ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

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Annales de la faculté des sciences de Toulouse 5e série, tome 12, nº 1 (1991), p. 37-60

<http://www.numdam.org/item?id=AFST_1991_5_12_1_37_0>

Annales de la Faculté des Sciences de Toulouse Vol. XII, n° 1, 1991

Periodic solutions of quadratic Lagrangian systems on p-convex sets

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RÉSUMÉ. — On prouve l'existence d'une infinité de solutions périodiques pour un système lagrangien quadratique sur une certaine classe d'ensembles non réguliers, c'est-à-dire les ensembles p-convexes. On emploie des méthodes variationnelles en analyse non linéaire et non régulière.

ABSTRACT. — We prove the existence of infinitely many periodic solutions for a quadratic Lagrangian system on a certain class of non-smooth sets, namely the p-convex sets. We use variational methods in non-smooth nonlinear analysis.

Introduction

If M is a compact submanifold without boundary in $\mathbb{R}^n$ and $N_x M$ denotes the normal subspace to M at x, the study of the Lagrangian system

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\nabla_v L(s,\gamma,\gamma')\right) - \nabla_q L(s,\gamma,\gamma') \in N_{\gamma(s)} M$$

has been carried out in [1], where the existence of infinitely many periodic solutions γ is proved under quite general assumptions.

The corresponding problem on manifolds with boundary has been treated in [18], where the existence of a periodic solution is proved and in [5], where the existence of infinitely many periodic solutions is shown. The feature of unilateral constraints (cf. [6], [7], [14], [15], [16], [17], [18]) is that, even if the manifold M is of class C^∞, the corresponding variational problem does not have a smooth structure.

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For this reason, it seems to be natural to allow for the set M itself a certain kind of irregularity. The aim of the paper is to treat the case in which M is a p-convex set (see Def. 1.3) and L is quadratic with respect to γ', namely

$$L(s,q,v) = \frac{1}{2}\left(a(s,q)v\,,v\right) - V(s,q)\,.$$

The particular case $a \equiv \mathrm{Id}$, $V \equiv 0$, which leads to the study of geodesics, was already treated in [2] and [3].

The main tools are the techniques of non-smooth nonlinear analysis developed in [8], [9], [10] and [11]. Actually, the main part of the paper, the second section, is devoted to the proof that these techniques can be applied to our situation.

1. Recalls of non-smooth analysis and the main result

We will recall some notions of non-smooth analysis as developed in [6], [7], [8], [9], [11].

From now on, H will be a real Hilbert space, $|\cdot|$ and $(\cdot\,,\cdot)$ its norm and scalar product, respectively.

If $u \in H$ and $r > 0$, we set $B(u,r) = \{v \in H \mid |v-u| < r\}$.

DEFINITION 1.1.— (see also [6] and [9]) *Let Ω be an open subset of H and $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ a map.*

We set

$$D(f) = \{u \in \Omega \mid f(u) < +\infty\}\,.$$

Let u belong to $D(f)$. The function f is said to be subdifferentiable at u if there exists $\alpha \in H$ such that

$$\liminf_{v \to u} \frac{f(v) - f(u) - (\alpha\,,\,v-u)}{|v-u|} \geq 0\,.$$

We denote by $\partial^- f(u)$ the (possibly empty) set of such α's and we set

$$D(\partial^- f) = \{u \in D(f) \mid \partial^- f(u) \neq \emptyset\}\,.$$

It is easy to check that $\partial^- f(u)$ is convex and closed $\forall\, u \in D(f)$.

If $u \in D(\partial^- f)$, $\text{grad}^- f(u)$ *will denote the element of minimal norm of* $\partial^- f(u)$.

Moreover, let M *be a subset of* H. *We denote by* I_M *the function:*

$$I_M(u) = \begin{cases} 0 & u \in M \\ +\infty & u \in H \setminus M. \end{cases}$$

It is easy to check that $\partial^- I_M(u)$ *is a closed convex cone* $\forall\, u \in M$.

We will call (outward) normal cone to M *at* u *the set* $\partial^- I_M(u)$ *and tangent cone to* M *at* u *its negative polar* $\left(\partial^- I_M(u)\right)^-$, *i.e.,*

$$\left(\partial^- I_M(u)\right)^- = \left\{v \in H \mid (v, w) \le 0\,,\ \forall\, w \in \partial^- I_M(u)\right\}.$$

Remark 1.2.— Let us suppose that $g : \Omega \to \mathbb{R}$ is Fréchet differentiable at $u \in \Omega$. Then:

$$\partial^-(f+g)(u) \neq \emptyset \text{ if and only if } \partial^- f(u) \neq \emptyset$$

and

$$\partial^-(f+g)(u) = \left\{\alpha + \text{grad}\, g(u) \mid \alpha \in \partial^- f(u)\right\}.$$

Let us introduce the class of p-convex sets as defined in [2] and [3]. An other characterization of this class is in [4].

DEFINITION 1.3.— *A subset* M *of* H *is said to be a* p*-convex set if there exists a continuous function* $p : M \to \mathbb{R}^+$ *such that*

$$(\alpha\,,\, v-u) \le p(u)|\alpha|\,|v-u|^2$$

whenever u, $v \in M$ *and* $\alpha \in \partial^- I_M(u)$.

Examples of p-convex sets are the following ones:

(1) The $C^{1,1}_{\text{loc}}$-submanifolds (possibly with boundary) of H;

(2) The convex subsets of H;

(3) The images under a $C^{1,1}_{\text{loc}}$- diffeomorphism of convex sets;

(4) The subset of $\mathbb{R}^n$: $\left\{x \mid \max|x_i| \le 1\,,\ \sum x_i^2 \ge 1\right\}$ (note that it is not included in the classes (1), (2), (3)).

Now, we can state the main result of the paper.

Let M be a p-convex subset of $\mathbb{R}^n$ and let $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a Lagrangian of the form

$$L(s,q,v) = \frac{1}{2}\left(a(s,q)v\,,\,v\right) - V(s,q)$$

where a, V are of class C^2 on $\mathbb{R} \times \mathbb{R}^n$ and the matrix $a(s,q)$ is symmetric and positive definite, that is there exists a constant $\nu > 0$ such that

$$\left(a(s,q)v\,,\,v\right) \geq \nu|v|^2\,, \quad \forall\, s \in \mathbb{R}\,,\ \forall\, q,\, v \in \mathbb{R}^n \tag{1.1}$$

Moreover, let us suppose that a and V are 1-periodic in the first variable:

$$a(s+1\,,\,q) = a(s,q)\,; \quad V(s+1\,,\,q) = V(s,q) \tag{1.2}$$

THEOREM 1.4.— *Let us suppose that M is compact, connected and noncontractible in itself and that either*

a) $\pi_1(M)$ has infinitely many conjugacy classes

or

b) $\pi_1(M)$ has a finite number of elements.

Then, there exists a sequence $\{\gamma_h\}_h \subset W^{2,\infty}(\mathbb{R}\,;\,\mathbb{R}^n)$ such that $\forall\, h \in \mathbb{N}$

i) γ_h is 1-periodic and $\gamma_h(s) \in M$

ii) $\dfrac{\mathrm{d}}{\mathrm{d}s}\left(\nabla_v L(s,\gamma_h,\gamma_h')\right) - \nabla_q L(s,\gamma_h,\gamma_h') \in \partial^- I_M(\gamma_h)$ a.e. *in* $]\,0\,,\,1\,[$

iii) $\lim_{h\to\infty} \int_0^1 L(s,\gamma_h,\gamma_h')\,\mathrm{d}s = +\infty$.

In order to apply the critical point theory for non-smooth functionals some other notions and results have to be recalled.

DEFINITION 1.5.— *Let Ω be an open subset of H and $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ a function. A point $u \in D(f)$ si said to be a lower critical point for f if $0 \in \partial^- f(u)$; $c \in \mathbb{R}$ is said to be a critical value of f if there exists $u \in D(f)$ such that*

$$0 \in \partial^- f(u) \quad \text{and} \quad f(u) = c\,.$$

DEFINITION 1.6.— (see also [8], [11]) *Let Ω be an open subset of H. A function $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ is said to have a φ-monotone subdifferential of order two if there exists a continuous function*

$$\chi : D(f)^2 \times \mathbb{R}^2 \to \mathbb{R}^+$$

such that

$$(\alpha - \beta\,,\,u - v) \geq -\chi\left(u, v, f(u), f(v)\right)\left(1 + |\alpha|^2 + |\beta|^2\right)|u - v|^2$$

whenever

$$u,\, v \in D(\partial^- f)\,, \quad \alpha \in \partial^- f(u) \quad \textit{and} \quad \beta \in \partial^- f(v)\,.$$

The notion of p-convex set is actually a particular case of the previous notion. In fact, it turns out (see [2]) that a subset M of H is p-convex if and only if I_M has a φ-monotone subdifferential of order two.

