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Smoothness effect and decay on a class of non linear evolution equation

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RÉSUMÉ. — Nous nous proposons dans ce travail de prouver l'existence de solutions du système $u_{tt} + M\left(\left\|A^{1/2}u(t)\right\|_{\|}^{2}\right) Au + Au_t = 0$, avec les conditions initiales $u(0) = u_0 \in D(A)$, $u_t(0) = u_1 \in H$, qui décroissent uniformément quand $t \to +\infty$ et ont la propriété de régularisation des conditions initiales, c'est-à-dire si $u_0 \in D(A)$ et $u_1 \in H$ alors la solution du système satisfait $u \in C^2\left(\,]\,0\,,\,T\,]\,;\,D(A^k)\right) \forall\, k \in \mathbb{N}$.

ABSTRACT. — Our aim in this paper is to prove the existence of solution for system $u_{tt} + M\left(\left\|A^{1/2}u(t)\right\|_{\|}^{2}\right) Au + Au_t = 0$, with initial data $u(0) = u_0$, $u_t(0) = u_1$ in $D(A)$ and H respectively, which decay uniformly as time goes to infinity and has the smoothness effect property on the initial data u_0 and u_1, that is if u_0 and u_1 belong to $D(A)$ and H respectively then the corresponding solution satisfies $u \in C^2\left(\,]\,0\,,\,T\,]\,;\,D(A^k)\right) \forall\, k \in \mathbb{N}$.

KEY-WORDS : Existence, smoothness effect, asymptotic behaviour.

1. Introduction

Several authors have studied the existence of solutions for the semilinear wave equation given by

$$u_{tt} - M\left(\left\|\nabla u(t)\right\|^{2}\right)\Delta u = 0\,, \quad u(0) = u_0\,,\ u_t(0) = u_1 \qquad (1.1)$$

but until now is not proved the existence of global solution when the initial data is taken in the usual Sobolev's spaces, in this direction all the literature

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concerning this equation is about the existence of local solution. Moreover when the initial data u_0 and u_1 are taken in $H_0^1(\Omega)$ and $L^2(\Omega)$ respectively it is not known the existence even for local weak solutions. The existence of global solutions is proved only for analytical initial data, see for example the work of Berstein [2], Pohozaev [6], and Arosio & Spagnolo [1].

In order to obtain global solution for a class of equations relative to (1.1), several authors ([7], [8], [9] to name but a few) have considered damping terms as A^2u, Au_t or $A^\alpha u_t$, which gives strong estimates resulting in the convergence of the nonlinear term, showing in this way global existence results. In Nishihara [9], for example, the author studies the equation

$$u_{tt} + M\left(\left\|A^{1/2}u(t)\right\|^2\right) Au + Au_t = 0\,,\; u(0) = u_0\,,\; u_t(0) = u_1 \qquad (1.2)$$

and proved the existence and uniqueness of global solution satisfying

$$u \in C^1\left([\,0\,,\infty[\,;\, H\right)$$
$$u \in L^\infty\left([\,0\,,\infty[\,;\, D(A)\right)$$
$$u_t \in L^\infty\left([\,0\,,\infty[\,;\, D(A^{1/2})\right) \cap L^2\left([\,0\,,\infty[\,;\, D(A)\right)$$
$$u_{tt} \in L^2\left([\,0\,,\infty[\,,\, H\right)$$

when H is a Hilbert space and the following hypotheses hold.

H1 .— *A is a positive selfadjoint operator of H such that $D(A)$ has compact imbedding in H.*

H2 .— *M is a nonnegative C^1-function.*

H3 .— *u_0 and u_1 are given in $D(A)$ and $D(A^{1/2})$ respectively.*

Other weaker dissipative mechanism was studied by Ikehata and Okazawa [3], the authors introduce the friccional damping term u_t and show, by using the Yosida's aproximation method together with compactness arguments, the existence of global solution provided the initial data is small enough. This case is different to others studied in [7], [8] and [9] because the compactness method is not enough to obtain global solutions for large data when considered the friccional damping u_t.

Concerning the asymptotic behaviour Nishihara [9] proves when $M(s) = s$ and u_0 and u_1 are given in $D(A^{3/2})$, that the solution $u(t)$ satisfies the

following decay properties.

$$\left\|A^{1/2}u(t)\right\|^2 \le C_\epsilon (1+t)^{-1+\epsilon}, \quad \left\|Au(t)\right\|^2 \le C_\epsilon (1+t)^{-\frac{1}{2}+\epsilon}$$
$$\left\|A^{1/2}u_t(t)\right\|^2 \le C_\epsilon (1+t)^{-3+\epsilon}, \quad \left\|u_{tt}(t)\right\|^2 \le C_\epsilon (1+t)^{-4+\epsilon}$$

where $C_\epsilon \to +\infty$ when $\epsilon \to 0$.

This rate of decay was improved in Matos and Pereira [6], by removing ϵ in the above inequalities. It is not possible to prove the exponential decay (as is expected when hypothesis H1 holds) because $M(0) = 0$, that is, the equation is degenerate. In this direction Nishihara [10] shows that the rate obtained in [6] is optimal, that is

$$\left\|A^{1/2}u(t)\right\|^2 \ge C(1+t)^{-1}, \quad \forall\, t \ge 0\,.$$

On the other hand, when $M(s) \ge m_0 > 0$, Nishihara [9] pointed out that the exponential decay holds, latter Zuazua [12] proved the exponential decay for a general class of hyperbolic equations with damping terms.

To the best of my knowledge there is no result concerning the smoothness effect for equation (1.2).

In this paper we will discuss about some properties which the damping term Au_t produces on the solution of equation (1.2), such as global existence, regularity, decay and also the smoothness effect property on the initial data, this means that the solution of (1.2) satisfies $u \in C^2\big([\,0\,,\infty[\,; D(A^\infty)\big)$ where $D(A^\infty) = \bigcap_{k\in\mathbb{N}} D(A^k)$, no matter the regularity of the initial data have, whenever they are taken in a suitable space (in $D(A) \times H$ for example) in order to obtain the existence result. Smoothness effect is not expected for any non negative continuous function M; in fact if u_0 is such that $M\big(\|A^{1/2}u_0\|^2\big) = 0$, then the function $t \mapsto u(t) = u_0$ is the solution of (1.2) when $u_1 = 0$, therefore we have no smoothness effect in this case.

In section 2 we will prove that the smoothness effect holds for any non negative continuous function satisfying $M\big(\|A^{1/2}u_0\|^2\big) > 0$. Existence as well as regularity result are also proved in this section.

Finally in section 3 we will prove that the solution of equation (1.2) decays uniformly as time goes to infinity.

2. Existence and regularity

Let $M : [\, 0 \,, +\infty\, [\, \to [\, 0 \,, +\infty\, [$ be a non negative continuous function and A satisfying H1. By a weak solution of system (1.2) we mean a function

$$u : [\, 0 \,, \infty\, [\, \to D(A^{1/2})$$

satisfying

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\,(u_t, w) + M\left(\left\|A^{1/2}u(t)\right\|^2\right)(A^{1/2}u\,,\,A^{1/2}w) + \\
+ (A^{1/2}u_t\,,\,A^{1/2}w) = 0
\end{aligned}
\tag{2.1}
$$

$$u(0) = u_0 \text{ in } D(A)\,, \quad u_t(0) = u_1 \text{ in } H\,,$$

for any w in $D(A^{1/2})$.

