  via  $\varphi$ , we have  $*V = V^\perp$ , where  $V^\perp$  is the oriented orthogonal complement of  $V$  in  $E^4$ . Since  $*$  is a symmetric involution,  $\bigwedge^2 E^4$  is decomposed into the following orthogonal direct sum :

$$\bigwedge^2 E^4 = \bigwedge_+^2 E^4 \oplus \bigwedge_-^2 E^4 \quad (2.4)$$

of eigenspaces of  $*$  with eigenvalues 1 and  $-1$ , respectively. Denote by  $\pi_+$  and  $\pi_-$  the natural projections from  $\bigwedge^2 E^4$  into  $\bigwedge_+^2 E^4$  and  $\bigwedge_-^2 E^4$ , respectively.

Given a positive orthonormal basis  $\{e_1, \dots, e_4\}$  of  $E^4$ , we put

$$\begin{aligned} \eta_1 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + e_3 \wedge e_4), & \eta_2 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_3 - e_2 \wedge e_4), \\ \eta_3 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_4 + e_2 \wedge e_3), & \eta_4 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_2 - e_3 \wedge e_4), \\ \eta_5 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_3 + e_2 \wedge e_4), & \eta_6 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_4 - e_2 \wedge e_3). \end{aligned} \quad (2.5)$$

Then  $\{\eta_1, \eta_2, \eta_3\}$  and  $\{\eta_4, \eta_5, \eta_6\}$  are orthonormal bases of  $\bigwedge_+^2 E^4$  and  $\bigwedge_-^2 E^4$  respectively. We shall orient  $\bigwedge_+^2 E^4$  and  $\bigwedge_-^2 E^4$  so that these two bases are positive.

For any  $\xi \in D_1(2, 4)$  we have

$$\pi_+(\xi) = \frac{1}{2}(\xi + *\xi) \quad \text{and} \quad \pi_-(\xi) = \frac{1}{2}(\xi - *\xi). \quad (2.6)$$

If we denote by  $S_+^2$  and  $S_-^2$  the 2-spheres centered at the origin with radius  $1/\sqrt{2}$  in  $\Lambda_+^2 E^4$  and  $\Lambda_-^2 E^4$ , respectively, then we have

$$\pi_+ : D_1(2, 4) \rightarrow S_+^2 \quad \text{and} \quad \pi_- : D_1(2, 4) \rightarrow S_-^2 \quad (2.7)$$

and

$$D_1(2, 4) = S_+^2 \times S_-^2. \quad (2.8)$$

### 3. Complex structures on $E^4$

Let  $\mathbb{C}^2 = (\mathbb{R}^2, \langle \cdot, \cdot \rangle, J_0)$  be the complex plane with the canonical (almost) complex structure  $J_0$  defined by  $J_0(a, b, c, d) = (-b, a, -d, c)$ .  $J_0$  is an orientation preserving isomorphism. We denote by  $\mathcal{G}$  the set of all (almost) complex structures on  $E^4$  which are compatible with the inner product  $\langle \cdot, \cdot \rangle$ , i.e.,

$$\mathcal{G} = \{J : E^4 \rightarrow E^4 \mid J \text{ is linear, } J^2 = -\text{Id}, \\ \langle JX, JY \rangle = \langle X, Y \rangle, \text{ for any } X, Y \in E^4\}.$$

For each  $J \in \mathcal{G}$ , we choose an (orthonormal)  $J$ -basis  $\{e_1, \dots, e_4\}$  so that  $Je_1 = e_2, Je_3 = e_4$ . Two  $J$ -bases of the same complex structure  $J$  have the same orientation. By using the canonical orientation  $\Psi = \epsilon_1 \wedge \dots \wedge \epsilon_4$ , we divide  $\mathcal{G}$  into two disjoint subsets :

$$\mathcal{G}^+ = \{J \in \mathcal{G} \mid J\text{-bases are positive,}\}$$

and

$$\mathcal{G}^- = \{J \in \mathcal{G} \mid J\text{-bases are negative.}\}$$

For each  $J \in \mathcal{G}$ , there is a unique 2-vector  $\zeta_J \in \Lambda^2 E^4$  defined as follows.