THEOREM 1.7.— (see [10]) *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function with a φ-monotone subdifferential of order two. We set*

$$d^*(u,v) = |u-v| + \big|f(u) - f(v)\big|\,, \quad \forall\, u,\, v \in D(f)\,.$$

Let us suppose that:

i) $\inf_H f > -\infty$

ii) every sequence $(u_h)_h \subset D(\partial^- f)$ with $\sup_h f(u_h) < +\infty$ and $\lim_h \operatorname{grad}^- f(u_h) = 0$ has a subsequence converging in H.

Then, f has at least $\operatorname{cat}\big(D(f)\,,\, d^*\big) = +\infty$*, then*

$$\sup\big\{f(u) \mid u \in D(\partial^- f)\,,\, 0 \in \partial^- f(u)\big\} = +\infty\,.$$

Let M be a p-convex subset of H.

DEFINITION 1.8.— *Let us denote by $\widehat{A}$ the set of u's $\in H$ with the two properties:*

i) $\delta_p(u,M) < 1$ *where* $\delta_p(u,M) = \limsup\limits_{\substack{|u-w| \to d(u,M) \\ w \in M}} 2p(w)|u-w|\,.$

ii) $\exists\, r \geq 0$ *such that* $M \cap \big\{v \in H \mid |v-u| \leq r\big\}$ *is closed in H and not empty.*

Obviously, $M \subset \widehat{A}$ and:

PROPOSITION 1.9.— (see prop. 2.9 in [2]) *Let $M \subset H$ be p-convex and locally closed. Then $\widehat{A}$ is open and $\forall\, u \in \widehat{A}$ there exists one and only one $w \in M$ such that $|u-w| = d(u,M)$.*

Moreover, if we set $\pi(u) = w$, then

i) $\big(u - \pi(u)\big) \in \partial^- I_M\big(\pi(u)\big)$ *and* $2p\big(\pi(u)\big)\big|u - \pi(u)\big| < 1$, $\forall\, u \in \widehat{A}$

ii) $|\pi(u_1) - \pi(u_2)| \leq \Big(1 - p(\pi(u_1))|u_1 - \pi(u_1)| +$

$$- p(\pi(u_2))|u_2 - \pi(u_2)|\Big)^{-1} |u_1 - u_2|,$$

$$\forall\, u_1,\, u_2 \in \widehat{A}$$

iii) $(t\pi(u) + (1-t)u) \in \widehat{A}$, $\forall\, u \in \widehat{A}$, $\forall\, t \in [0\,,1]$.

Remark 1.10.— Let us set $A = \Big\{u \in \widehat{A} \mid 4p(\pi(u))|u - \pi(u)| < 1\Big\}$. Then A is an open set containing M and, by proposition 1.9 ii), $\pi : A \to M$ is Lipschitz continuous of constant two.

PROPOSITION 1.11.— (see prop. 2.2 in [2]) *Let $M \subset H$ be p-convex. If $\{u_h\}_h \subset M$ is a sequence converging to $u \in M$ and $\{\alpha_h\}_h \subset H$ is a sequence converging weakly to α with $\alpha_h \in \partial^- I_M(u_h)$, then $\alpha \in \partial^- I_M(u)$.*

PROPOSITION 1.12.— (see prop. 2.12 in [2]) *Let $M \subset H$ be locally closed and p-convex. Then*

$$\lim_{t \to 0^+} \frac{\pi(u+tv) - u}{t} = P_u v\,, \quad \forall\, u \in M \quad \textit{and} \quad \forall\, v \in H$$

where P_u is the projection on the tangent cone to M at u.

PROPOSITION 1.13.— *Let $M \subset H$ be locally closed and p-convex. Let $\{u_h\}_h$ be a sequence in M converging to $u \in M$ and let $\tau \in \big(\partial^- I_M(u)\big)^-$. Then*

$$\lim_h P_{u_h}\tau = \tau\,.$$

Proof.— Since $\{P_{u_h}\tau\}_h$ is bounded, up to a subsequence $\{P_{u_h}\tau\}_h$ is weakly convergent to some $\xi \in H$. Since $(\tau - P_{u_h}\tau) \in \partial^- I_M(u_h)$, by proposition 1.11 we have $(\tau - \xi) \in \partial^- I_M(u)$. Therefore $(\tau - \xi\,,\tau) \leq 0$, which implies $|\tau| \leq |\xi|$, hence

$$\liminf_h |P_{u_h}\tau| \geq |\tau|\,.$$

From the equality

$$|\tau|^2 = |P_{u_h}\tau|^2 + |\tau - P_{u_h}\tau|^2$$

the thesis follows. □

2. The variational structure of the problem

In this section, we want to supply our problem, that is the research of periodic orbits of the considered Lagrangian system, with a variational structure. Our aim is to characterize such periodic orbits as lower critical points of the functional

$$f : L^2(0,1;\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$$

defined in the following way:

$$f(\gamma) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 \big(a(s,\gamma)\gamma' \, , \gamma'\big)\, \mathrm{d}s - \int_0^1 V(s,\gamma)\, \mathrm{d}s & \gamma \in X \\ +\infty & \gamma \in L^2(0,1;\mathbb{R}^n)\backslash X \end{cases}$$

where the space of the admissible paths is:

$$X = \left\{\gamma \in W^{1,2}(0,1;\mathbb{R}^n) \mid \gamma(s) \in M \,,\ \gamma(0) = \gamma(1)\right\}.$$

Since M is compact, we shall assume the function p of definition 1.3 to be constant.

Moreover, if $\gamma \in W^{1,2}(0,1;\mathbb{R}^n)$ with $\gamma(s) \in M$ and $\delta \in L^2(0,1;\mathbb{R}^n)$, we set

$$(P_\gamma \delta)(s) = P_{\gamma(s)}\delta(s)$$

where $P_{\gamma(s)}$ is the projection on the tangent cone to M at $\gamma(s)$, according to the scalar product

$$\big(u, v\big)_s = \big(a\big(s, \gamma(s)\big)u \, , \, v\big) \ .$$

Let us also denote by π_s the projection on M according to the scalar product $\big(u, v\big)_s$. By remark 1.8 and the assumptions on a, there exists an open set A containing M such that each π_s is defined on A and is Lipschitz continuous of constant 2.

Let us begin with a regularity result.

THEOREM 2.1.— *Let us take $\gamma \in X$. If $\partial^- f(\gamma) \neq \emptyset$ then*

$$\gamma \in W^{2,2}(0,1;\mathbb{R}^n)\,, \quad \gamma'_+(0) = \gamma'_-(1)$$

and

$$\left|\gamma''(s)\right| \leq \text{const}\left(1 + \left|\alpha(s)\right| + \left|\gamma'(s)\right|^2\right) \text{ a.e. }, \quad \forall\, \alpha \in \partial^- f(\gamma)\,.$$

Moreover, if $0 \in \partial^- f(\gamma)$ *then* $\gamma \in W^{2,\infty}(0,1;\mathbb{R}^n)$.

For the proof of this theorem, we need some lemmas.

LEMMA 2.2.— *Let us take* $\delta \in W^{1,2}(0,1;\mathbb{R}^n)$ *and* $\gamma \in W^{1,2}(0,1;\mathbb{R}^n)$ *such that* $\gamma(s) \in M$, $\forall\, s \in [\,0\,,\,1\,]$.

Then the following facts hold:

a) $\displaystyle\lim_{t\to 0^+} \frac{\pi_s(\gamma + t\delta) - \gamma}{t} = P_\gamma \delta, \quad \forall\, s \in [\,0\,,\,1\,]$

and

$\displaystyle\limsup_{t\to 0^+} \left\| \frac{\pi_s(\gamma + t\delta) - \gamma}{t} \right\|_{L^\infty} < \infty$

b) for every sufficiently small $t > 0$, *we have*

$\pi_s(\gamma + t\delta) \in W^{1,2}(0,1;\mathbb{R}^n)$

and a.e. in $]\,0\,,\,1\,[$

$\left|\left(\pi_s(\gamma + t\delta)\right)'\right|_s \leq \left|\gamma' + t\delta'\right|_s + \text{const}\left|(\gamma + t\delta) - \pi_s(\gamma + t\delta)\right|\left(1 + |\gamma'| + t|\delta'|\right)$

c) $\displaystyle\lim_{t\to 0^+} \left(\pi_s(\gamma + t\delta)\right)' = \gamma'$ *in* $L^2(0,1;\mathbb{R}^n)$.