It is well known that there exists sequences $(w_i)_{i \in \mathbb{N}}$ and $(\lambda_i)_{i \in \mathbb{N}}$ of eigenfunctions and eigenvalues of A such that $\{w_i\,;\, i \in \mathbb{N}\}$ is an orthonormal set and $\lambda_1 < \lambda_2 \le \cdots \le \lambda_i \to \infty$ as $i \to \infty$. Let us denote by

$$u_0^{(m)} = \sum_{i=1}^{m}(u_0, w_i)w_i \quad \text{and} \quad u_1^{(m)} = \sum_{i=1}^{m}(u_1, w_i)w_i\,.$$

If u_0 and u_1 belongs to $D(A)$ and H respectively we have that the following convergence holds

$$(u_0^{(m)}, u_1^{(m)}) \to (u_0, u_1) \text{ in } D(A) \times H\,.$$

Finally by V_m we will denote the space generated by the first m eigenvectors. In this conditions there exists a local solution $u^{(m)}(t)$ on some interval $[\, 0 \,, T_m\, [$, for the ordinary differential equation given by

$$(u_{tt}^{(m)}, w) + M\left(\left\|A^{1/2}u^{(m)}(t)\right\|^2\right)(Au^{(m)}, w) + (Au_t^{(m)}, w) = 0 \tag{2.2}$$

$$u^{(m)}(0) = u_0^{(m)}\,, \quad u_t^{(m)}(0) = u_1^{(m)}\,, \tag{2.3}$$

for any w and V_m and $u^{(m)}(t) = \sum_{i=1}^{m} g_{i,m}(t)w_i$.

Moreover if M is locally Lipschitz this solution is unique.

THEOREM 2.1 (Existence). — *Let M be a non negative locally Lipschitz function. If $u_0 \in D(A)$ and $u_1 \in H$ then there exist a unique global weak solution of (1.2) in the sence of (2.1) satisfying*

$$u \in C\left([\,0\,,\,\infty\,]\,;\,D(A)\right), \quad u_t \in C\left([\,0\,,\,\infty\,]\,;\,H\right) \cap L^2\left([\,0\,,\,\infty\,]\,;\,D(A^{1/2})\right).$$

Moreover the identity

$$E(t) + \int_0^t \left\| A^{1/2} u_t(s) \right\|^2 \mathrm{d}s = E(0) \tag{2.4}$$

holds for any $t \geq 0$, where E is given by

$$E(t) = \frac{1}{2} \left\| u_t(t) \right\|^2 + \frac{1}{2} \widetilde{M}\left(\left\| A^{1/2} u(t) \right\|^2 \right)$$

and

$$\widetilde{M}(s) = \int_0^s M(\xi)\,\mathrm{d}\xi.$$

Proof. — In order to facilitate our analysis we will denote by

$$E_m(t) = \frac{1}{2} \left\| u_t^{(m)}(t) \right\|^2 + \frac{1}{2} \widetilde{M}\left(\left\| A^{1/2} u^{(m)}(t) \right\|^2 \right).$$

Taking $w = u_t^{(m)}$ in (2.2) and integrating the expresion result we have

$$E_m(t) + \int_0^t \left\| A^{1/2} u_t^{(m)}(s) \right\|^2 \mathrm{d}s = E_m(0) \tag{2.5}$$

for $t \in [\,0\,,\,T_m\,[$. Taking $u_t^{(m)}$ instead of w in (2.2) we obtain

$$(u_{tt}^{(m)}, u^{(m)}) - M\left(\|A^{1/2} u^{(m)}\|^2 \right)(Au^{(m)}, u^{(m)}) + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|A^{1/2} u^{(m)}\|^2 = 0. \tag{2.6}$$

Substitution of

$$\left(u_{tt}^{(m)}(t),\, u^{(m)}(t)\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(u_t^{(m)}(t),\, u^{(m)}(t)\right) - \left\| u_t^{(m)}(t) \right\|^2$$

in (2.6) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{ \left(u_t^{(m)}(t),\, u^{(m)}(t)\right) + \frac{1}{2} \left\| A^{1/2} u^{(m)}(t) \right\|^2 \right\} =$$

$$= \left\| u_t^{(m)}(t) \right\|^2 - M\left(\left\| A^{1/2} u^{(m)}(t) \right\|^2 \right) \left\| A^{1/2} u^{(m)}(t) \right\|^2 \leq \tag{2.7}$$

$$\leq \left\| u_t^{(m)}(t) \right\|^2$$

for $t \in [\,0\,,\,T_m\,[$. From relation (2.7) we obtain

$$\frac{1}{2} \left\| A^{1/2} u^{(m)}(t) \right\|^2 \leq \left(u_1^{(m)},\, u_0^{(m)} \right) + \frac{1}{2} \left\| A^{1/2} u_0^{(m)} \right\|^2 +$$

$$- \left(u_t^{(m)}(t),\, u^{(m)}(t) \right) + \frac{1}{\lambda_1} \int_0^t \left\| A^{1/2} u_t^{(m)}(s) \right\|^2 \mathrm{d}s\,,$$

so we have

$$\frac{1}{2} \left\| A^{1/2} u^{(m)}(t) \right\|^2 \leq \left(u_1^{(m)},\, u_0^{(m)} \right) + \frac{1}{2} \left\| A^{1/2} u_0^{(m)} \right\|^2 +$$

$$+ \lambda_1^{-1/2} \left\| u_t^{(m)}(t) \right\| \left\| A^{1/2} u^{(m)}(t) \right\|$$

$$+ \frac{1}{\lambda_1} \int_0^t \left\| A^{1/2} u_t^{(m)}(s) \right\|^2 \mathrm{d}s\,.$$

By using (2.5) in the above inequality we obtain

$$\frac{1}{4} \left\| A^{1/2} u^{(m)}(t) \right\|^2 \leq \left(u_1^{(m)},\, u_0^{(m)} \right) + \frac{1}{2} \left\| A^{1/2} u_0^{(m)} \right\|^2 + \frac{2}{\lambda_1}\, E_m(0)\,, \quad (2.8)$$

for $t \in [\,0\,,\,T_m\,[$. Let's take $w = A u^{(m)}$ in (2.2) then it follows

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\{ \left(u_t^{(m)}(t),\, A u^{(m)}(t) \right) + \frac{1}{2} \left\| A u^{(m)}(t) \right\|^2 \right\} =$$

$$= \left\| A^{1/2} u_t^{(m)}(t) \right\|^2 - M\left(\left\| A^{1/2} u^{(m)}(t) \right\|^2 \right) \left\| A u^{(m)}(t) \right\|^2 \leq$$

$$\leq \left\| A^{1/2} u_t^{(m)}(t) \right\|^2\,.$$

Integration in time yields

$$\frac{1}{2} \left\| A u^{(m)}(t) \right\|^2 \leq - \left(u_t^{(m)}(t),\, A u^{(m)}(t) \right) \times$$

$$\times \int_0^t \left\| A^{1/2} u_t^{(m)}(\tau) \right\|^2 \mathrm{d}\tau + \left(u_1^{(m)},\, A u_0^{(m)} \right) + \frac{1}{2} \left\| A u_0^{(m)} \right\|^2$$

which implies that

$$\frac{1}{4} \left\| A u^{(m)}(t) \right\|^2 \leq \left\| u_t^{(m)}(t) \right\|^2 + \int_0^t \left\| A^{1/2} u_t^{(m)}(\sigma) \right\|^2 \mathrm{d}\sigma +$$

$$+ \left(u_1^{(m)},\, A u_0^{(m)} \right) + \frac{1}{2} \left\| A u_0^{(m)} \right\|^2\,. \tag{2.9}$$

