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Interpolation on plane sets in $\mathbb{C}^{2(*)}$

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RÉSUMÉ. — On donne les conditions analytiques et géométriques pour le prolongement analytique de fonctions par ensembles plates avec l'estimation de l'indicatrice radiale.

ABSTRACT. — Analytic and geometric conditions are given, for holomorphic extension of functions from plane sets with estimates of radial indicator.

Introduction

We consider the problem of extension of a function holomorphic on an analytic variety in $\mathbb{C}^2$ to an entire function with estimates of growth. The varieties we deal with are unions of countable families of hyperplanes. We are interested mainly in the question of existence of an extension in the class $[\rho, h(z)]$ of functions whose radial indicator $L_f(z) \stackrel{\text{def}}{=} \limsup_{t\to\infty} t^{-\rho} \log|f(tz)|$ with respect to order ρ does not exceed a given function $h(z)$, although the general character of estimates in our theorem 4 allows to consider the problem also in other classes.

Interpolation problems in classes of functions defined by growth restrictions are traditional, and we have no possibility to give a complete historical overview of the question. Therefore we would like to mention here just some of the papers, to which, as it seems to us, the present one is close in setting of the problems and character of the estimates. One must begin here with L. Hörmander's theorem [Hö] on extension from a subspace in $\mathbb{C}^N$. Hörmander's method, based on the solution of $\bar{\partial}$-problem with bounds, was used

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afterwards for solving interpolation problems in many papers (including the present one). C. A. Berenstein and B. A. Taylor [BT1], [BT2] obtained a number of results on extension with bounds from analytic varieties of rather general form. Their results, however, do not apply to the class $[\rho, h(z)]$, because one needs more precise estimates here. Interpolation problems in this class were treated in the papers of L. I. Ronkin [Ro1], [Ro2] and of L. I. Ronkin and one of the authors [RoRu]. The analytic varieties considered there were either algebraic or pseudoalgebraic. A solution of extension problem in the class $[\rho, h(z)]$ from smooth varieties of codimension 1 in $\mathbb{C}^N$ was constructed in [Ru]. The problem of extension in the mentioned class from a discrete set in $\mathbb{C}^N$ was solved in [Pa1]. In the same paper for the first time in questions of interpolation were used entire functions, whose zero set was a union of hyperplanes (the so-called functions with " plane " zeros). Our paper deals exactly with sets of this type; the difficulties that arise here are due to the fact that such a set is not smooth (with the exception of the trivial case, when all the planes in the family are parallel). We note here, that all the results we know, where singularities are allowed, apply to either algebraic or pseudoalgebraic varieties.

Plane analytic sets possess a number of properties, which simplify the construction of the extension. The main of them are the continuity of the radial indicator of an entire function with plane zeros, proved by L. Gruman [Gru], and also the simple geometric structure, investigation of which allows to find practically checkable sufficient conditions of the existence of the extension in our class. We also note, that we use as a tool theory of functions of regular growth with plane zeros, developed in [Pa2].

In general, for entire functions of several variables, there exist two different definitions of completely regular growth (see [LGru], [Ro4]) which are nonequivalent. However, for functions with plane zeros, it was shown in [Pa3], that both definitions coincide. We remind the one that we actually use (see, e.g. [Az], [Ro4]).

An entire function $f(z)$ with radial indicator $L(z)$ is called the function of completely regular growth if the functions $u_t(z) = t^{-\rho} \cdot \log|f(tz)|$ tend to $L(z)$ in the sense of distributions in $\mathbb{R}^{2N}$ when $t \to \infty$. By a theorem of Azarin, this implies, in particular, that

$$\lim_{\substack{|z|\to\infty \\ z\notin E}} \frac{\big|\log|f(z)| - L(z)\big|}{|z|^{\rho}} = 0$$

for some C_0^0-set E. Here C_0^0 means C_0^ε for each $\varepsilon > 0$, and C_0^ε is a set in $\mathbb{R}^{2N}$, which may be covered by balls $B_{r_k}(z_k)$ of radii r_k centered at z_k so that

$$\frac{1}{R^{2N-2+\varepsilon}} \sum_{|z_k|\leq R} r_k^{2N-2+\varepsilon} \to 0\,, \quad R \to \infty\,.$$

The structure of a C_0^0-set is not well investigated. However, our lemma 5 below gives some information about it. Namely, it is proved that, given a point z and a C_0^0-set E, one can always find a circumference centered at z of radius not greater than $\delta|z|$ lying outside E.

Consider a set of hyperplanes Λ in $\mathbb{C}^N$. Each hyperplane H is uniquely defined by its perpendicular vector. Denote by $n_\Lambda(K)$ the number of all perpendicular vectors of Λ contained in a set K. Put $K_t = \{z : (z/t) \in K\}$. One says that the family of hyperplanes Λ is regularly distributed, if there exists a density

$$d_\Lambda(K) = \lim_{t\to\infty} t^{-\rho} n_\Lambda(K_t)$$

for almost all compacts K in $\mathbb{C}^N$. If the order ρ is an integer one has to add one more condition implying some symmetry in the distribution of hyperplanes.