Let  $\Omega_J$  be the Kaehler form of  $J$ , i.e.,

$$\Omega_J(X, Y) = \langle X, JY \rangle, \quad (3.1)$$

for any  $X, Y \in E^4$ . The 2-vector  $\zeta_J$  associated with  $J$  is defined to be the unique 2-vector satisfying

$$\langle \zeta_J, X \wedge Y \rangle = -\Omega_J(X, Y), \quad \text{for any } X, Y \in E^4 \quad (3.2)$$

LEMMA 3.1.. — *The mapping*

$$\zeta : \mathcal{G} \rightarrow \bigwedge^2 E^4 \quad (3.3)$$

defined by  $\zeta(J) = \zeta_J$  gives rise to two bijections :

$$\zeta : \mathcal{G}^+ \rightarrow S_+^2(\sqrt{2}) \quad \text{and} \quad \zeta : \mathcal{G}^- \rightarrow S_-^2(\sqrt{2}), \quad (3.4)$$

where  $S_+^2(\sqrt{2})$  and  $S_-^2(\sqrt{2})$  are the 2-spheres centered at the origin with radius  $\sqrt{2}$  in  $\bigwedge_+^2 E^4$  and  $\bigwedge_-^2 E^4$ , respectively.

*Proof.* — Let  $J \in \mathcal{G}$  and  $\{e_1, \dots, e_4\}$  be a  $J$ -basis. If  $J \in \mathcal{G}^+$  (respectively,  $J \in \mathcal{G}^-$ ), then  $\{e_1, \dots, e_4\}$  is positive (respectively, negative). Since  $\zeta_J = e_1 \wedge e_2 + e_3 \wedge e_4$  by (3.1),  $\zeta$  maps  $\mathcal{G}^+$  into  $S_+^2(\sqrt{2})$  and maps  $\mathcal{G}^-$  into  $S_-^2(\sqrt{2})$ . Their injectivity are clear.

Conversely, for each  $\xi \in S_+^2(\sqrt{2})$ , we have  $\frac{1}{2}\zeta \in S_+^2$ . Hence, we can pick an oriented 2-plane  $V$  such that  $V \in \pi_+^{-1}(\frac{1}{2}\zeta)$ . Now we choose a positive orthonormal basis  $\{e_1, \dots, e_4\}$  such that  $\{e_1, e_2\}$  is a positive basis of  $V$ . Let  $J$  be the complex structure on  $E^4$  such that  $Je_1 = e_2, Je_2 = -e_1, Je_3 = e_4, Je_4 = -e_3$ . Then  $J \in \mathcal{G}^+$  and  $\zeta_J = \xi$ .

If  $\xi \in S_-^2(\sqrt{2})$ , we pick  $V \in \pi_-^{-1}(\frac{1}{2}\zeta)$  and define  $J$  by  $Je_1 = e_2, Je_2 = -e_1, Je_3 = -e_4, Je_4 = e_3$ . Then we have a similar result.  $\square$

In the following we identify  $\mathcal{G}, \mathcal{G}^+$  and  $\mathcal{G}^-$  with  $S_+^2(\sqrt{2}) \cup S_-^2(\sqrt{2}), S_+^2(\sqrt{2})$ , and  $S_-^2(\sqrt{2})$ , respectively, via  $\zeta$ .

For each  $V \in G(2, 4)$  and each  $J \in \mathcal{G}$ , we define

$$\alpha_J(V) = \cos^{-1}(-\Omega_J(V)). \quad (3.5)$$

Then  $\alpha_J(V) \in [0, \pi]$ . A 2-plane  $V$  is said to be *a-slant* if  $\alpha_J(V) = a$ .

The relation between  $\theta(X)$  defined in paragraph 1 and  $\alpha_J(V)$  is as follows. Let  $x$  be an isometric immersion of  $M$  into  $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ . If we regard  $(T_p \widetilde{M}, \widetilde{g}, \widetilde{J})$  as a complex plane with the induced inner product  $\langle \cdot, \cdot \rangle$ , then we have

$$\theta(X) = \min\{\alpha_{\widetilde{J}}(T_p M), \pi - \alpha_{\widetilde{J}}(T_p M)\} \quad (3.6)$$

for any non-zero vector  $X \in T_p M$ .

If  $M$  is oriented, then  $M$  has a unique complex structure  $J$  determined by its orientation and its metric induced from  $\tilde{g}$ . With respect to  $\alpha_J$ , we have:

$$x \text{ is holomorphic} \Leftrightarrow \alpha_{\tilde{J}}(T_p M) \equiv 0, \quad (3.7)$$

$$x \text{ is anti-holomorphic} \Leftrightarrow \alpha_{\tilde{J}}(T_p M) \equiv \pi, \quad (3.8)$$

$$x \text{ is totally real} \Leftrightarrow \alpha_{\tilde{J}}(T_p M) \equiv \frac{\pi}{2}. \quad (3.9)$$

The argument above also hold for the case  $\dim \tilde{M} > 4$ . We note that  $\alpha_J$  coincides with the angle defined in [6].

LEMMA 3.2

(i) If  $J \in \mathcal{G}^+$ , then  $\alpha_J(V)$  is the angle between  $\pi_+(V)$  and  $\zeta_J$ .