Proof.— a) First of all, let us remark that $P_\gamma\delta$ is measurable. By proposition 1.12, we have

$$\lim_{t\to 0^+} \frac{\pi_s(\gamma + t\delta) - \gamma}{t} = P_\gamma \delta$$

and, also

$$\left|\frac{\pi_s(\gamma + t\delta) - \gamma}{t}\right|_s \leq 2\left|\frac{\gamma + t\delta - \gamma}{t}\right|_s \leq 2\left|\delta\right|_s \tag{2.2.1}$$

Then, by (2.2.1), we get

$$\begin{aligned} \left|\frac{\pi_s(\gamma + t\delta) - \gamma}{t}\right| &\leq \text{const}\left|\frac{\pi_s(\gamma + t\delta) - \gamma}{t}\right|_s \leq \\ &\leq \text{const}\, 2\left|\delta\right|_s \leq \\ &\leq \text{const}\, |\delta| \end{aligned} \tag{2.2.2}$$

so, the proof of a) is over.

b) Let us consider the two scalar products

$$\big(u, v\big)_{s_i} = (A_{s_i} u,\, v) = \big(a(s_i,\, \gamma(s_i))u,\, v\big)\ , \quad i = 1,\, 2\, .$$

For every $u \in A$, let $w_{s_i} = \pi_{s_i} u$ the projection of u according to the scalar product $\big(u, v\big)_{s_i}$, $i = 1, 2$. We want to prove that

$$|w_{s_1} - w_{s_2}| \le \frac{2}{\nu}\, |u - w_{s_2}||A_{s_1} - A_{s_2}|\, . \tag{2.2.3}$$

Let us observe that, by proposition 1.9, $u - w_{s_1} \in \partial^-_{s_1} I_M(w_{s_1})$, that is

$$\liminf_{v \to u,\, v \in M} \frac{\big(u - w_{s_1}\,,\, w_{s_1} - v\big)_{s_1}}{\big|w_{s_1} - v\big|_{s_1}} \ge 0\, . \tag{2.2.4}$$

Passing to the usual metric in $\mathbb{R}^n$, (2.2.4) is equivalent to

$$\liminf_{v \to u,\, v \in M} \frac{\big(A_{s_1}(u) - A_{s_1}(w_{s_1})\,,\, w_{s_1} - v\big)}{\big|w_{s_1} - v\big|} \ge 0\, . \tag{2.2.5}$$

By (2.2.5), it is easy to deduce that

$$A_{s_1}(u) - A_{s_1}(w_{s_1}) \in \partial^- I_M(w_{s_1})\, .$$

Analogously,

$$A_{s_2}(u) - A_{s_2}(w_{s_2}) \in \partial^- I_M(w_{s_2})\, .$$

Since M is a p-convex set, we have:

$$\begin{aligned}&\big(A_{s_1}(u) - A_{s_1}(w_{s_1}) - A_{s_2}(u) + A_{s_2}(w_{s_2})\,,\, w_{s_1} - w_{s_2}\big) \ge \\ &\quad \ge -p\left(\big|A_{s_1}(u) - A_{s_1}(w_{s_1})\big| + \big|A_{s_2}(u) - A_{s_2}(w_{s_2})\big|\right) |w_{s_1} - w_{s_2}|^2.\end{aligned} \tag{2.2.6}$$

On the other hand, we have

$$\begin{aligned}&\big(A_{s_1}(u) - A_{s_1}(w_{s_1}) - A_{s_2}(u) + A_{s_2}(w_{s_2})\,,\, w_{s_1} - w_{s_2}\big) \le \\ &\quad \le \big(A_{s_1}(u) - A_{s_2}(u) + A_{s_2}(w_{s_2}) - A_{s_1}(w_{s_2})\,,\, w_{s_1} - w_{s_2}\big) + \\ &\quad\quad + \big(A_{s_1}(w_{s_2}) - A_{s_1}(w_{s_1})\,,\, w_{s_1} - w_{s_2}\big) \le \\ &\quad \le \big((A_{s_1} - A_{s_2})(u - w_{s_2})\,,\, w_{s_1} - w_{s_2}\big) - \big|w_{s_1} - w_{s_2}\big|^2_{s_1}\, .\end{aligned} \tag{2.2.7}$$

By (2.2.6) and (2.2.7), we obtain

$$\begin{aligned}&|w_{s_1}-w_{s_2}|_{s_1}^2+\\ &\quad -p\Big(\big|A_{s_1}(u)-A_{s_1}(w_{s_1})\big|+\big|A_{s_2}(u)-A_{s_2}(w_{s_2})\big|\Big)\,|w_{s_1}-w_{s_2}|^2\le\\ &\quad\le |u-w_{s_2}|\,|A_{s_1}-A_{s_2}|\,|w_{s_1}-w_{s_2}|\,.\end{aligned}$$

By hypothesis (1.1), it follows

$$\big|w_{s_1}-w_{s_2}\big|_{s_1}^2\ge\nu\big|w_{s_1}-w_{s_2}\big|^2$$

and then

$$\begin{aligned}&\Big[\nu-p\Big(\big|A_{s_1}(u)-A_{s_1}(w_{s_1})\big|+\big|A_{s_2}(u)-A_{s_2}(w_{s_2})\big|\Big)\Big]\,|w_{s_1}-w_{s_2}|^2\le\\ &\quad\le |u-w_{s_2}|\,|A_{s_1}-A_{s_2}|\,|w_{s_1}-w_{s_2}|\,.\end{aligned}\tag{2.2.8}$$

By substituting A with a smaller open set containing M, we can assume that

$$\Big[\nu-p\Big(\big|A_{s_1}(u)-A_{s_1}(w_{s_1})\big|+\big|A_{s_2}(u)-A_{s_2}(w_{s_2})\big|\Big)\Big]\ge\frac{\nu}{2}\,,$$

hence (2.2.3) follows.

Now, let us consider $s_1,\ s_2\in\,]\,0\,,1\,[$. We have

$$\begin{aligned}&\frac{\big|\pi_{s_1}(\gamma+t\delta)(s_1)-\pi_{s_2}(\gamma+t\delta)(s_2)\big|_{s_1}}{\big|s_2-s_1\big|_{s_1}}\le\\ &\quad\le\frac{\big|\pi_{s_1}(\gamma+t\delta)(s_1)-\pi_{s_1}(\gamma+t\delta)(s_2)\big|_{s_1}}{\big|s_2-s_1\big|_{s_1}}+\\ &\quad+\frac{\big|\pi_{s_1}(\gamma+t\delta)(s_2)-\pi_{s_2}(\gamma+t\delta)(s_2)\big|_{s_1}}{\big|s_2-s_1\big|_{s_1}}\,.\end{aligned}$$

By i) and ii) of proposition 1.9, we have

$$\big|\pi_{s_1}(\gamma+t\delta)(s_1)-\pi_{s_1}(\gamma+t\delta)(s_2)\big|_{s_1}\le\frac{\big|(\gamma+t\delta)(s_1)-(\gamma+t\delta)(s_2)\big|_{s_1}}{\mathfrak{D}}$$

where

$$\begin{aligned}\mathfrak{D}=\Big(1-p\big|(\gamma+t\delta)(s_1)-\pi_{s_1}(\gamma+t\delta)(s_1)\big|_{s_1}+\\ -p\big|(\gamma+t\delta)(s_2)-\pi_{s_1}(\gamma+t\delta)(s_2)\big|_{s_1}\Big)\,.\end{aligned}$$

Hence

$$\begin{aligned}&\left|\pi_{s_1}(\gamma+t\delta)(s_1)-\pi_{s_1}(\gamma+t\delta)(s_2)\right|_{s_1}\le\\&\quad\le \text{const}\,\left(\left|\gamma(s_1)-\gamma(s_2)\right|+t\left|\delta(s_1)-\gamma(s_2)\right|\right)\,.\end{aligned}$$

Moreover, by applying (2.2.3), we get

$$\begin{aligned}&\left|\pi_{s_1}(\gamma+t\delta)(s_2)-\pi_{s_2}(\gamma+t\delta)(s_2)\right|_{s_1}\le\\&\quad\le\frac{2}{\nu}\left|(\gamma+t\delta)(s_2)-\pi_{s_2}(\gamma+t\delta)(s_2)\right|\left|a\big(s_1,\gamma(s_1)\big)-a\big(s_2,\gamma(s_2)\big)\right|\le\\&\quad\le\text{const}\left|(\gamma+t\delta)(s_2)-\pi_{s_2}(\gamma+t\delta)(s_2)\right|\left(\left|s_2-s_1\right|+\left|\gamma(s_2)-\gamma(s_1)\right|\right)\le\\&\quad\le\text{const}\,\left(\left|s_2-s_1\right|+\left|\gamma(s_2)-\gamma(s_1)\right|\right)\,.\end{aligned}$$