From (2.5) and (2.9) we conclude that the interval $[0, T_m[$ can be extended to the whole interval $[0, +\infty[$. Moreover we have that

$$u_t^{(m)} \text{ is bounded in } L^2([0,T]; D(A^{1/2})) \qquad (2.10)$$

$$u^{(m)} \text{ is bounded in } L^\infty([0,T]; D(A)). \qquad (2.11)$$

For any $T > 0$. From Lions-Aubin's theorem (see [4] p. 58, th 5.1) we conclude that there exists a subsequence (which we still denoting in the same way) satisfying

$$u^{(m)} \to u \text{ strongly in } L^2([0,T]; D(A^{1/2}))$$
$$\|A^{1/2} u^{(m)}(t)\| \to \|A^{1/2} u(t)\| \text{ a.e. in } [0,T].$$

Since $t \mapsto \|A^{1/2} u^{(m)}(t)\|$ is bounded in $L^\infty([0,T])$, by the last convergence and the Lebesgue's dominated convergence theorem we have that

$$M\left(\|A^{1/2} u^{(m)}(t)\|^2\right) \to M\left(\|A^{1/2} u(t)\|^2\right) \text{ strongly in } L^2([0,T])$$

so $M\left(\|A^{1/2} u^{(m)}(t)\|^2\right)$ is a Cauchy sequence in $L^2([0,T])$ therefore, for any $\epsilon > 0$, there exists $N > 0$, such that

$$\int_0^T \left| M\left(\|A^{1/2} u^{(m)}(\sigma)\|^2\right) - M\left(\|A^{1/2} u^{(\mu)}(\sigma)\|^2\right)\right|^2 d\sigma \le \epsilon, \qquad (2.12)$$
$$\forall\, m,\, \mu \ge N.$$

Our next goal is to prove that $u^{(m)}$ and $u_t^{(m)}$ are Cauchy sequences in $C([0,T]; D(A))$ and $C([0,T]; H)$ respectively. In order to prove this fact let us take m and μ any positive natural numbers such that $m > \mu$ and let us define $g_{i,\mu}(t) = 0$ for $\mu \le i \le m$. In this conditions we can conclude that both $u^{(m)}$ and $u^{(\mu)}$ are aproximation solutions of system (1.2) in V_m. Let us denote by $U^{(m)}$ the Cauchy difference given by

$$U^{(m)} = u^{(m)} - u^{(\mu)}.$$

Since $u_0^{(m)}$ and $u_1^{(m)}$ are Cauchy sequences in $D(A)$ and H respectively, then there exist $N > 0$, such that

$$\|A U^{(m)}(0)\| < \epsilon; \quad \|U_t^{(m)}(0)\| < \epsilon, \quad \forall\, m,\, \mu \ge N. \qquad (2.13)$$

On the other hand, from (2.2) we have that $U^{(m)}$ satisfies

$$\left(U_{tt}^{(m)}, w\right) + M\left(\left\|A^{1/2} u^{(m)}(t)\right\|^2\right)\left(AU^{(m)}, w\right) + \left(AU_t^{(m)}, w\right) =$$

$$= \left(M\left(\left\|A^{1/2} u^{(\mu)}(t)\right\|^2\right) - M\left(\left\|A^{1/2} u^{(m)}(t)\right\|^2\right)\right)\left(Au^{(\mu)}(t), w\right).$$

$$(2.14)$$

Since $u^{(m)}$ is bounded in H so is $U^{(m)}$, taking $w = U_t^{(m)}$ in (2.14) and using (2.12) we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|U_t^{(m)}\right\|^2 + \frac{1}{2}\left\|A^{1/2} U_t^{(m)}\right\|^2 \leq$$

$$\leq M\left(\left\|A^{1/2} u^{(m)}(t)\right\|^2\right)\left(A^{1/2} U^{(m)}, A^{1/2} U_t^{(m)}\right) +$$

$$+ \epsilon\left(A^{1/2} u^{(\mu)}(t), A^{1/2} U_t^{(m)}\right),$$

from where it follows

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|U_t^{(m)}\right\|^2 + \frac{1}{4}\left\|A^{1/2} U_t^{(m)}\right\|^2 \leq$$

$$\leq 4M\left(\left\|A^{1/2} u^{(m)}(t)\right\|^2\right)^2\left\|A^{1/2} U^{(m)}\right\|^2 + 4\,\epsilon^2\left\|A^{1/2} u^{(\mu)}(t)\right\|^2.$$

Integrating the above inequality and applying the relation

$$\left\|w(t)\right\|^2 \leq 2\left\|w(0)\right\|^2 + 2t\int_0^t \left\|w_t(\sigma)\right\|^2 \mathrm{d}\sigma$$

for $A^{1/2} U^{(m)}$ instead of w, we obtain

$$\frac{1}{2}\left\|U_t^{(m)}\right\|^2 + \frac{1}{4}\int_0^t\left\|A^{1/2} U_t^{(m)}\right\|^2 \leq \frac{1}{2}\left\|U_t^{(m)}(0)\right\|^2 + \frac{1}{2}C\left\|A^{1/2} U_t^{(m)}(0)\right\| +$$

$$+ \frac{CT}{2}\int_0^t\int_0^\tau\left\|A^{1/2} U_t^{(m)}(\sigma)\right\|^2 \mathrm{d}\sigma\,\mathrm{d}\tau$$

$$+ 4\,\epsilon^2 t\left\|A^{1/2} u^{(\mu)}(t)\right\|^2.$$

For some positive constant C depending only on the initial data. Applying Gronwall's inequality and relation (2.13) we conclude that $u_t^{(m)}$ is a Cauchy sequence, then we have

$$u_t^{(m)} \to u_t \text{ strongly in } L^2\big([\,0\,,\,T\,]\,;\,D(A^{1/2})\big)$$

$$u_t^{(m)} \to u_t \text{ strongly in } C\big([\,0\,,\,T\,]\,;\,H\big).$$

Taking $w = AU^{(m)}$ in relation (2.14) and applying the same above reasoning we conclude that $u^{(m)}$ is a Cauchy sequence in $C([0,T];D(A))$ then

$$u^{(m)} \rightarrow u \text{ strongly in } C([0,T];D(A)).$$

From the uniform convergence follows that u and u_t satisfies the initial conditions therefore u is a weak solution of equation (1.2). By using the same method as in [9] we conclude the uniqueness. Finally since identity (2.5) holds for any $m \in \mathbb{N}$, by the strong convergence of $u^{(m)}$ and $u_t^{(m)}$, easily follows that the energy identity (2.4) holds for any positive value of t. The proof is now complete. $\square$

Remark 2.1.— It is well known that for any $v \in H$ the following properties holds

$$v = \sum_{i=1}^{\infty} (v, w_i) w_i$$

$$v \in D(A) \Leftrightarrow \sum_{i=1}^{\infty} \lambda_i^2 (v, w_i)^2 < \infty.$$

Therefore if u denotes the solution of (1.2) in the sence of theorem 2.1 we have

$$u(t) = \sum_{i=1}^{\infty} (u(t), w_i) w_i.$$

It is not difficult to prove that the convergence of the above series is uniformly on bounded set of $\mathbb{R}_+$.