(ii) If  $J \in \mathcal{G}^-$ , then  $\alpha_J(V)$  is the angle between  $\pi_-(V)$  and  $\zeta_J$ .

*Proof.* — If  $J \in \mathcal{G}^+$ , then, by (3.2), (3.5) and lemma 3.1, we have

$$\cos(\alpha_J(V)) = -\Omega_J(V) = \langle \zeta_J, V \rangle = \langle \zeta_J, \pi_+(V) + \pi_-(V) \rangle = \langle \zeta_J, \pi_+(V) \rangle$$

which is the cosine of the angle between  $\pi_+(V)$  and  $\zeta_J$ , since  $\|\zeta_J\| = \sqrt{2}$  and  $\|\pi_+(V)\| = 1/\sqrt{2}$ . Similar argument applies to the case  $J \in \mathcal{G}^-$ .  $\square$

For each  $a \in [0, \pi]$  and  $J \in \mathcal{G}$ , we define

$$G_{J,a} = \{V \in G(2, 4) \mid \alpha_J(V) = a\}, \quad (3.10)$$

i.e.,  $G_{J,a}$  is the set of all oriented 2-planes in  $E^4$  which are  $a$ -slant with respect to  $J$ . Also for each  $a \in [0, \pi]$  and  $V \in G(2, 4)$  we define

$$\mathcal{G}_{V,a} = \{J \in \mathcal{G} \mid \alpha_J(V) = a\}, \quad (3.11)$$

i.e.,  $\mathcal{G}_{V,a}$  is the set of all compatible complex structure on  $E^4$  with respect to which  $V$  is  $a$ -slant. We put

$$\mathcal{G}_{V,a}^+ = \mathcal{G}_{V,a} \cap \mathcal{G}^+ \quad \text{and} \quad \mathcal{G}_{V,a}^- = \mathcal{G}_{V,a} \cap \mathcal{G}^-. \quad (3.12)$$

By applying lemma 3.2 we obtain the following generalization of proposition 2 of [3].

PROPOSITION 3.3

- (i) If  $J \in \mathcal{G}^+$ , then  $G_{J,\alpha} = S_{J,\alpha}^+ \times S_-^2$  where  $S_{J,\alpha}^+$  is the circle on  $S_+^2$  consisting of 2-vectors which make constant angle  $\alpha$  with  $\zeta_J$ .
- (ii) If  $J \in \mathcal{G}^-$ , then  $G_{J,\alpha} = S_+^2 \times S_{J,\alpha}^-$  where  $S_{J,\alpha}^-$  is the circle on  $S_-^2$  consisting of 2-vectors which make constant angle  $\alpha$  with  $\zeta_J$ .
- (iii) Via the identification  $\zeta$  given in (3.4),  $\mathcal{G}_{V,\alpha}^+$  is a circle on  $S_+^2(\sqrt{2})$  consisting of 2-vectors in  $S_+^2(\sqrt{2})$  which make constant angle  $\alpha$  with  $\pi_+(V)$ . Similarly,  $\mathcal{G}_{V,\alpha}^-$  is a circle on  $S_-^2(\sqrt{2})$  obtained in a similar way.

For simplicity, for each  $V \in G(2, 4)$ , we define  $J_V^+$  and  $J_V^-$  as the complex structures given by

$$J_V^+ = \zeta^{-1}(\pi_+(V)) \quad \text{and} \quad J_V^- = \zeta^{-1}(\pi_-(V)), \quad (3.13)$$

where  $\zeta$  is the bijection given in lemma 3.1. It is clear that  $J_V^+ \in \mathcal{G}^+$  and  $J_V^- \in \mathcal{G}^-$ .

#### 4. Slant surfaces and Gauss map

Let  $x : M \rightarrow E^4$  be an isometric immersion from an oriented surface  $M$  into  $E^4$ . We denote by  $\nabla$  and  $\tilde{\nabla}$  the Riemannian connections of  $M$  and  $E^4$ , respectively. We choose a positive orthonormal local frame field  $\{e_1, \dots, e_4\}$  in  $E^4$  such that, restricted to  $M$ , the vectors  $e_1, e_2$  give a positive frame field tangent to  $M$  (and hence  $e_3, e_4$  are normal to  $M$ ). Let  $\omega^1, \dots, \omega^4$  be the field of dual frame field. The structure equations of  $E^4$  are then given by

$$\tilde{\nabla}e_A = \sum \omega_A^B \otimes e_B, \quad \omega_A^B + \omega_B^A = 0, \quad (4.1)$$

$$d\omega^B = \sum \omega_A^B \otimes \omega^A, \quad (4.2)$$

$$d\omega_B^A = - \sum \omega_C^A \wedge \omega_B^C, \quad A, B, C = 1, 2, 3, 4. \quad (4.3)$$