Therefore

$$\pi_s(\gamma+t\delta)\in W^{1,2}(0,1;\mathbb{R}^n)$$

and we have a.e. in $]\,0\,,\,1\,[$

$$\begin{aligned}&\lim_{s_2\to s_1}\frac{\left|\pi_{s_1}(\gamma+t\delta)(s_1)-\pi_{s_1}(\gamma+t\delta)(s_2)\right|_{s_1}}{\left|s_2-s_1\right|_{s_1}}\le\\&\quad\le\frac{\left|(\gamma+t\delta)'(s_1)\right|_{s_1}}{\left(1-2p\left|(\gamma+t\delta)(s_1)-\pi_{s_1}(\gamma+t\delta)(s_1)\right|_{s_1}\right)}\le\\&\quad\le\left|(\gamma+t\delta)'(s_1)\right|_{s_1}+\\&\qquad+\text{const}\left|(\gamma+t\delta)(s_1)-\pi_{s_1}(\gamma+t\delta)(s_1)\right|\left|(\gamma+t\delta)'(s_1)\right|\end{aligned}\tag{2.2.9}$$

and

$$\begin{aligned}&\lim_{s_2\to s_1}\frac{\left|\pi_{s_1}(\gamma+t\delta)(s_2)-\pi_{s_2}(\gamma+t\delta)(s_2)\right|_{s_1}}{\left|s_2-s_1\right|_{s_1}}\le\\&\quad\le\text{const}\lim_{s_2\to s_1}\left|(\gamma+t\delta)(s_2)-\pi_{s_2}(\gamma+t\delta)(s_2)\right|\times\\&\qquad\times\frac{\left(\left|s_2-s_1\right|+\left|\gamma(s_2)-\gamma(s_1)\right|\right)}{\left|s_2-s_1\right|}\le\\&\quad\le\text{const}\left|(\gamma+t\delta)(s_1)-\pi_{s_1}(\gamma+t\delta)(s_1)\right|\left(1+\left|\gamma'(s_1)\right|\right)\,.\end{aligned}\tag{2.2.10}$$

Hence we have a.e. in $]\,0\,,\,1\,[$

$$\left|\big(\pi_s(\gamma+t\delta)\big)'\right|_s\le\left|\gamma'+t\delta'\right|_s+\text{const}\left|(\gamma+t\delta)-\pi_s(\gamma+t\delta)\right|\left(1+|\gamma'|+t|\delta'|\right).$$

c) In $L^2(0,1;\mathbb{R}^n)$ we can consider the following scalar product

$$(\eta,\xi)=\int_0^1 a\big(s,\gamma(s)\big)\eta(s)\xi(s)\,\mathrm{d}s=\int_0^1\big(\eta(s)\,,\,\xi(s)\big)_s\,\mathrm{d}s\,.$$

Since, by a) $\pi_s(\gamma+t\delta)\to\gamma$ in L^∞ and $\big(\pi_s(\gamma+t\delta)\big)'$ is bounded in $L^2(0,1;\mathbb{R}^n)$ as $t\to 0$, we have that

$$\big(\pi_s(\gamma+t\delta)\big)' \text{ weakly converges to } \gamma' \text{ in } L^2(0,1;\mathbb{R}^n)\,. \tag{2.2.11}$$

Moreover

$$\lim_{t\to 0^+}\big|(\gamma+t\delta)-\pi_s(\gamma+t\delta)\big|=0 \text{ uniformly on } [\,0\,,\,1\,]\,,$$

and by b) we have

$$\limsup_{t\to 0}\int_0^1\Big|\big(\pi_s(\gamma+t\delta)\big)'\Big|_s^2\,\mathrm{d}s\le\int_0^1|\gamma'|_s^2\,\mathrm{d}s\,. \tag{2.2.12}$$

Combining (2.2.11) and (2.2.12), we get

$$\lim_{t\to 0^+}\big(\pi_s(\gamma+t\delta)\big)'=\gamma' \text{ in } L^2(0,1;\mathbb{R}^n)\,.\ \square$$

LEMMA 2.3.— *Let us take $\delta\in W^{1,2}(0,1;\mathbb{R}^n)$ and $\gamma\in W^{1,2}(0,1;\mathbb{R}^n)$ such that $\gamma(s)\in M$, $\forall\ s\in[\,0\,,\,1\,]$.*

Then

$$\begin{aligned}\liminf_{t\to 0^+}\frac{1}{t}\bigg\{\frac{1}{2}\int_0^1&\big(a(s,\gamma)(\gamma+t\delta)'\,,\,(\gamma+t\delta)'\big)\,\mathrm{d}s\,+\\ &-\frac{1}{2}\int_0^1\Big(a(s,\gamma)\big(\pi_s(\gamma+t\delta)\big)'\,,\,\big(\pi_s(\gamma+t\delta)\big)'\Big)\,\mathrm{d}s\bigg\}\ge\\ &\ge-\mathrm{const}\int_0^1|\gamma'|\big(1+|\gamma'|\big)|\delta-P_\gamma\delta|\,\mathrm{d}s\,.\end{aligned} \tag{2.3.1}$$

Proof.— Let us fix $t>0$ and let us take the path $\gamma+t\delta$. If t is small, $(\gamma+t\delta)(s)\in A$, $\forall\ s\in[\,0\,,\,1\,]$.

By lemma 2.2 b), we have $\pi_s(\gamma+t\delta)\in W^{1,2}(0,1;\,M)$ and

$$\begin{aligned}\frac{1}{2}\int_0^1\frac{1}{t}&\bigg\{\big|(\gamma+t\delta)'\big|_s^2-\Big|\big(\pi_s(\gamma+t\delta)\big)'\Big|_s^2\bigg\}\,\mathrm{d}s\ge\\ &\ge\frac{1}{2}\int_0^1\frac{1}{t}\Big\{-\mathrm{const}\big|(\gamma+t\delta)-\pi_s(\gamma+t\delta)\big|^2\big(1+|\gamma'|+t|\delta'|\big)^2\,+\\ &\quad-\mathrm{const}\big|(\gamma+t\delta)'\big|\big|(\gamma+t\delta)-\pi_s(\gamma+t\delta)\big|\big(1+|\gamma'|+t|\delta'|\big)\Big\}\,\mathrm{d}s\,.\end{aligned}$$

Combining lemma 2.2 c) with Lebesgue theorem, we get:

$$\liminf_{t\to 0^+} \frac{1}{2}\int_0^1 \frac{1}{t}\left\{\left|(\gamma+t\delta)'\right|_s^2 - \left|\left(\pi_s(\gamma+t\delta)\right)'\right|_s^2\right\} \mathrm{d}s \geq$$
$$\geq -\operatorname{const}\int_0^1 |\gamma'|\big(1+|\gamma'|\big)|\delta - P_\gamma\delta| \,\mathrm{d}s\,. \ \square$$

LEMMA 2.4.— *Let us take* $\delta \in W^{1,2}(0,1;\mathbb{R}^n)$ *with* $\delta(0) = \delta(1)$ *and* $\alpha \in \partial^- f(\gamma)$. *Then*

$$\int_0^1 \big(a(s,\gamma)\gamma'\,,\,\delta'\big)\,\mathrm{d}s - \int_0^1 (\alpha,\, P_\gamma\delta)\,\mathrm{d}s \geq$$
$$\geq -\operatorname{const}\int_0^1 |\gamma'|\big(1+|\gamma'|\big)|\delta - P_\gamma\delta|\,\mathrm{d}s +$$
$$+\int_0^1 \big(\nabla_q V(s,\gamma)\,,\,P_\gamma\delta\big)\,\mathrm{d}s - \frac{1}{2}\int_0^1 \left(\frac{\partial a(s,\gamma)}{\partial q}(P_\gamma\delta)\gamma'\,,\,\gamma'\right)\mathrm{d}s\,. \quad (2.4.1)$$

Proof.— Let us take $\delta \in W^{1,2}(0,1;\mathbb{R}^n)$ with $\delta(0) = \delta(1)$ and $t > 0$ small enough that

$$\pi_s(\gamma + t\delta) \in M\,.$$

Let us observe that, by setting

$$f_1(\gamma) = \frac{1}{2}\int_0^1 \big(a(s,\gamma)\gamma'\,,\,\gamma'\big)\,\mathrm{d}s\,,$$

from remark 1.2, we deduce that $\alpha \in \partial^- f(\gamma)$ if and only if $\alpha = \widetilde{\alpha} - \nabla_q V(s,\gamma)$ where $\widetilde{\alpha} \in \partial^- f_1(\gamma)$.