From now on $g_i(t) = (u(t), w_i)$ and $g_i'(t) = (u_t(t), w_i)$ where u is the solution of (1.2). Since $u_1^{(m)}$ and $u_0^{(m)}$ are Cauchy sequences in H and $D(A)$ respectively we conclude that there exists $\sigma > 0$ satisfying

$$\sigma > (u_1^{(m)}, u_0^{(m)}) + \frac{1}{2} \|A^{1/2} u_0^{(m)}\|^2 + \frac{2}{\lambda_1} E_m(0), \quad \forall m \in \mathbb{N}.$$

In this conditions we will define M_0 as

$$M_0 = \sup\{M(s); s \in [0,\sigma]\}.$$

Let us denote by $H_i(t)$ the following function

$$H_i(t) = \frac{1}{2} g_i'(t)^2 + 2M_0 g_i'(t) g_i(t).$$

Finally by u we will denote the solution of (1.2) or its restriction over any subinterval of $[\,0\,,\infty\,[$.

We will conclude the smoothness effect property of u by studing the behaviour of the functions g_i defined above.

LEMMA 2.1.— *Under the hypotheses of theorem 2.1 and for any $i \geq I$, where I is such that $\lambda_I \geq 16\, M_0$, the following inequality holds*

$$H_i(t) \leq H_i(0)\, e^{-\lambda_i t} \quad \text{for any } t \geq 0\,. \tag{2.15}$$

Proof. — For simplicity we will denote by $a(t) = M\left(\left\|A^{1/2}u(t)\right\|^2\right)$, then projecting equation (1.2) on $\mathbb{R}w_i$ we have

$$g_i''(t) + \lambda_i a(t) g_i(t) + \lambda_i g_i'(t) = 0 \tag{2.16}$$

multiplying equation (2.16) by $g_i'(t)$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\, g_i'(t)^2 = -\lambda_i a(t) g_i(t) g_i'(t) - \lambda_i g_i'(t)^2 \leq$$

$$\leq \left(2M_0 \lambda_i a(t)\right)^{\frac{1}{2}}\left| g_i(t)\left(\frac{1}{2}\,\lambda_i\right)^{\frac{1}{2}} g_i'(t)\right| - \lambda_i g_i'(t)^2\,.$$

From where it follows

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\, g_i'(t)^2 \leq M_0 \lambda_i a(t) g_i(t)^2 - \frac{3}{4}\,\lambda_i g_i'(t)^2\,. \tag{2.17}$$

Let us denote by $\phi_i(t) = g_i(t)g_i'(t)$. By (2.16) we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\phi_i(t) = g_i'(t)^2 \lambda_i a(t) g_i(t)^2 - \lambda_i \phi_i(t)\,. \tag{2.18}$$

Multiplying equation (2.18) by $2M_0$ and adding the product result to inequality (2.17), for $\lambda_i \geq 16\, M_0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\,H_i(t) \leq -\frac{1}{8}\,\lambda_i g_i'(t)^2 - \lambda_i M_0\, a(t) g_i(t)^2 - \lambda_i H_i(t)\,. \tag{2.19}$$

In particular

$$\frac{d}{dt}\,H_i(t) \le -\lambda_i H_i(t)$$

multiplying by this inequality by $e^{-\lambda_i t}$ and integrating in time, the result follows. $\square$

In order to conclude the regularity result let us introduce the following space

$$\mathcal{W}(a,b) = \Big\{ u \in L^2\big([\,a\,,\,b\,]\,;\,D(A^\alpha)\big) \mid u_t \in L^2\big([\,a\,,\,b\,]\,;\,D(A^\beta)\big) \Big\}$$

where α and β are positive real numbers, this space with the norm

$$\|w\|^2_{\mathcal{W}(a,b)} = \int_a^b \|A^\alpha w(\tau)\|^2_H \,d\tau + \int_a^b \|A^\beta w_t(\tau)\|^2_H \,d\tau$$

is a Hilbert space (see [5]). The following lemma will play and important role in the sequel

LEMMA 2.2. — *With the above notations we have*

$$\mathcal{W}(a,b) \subset C\big([\,a\,,\,b\,]\,;\,D(A^{(\alpha+\beta)/2})\big)$$

and

$$\big\|A^{(\alpha+\beta)/2} u(t)\big\| \le C\|u\|_{\mathcal{W}(a,b)}\,, \qquad \forall\, t \in [\,a\,,\,b\,]. \tag{2.20}$$

Moreover if $\beta > \alpha$ then for any $\epsilon > 0$ we have

$$\mathcal{W}(a,b) \subset C\big([\,a\,,\,b\,]\,;\,D(A^{\beta-\epsilon})\big)$$

and

$$\big\|A^{\beta-\epsilon} u(t)\big\| \le C_\epsilon\|u\|_{\mathcal{W}(a,b)}\,, \qquad \forall\, t \in [\,a\,,\,b\,].$$

Proof. — Since there exist an extension operator $\mathcal{P}$ defined in $\mathcal{W}(a,b)$ into $\mathcal{W}(\mathbb{R})$, we can suppose that $a = -\infty$ and $b = +\infty$ (see Lions [5]). Since the space $C_0^\infty\big(\mathbb{R}\,;\,D(A^\gamma)\big)$ is dense in $\mathcal{W}(\mathbb{R})$ (where $\gamma = \max\{\alpha,\beta\}$) it is sufficient to prove that for any u in $C_0^\infty\big(\mathbb{R}\,;\,D(A^\gamma)\big)$ relation (2.20) holds. First note that the function $t \mapsto \big(u(t)\,,\,w_i\big)$ belongs to $L^2(\mathbb{R})$ for any $i \in \mathbb{N}$, then by using the Fourier transform

$$\mathcal{F}v(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} v(\tau)\,e^{i\tau\xi}\,d\tau$$

we have

$$\big(u(t)\,,\,w_i\big)^2 = \left| \frac{1}{\sqrt{2\pi}} \int_{\mathrm{IR}} e^{-it\tau} \mathcal{F}\left\{\big(u(\tau)\,,\,w_i\big)\right\} \,\mathrm{d}\tau \right|^2 \le$$

$$\le \frac{1}{\sqrt{2\pi}} \left| \int_{\mathrm{IR}} \big(\lambda_i^{2\alpha} + \tau^2\lambda_i^{2\beta}\big)^{-\frac{1}{2}} \big(\lambda_i^{2\alpha} + \tau^2\lambda_i^{2\beta}\big)^{\frac{1}{2}} \mathcal{F}\left\{\big(u(\tau)\,,\,w_i\big)\right\} \,\mathrm{d}\tau \right|^2$$

$$\le \frac{1}{\sqrt{2\pi}} \int_{\mathrm{IR}} \big(\lambda_i^{2\alpha} + \tau^2\lambda_i^{2\beta}\big)^{-1} \,\mathrm{d}\tau \times$$

$$\times \left\{ \int_{\mathrm{IR}} \big(\lambda_i^{2\alpha} + \tau^2\lambda_i^{2\beta}\big) \left| \mathcal{F}\left\{\big(u(\tau)\,,\,w_i\big)\right\}\right|^2 \,\mathrm{d}\tau \right\}$$

$$\le \frac{1}{\sqrt{2\pi}} \int_{\mathrm{IR}} \big(\lambda_i^{2\alpha} + \tau^2\lambda_i^{2\beta}\big)^{-1} \,\mathrm{d}\tau \times$$