If we restrict these forms to  $M$ , then  $\omega^3 = \omega^4 = 0$ . From  $d\omega^3 = d\omega^4 = 0$ , (4.2), and Cartan's lemma, we have

$$\omega_i^r = \sum h_{ij}^r \omega^j, \quad h_{ij}^r = h_{ji}^r, \quad r = 3, 4; \quad i, j = 1, 2. \quad (4.4)$$



The Gauss curvature  $G$  and the normal curvature  $G^D$  of  $M$  in  $E^4$  are given respectively by

$$G = \sum (h_{11}^r h_{22}^r - h_{12}^r h_{12}^r), \quad (4.5)$$

$$G^D = (h_{11}^3 - h_{22}^3)h_{12}^4 - (h_{11}^4 - h_{22}^4)h_{12}^3. \quad (4.6)$$

In the following we denote by  $\nu : M \rightarrow G(2, 4)$  the *Gauss map* of the immersion  $x$  defined by

$$\nu(p) = (e_1 \wedge e_2)(p), \quad (4.7)$$

where we identify  $G(2, 4)$  with  $D_1(2, 4)$  consisting of unit decomposable 2-vectors in  $\bigwedge^2 E^4$ . We define two maps  $\nu_+$  and  $\nu_-$  by

$$\nu_+ = \pi_+ \circ \nu \quad \text{and} \quad \nu_- = \pi_- \circ \nu, \quad (4.8)$$

where  $\pi_+$  and  $\pi_-$  are the projections given in paragraph 2. Then  $\nu_+$  and  $\nu_-$  map  $M$  into  $S_+^2$  and  $S_-^2$ , respectively.

In terms of  $\nu_+$  and  $\nu_-$ , we have the following characterization of slant surfaces in  $\mathbb{C}^2$ .

**PROPOSITION 4.1.** — *Let  $x : M \rightarrow E^4$  be an isometric immersion from an oriented surface  $M$  into  $E^4$ . Then  $x$  is a slant immersion with respect to a complex structure  $J \in \mathcal{G}^+$  (respectively,  $J \in \mathcal{G}^-$ ) if and only if  $\nu_+(M)$  is contained in a circle on  $S_+^2$  (respectively,  $\nu_-(M)$  is contained in a circle on  $S_-^2$ ).*

*Moreover,  $x$  is  $\alpha$ -slant with respect to  $J \in \mathcal{G}^+$  (respectively,  $J \in \mathcal{G}^-$ ) if and only if  $\nu_+(M)$  is contained in a circle  $S_{J,\alpha}^+$  on  $S_+^2$  (respectively,  $\nu_-(M)$  is contained in a circle  $S_{J,\alpha}^-$  on  $S_-^2$ ), where  $S_{J,\alpha}^+$  and  $S_{J,\alpha}^-$  are the circles defined in proposition 3.3.*

*Proof.* — If  $x : M \rightarrow E^4$  is  $\alpha$ -slant with respect to  $J \in \mathcal{G}^+$ , then by (3.11) and proposition 3.3, we have  $\nu(T_p M) \in S_{J,\alpha}^+ \times S_-^2$  for any  $p \in M$ . Thus  $\nu_+(M)$  is contained in a circle  $S_{J,\alpha}^+$  on  $S_+^2$  consisting of 2-vectors in  $S_+^2$  which make angle  $\alpha$  with  $\zeta_J$ .

Conversely, if  $x : M \rightarrow E^4$  is an immersion such that  $\nu_+(M)$  is contained in a circle  $S^1$  on  $S_+^2$ . Let  $\eta$  be a vector in  $\bigwedge^2 E^4$  with length  $\sqrt{2}$  which is normal to the 2-plane in  $\bigwedge^2 E^4$  containing  $S^1$ . Then  $\eta \in S_+^2(\sqrt{2})$ . By

lemma 3.1, there is a unique  $J \in \mathcal{G}^+$  such that  $\zeta_J = \eta$ . It is clear that  $S^1$  is a  $S_{J,a}^+$  for some constant angle  $a$ . Therefore, by proposition 3.3, the immersion  $x$  is  $a$ -slant with respect to  $J \in \mathcal{G}^+$ . Similar argument applies to the other case.  $\square$

The following lemma was obtained in [1] (given in the proof of theorem 1 of [1]). Here we reprove it by using Gauss map.