Now, let us take $\alpha \in \partial^- f(\gamma)$. By proposition 1.12, we have:

$$\int_0^1 \big(a(s,\gamma)\gamma'\,,\,\delta'\big)\,\mathrm{d}s - \int_0^1 (\alpha,\, P_\gamma\delta)\,\mathrm{d}s =$$
$$= \lim_{t\to 0^+}\frac{1}{t}\int_0^1\left\{\frac{1}{2}\big(a(s,\gamma)(\gamma+t\delta)',\,(\gamma+t\delta)'\big) - \frac{1}{2}\big(a(s,\gamma)\gamma'\,,\,\gamma'\big) + \right.$$
$$\left. -\,\widetilde{\alpha}\big(\pi_s(\gamma+t\delta)-\gamma\big)\right\}\mathrm{d}s + \int_0^1 \big(\nabla_q V(s,\gamma)\,,\,P_\gamma\delta\big)\,\mathrm{d}s \geq$$
$$\geq \liminf_{t\to 0^+}\frac{1}{t}\int_0^1\left\{\frac{1}{2}\Big(a\big(s,\ \pi_s(\gamma+t\delta)\big)\big(\pi_s(\gamma+t\delta)\big)',\,\big(\pi_s(\gamma+t\delta)\big)'\Big) + \right.$$

$$- \frac{1}{2}\big(a(s,\gamma)\gamma' \,,\, \gamma'\big) - \widetilde{\alpha}\big(\pi_s(\gamma + t\delta) - \gamma\big)\Big\} \,\mathrm{d}s \,+$$

$$+ \liminf_{t\to 0^+} \frac{1}{t} \int_0^1 \Big\{ \frac{1}{2} \Big(a(s,\gamma)\big(\pi_s(\gamma + t\delta)\big)' \,,\, \big(\pi_s(\gamma + t\delta)\big)'\Big) \,+$$

$$- \frac{1}{2} \Big(a\big(s \,,\, \pi_s(\gamma + t\delta)\big)\big(\pi_s(\gamma + t\delta)\big)' \,,\, \big(\pi_s(\gamma + t\delta)\big)'\Big)\Big\} \,\mathrm{d}s \,+$$

$$+ \liminf_{t\to 0^+} \frac{1}{t} \int_0^1 \Big\{ \frac{1}{2}\big(a(s,\gamma)(\gamma + t\delta)' \,,\, (\gamma + t\delta)'\big) \,+$$

$$- \frac{1}{2} \Big(a(s,\gamma)\big(\pi_s(\gamma + t\delta)\big)' \,,\, \big(\pi_s(\gamma + t\delta)\big)'\Big)\Big\} \,\mathrm{d}s \,+$$

$$+ \int_0^1 \big(\nabla_q V(s,\gamma) \,,\, P_\gamma \delta\big) \,\mathrm{d}s \,.$$

Recalling that $\big(\pi_s(\gamma + t\delta) - \gamma\big)/t$ is bounded in $L^2(0,1;\mathbb{R}^n)$, by lemma 2.2 c), proposition 1.12 and Lebesgue theorem, we get

$$\liminf_{t\to 0^+} \frac{1}{t} \int_0^1 \Big\{ \frac{1}{2} \Big(a(s,\gamma)\big(\pi_s(\gamma + t\delta)\big)' \,,\, \big(\pi_s(\gamma + t\delta)\big)'\Big) \,+$$

$$- \frac{1}{2} \Big(a\big(s \,,\, \pi_s(\gamma + t\delta)\big)\big(\pi_s(\gamma + t\delta)\big)' \,,\, \big(\pi_s(\gamma + t\delta)\big)'\Big)\Big\} \,\mathrm{d}s \geq$$

$$\geq -\frac{1}{2} \int_0^1 \Big(\frac{\partial a(s,\gamma)}{\partial q} \,(P_\gamma \delta)\gamma' \,,\, \gamma' \Big) \,\mathrm{d}s \,.$$

Then, by definition 1.1 and lemma 2.3, we get the thesis. □

LEMMA 2.5.— *Let us take* $\alpha \in L^2(0,1;\mathbb{R}^n)$ *and* $\gamma \in W^{1,2}(0,1;\mathbb{R}^n)$ *such that* $\gamma(s) \in M$, $\forall\, s \in [0\,,1]$. *Let us suppose that (2.4.1) holds* $\forall\, \delta \in W^{1,2}(0,1;\mathbb{R}^n)$ *with* $\delta(0) = \delta(1)$.

Then

i) $\gamma \in W^{2,2}(0,1;\mathbb{R}^n)\,;\quad \gamma'_+(0) = \gamma'_-(1)$

ii) $\alpha + \frac{\mathrm{d}}{\mathrm{d}s}\big(a(s,\gamma)\gamma'\big) + \nabla_q V(s,\gamma) - \frac{1}{2}\,\frac{\partial a(s,\gamma)}{\partial q}\,(\gamma',\gamma') \in \partial^- I_M\big(\gamma(s)\big)$ a.e.

iii) $|\gamma''(s)| \leq \mathrm{const}\Big(1 + |\alpha(s)| + |\gamma'(s)|^2\Big)$ a.e.

Proof. — Since $|\delta - P_\gamma \delta| \le |\delta|$, by applying Cauchy-Schwartz inequality to (2.4.1), we obtain $\forall\ \delta \in W^{1,2}(0,1;\mathbb{R}^n)$ with $\delta(0)=\delta(1)$

$$\int_0^1 \big(a(s,\gamma)\gamma'\,,\,\delta'\big)\,\mathrm{d}s \ge -\|\alpha\|_{L^2}\|\delta\|_{L^2} - \mathrm{const}\int_0^1 |\gamma'|\,\big|\big(1+|\gamma'|\big)\big|\,|\delta|\,\mathrm{d}s + \\ - \int_0^1 \big|\nabla_q V(s,\gamma)\big|\,|\delta|\,\mathrm{d}s - \frac{1}{2}\int_0^1 \left|\frac{\partial a(s,\gamma)}{\partial q}\right| |\delta|\,|\gamma'|^2\,\mathrm{d}s$$

and then

$$\left|\int_0^1 \big(a(s,\gamma)\gamma'\,,\,\delta'\big)\right|\mathrm{d}s \le \left(\|\alpha\|_{L^2} + \mathrm{const}\int_0^1 |\gamma'|\,\mathrm{d}s + \right. \\ \left. + \ \mathrm{const}\int_0^1 |\gamma'|^2\,\mathrm{d}s + \mathrm{const}\right)\|\delta\|_{L^\infty}\,. \quad (2.5.1)$$

By (2.5.1), we deduce that $a(s,\gamma)\gamma' \in L^\infty$ and

$$\|a(s,\gamma)\gamma'\|_{L^\infty} \le \|a(s,\gamma)\gamma'\|_{L^1} + \|\alpha\|_{L^2} + \\ + \mathrm{const}\int_0^1 |\gamma'|\,\mathrm{d}s + \mathrm{const}\int_0^1 |\gamma'|^2\,\mathrm{d}s + \mathrm{const}\,. \quad (2.5.2)$$

On the hand, by hypothesis (1.1)

$$\nu|\gamma'|^2 \le \big(a(s,\gamma)\gamma'\,,\,\gamma'\big) \le \big|a(s,\gamma)\gamma'\big|\,|\gamma'|$$

which implies

$$\nu\|\gamma'\|_{L^\infty} \le \|a(s,\gamma)\gamma'\|_{L^\infty}\,.$$

Thus, $\gamma' \in L^\infty$ and

$$\|\gamma'\|_{L^\infty} \le \frac{1}{\nu}\left(\|a(s,\gamma)\gamma'\|_{L^1} + \|\alpha\|_{L^2} + \mathrm{const}\int_0^1 |\gamma'|\,\mathrm{d}s + \right. \\ \left. + \mathrm{const}\int_0^1 |\gamma'|^2\,\mathrm{d}s + \mathrm{const}\right)\,. \quad (2.5.3)$$