$$\times \left\{ \int_{\mathrm{IR}} \lambda_i^{2\alpha} \left|\big(u(\tau)\,,\,w_i\big)\right|^2 + \lambda_i^{2\beta} \left|\big(u_t(\tau)\,,\,w_i\big)\right|^2 \,\mathrm{d}\tau \right\}$$

taking $\sigma = \lambda_i^{\beta-\alpha}\tau$ we have

$$\lambda_i^{\alpha+\beta}\big(u(t)\,,\,w_i\big)^2 \le \frac{1}{\sqrt{2\pi}} \int_{\mathrm{IR}} \big(1 + \sigma^2\big)^{-1} \,\mathrm{d}\sigma \times$$

$$\times \left\{ \int_{\mathrm{IR}} \lambda_i^{2\alpha} \left|\big(u(\tau)\,,\,w_i\big)\right|^2 + \int_{\mathrm{IR}} \lambda_i^{2\beta} \left|\big(u_t(\tau)\,,\,w_i\big)\right|^2 \,\mathrm{d}\tau \right\}$$

summing up from 1 to m letting $m \to \infty$, (2.20) follows. Finally let us suppose that $\beta > \alpha$ and $a, b \in \mathrm{IR}$ from (2.20) we obtain that

$$u \in C\big([a\,,\,b]\,;\,D(A^{(\alpha+\beta)/2})\big)\,,$$

since

$$u_t \in L^2\big([a\,,\,b]\,;\,D(A^\beta)\big)\,.$$

Aplication of (2.20) yields

$$\left\| A^{\beta+2^{-2}(\alpha-\beta)}u(t) \right\| \le$$

$$\le C\left\{ \int_a^b \left\| A^{(\alpha+\beta)/2}u(\tau) \right\|_H^2 \,\mathrm{d}\tau + \int_a^b \left\| A^\beta u_t(\tau) \right\|_H^2 \,\mathrm{d}\tau \right\}$$

$$\le C_2\left\{ \int_a^b \left\| A^\alpha u(\tau) \right\|_H^2 \,\mathrm{d}\tau + \int_a^b \left\| A^\beta u_t(\tau) \right\|_H^2 \,\mathrm{d}\tau \right\}$$

where $C_2 = C[C(b-a)+1]$. Applying (2.20) for $\beta + 2^{-2}(\alpha - \beta)$ instead of α we have

$$\|A^{\beta+2^{-3}(\alpha-\beta)}u(t)\| \le$$

$$\le C\left\{\int_a^b \|A^{\beta+2^{-2}(\alpha-\beta)}u(\tau)\|_H^2\, d\tau + \int_a^b \|A^\beta u_t(\tau)\|_H^2\, d\tau\right\}$$

$$\le C_3\left\{\int_a^b \|A^\alpha u(\tau)\|_H^2\, d\tau + \int_a^b \|A^\beta u_t(\tau)\|_H^2\, d\tau\right\}$$

where $C_3 = C[C_2(b-a)+1]$. Repeating this process n times we conclude

$$\|A^{\beta+2^{-n}(\alpha-\beta)}u(t)\| \le$$

$$\le C_n\left\{\int_a^b \|A^\alpha u(\tau)\|_H^2\, d\tau + \int_a^b \|A^\beta u_t(\tau)\|_H^2\, d\tau\right\}$$

where $C_n = C[C_{n-1}(b-a)+1]$. Finally taking $\epsilon > 0$, there exist n such that $2^{-n}(\beta - \alpha) < \epsilon$, so we have

$$\beta + 2^{-n}(\alpha - \beta) > \beta - \epsilon\,,$$

then the result follows. $\square$

Remark 2.2. — Let I be any natural number, and v_1, v_2 in $L^2([a,b]; H)$ and $L^\infty([a,b]; H)$ respectively such that

$$\sum_{i=I}^m \int_a^b \lambda_i^{2\ell}(v_1(\tau),\, w_i)^2\, d\tau + \sum_{i=I}^m \lambda_i^{2\ell}(v_2(t),\, w_i)^2 \le C\,, \quad \forall\, m \ge I$$

where C is a positive constant which does not depend on m. Then it follows that

$$v_1 \in L^2([a,b]; D(A^\ell)) \quad \text{and} \quad v_2 \in L^\infty([a,b]; D(A^\ell))\,.$$

In fact since

$$\sum_{i=1}^I \lambda_i^{2\ell}\left\{\int_a^b (v_1(\tau),\, w_i)^2\, d\tau + (v_2(t),\, w_i)^2\right\} \le$$

$$\le \lambda_I^{2\ell}\sum_{i=1}^I\left\{\int_a^b (v_1(\tau),\, w_i)^2\, d\tau + (v_2(t),\, w_i)^2\right\}\,.$$

By hypoteses on v_1 and v_2 the second member of the last inequality is bounded then there exist a positive constant C_1 which does not depend on m such that

$$\int_a^b \sum_{i=1}^m \lambda_i^{2\ell} \big(v_1(\tau),\, w_i\big)^2 \,\mathrm{d}\tau + \sum_{i=1}^m \lambda_i^{2\ell} \big(v_2(t),\, w_i\big)^2 \leq C_1\,, \quad \forall\, m \in \mathbb{N}\,.$$

Letting $m \to \infty$ and using remark 2.1 our assertion follows. $\square$

COROLLARY 2.1.— *Let M be a non negative continuous function and $\ell \geq 1$. If $u_0 \in D(A^\ell)$ and $u_1 \in D(A^{\ell-1/2})$, then the solution of equation (1.2) satisfies*

$$u \in C\big([\,0\,,\,+\infty\,]\,;\, D(A^\ell)\big) \cap L^\infty\big([\,0\,,\,+\infty\,]\,;\, D(A^\ell)\big)$$

and

$$u_t \in C\big([\,0\,,\,+\infty\,]\,;\, D(A^{\ell-1/2})\big) \cap L^\infty\big([\,0\,,\,+\infty\,]\,;\, D(A^{\ell-1/2})\big)\,.$$

Proof.— Multiplying relation (2.15) by $\lambda_i^{2\ell}$ and integrating in time the product result we obtain

$$\frac{1}{2}\int_0^t \lambda_i^{2\ell} g_i'(\tau)^2 \,\mathrm{d}\tau + M_0 \lambda_i^{2\ell} g_i(t)^2 \leq M_0 \lambda_i^{2\ell} g_i(0)^2 + \lambda_i^{2\ell-1} \max\{0,\, H_i(0)\}\,.$$

Let us denote by I the first natural number for which we have

$$\lambda_I \geq 16\, M_0\,.$$

Then summing up the above inequality fomr I to m we have

$$\frac{1}{2}\sum_{i=I}^m \int_0^t \lambda_i^{2\ell} g_i'(\tau)^2 \,\mathrm{d}\tau + \sum_{i=I}^m M_0 \lambda_i^{2\ell} g_i(t)^2 \leq$$

$$\leq C\left\{ \big\|A^\ell u_C^{(m)}\big\|^2 + \big\|A^{\ell-1/2} u_1^{(m)}\big\|^2 \right\}\,.$$

For some positive constant C. Since $u,\, u_t \in L^\infty\big([\,0\,,\,T\,]\,;\, H\big)$ remark 2.2 implies

$$u_t \in L^2\big([\,0\,,\,T\,]\,;\, D(A^\ell)\big) \quad \text{and} \quad u \in L^\infty\big([\,0\,,\,T\,]\,;\, D(A^\ell)\big)\,.$$