LEMMA 4.2. — *Let  $x : M \rightarrow E^4$  be an isometric immersion from a surface  $M$  into  $E^4$ . Then  $M$  is minimal and slant with respect to some  $J \in \mathcal{G}^+$  (respectively,  $J \in \mathcal{G}^-$ ) if and only if  $\nu_+(M)$  (respectively,  $\nu_-(M)$ ) is a singleton.*

*Proof.* — If  $x$  is minimal, then both  $\nu_+$  and  $\nu_-$  are anti-holomorphic [5]. In particular,  $\nu_+$  and  $\nu_-$  are open maps if they are not constant maps. However, if  $x$  is slant with respect to  $J \in \mathcal{G}^+$  (respectively,  $J \in \mathcal{G}^-$ ), then, by proposition 4.1,  $\nu_+$  (respectively,  $\nu_-$ ) cannot be an open map. Hence,  $\nu_+(M)$  (respectively,  $\nu_-(M)$ ) is a singleton.

Conversely, if  $\nu_+(M)$  (respectively,  $\nu_-(M)$ ) is a singleton, say  $\{\xi\}$ . Then  $2\xi \in S_+^2(\sqrt{2})$ . Thus, by lemma 3.1, there is a unique  $J = \zeta^{-1}(2\xi) \in \mathcal{G}^+$  such that  $\nu_+(M)$  is contained in  $S_{J,0}^+$ . Thus  $x$  is holomorphic with respect to  $J$  (see proposition 2 of [3]), in particular,  $x$  is a minimal immersion. Because a singleton  $\{\xi\}$  lies in every circle  $S^1$  on  $S_+^2$  through  $\xi$ , proposition 4.1 implies that for any  $a \in [0, \pi]$ , there exists a  $J_a \in \mathcal{G}^+$  (respectively,  $J_a \in \mathcal{G}^-$ ), such that  $x$  is  $a$ -slant with respect to  $J_a$ .  $\square$

By applying proposition 4.1 and lemma 4.2 we have the following result concerning non-minimal surfaces.

THEOREM 4.3. — *If  $x : M \rightarrow E^4$  is not minimal, then there exist at most two complex structures  $\pm J \in \mathcal{G}^+$  and at most two complex structures  $\pm J' \in \mathcal{G}^-$  such that  $x$  is slant with respect to them.*

*Proof.* — If  $x$  is a non-minimal,  $a$ -slant immersion with respect to a complex structure  $J \in \mathcal{G}^+$  (respectively,  $J' \in \mathcal{G}^-$ ), then, by proposition 4.1 and lemma 4.2,  $\nu_+(M)$  (respectively,  $\nu_-(M)$ ) contains an arc of the circle  $S_{J,a}^+$  (respectively,  $S_{J',a}^-$ ).

Thus,  $\pm J$  and  $\pm J'$  are the only possible complex structures which make  $x$  to be slant according to proposition 4.1.  $\square$

From proposition 4.1, lemma 4.2 and theorem 4.3 we have the following.

PROPOSITION 4.4. — *Let  $x : M \rightarrow E^4$  be an isometric immersion from a surface  $M$  into  $E^4$ . Then the following statements are equivalent:*

- (i)  *$x$  is minimal and slant with respect to some complex structure  $J \in \mathcal{G}^+$  (respectively,  $J \in \mathcal{G}^-$ );*
- (ii)  *$\nu_+(M)$  (respectively,  $\nu_-(M)$ ) is a singleton;*
- (iii)  *$x$  is holomorphic with respect to some complex structure  $J \in \mathcal{G}^+$  (respectively,  $J \in \mathcal{G}^-$ );*
- (iv) *for each  $a \in [0, \pi]$ , there exist  $J_a \in \mathcal{G}^+$  (respectively,  $J_a \in \mathcal{G}^-$ ) such that  $x$  is  $a$ -slant with respect to  $J_a$ .*

Theorem 4.3 and proposition 4.4 provide us a clear geometric understanding of lemmas 5 and 6 and theorems 1, 3 and 4 of [1].

## 5. Slant surfaces in 4-dimensional almost Hermitian manifolds

Let  $x : M \rightarrow (\widetilde{M}, J)$  be an immersion of a manifold  $M$  into an almost complex manifold  $(\widetilde{M}, J)$ . Then a point  $p \in M$  is called a *complex tangent point* if the tangent plane of  $M$  at  $p$  is invariant under the action of  $J$ . The purpose of this section is to prove the following.

THEOREM 5.1. — *Let  $x : M \rightarrow (\widetilde{M}, g, J)$  be an imbedding from an oriented surface  $M$  into an almost Hermitian manifold  $(\widetilde{M}, g, J)$ . If  $x$  has no complex tangent point, for any prescribed angle  $a \in (0, \pi)$ , there exists an almost complex structure  $\widetilde{J}$  on  $\widetilde{M}$  satisfying the following conditions:*

- (i)  *$(\widetilde{M}, g, \widetilde{J})$  is an almost Hermitian manifold, and*
- (ii)  *$x$  is  $a$ -slant with respect to  $\widetilde{J}$ .*

*Proof.* —  $\widetilde{M}$  has a natural orientation determined by  $J$ . At each point  $p \in \widetilde{M}$ ,  $(T_p \widetilde{M}, g_p)$  is a Euclidean 4-space and so we can apply the argument given in paragraphs 2 and 3.