By using (2.5.3), we have

$$\int_0^1 |\gamma'|\big(1+|\gamma'|\big)|\delta|\,\mathrm{d}s \le \|\delta\|_{L^2}\|\gamma'\|_{L^2} + \|\delta\|_{L^2}\|\gamma'\|_{L^4}^2 \le \\ \le \|\delta\|_{L^2}\|\gamma'\|_{L^2} + \|\delta\|_{L^2}\|\gamma'\|_{L^2}\|\gamma'\|_{L^\infty} \le \\ \le \|\gamma'\|_{L^2}\left(1+\|\gamma'\|_{L^\infty}\right)\|\delta\|_{L^2} \le \\ \le \|\gamma'\|_{L^2}\left(1 + \frac{1}{\nu}\left(\mathrm{const}\|\gamma'\|_{L^1} + \|\alpha\|_{L^2} + \right.\right. \\ \left.\left. + \ \mathrm{const}\|\gamma'\|_{L^2}^2 + \mathrm{const}\right)\right)\|\delta\|_{L^2}\,. \quad (2.5.4)$$

Since,

$$\frac{1}{2}\int_0^1 \left|\frac{\partial a(s,\gamma)}{\partial q}\right| |\delta|\,|\gamma'|^2\,\mathrm{d}s \le \mathrm{const}\|\delta\|_{L^2}\|\gamma'\|_{L^4}^2$$

by (2.4.1) and (2.5.4), we have

$$\begin{aligned}
&\left|\int_0^1 \big(a(s,\gamma)\gamma'\,,\,\delta'\big)\right| \mathrm{d}s \le \\
&\quad \le \Big[\|\alpha\|_{L^2} + \mathrm{const}\|\gamma'\|_{L^2}\Big(1+\|\gamma'\|_{L^1}+\|\alpha\|_{L^2}+\|\gamma'\|_{L^2}^2\Big) + \\
&\qquad + \ \mathrm{const}\Big]\|\delta\|_{L^2} \le \\
&\quad \le \Big[\,\big(1+\mathrm{const}\|\gamma'\|_{L^2}\big)\,\|\alpha\|_{L^2} + \\
&\qquad + \mathrm{const}\|\gamma'\|_{L^2}\Big(1+\|\gamma'\|_{L^1}+\|\gamma'\|_{L^2}^2\Big) + \mathrm{const}\Big]\,\|\delta\|_{L^2}\,.
\end{aligned}$$

So, we can conclude that

$$a(s,\gamma)\gamma' \in W^{1,2}(0,1,\mathbb{R}^n)\,.$$

Moreover, by (2.4.1), we deduce that

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\big(a(s,\gamma)\gamma'\big)\right| \le \mathrm{const}\big(1+|\alpha|+|\gamma'|^2\big) \quad \text{a.e.} \quad \text{in }]\,0\,,\,1\,[\,.$$

It remains to prove that $\gamma' \in W^{1,2}(0,1;\mathbb{R}^n)$. By hypothesis (1.1), we have

$$\begin{aligned}
&\nu^2\frac{|\gamma'(s_2)-\gamma'(s_1)|^2}{|s_2-s_1|} \le \\
&\quad \le \frac{\big|a\big(s_1,\gamma(s_1)\big)\big(\gamma'(s_2)-\gamma'(s_1)\big)\big|^2}{|s_2-s_1|} = \frac{\mathfrak{N}}{|s_2-s_1|} \le \\
&\quad \le 2\frac{\big|a\big(s_2,\gamma(s_2)\big)\gamma'(s_2)-a\big(s_1,\gamma(s_1)\big)\gamma'(s_1)\big|^2}{|s_2-s_1|} + \\
&\qquad + 2\frac{\big|a\big(s_2,\gamma(s_2)\big)-a\big(s_1,\gamma(s_1)\big)\big|\,\big|\gamma'(s_2)\big|^2}{|s_2-s_1|}\,.
\end{aligned} \tag{2.5.6}$$

where

$$\begin{aligned}
\mathfrak{N} = \big|&a\big(s_2,\gamma(s_2)\big)\gamma'(s_2) - a\big(s_1,\gamma(s_1)\big)\gamma'(s_1) + \\
&- \big[a\big(s_2,\gamma(s_2)\big) - a\big(s_1,\gamma(s_1)\big)\big]\,\gamma'(s_2)\big|^2\,.
\end{aligned}$$

Since $a(\gamma)$, $a(s,\gamma)\gamma' \in W^{1,2}(0,1;\mathbb{R}^n)$ and $\gamma' \in L^\infty(0,1;\mathbb{R}^n)$, we deduce that

$$\gamma' \in W^{1,2}(0,1;\mathbb{R}^n)$$

and

$$|\gamma''| \leq \frac{1}{\nu}\left(\left|\frac{\mathrm{d}}{\mathrm{d}s}\big(a(s,\gamma)\gamma'\big)\right| + \left|\frac{\mathrm{d}}{\mathrm{d}s}\big(a(s,\gamma)\big)\right| |\gamma'|\right) \leq$$
$$\leq \mathrm{const}\big(1+|\alpha|+|\gamma'|^2\big) \quad \text{a.e.} \quad \text{in }]\,0\,,\,1\,[\,.$$

If we set

$$\widetilde{\gamma}(s) = \begin{cases} \gamma(s+\frac{1}{2}) & 0 \leq s \leq \frac{1}{2} \\ \gamma(s-\frac{1}{2}) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

it turns out that $\widetilde{\gamma}$ also satisfies (2.4.1) with a, V and α substituted by other suitable maps. It follows that

$$\widetilde{\gamma} \in W^{2,2}(0,1;\mathbb{R}^n)\,,$$

hence

$$\gamma'_+(0) = \gamma'_-(1)\,.$$

Since $\partial^- I_M\big(\gamma(s)\big)$ is a closed convex cone, to prove ii) is equivalent to prove that a.e. $\forall\ \eta \in \big(\partial^- I_M\big(\gamma(s)\big)\big)^-$

$$\left(\alpha + \frac{\mathrm{d}}{\mathrm{d}s}\big(a(s,\gamma)\gamma'\big) + \nabla_q V(s,\gamma) - \frac{1}{2}\,\frac{\partial a(s,\gamma)}{\partial q}(\gamma',\gamma')\,,\,\eta\right) \leq 0\,.$$

Let us define the following functions:

$$\delta_n(s) = \rho_n(s-s_0)\eta\,, \quad \forall\ n \in \mathbb{N}$$

where

$$\begin{cases} s_0 \in\,]\,0\,,\,1\,[\text{ is a Lebesgue point for } \alpha \text{ and for } \dfrac{\mathrm{d}}{\mathrm{d}s}\big(a(s,\gamma)\gamma'\big)\,; \\ \eta \in \big(\partial^- I_M\big(\gamma(s_0)\big)\big)^-\ ; \\ \rho_n \in C_0^\infty(\mathbb{R})\,, \quad \rho_n \geq 0\,, \quad \displaystyle\int_0^1 \rho_n\,\mathrm{d}s = 1 \quad \text{and} \quad \operatorname{supt}\rho_n \subset \left[-\frac{1}{n}\,,\,\frac{1}{n}\right]. \end{cases}$$

Clearly, $\delta_n \in W_0^{1,2}(0,1;\mathbb{R}^n)$ and then by (2.4.1) we get

$$\begin{aligned}
&-\left(\eta\,,\int_0^1 \rho_n(s_0-s)\frac{\mathrm{d}}{\mathrm{d}s}\big(a(s,\gamma)\gamma'\big)\,\mathrm{d}s\right) \geq \\
&\geq \int_0^1 \rho_n(s_0-s)\big(\alpha(s),P_\gamma\eta\big)(s)\,\mathrm{d}s + \\
&\quad - \text{const}\int_0^1 \rho_n(s_0-s)\big|\eta-P_\gamma\eta\big|\,|\gamma'|\,\mathrm{d}s + \\
&\quad - \text{const}\int_0^1 \rho_n(s_0-s)\big|\eta-P_\gamma\eta\big|\,|\gamma'|^2\,\mathrm{d}s + \\
&\quad + \int_0^1 \rho_n(s_0-s)\big(\nabla_q V(s,\gamma)\,,\,P_\gamma\eta\big)(s)\,\mathrm{d}s + \\
&\quad - \frac{1}{2}\int_0^1 \rho_n(s_0-s)\left(\frac{\partial a(s,\gamma)}{\partial q}\,(P_\gamma\eta)\gamma'\,,\,\gamma'\right)\,\mathrm{d}s\,. \qquad (2.5.7)
\end{aligned}$$