By lemma 2.2 we have $u \in C([0, T]; D(A^\ell))$. Finally (1.2) yields $u_{tt} \in L^2([0, T]; D(A^{\ell-1}))$ and since $u_t \in L^2([0, T]; D(A^\ell))$ from lemma 2.2 the result follows. $\square$

In order to prove the smoothness effect we will define the auxiliarity function $J_i(t)$ as

$$J_i(t) = \frac{1}{2} g_i''(t)^2 + 4 M_0 g_i''(t) g_i'(t).$$

In this conditions we have

LEMMA 2.3. — *Under the hypotheses of theorem 2.1 and $M \in C^1$, such that $M(\|A^{1/2}u_0\|) > 0$, there exist $C > 0$ and $\eta > 0$ for which the following inequality holds*

$$J_i(t) + C H_i(t) \leq \{J_i(0) + C H_i(0)\} e^{-\lambda_i t}, \quad \forall \, t \in [0, \eta]$$

for any $i \geq I$, where $\lambda_I \geq 32 M_0$.

Proof. — Differentiating in time equation (2.16) we obtain

$$g_i''' + \lambda_i a(t) g_i' + \lambda_i g_i'' + \lambda_i a'(t) g_i = 0 \qquad (2.21)$$

Multiplying (2.21) by $g_i''(t)$ we have

$$\frac{1}{2} \frac{d}{dt} g_i''(t)^2 = -\lambda_i a(t) g_i'(t) g_i''(t) - \lambda_i g_i''(t)^2 - \lambda_i a'(t) g_i(t) g_i''(t),$$

but the inequalities

$$a(t) g_i'(t) g_i''(t) \leq 4 a(t)^2 g_i'(t)^2 + \frac{1}{16} g_i''(t)^2$$

and

$$a'(t) g_i(t) g_i''(t) \leq 4 a'(t)^2 g_i(t)^2 + \frac{1}{16} g_i''(t)^2,$$

holds for any $t > 0$; therefore using the fact that $a(t) \leq M_0$ we conclude that

$$\frac{1}{2} \frac{d}{dt} g_i''(t)^2 \leq 4 M_0 \lambda_i a(t) g_i'(t)^2 - \frac{7}{8} \lambda_i g_i''(t)^2 + 4 \lambda_i a'(t)^2 g_i(t)^2. \qquad (2.22)$$

Let us denote by $\psi_i(t) = g_i'(t) g_i''(t)$. Differentiating ψ_i and using (2.21) we obtain

$$\frac{d}{dt} \psi_i(t) = g_i''(t)^2 - \lambda_i a(t) g_i'(t)^2 - \lambda_i \psi_i(t) - \lambda_i a'(t) g_i(t) g_i'(t). \qquad (2.23)$$

Multiplying equation (2.23) by $4M_0$ and adding up the relation result to inequality (2.22) we obtain

$$\frac{d}{dt} J_i(t) \leq \left\{ 4M_0 - \frac{3}{8} \lambda_i \right\} g_i''(t)^2 - \lambda_i J_i(t) + 4\lambda_i a'(t)^2 g_i(t)^2 +$$
$$- 4M_0 \lambda_i a'(t) g_i(t) g_i'(t).$$

For $\lambda_i \geq \dfrac{32}{3} M_0$, we have that

$$\frac{d}{dt} J_i(t) \leq -\lambda_i J_i(t) + 4\lambda_i (1 + 16 M_0^2) a'(t)^2 g_i(t)^2 + \frac{1}{16} \lambda_i g_i'(t)^2.$$

From (2.19) we conclude that

$$\frac{d}{dt} \{ J_i(t) + C H_i(t) \} \leq$$
$$\leq -\lambda_i J_i(t) - C\lambda_i H_i(t) + \lambda_i \{ 4(1 + 16 M_0^2) a'(t)^2 - C M_0 a(t) \} g_i(t)^2 +$$
$$+ \lambda_i \left(-\frac{C}{8} + \frac{1}{16} \right) g_i'(t)^2.$$

On the other hand, since $a(0) = M\big(\|A^{1/2} u_0\|^2\big) > 0$ there exist $\eta > 0$ for which we have

$$a(t) > \frac{1}{2} M\big(\|A^{1/2} u_0\|^2\big), \quad \forall \, t \in [\, 0\, , \eta\,].$$

Taking $C = \max\{r, 1\}$ where $r = 8(1 + 16 M_0^2)\|a'\|_\infty^2 / \{a(0) M_0\}$ and $\|a'\|_\infty = \sup\left\{ |a'(t)|^2 ; t \in [\, 0\, , T\,] \right\}$ we have

$$4(1 + 16 M_0^2) a'(t)^2 - C M_0 a(t) \leq 0, \quad \forall \, t \in [\, 0\, , \eta\,]$$

then

$$\frac{d}{dt} \{ J_i(t) + C H_i(t) \} \leq -\lambda_i J_i(t) - C\lambda_i H_i(t), \quad \forall \, t \in [\, 0\, , \eta\,]$$

multiplying by $e^{-\lambda_i t}$ and intregrating in time, the result follows. $\square$

Our next goal is to prove that for any $\delta > 0$, small enough, the solution u of equation (1.2) satisfies the following property

$$u(\delta),\ u_t(\delta) \in D(A^\infty),$$

which together with corollary 2.1 implies that the solution of equation (2.2) with initial datum given by $u(\delta)$, $u_t(\delta)$ satisfies

$$u \in C^1([\delta,\, T]\,;\, D(A^\infty))$$

that implies, by the uniqueness and the arbitrary choose of δ, that the solution u satisfies the smoothness effect property.

LEMMA 2.4. — *Under the hypotheses and notations of lemma 2.3 let us suppose that*

$$u \in L^\infty([\delta,\, \eta]\,;\, D(A^s))$$

for $0 < \delta < \eta$ and s a positive real number. In this conditions we have

$$u_t\ and\ u_{tt} \in L^\infty([\delta,\, \eta]\,;\, D(A^s))\,.$$

Proof. — It is easy to see that

$$4 M_0 g_i''(t) g_i'(t) \geq -\frac{1}{4}\, g_i''(t)^2 - 16\, M_0^2 g_i'(t)^2$$

$$2 C M_0 g_i'(t) g_i(t) \geq -2 C^2 M_0^2 g_i'(t)^2 - \frac{1}{2}\, g_i(t)^2\,.$$

Recalling the definition of J_i and H_i we have

$$J_i(t) + C H_i(t) = \frac{1}{2}\, g_i''(t)^2 + 4 M_0 g_i''(t) g_i'(t) + \frac{C}{2}\, g_i'(t)^2 + 2 C M_0 g_i'(t) g_i(t)$$

therefore

$$J_i(t) + C H_i(t) \geq \frac{1}{4}\, g_i''(t)^2 - \left(16 + 2 C^2 - \frac{C}{2}\right) M_0^2 g_i'(t)^2 - \frac{1}{2}\, g_i(t)^2\,.$$