According to (2.4) the vector bundle  $\Lambda^2(\widetilde{M})$  of 2-vectors on  $\widetilde{M}$  is a direct sum of two vector subbundles.

$$\Lambda^2(\widetilde{M}) = \Lambda_+^2(\widetilde{M}) \oplus \Lambda_-^2(\widetilde{M}). \quad (5.1)$$

We define two sphere-bundles over  $\widetilde{M}$  by

$$\begin{aligned} S_+^2(\widetilde{M}) &= \left\{ \xi \in \Lambda_+^2(\widetilde{M}) \mid \|\xi\| = \frac{1}{\sqrt{2}} \right\}, \\ \overline{S}_+^2(\widetilde{M}) &= \left\{ \xi \in \Lambda_-^2(\widetilde{M}) \mid \|\xi\| = \sqrt{2} \right\}. \end{aligned} \quad (5.2)$$

By applying lemma 3.1 we can identify a cross-section

$$\gamma : \widetilde{M} \rightarrow \overline{S}_+^2(\widetilde{M}) \quad (5.3)$$

with an almost complex structure  $J_\gamma$  on  $\widetilde{M}$  such that  $(\widetilde{M}, g, J_\gamma)$  is an almost Hermitian manifold. In the following we denote by  $\rho$  the cross-section corresponding to  $J$  and we want to construct another cross-section  $\tilde{\sigma}$  to obtain the desired almost complex structure  $\tilde{J}$ .

We consider the pull-backs of these bundles via the immersion  $x$ , i.e.,

$$\begin{aligned} \Lambda_+^2(M) &= x^*(\Lambda_+^2(\widetilde{M})), \\ S_+^2(M) &= x^*(S_+^2(\widetilde{M})), \\ \overline{S}_+^2(M) &= x^*(\overline{S}_+^2(\widetilde{M})). \end{aligned} \quad (5.4)$$

The tangent bundle  $TM$  determines a cross-section  $\tau : M \rightarrow S_+^2(M)$  given by

$$\tau(p) = \pi_+(T_p M), \quad \text{for any } p \in M, \quad (5.5)$$

where  $\pi_+$  is the projection of  $\Lambda^2(T_p \widetilde{M})$  onto  $\Lambda_+^2(T_p \widetilde{M})$ . Note that  $2\tau$  is a cross-section of  $\overline{S}_+^2(M)$ ;

$$2\tau : M \rightarrow \overline{S}_+^2(M). \quad (5.6)$$

We denote  $x^*\rho$  also by  $\rho$ , which gives us another cross-section:

$$\rho = x^*\rho : M \rightarrow \overline{S}_+^2(M). \quad (5.7)$$

Since  $x$  has no complex tangent point,

$$\rho(p) \neq \pm 2\tau(p), \quad \text{for any } p \in M. \quad (5.8)$$

Therefore,  $\rho(p)$  and  $2\tau(p)$  determine a 2-plane in  $\Lambda_+^2(T_p \widetilde{M})$  which intersects the circle  $(\mathcal{G}_{\tau,a}^+)_p$  at two points, where  $(\mathcal{G}_{\tau,a}^+)_p$  is the circle on  $(\overline{S}_+^2(M))_p$  determined in proposition 3.3 with  $V = T_p M$ . Let  $\sigma(p)$  be one of the two points which lies on the half-great-circle on  $(\overline{S}_+^2(M))_p$  starting from  $2\tau(p)$  and passing through  $\rho(p)$ . Since  $\rho$  and  $\tau$  are differentiable, so is  $\sigma$ . Thus we get the third cross-section:

$$\sigma : M \rightarrow \overline{S}_+^2(M) \quad (5.9)$$

and we want to extend  $\sigma$  to a cross-section  $\tilde{\sigma}$  of  $\overline{S}_+^2(\widetilde{M})$ .