By proposition 1.13, $P_\gamma\delta$ is continuous at s_0, hence passing to the limit as $s \to s_0$ in (2.5.7), we obtain

$$\begin{aligned}
&-\left(\eta\,,\frac{\mathrm{d}}{\mathrm{d}s}\Big(a\big(s_0,\gamma(s_0)\big)\gamma'(s_0)\Big)\right) \geq \\
&\geq \big(\alpha(s_0)\,,\,(P_\gamma\eta)(s_0)\big) - \text{const}\,|\eta-P_\gamma\eta|(s_0)|\gamma'|^2(s_0) + \\
&\quad - \text{const}\,|\eta-P_\gamma\eta|(s_0)|\gamma'|(s_0) + \Big(\nabla_q V\big(s_0,\gamma(s_0)\big)\,,\,P_\gamma\eta\Big)(s_0) + \\
&\quad - \frac{1}{2}\left(\frac{\partial a\big(s_0,\gamma(s_0)\big)}{\partial q}\,(P_\gamma\eta)(s_0)\gamma'(s_0)\,,\,\gamma'(s_0)\right) = \\
&= \big(\alpha(s_0)\,,\,\eta(s_0)\big) + \Big(\nabla_q V\big(s_0,\gamma(s_0)\big)\,,\,P_\gamma\eta\Big)(s_0) + \\
&\quad - \frac{1}{2}\left(\frac{\partial a\big(s_0,\gamma(s_0)\big)}{\partial q}\,(P_\gamma\eta)(s_0)\gamma'(s_0)\,,\,\gamma'(s_0)\right) \quad \text{a.e. in }]\,0\,,1\,[.\ \square
\end{aligned}$$

Finally, we are able to prove theorem 2.1.

Proof of theorem 2.1

As a direct consequence of lemmas 2.4 and 2.5, we get

$$\gamma \in W^{2,2}(0,1;\mathbb{R}^n)\,,\quad \gamma'_+(0) = \gamma'_-(1)$$

and

$$\big|\gamma''(s)\big| \leq \text{const}\,\Big(1+\big|\alpha(s)\big|+\big|\gamma'(s)\big|^2\Big) \quad \text{a.e. in }]\,0\,,1\,[\,.$$

If $0 \in \partial^- f(\gamma)$, it is evident that

$$\gamma'' \in L^\infty(0,1;\mathbb{R}^n)\,. \ \square$$

Now, let us prove two properties of f.

THEOREM 2.6.— *The functional $f : L^2(0,1;\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and there exists a continuous function $\theta : \mathbb{R} \to \mathbb{R}$ such that*

$$f(\gamma+\delta) - f(\gamma) - \int_0^1 (\alpha, \delta)\,\mathrm{d}s \geq -\theta\big(f(\gamma)\big)\left(1 + \|\alpha\|_{L^2}^2\right)\|\delta\|_{L^2}^2$$

whenever γ, $\gamma+\delta \in X$ and $\alpha \in \partial^- f(\gamma)$.

In particular, f has a φ-monotone subdifferential of order two.

Proof.— Let us take a sequence $\{\gamma_n\}_n \subset X$ such that

$$\lim_n \gamma_n = \gamma \text{ in } L^2(0,1;\mathbb{R}^n) \text{ and } f(\gamma_n) \leq c\,.$$

In order to prove that f is lower semicontinuous, it is enough to prove that $f(\gamma) \leq c$. Since,

$$\frac{1}{2}\int_0^1 \big(a(s,\gamma_n)\gamma_n', \gamma_n'\big)\,\mathrm{d}s - \int_0^1 V(s,\gamma_n)\,\mathrm{d}s \leq c$$

and by hypothesis (1.1)

$$\int_0^1 \big(a(s,\gamma_n)\gamma_n', \gamma_n'\big)\,\mathrm{d}s \geq \nu \int_0^1 |\gamma_n'|^2\,\mathrm{d}s$$

we can deduce that $\{\gamma_n'\}_n$ is bounded in $L^2(0,1;\mathbb{R}^n)$ and thus, by the compactness of M, γ_n weakly converges to γ in $W^{1,2}(0,1;\mathbb{R}^n)$. Besides, L is continuous in the three variables and convex in the third one, so it is weakly lower semicontinuous in $W^{1,2}(0,1;\mathbb{R}^n)$. This implies

$$\frac{1}{2}\int_0^1 \big(a(s,\gamma)\gamma', \gamma'\big)\,\mathrm{d}s - \int_0^1 V(s,\gamma)\,\mathrm{d}s \leq$$
$$\leq \liminf_n \frac{1}{2}\int_0^1 \big(a(s,\gamma_n)\gamma_n', \gamma_n'\big)\,\mathrm{d}s - \int_0^1 V(s,\gamma_n)\,\mathrm{d}s \leq c\,.$$

Moreover, $\{\gamma_n\}_n$ converges uniformly to γ. Since M is closed, then $\gamma \in X$.

Now, let us take $\gamma \in X \cap W^{2,2}(0,1;\mathbb{R}^n)$, with $\gamma'_+(0) = \gamma'_-(1)$ and $\alpha \in \partial^- f(\gamma)$. Let $\delta \in W^{1,2}(0,1;\mathbb{R}^n)$ with $\delta(0) = \delta(1)$ be such that $\gamma+\delta \in X$. Then, by Taylor's formula, we have:

$$
\begin{aligned}
&f(\gamma+\delta) - f(\gamma) - \int_0^1 (\alpha, \delta)\,\mathrm{d}s = \\
&= \frac{1}{2}\int_0^1 \big(a(s,\gamma+\delta)(\gamma'+\delta')\,,\,(\gamma'+\delta')\big)\,\mathrm{d}s - \frac{1}{2}\int_0^1 \big(a(s,\gamma)\gamma'\,,\,\gamma'\big)\,\mathrm{d}s + \\
&\quad - \int_0^1 V(s,\gamma+\delta)\,\mathrm{d}s + \int_0^1 V(s,\gamma)\,\mathrm{d}s - \int_0^1 (\alpha,\delta)\,\mathrm{d}s = \\
&= \frac{1}{2}\int_0^1 \left(\frac{\partial a(s,\gamma)}{\partial q}(\delta)\gamma'\,,\,\gamma'\right)\mathrm{d}s - \int_0^1 \big(\nabla_q V(s,\gamma)\,,\,\delta\big)\,\mathrm{d}s + \\
&\quad + \int_0^1 \big(a(s,\gamma)\gamma'\,,\,\delta'\big)\,\mathrm{d}s - \int_0^1 (\alpha,\delta)\,\mathrm{d}s + \\
&\quad + \frac{1}{4}\int_0^1 \left(\frac{\partial^2 a(s,\widetilde{\gamma})}{\partial q^2}(\delta)^2\gamma'\,,\,\gamma'\right)\mathrm{d}s - \frac{1}{2}\int_0^1 \big(\nabla^2_{qq}V(s,\gamma)\delta\,,\,\delta\big)\,\mathrm{d}s + \\
&\quad + \frac{1}{4}\int_0^1 \left(\frac{\partial a(s,\widetilde{\gamma})}{\partial q}(\delta)\,,\,\delta'\right)\mathrm{d}s + \frac{1}{4}\int_0^1 \big(a(s,\gamma)\delta'\,,\,\delta'\big)\,\mathrm{d}s
\end{aligned}
$$

where $\widetilde{\gamma} = \gamma + t\delta$ for some $t = t(s) \in \,]\,0\,,\,1\,[$. Reordering terms, by hypothesis (1.1), we get:

$$
\begin{aligned}
&f(\gamma+\delta) - f(\gamma) - \int_0^1 (\alpha,\delta)\,\mathrm{d}s \geq \\
&\geq \frac{1}{2}\int_0^1 \left(\frac{\partial a(s,\gamma)}{\partial q}(\gamma',\gamma') - \nabla_q V(s,\gamma) - \frac{\mathrm{d}}{\mathrm{d}s}\big(a(s,\gamma)\gamma'\big) - \alpha\,,\,\delta\right)\mathrm{d}s + \\
&\quad - \left(\frac{1}{4}\left\|\frac{\delta^2 a(s,\gamma)}{\partial q^2}\right\|_{L^\infty}\|\gamma'\|^2_{L^2} + \frac{1}{2}\|\nabla_{qq}V(s,\gamma)\|_{L^\infty}\right)\|\delta\|^2_{L^\infty} + \\
&\quad - \frac{1}{4}\left\|\frac{\delta a(s,\gamma)}{\partial q}\right\|_{L^\infty}\|\delta\|_{L^2}\|\delta'\|_{L^2} + \frac{\nu}{2}\|\delta'\|^2_{L^2}\,.
\end{aligned}
$$