There is a relation between the density d and the indicator of the canonical product associated to the hyperplane set Λ, see [Pa2]. We will write $\Lambda \in \mathrm{Reg}\big[\,\rho\,,\, h(z)\,\big]$ if Λ is regularly distributed and the corresponding indicator is $h(z)$.

It was shown in [Pa2], that if an entire function has plane zeros which are regularly distributed, then it has completely regular growth.

In what follows $h(z)$ will be a continuous positively ρ-homogeneous plurisubharmonic function.

Remind that the radial indicator $L_{\varphi,\Lambda}(z)$ (with respect to order ρ) of a function $\varphi(z)$ holomorphic on an analytic variety $\Lambda \subset \mathbb{C}^N$ is defined (see [Ro1]) as follows:

$$L_{\varphi,\Lambda}(z) \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0} \limsup_{t\to\infty} t^{-\rho} \sup\big\{\log\big|f(tz')\big| : |z - z'| < \varepsilon\,,\; tz' \in \Lambda\big\}\,.$$

A divisor Λ is called interpolatory for the class $\big[\,\rho\,,\, h(z)\,\big]$, if the problem of free interpolation on Λ (i.e. of extending a function φ analytic on Λ with

$L_{\varphi,\Lambda}(z) \le h(z)$ to an entire function Φ with $L_\Phi(z) \le h(z)$) is solvable in this class.

To give an idea of our aims, we recall some one-dimensional results on interpolation in the class $[\rho, h(z)]$. We use some notions from [Gr] and [GrRu].

Let Λ be a set of points (divisor) in $\mathbb{C}$. Denote by $n_\Lambda(K)$ the number of points in $\Lambda \cap K$. For a compact set K put again $K_t = \{z : (z/t) \in K\}$ and $K^\sigma = \{z : \operatorname{dist}(z, K) \le \sigma\}$. For a point $z \in \mathbb{C}$ we write $n_z(t)$ for $n_\Lambda(B_t(z))$, $\widetilde{n}_z(t)$ for $[n_z(t) - 1]^+$ and $\Phi_z(\alpha)$ for $|z|^{-\rho}\widetilde{n}_z(\alpha|z|)$.

Define the (upper) density $\overline{d}_\Lambda(K)$ by the formula

$$\overline{d}_\Lambda(K) = \lim_{\sigma\to 0} \limsup_{t\to\infty} t^{-\rho} n_\Lambda\big((K_t)^\sigma\big)$$

and the " concentration "

$$c(\Lambda) = \lim_{\delta\to 0} \sup_{z\in\Lambda} \int_0^\delta \frac{\Phi_z(\alpha)}{\alpha}\, d\alpha\,.$$

THEOREM ([Gr], see also [GrRu]).— *The following statements are equivalent:*

i) *Λ is an interpolatory divisor for the class $[\rho, h(z)]$;*

ii) *there exists an entire function $f(z)$, of completely regular growth with indicator $h(z)$, with divisor containing Λ and with the property*

$$\lim_{\substack{|z|\to\infty \\ z\in\Lambda}} \frac{\big|\log|f'(z)| - h(z)\big|}{|z|^\rho} = 0\,; \tag{a}$$

iii) *the following two conditions hold:*

$$\forall\, K \subset \mathbb{C}: \quad \overline{d}_\Lambda(K) \le \mu_h(K)\,, \tag{g1}$$

where μ_h is the Riesz measure associated to the subharmonic function $h(z)$;

$$c(\Lambda) = 0\,. \tag{g2}$$

We will call (a) a condition of " analytic type " and (g1)-(g2) conditions of " geometric type ". Below we obtain only sufficient conditions of interpolation, but of both analytic and geometric type.

Our main results are stated for $\mathbb{C}^2$. However, many of the assertions hold also for $\mathbb{C}^N$ with $N \geq 2$ (some of them are formulated and proved in the general case). We believe, that the extension of all the theorems to the general case meets the difficulties of purely technical (but not of conceptional) character.

Results

It will be convenient to call a function $g(z)$ negligible if

$$\limsup_{|z|\to\infty} |z|^{-\rho} g(z) \leq 0 .$$

Consider a hyperplane divisor Λ in $\mathbb{C}^2$. Let

$$\Lambda \stackrel{\text{def}}{=} \left\{z : f(z) = 0\right\} = \bigcup_{k=1}^{\infty} H_k = \bigcup_{k=1}^{\infty} \left\{z \in \mathbb{C}^2 : \left\langle z\,,\, a^{(k)}\right\rangle = \left|a^{(k)}\right|^2\right\}\,, \quad (1)$$

and let $s_{kj} = H_k \cap H_j$.