For each  $p \in M$ , we choose an open neighborhood  $U_p$  of  $p$  in  $\widetilde{M}$  such that  $\sigma|_{U_p \cap M}$  can be extended to a cross-section of  $\overline{S}_+^2(\widetilde{M})$  on  $U_p$ :

$$\sigma_p : U_p \rightarrow \overline{S}_+^2(\widetilde{M})|_{U_p}. \quad (5.10)$$

We identify here the manifold  $M$  with its image  $x(M)$  via the imbedding. We put

$$U = \bigcup_{p \in M} U_p \quad (5.11)$$

and pick a locally finitely countable refinement  $\{U_i\}_{i=1}^\infty$  of the open covering  $\{U_p\}_{p \in M}$  of  $U$ . For each  $i$  we pick a point  $p \in M$  such that  $U_i$  is contained in  $U_p$  and put

$$\sigma_i = \sigma_p|_{U_i}. \quad (5.12)$$

Let  $\{\phi_i\}$  be a differentiable partition of unity on  $U$  subordinated to the covering  $\{U_i\}$ . We define a cross-section  $\overline{\sigma}$  of  $\Lambda_+^2(\widetilde{M})|_U$  by

$$\overline{\sigma} : U \rightarrow \Lambda_+^2(\widetilde{M})|_U \quad ; \quad \overline{\sigma} = \sum \phi_i \sigma_i. \quad (5.13)$$

By the construction of  $\sigma_i$  and  $\overline{\sigma}$  we have

$$\overline{\sigma}|_M = \sigma. \quad (5.14)$$

Since the angle  $\angle(\overline{\sigma}(p), \rho(p)) < \pi$  for any  $p \in M$ , we have

$$\overline{\sigma}(p) \neq 0, \quad \angle(\overline{\sigma}(p), \rho(p)) < \pi \text{ for any } p \in M. \quad (5.15)$$

By continuity of  $\overline{\sigma}$  we can pick an open neighborhood  $W$  of  $M$  contained in  $U$  such that

$$\overline{\sigma}(q) \neq 0, \quad \angle(\overline{\sigma}(q), \rho(q)) < \pi \text{ for any } q \in W. \quad (5.16)$$

We define a cross-section  $\widehat{\sigma}$  of  $\overline{S}_+^2(\widetilde{M})|_W$  by

$$\widehat{\sigma} : W \rightarrow \overline{S}_+^2(\widetilde{M})|_W; \quad \widehat{\sigma} = \frac{\overline{\sigma}}{\sqrt{2\|\overline{\sigma}\|}}. \quad (5.17)$$

Then we have  $\angle(\hat{\sigma}(q), \rho(q)) < \pi$  for any  $q \in M$  too. Finally, we consider the open covering  $\{W, \widetilde{M} - M\}$  of  $\widetilde{M}$  and local cross-section  $\hat{\sigma}$  and  $\rho|_{\widetilde{M}-M}$  and repeat the same argument using a partition of unity subordinate to  $\{W, \widetilde{M} - M\}$  to get a cross-section  $\tilde{\sigma} : \widetilde{M} \rightarrow \widetilde{S}_+^2(\widetilde{M})$  satisfying  $\tilde{\sigma}|_M = \sigma$ . Now, it is clear that the almost complex structure  $\tilde{J}$  corresponding to  $\tilde{\sigma}$  is the desired almost complex structure.  $\square$

## 6. Appendix : double slant surfaces in $\mathbb{C}^2$

An immersion  $x : M^2 \rightarrow E^4$  is called *doubly slant* if it is slant with respect to a complex structure  $J \in \mathcal{G}^+$  and at the same time it is slant with respect to another complex structure  $\hat{J} \in \mathcal{G}^-$ .

Equivalently, the immersion  $x$  is doubly slant if and only if there exists a  $V \in G(2, 4)$  such that  $x$  is slant with respect to both  $J_V^+$  and  $J_V^-$ , where  $J_V^+$  and  $J_V^-$  are defined by (3.13).

PROPOSITION 6.1. — *If  $x : M^2 \rightarrow E^4$  is a doubly slant immersion, then  $G = G^D = 0$ .*

*Proof.* — If  $x$  is doubly slant, then, by proposition 4.1, we know that  $\nu_+(M)$  and  $\nu_-(M)$  both lie in some circles on  $S_+^2$  and  $S_-^2$ , respectively. Thus, both  $(\nu_+)_*$  and  $(\nu_-)_*$  are singular at every point  $p \in M$ . The result then follows from the following lemma of [7].

LEMMA 6.2. — *For an isometric immersion  $x : M^2 \rightarrow E^4$ , we have*

$$\text{Jacobian of } \nu_+ = \frac{G^D + G}{2} \quad ; \quad \text{Jacobian of } \nu_- = \frac{G^D - D}{2}.$$

This lemma can be proved in our notation as follows.