Thus, by p-convexity of M, we get:

$$
\begin{aligned}
&f(\gamma+\delta) - f(\gamma) - \int_0^1 (\alpha,\delta)\,\mathrm{d}s \geq \\
&\geq -\left(\frac{1}{2}\left\|\frac{\partial a(s,\gamma)}{\partial q}\right\|_{L^\infty}\|\gamma'\|_{L^2} + \|\nabla_q V(s,\gamma)\| + \right. \\
&\quad \left. + \left\|\frac{\mathrm{d}}{\mathrm{d}s}\big(a(s,\gamma)\big)\gamma' + a(s,\gamma)\gamma''\right\|_{L^1} + \|\alpha\|_{L^1}\right)\|\delta\|_{L^\infty} +
\end{aligned}
$$

$$
\begin{aligned}
&- \text{const}\|\gamma'\|_{L^2}^2\|\delta\|_{L^\infty}^2 - \text{const}\|\delta\|_{L^\infty}\|\delta'\|_{L^2} + \frac{\nu}{2}\|\delta'\|_{L^2}^2 \geq \\
&\geq - \text{const}\left(\|\gamma''\|_{L^1} + \|\alpha\|_{L^1}\right)\|\delta\|_{L^\infty} - \text{const}\|\gamma'\|_{L^2}^2\|\delta\|_{L^\infty}^2 + \\
&- \text{const}\|\delta\|_{L^\infty}\|\delta'\|_{L^2} + \frac{\nu}{2}\|\delta'\|_{L^2}^2 .
\end{aligned}
$$

Using the interpolation inequality:

$$
\|\delta\|_{L^\infty}^2 \leq \|\delta\|_{L^2}^2 + 2\|\delta\|_{L^2}\|\delta'\|_{L^2}
$$

we obtain:

$$
\begin{aligned}
&f(\gamma+\delta) - f(\gamma) - \int_0^1 (\alpha, \delta)\,\mathrm{d}s \geq \\
&\geq - \text{const}\left(\|\gamma''\|_{L^1} + \|\alpha\|_{L^1}\right)\left(\|\delta\|_{L^2}^2 + 2\|\delta\|_{L^2}\|\delta'\|_{L^2}\right) + \\
&- \text{const}\|\gamma'\|_{L^2}\left(\|\delta\|_{L^2}^2 + 2\|\delta\|_{L^2}\|\delta'\|_{L^2}\right) + \\
&- \text{const}\left(\|\delta\|_{L^2}\|\delta'\|_{L^2} + \|\delta\|_{L^2}^{1/2}\|\delta'\|_{L^2}^{3/2}\right) + \frac{\nu}{2}\|\delta'\|_{L^2}^2 .
\end{aligned}
$$

Thus, by theorem 2.1 and hypothesis 1.1, we are able to conclude that:

$$
f(\gamma+\delta) - f(\gamma) - \int_0^1 (\alpha, \delta)\,\mathrm{d}s \geq -\theta\big(f(\gamma)\big)\left(1 + \|\alpha\|^2\right)\|\delta\|_{L^2}^2 . \ \square
$$

THEOREM 2.7.— *Let us consider* $\alpha \in L^2(0,1;\mathbb{R}^n)$ *and* $\gamma \in X \cap W^{2,2}(0,1;\mathbb{R}^n)$ *with* $\gamma^+(0) = \gamma^-(1)$. *Then* $\alpha \in \partial^- f(\gamma)$ *if and only if*

$$
\alpha(s) + \frac{\mathrm{d}}{\mathrm{d}s}\big(a(s,\gamma)\gamma'\big) + \nabla_q V(s,\gamma) - \frac{1}{2}\frac{\partial a(s,\gamma)}{\partial q}(\gamma',\gamma') \in \partial^- I_M\big(\gamma(s)\big) \text{ a.e.}
$$

Proof.— If $\alpha \in \partial^- f(\gamma)$, we have the thesis by lemmas 2.4 and 2.5. Viceversa, if

$$
\alpha(s) + \frac{\mathrm{d}}{\mathrm{d}s}\big(a(s,\gamma)\gamma'\big) + \nabla_q V(s,\gamma) - \frac{1}{2}\frac{\partial a(s,\gamma)}{\partial q}(\gamma',\gamma') \in \partial^- I_M\big(\gamma(s)\big) \text{ a.e.}
$$

the proof of theorem 2.6 shows that $\alpha \in \partial^- f(\gamma)$. $\square$

Finally, we can state the already quoted characterization.

THEOREM 2.8.— *Let us consider* $\gamma \in X$. *Then* $0 \in \partial^- f(\gamma)$ *if and only if* $\gamma \in W^{2,\infty}$, $\gamma'_+(0) = \gamma'_-(1)$ *and*

$$\alpha(s) + \frac{\mathrm{d}}{\mathrm{d}s}\big(a(s,\gamma)\gamma'\big) + \nabla_q V(s,\gamma) - \frac{1}{2}\frac{\partial a(s,\gamma)}{\partial q}(\gamma',\gamma') \in \partial^- I_M\big(\gamma(s)\big) \text{ a.e.}$$

Proof.— If $0 \in \partial^- f(\gamma)$, from theorem 2.1 $\gamma \in W^{2,\infty}(0,1;\mathbb{R}^n)$ and $\gamma^+(0) = \gamma^-(1)$. Moreover from theorem 2.7

$$\alpha(s) + \frac{\mathrm{d}}{\mathrm{d}s}\big(a(s,\gamma)\gamma'\big) + \nabla_q V(s,\gamma) - \frac{1}{2}\frac{\partial a(s,\gamma)}{\partial q}(\gamma',\gamma') \in \partial^- I_M\big(\gamma(s)\big) \text{ a.e.}$$

Viceversa, it is enough to apply theorem 2.7 with $\alpha = 0$. □

3. The category of the space of the admissible paths

After theorem 2.8, our goal is to prove the existence of infinitely many lower critical points for f on X by means to theorem 1.7. Therefore, let us investigate the topological properties of X.

If Y is a topological space, we will denote by $\Lambda(Y)$ the free loop space of Y.

Let us recall that in [5], using results contained in [12], [13] and in [19], it is proved the following theorem.

THEOREM 3.1.— (see theorem 3.3 in [5]) *Let* A *be an open subset of* $\mathbb{R}^n$, *connected and non-contractible in itself. Moreover, let us suppose that either*

i) $\pi_1(A)$ *has infinitely many conjugacy classes*

or

ii) $\pi_1(A)$ *has a finite number of elements.*

Then $\operatorname{cat}\Lambda(A) = +\infty$.

Now, let us consider X endowed with the $W^{1,2}$-topology and the space

$$\Lambda(M) = \big\{\gamma \mid [\,0\,,\,1\,] \to M,\ \gamma \text{ is continuous and } \gamma(0) = \gamma(1)\big\}$$

endowed with the uniform topology.

THEOREM 3.2.— (see theorem 4.5 in [4]) *The inclusion map* $i : X \to \Lambda(M)$ *is a homotopy equivalence.*

Now, we are to able evaluate the category of X.

THEOREM 3.3.— *Let* $M \subset \mathbb{R}^n$ *be a connected, non-contractible in itself, compact p-convex set. Let us suppose that either*

i) $\pi_1(M)$ *has infinitely many conjugacy classes*

or

ii) $\pi_1(M)$ *has a finite number of elements.*

Then $\text{cat}(X) = +\infty$.

Proof.— Let us consider A, the open subset of $\mathbb{R}^n$ defined in Remark 1.10. Clearly, M is a deformation retract of it. Then, A is homotopically equivalent to M and $\Lambda(A)$ is homotopically equivalent to $\Lambda(M)$. By applying theorems 3.1 and 3.2, the proof is over. □

Finally we are able to prove the main theorem.

Proof of theorem 1.4

We want to apply theorem 1.7. Let us consider the functional f defined in section 2. By hypothesis (1.1) and theorem 2.6, f is a lower semicontinuous function, bounded below and it has a φ-monotone subdifferential of order two.

Moreover, let us observe that $D(f) = X$ and that d^* induces the $W^{1,2}$-topology on X. By theorem 3.3, $\text{cat}\big(D(f), d^*\big) = +\infty$.

Now, we will consider a sequence $\{\gamma_h\}_h \subset D(\partial^- f)$ with $\sup_h f(\gamma_h) < +\infty$ and $\lim_h \text{grad}^- f(\gamma_h) = 0$. Since M is compact, $\{\gamma_h\}_h$ is bounded in $L^2(0,1;\mathbb{R}^n)$. But also $\{\gamma'_h\}_h$ is bounded in $L^2(0,1;\mathbb{R}^n)$. Thus, by Rellich's theorem, $\{\gamma_h\}_h$ has a subsequence converging in $L^2(0,1;\mathbb{R}^n)$.

So, applying theorem 1.7 and theorem 2.8, the thesis follows. □

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