From lemma 2.2 and lemma 2.3 we have

$$\frac{1}{4}\, g_i''(t)^2 \leq \left(16 + 2 C^2 - \frac{C}{2}\right) M_0^2 g_i'(t)^2 +$$

$$+ \frac{1}{2}\, g_i(t)^2 + \{J_i(0) + C H_i(0)\}\, e^{-\lambda_i t} \qquad (2.24)$$

$$\frac{1}{4}\, g_i'(t)^2 \leq 4 M_0^2 g_i(t)^2 + H_i(0)\, e^{-\lambda_i t}\,. \qquad (2.25)$$

Denoting by

$$c_1 = \left(16 + 2C^2 - \frac{C}{2}\right) M_0^2$$

inequalities (2.24) and (2.25) yields

$$\frac{1}{4} g_i''(t)^2 \leq \left\{16c_1 M_0^2 + \frac{1}{2}\right\} g_i(t)^2 + \\ + \left\{J_i(0) + (c_1 + C)H_i(0)\right\} e^{-\lambda_i t} . \qquad (2.26)$$

On the other hand the hypotheses on u implies that there exist a constant C which does not depend on m satisfying

$$\sum_{i=I}^{m} \lambda_i^{2s} g_i(t)^2 < C, \quad \forall \, m \geq I .$$

Therefore multiplying inequality (2.25) and (2.26) by λ_i^{2s}, summing up the product result from I to m and using remark 2.2 we obtain

$$u_t \text{ and } u_{tt} \in L^\infty\left([\delta, \eta]; D(A^s)\right),$$

as requerided. $\square$

Now we are in condition to prove the main result of this paper.

THEOREM 2.2. — *Under the hypotheses of lemma 2.3, the solution u of (1.2) satisfies*

$$u \in C^2\left(]\,0, +\infty\,[; D(A^\infty)\right).$$

Proof. — We will prove that $u \in C^2\left([\delta, T]; D(A^n)\right)$ for any natural number n. In fact since $u \in C\left([0, T]; D(A)\right)$, from lemma 2.4 we conclude that

$$u_t, \ u_{tt} \in L^\infty\left([\delta, \eta]; D(A)\right)$$

from where it follows that

$$u_{tt} = -a(t)Au - Au_t \in L^\infty\left([\delta, \eta]; D(A)\right) \text{ for } \delta \leq t \leq \eta$$

which is equivalent to

$$e^{\int_0^t a(\sigma)\,\mathrm{d}\sigma} u_{tt} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{e^{\int_0^t a(\sigma)\,\mathrm{d}\sigma} Au\right\} \in L^\infty\left([\delta, \eta]; D(A)\right).$$

Denoting by

$$v = e^{\int_0^t a(\sigma)\,\mathrm{d}\sigma}\, u$$

we have that

$$\frac{\mathrm{d}v}{\mathrm{d}t} \in L^\infty\big([\delta,\eta]; D(A)\big) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\,Av \in L^\infty\big([\delta,\eta]; D(A)\big)$$

therefore it follows that

$$\frac{\mathrm{d}v}{\mathrm{d}t} \in L^\infty\big([\delta,\eta]; D(A^2)\big). \tag{2.27}$$

On the other hand since

$$v \in L^\infty\big([\delta,\eta]; D(A)\big) \tag{2.28}$$

from (2.27), (2.28) and the second part of lemma 2.2 we conclude that

$$v \in C\big([\delta,\eta]; D(A^{2-2^{-1}\epsilon})\big),$$

for a small $\epsilon > 0$. That is, we have that

$$u \in C\big([\delta,\eta]; D(A^{2-2^{-1}\epsilon})\big).$$

Repeating this reasoning, from lemma 2.4, we obtain that

$$u_t,\, u_{tt} \in L^\infty\big([\delta,\eta]; D(A^{2-2^{-1}\epsilon})\big);$$

by (1.2) we have that

$$\frac{\mathrm{d}v}{\mathrm{d}t} \in L^\infty\big([\delta,\eta]; D(A^{2-2^{-1}\epsilon})\big)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\,Av \in L^\infty\big([\delta,\eta]; D(A^{2-2^{-1}\epsilon})\big)$$

therefore we have that

$$\frac{\mathrm{d}v}{\mathrm{d}t} \in L^\infty\big([\delta,\eta]; D(A^{3-2^{-1}\epsilon})\big). \tag{2.29}$$

From (2.28), (2.29), and the second part of lemma 2.1, we obtain

$$u \in C\big([\delta,\eta]; D(A^{3-2^{-1}\epsilon-4^{-1}\epsilon})\big).$$

Repeating this reasoning $n + 1$ times we conclude that

$$u \in C\big([\delta \,, \eta]\,; D(A^{n+2-\rho})\big)\,,$$

where

$$\rho = \epsilon \sum_{i=1}^{n+2} \left(\frac{1}{2}\right)^i.$$

By lemma 2.4, we obtain that

$$u_t\,, \ u_{tt} \in L^{\infty}\big([\delta \,, \eta]\,; D(A^{n+2-\rho})\big)\,,$$

then by using the first part of lemma 2.1, we conclude that

$$u \in C^1\big([\delta \,, \eta]\,; D(A^{n+1})\big)\,.$$

From corollary 2.1 and for the uniqueness of solutions, we conclude that

$$u \in C^1\big([\delta \,, +\infty\,[\,; D(A^{n+1})\big)$$

and from (1.2), we obtain that $u \in C^2\big([\delta \,, +\infty\,[\,; D(A^n)\big)$ as requerided. $\square$

3. Asymtotic Behaviour

In this section we will prove that the solution of equation (1.2) decay uniformly to zero as time goes to infinity. The method we use here is based on the construction of a Liapunov functional which its derivativee is negative proporcional to a power of itself, therefore an uniformly decay holds.

First of all we will prove the following lemma.

LEMMA 3.1. — *Let f be a C^1 function satisfying*

$$\frac{\mathrm{d}}{\mathrm{d}t}\, f(t) \leq -cf(t) + C(1+t)^{-\beta}$$

where c, C and β are positives constants. Then there exist $T > 0$ for which we have

$$f(t) \leq C(1+t)^{-\beta}\,, \quad \forall\, t \geq T.$$

Proof. — Multiplying the condition on f by e^{ct} and integrating by part the product result yields

$$e^{ct} f(t) \leq f(0) - \frac{C}{c} + \frac{C}{c} e^{ct}(1+t)^{-\beta} + \beta \frac{C}{c} \int_0^t e^{c\sigma}(1+\sigma)^{-\beta-1} \, d\sigma \, .$$

Since the function $\sigma \mapsto e^{c\sigma}(1+\sigma)^{-\beta-1}$ is non decreasing for $\sigma \geq ((\beta+1)/c) - 1$ we have that

$$e^{ct} f(t) \leq f(0) - \frac{C}{c} + \frac{C}{c} e^{ct}(1+t)^{-\beta} + \beta \frac{C}{c} 2 \, e^{c\beta} + \beta \frac{C}{c} t e^{ct}(1+t)^{-\beta-1}$$

$$\leq f(0) - \frac{C}{c} + (1+\beta)\frac{C}{c} e^{ct}(1+t)^{-\beta} + \beta \frac{C}{c} 2 e^{c\beta} \, .$$

From this inequality our result follows. $\square$

THEOREM 3.1. — *Under the hypotheses of theorem 2.1 and M a C^1-non negative monotonic function satisfying*

$$\liminf_{s \to 0^+} \frac{M(s)}{s^\alpha} > 0 \, ; \quad M(s) = O(s^\alpha), \quad s \to 0 \, ;$$

there exist a positive constant C satisfying

$$\left\| A^{1/2} u(t) \right\|^2 \leq c \left(1+t\right)^{-\frac{1}{\alpha}} \tag{3.1}$$

$$\left\| A^{1/2} u_t(t) \right\|^2 \leq C \left(1+t\right)^{-2-\frac{1}{\alpha}} \tag{3.2}$$

$$\left\| u_{tt}(t) \right\|^2 \leq C \left(1+t\right)^{-\gamma} \tag{3.3}$$

where $\gamma = \min \left\{ 4 + \dfrac{1}{\alpha}, \ 2 + \dfrac{3}{\alpha} \right\}$.