From (4.8) we have  $\nu_*X = (\tilde{\nabla}_X e_1) \wedge e_2 + e_1 \wedge (\tilde{\nabla}_X e_2)$  for any  $X$  tangent to  $M$ . Thus, by applying (2.5), (4.1) and (4.4), we find (see [3])

$$\begin{aligned} \nu_*X = \frac{1}{\sqrt{2}} \left\{ (\omega_1^4 + \omega_2^3)(X)\eta_2 + (-\omega_1^3 + \omega_2^4)(X)\eta_3 + \right. \\ \left. + (-\omega_1^4 + \omega_2^3)(X)\eta_5 + (\omega_1^3 + \omega_2^4)(X)\eta_6 \right\}. \end{aligned} \quad (6.1)$$

Since  $\{\eta_2, \eta_3\}$  is a positive orthonormal basis of  $T_{\nu_+(p)}S_+^2$  and  $\{\eta_5, \eta_6\}$  is a positive orthonormal basis of  $T_{\nu_-(p)}S_-^2$ , we obtain

$$\begin{aligned} (\nu_+)_*X &= \frac{1}{\sqrt{2}} \left\{ (\omega_1^4 + \omega_2^3)(X)\eta_2 + (-\omega_1^3 + \omega_2^4)(X)\eta_3 \right\}, \\ (\nu_-)_*X &= \frac{1}{\sqrt{2}} \left\{ (-\omega_1^4 + \omega_2^3)(X)\eta_5 + (\omega_1^3 + \omega_2^4)(X)\eta_6 \right\}. \end{aligned} \tag{6.2}$$

By applying (4.4), (4.6), (4.7) and (6.2) we obtain the lemma.

*Remark 6.3.* — Examples 1 through 6 of [1] are examples of doubly slant surfaces in  $\mathbb{C}^2$ . Here we give another example of doubly slant surfaces.

*Example.* — For any two nonzero real numbers  $p$  and  $q$ , we consider the following immersion from  $\mathbb{R} \times (0, \infty)$  into  $\mathbb{C}^2$  defined by

$$x(u, \nu) = (p\nu \sin u, p\nu \cos u, \nu \sin qu, \nu \cos qu). \tag{6.3}$$

The slant angles and the ranks of  $\nu, \nu_+$  and  $\nu_-$  of those examples can be determined by direct computation. We list them as the following table.

	Slant angles	rank $\nu$	rank $\nu_+$	rank $\nu_-$
Example 1	$a = b = \frac{\pi}{2}$	2	1	1
Example 2	$a = b = \frac{\pi}{2}$	1	1	1
Example 3	$a = b = \frac{\pi}{2}$	2	1	1
Example 4	$a = b \in \left[0, \frac{\pi}{2}\right]$	0	0	0
Example 5	$a \in \left[0, \frac{\pi}{2}\right], b = \frac{\pi}{2}$	2	1	1
Example 6	$a = b \in \left(0, \frac{\pi}{2}\right)$	non-constant	1	1
Example A	$a, b \in \left(0, \frac{\pi}{2}\right)$	1	1	1

Except for example 4,  $a$  and  $b$  above are the slant angles with respect to  $J_0 = J_{\epsilon_1 \wedge \epsilon_2}^+$  and  $J_1 = J_{\epsilon_1 \wedge \epsilon_2}^-$  (cf. (3.13)). For example 4,  $a$  is the slant angle with respect to  $J_0$  and  $b$  is the slant angle with respect to  $J_{\epsilon_1 \wedge \epsilon_3}^-$ .

In [4] further results on slant immersions have been obtained.

### References

- [1] CHEN (B.Y.) .— *Slant immersions*,  
Bull. Austral. Math. Soc., **41** (1990), pp. 135-147.
- [2] CHEN (B.Y.) and MORVAN (J.M.) .— *Propriétés riemanniennes des surfaces lagrangiennes*,  
C. R. Acad. Sc. Paris, Ser. I, **301** (1985), pp. 209-212.
- [3] CHEN (B.Y.) and MORVAN (J.M.) .— *Géométrie des surfaces lagrangiennes de  $\mathbb{C}^2$* ,  
J. Math. Pures et Appl., **66** (1987), pp. 321-335.
- [4] CHEN (B.Y.) and TAZAWA (Y.) .— *Slant submanifolds in complex Euclidean spaces*,  
preprint.
- [5] CHERN (S.S.) .— *Minimal surfaces in a Euclidean space of  $N$  dimensions*,  
Differential and Combinatorial Topology, Princeton Univ. Press., (1965),  
pp. 187-198.
- [6] CHERN (S.S.) and WOLFSON (J.G.) .— *Minimal surfaces by moving frames*,  
Amer. J. Math., **105** (1983), pp. 59-83.
- [7] HOFFMAN (D.A) and OSSERMAN (R.) .— *The Gauss map of surfaces in  $\mathbb{R}^3$  and  $\mathbb{R}^4$* ,  
Proc. London Math. Soc., (3) **50** (1985), pp. 27-56.
- [8] SINGER (I.M.) and THORPE (J.A.) .— *The curvature of 4-dimensional Einstein spaces*,  
Global Analysis, Princeton University Press, 1969, pp. 73-114.