Proof. — Let us multiply equation (1.2) by u_t then we have

$$\frac{d}{dt} E(t) = -\left\| A^{1/2} u_t(t) \right\|^2 \, . \tag{3.4}$$

Let us define the auxiliarity function ρ as

$$\rho(t) = \left(u_t(t), \, u(t) \right) + \frac{1}{2} \left\| A^{1/2} u(t) \right\|^2$$

differentiating ρ and using (1.2), we obtain

$$\frac{d}{dt} \rho(t) = \left\| u_t(t) \right\|^2 - M \left(\left\| A^{1/2} u(t) \right\|^2 \right) \left\| A^{1/2} u(t) \right\|^2 . \tag{3.5}$$

From (3.4), (3.5) and ϵ small enough we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{E(t)+\epsilon\rho(t)\right\} \leq -(1-c\epsilon)\left\|A^{1/2}u_t(t)\right\|^2 - \epsilon\widetilde{M}\left(\left\|A^{1/2}u(t)\right\|^2\right)$$

$$\leq -\left[1-\left(c+\frac{1}{2}\lambda_1^{-1}\right)\epsilon\right]\left\|A^{1/2}u_t(t)\right\|^2 - \epsilon E(t) \quad (3.6)$$

$$\leq -\epsilon E(t)\,.$$

From this inequality we conclude that $E \in L^1(\mathbb{R})$. On the other hand from (3.4) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{(1+t)E(t)\right\} \leq E(t)\,.$$

Integration in time implies that there exist a positive constant C satisfying

$$E(t) \leq C(1+t)^{-1}\,.$$

Here C will denote a generic constant which does not depend on ϵ. Since M is a increasing function the last inequality implies that $\left\|A^{1/2}u(t)\right\| \to 0$ as $t \to \infty$. Then by hypothesis on M we conclude that there exist $N > 0$ such that

$$\left\|A^{1/2}u(t)\right\|^{2(\alpha+1)} \leq C\widetilde{M}\left(\left\|A^{1/2}u(t)\right\|^2\right) \quad \text{for } t \geq N\,,$$

then it follows that

$$\left[E(t)+\epsilon\rho(t)\right]^{\alpha+1} \leq CE(t)\,. \quad (3.7)$$

Multiplying (3.6) by C and (3.7) by ϵ and adding the products results, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{E(t)+\epsilon\rho(t)\right\} + \frac{C}{\epsilon}\left\{E(t)+\epsilon\rho(t)\right\}^\alpha \leq 0\,.$$

From where it follows that

$$E(t)+\epsilon\rho(t) \leq C\left(1+t\right)^{-\frac{1}{\alpha}}\,. \quad (3.8)$$

On the other hand taking ϵ small enough we have that there exist a positive constant c such that

$$E(t)+\epsilon\rho(t) \geq c\left\{\left\|u_t(t)\right\|^2 + \widetilde{M}\left(\left\|A^{1/2}u(t)\right\|^2\right) + \left\|A^{1/2}u(t)\right\|^2\right\}\,. \quad (3.9)$$

From (3.8) and (3.9), (3.1) follows. Since $M \in C^1$, we can differentiate equation (1.2) and multiplying by u_{tt} the equation result for $t > 0$, we have

$$
\begin{aligned}
\frac{d}{dt} \left\| u_{tt}(t) \right\|^2 = &-\left\| A^{1/2} u_{tt}(t) \right\|^2 + \\
&- M \left(\left\| A^{1/2} u(t) \right\|^2 \right) \left(A u_t(t), \, u_{tt}(t) \right) + \\
&- M' \left(\left\| A^{1/2} u(t) \right\|^2 \right) \left(u(t), \, A u_t(t) \right) \left(A^{1/2} u(t), \, A^{1/2} u_{tt}(t) \right).
\end{aligned}
\tag{3.10}
$$

Since

$$
A u_t(t) = -u_{tt} - M \left(\left\| A^{1/2} u(t) \right\|^2 \right) A u(t)
$$

sustitution in (3.10) yields

$$
\begin{aligned}
\frac{d}{dt} \left\| u_{tt}(t) \right\|^2 = &-\left\| A^{1/2} u_{tt}(t) \right\|^2 + \\
&+ M \left(\left\| A^{1/2} u(t) \right\|^2 \right) \left\| u_{tt}(t) \right\|^2 + \\
&+ M \left(\left\| A^{1/2} u(t) \right\|^2 \right)^2 \left(A^{1/2} u(t), \, A^{1/2} u_{tt}(t) \right) \times \\
\times \quad &M' \left(\left\| A^{1/2} u(t) \right\|^2 \right) \left(u(t), \, u_{tt}(t) \right) \left(A^{1/2} u(t), \, A^{1/2} u_{tt}(t) \right) + \\
&+ M \left(\left\| A^{1/2} u(t) \right\|^2 \right) M' \left(\left\| A^{1/2} u(t) \right\|^2 \right) \left\| A^{1/2} u(t) \right\|^2 \times \\
\times \quad &\left(A^{1/2} u(t), \, A^{1/2} u_{tt}(t) \right).
\end{aligned}
\tag{3.11}
$$

Using (3.1) for $t \geq N$, N big enough, we have

$$
\begin{aligned}
\frac{d}{dt} \left\| u_{tt}(t) \right\|^2 = &-\frac{1}{2} \left\| A^{1/2} u_{tt}(t) \right\|^2 + \\
&+ C M \left(\left\| A^{1/2} u(t) \right\|^2 \right)^4 \left\| A^{1/2} u(t) \right\|^2 + \\
&+ C M \left(\left\| A^{1/2} u(t) \right\|^2 \right)^2 \left\| A^{1/2} u(t) \right\|^4.
\end{aligned}
$$

Using the above decay result, lemma 3.1 and since

$$
M \left(\left\| A^{1/2} u(t) \right\|^2 \right) \leq C (1 + t)^{-1}, \quad \forall\, t \geq N
$$

we obtain (3.3). Finally multiplying (1.2) by u_t we have that

$$
\begin{aligned}
\left\| A^{1/2} u_t(t) \right\|^2 \leq &M \left(\left\| A^{1/2} u(t) \right\|^2 \right) \left\| A^{1/2} u_t(t) \right\| \left\| A^{1/2} u(t) \right\| + \\
&+ \left\| u_{tt}(t) \right\| \left\| u_t(t) \right\|
\end{aligned}
$$

from where it follows

$$\frac{1}{2} \left\| A^{1/2} u_t(t) \right\|^2 \leq 4M \left(\left\| A^{1/2} u(t) \right\|^2 \right)^2 \left\| A^{1/2} u_t(t) \right\|^2 +$$

$$+ \frac{4}{\lambda_1} \left\| u_{tt}(t) \right\|^2 ;$$

using the above decay rates we obtain (3.2). Then our result follows. □

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