Denote by t_{kj} the maximal radius of a polydisk $U_{t_{kj}}(s_{kj})$ not intersected by hyperplanes different from H_k and H_j:

$$U_{t_{kj}}(s_{kj}) \cap \Lambda \setminus (H_k \cup H_j) = \emptyset .$$

We make an important assumption:

$$\exists\ g\text{-negligible} : \forall\ k,\, j\ \log \frac{1}{t_{kj}} \leq g(s_{kj}) . \quad (2)$$

Put $r_{kj} = e^{-g(s_{kj})}$, $U_{kj} = U_{r_{kj}}(s_{kj})$, $\widetilde{U}_{kj} = U_{r_{kj}/16}(s_{kj})$, $U = \bigcup_{j,k=1}^{\infty} \widetilde{U}_{kj}$.

We fix a continuous positively ρ-homogeneous plurisubharmonic function $h(z)$ and state our analytic conditions of interpolation.

THEOREM 1.— *Let $f(z)$ be an entire function in $\mathbb{C}^2$ with radial indicator $L_f(z) = h(z)$, with zero set of the form (1) satisfying (2), and such that*

$$\lim_{\substack{|z|\to\infty \\ z\in\Lambda\setminus U}} \frac{\left|\log\left|\nabla f(z)\right| - L_f(z)\right|}{|z|^{\rho}} = 0 . \quad (\mathrm{A})$$

Then Λ is an interpolatory divisor for $\left[\,\rho\,,\, h(z)\,\right]$.

This result can be somewhat generalized by letting $f(z)$ have indicator " smaller " than $h(z)$.

THEOREM 2.— *Let $f(z)$ be the same as in theorem 1 with the only difference that $h(z) - L_f(z)$ is a plurisubharmonic function in $\mathbb{C}^2$. Suppose that (A) holds.*

Then Λ is interpolatory for $[\rho, h(z)]$.

Remark.— It is clear that the conditions (2) and (A) are not independent. As we will see later, (A) will imply some estimates of $g(z)$. We also note that the estimate from above in (A) always holds in our case, and the nontrivial part of (A) consists in the estimate from below of $|\nabla f|$.

Theorems 1-2 give some analytic sufficient conditions for interpolation. If we assume that our divisor Λ is regularly distributed, it is possible to give geometric sufficient conditions.

We denote by $n_z(t)$ the number of hyperplanes from Λ intersecting the ball $B_t(z)$; put $\widetilde{n}_z(t) = [n_z(t) - 1]^+$ and $\Phi_z(\alpha) = |z|^\rho \widetilde{n}_z(\alpha|z|)$. As in one-dimensional case we define a kind of " concentration ":

$$c(\Lambda) = \limsup_{\delta \to 0} \sup_{z \in \Lambda \setminus U} \int_0^\delta \frac{\Phi_z(\alpha)}{\alpha} \, d\alpha .$$

THEOREM 3.— *Let Λ be a set of hyperplanes in $\mathbb{C}^2$ satisfying (2). Let, further,*

$$\Lambda \in \mathbf{Reg}[\rho, h(z)] ; \tag{G1}$$

$$c(\Lambda) = 0 . \tag{G2}$$

Then Λ is interpolatory for $[\rho, h(z)]$.

Examples of sets satisfying (G1)-(G2) may be given by either considering a finite collection of parallel families of hyperplanes, each of them being regularly distributed and interpolatory when restricted to the corresponding perpendicular complex line, or by small perturbations of a parallel family satisfying the same requirements, if we need to obtain infinite set of directions.

Our theorems on interpolation in $[\rho, h(z)]$ are derived from a rather general result, which we present below.

We denote by PSH(Ω) the set of functions plurisubharmonic on a set $\Omega \subset \mathbb{C}^N$, and by $A(\Omega)$ the set of analytic functions in Ω. For a function $g(z)$ we use the denotions $g^{[r]}(z) = \sup_{|z-\zeta|\leq r} g(\zeta)$ and $M_g(r) = g^{[r]}(0)$.

Let functions u, v_j $(j = 1, 2, 3)$, $v(z) = v_1^{[1]}(z) + v_2^{[1]}(z)$, $z \in \mathbb{C}^2$, be such that

$$u,\ v_1,\ v_3,\ v,\ u^{[1]}(z) + v_2^{[1]}(z) \quad \text{belong to} \quad \mathrm{PSH}(\mathbb{C}^2)\,;$$
$$v \geq v_3 \geq 0\,;$$
$$\left|\bar{\partial} v(z)\right| \leq e^{v(z)}\,. \tag{3}$$

Let now $f(z)$ be an entire function in $\mathbb{C}^2$ with plane zeroes, and let $A = \{a^{(k)}\}_{k=1}^{\infty}$ be the sequence of feet of perpendiculars dropped from the origin onto zero hyperplanes H_k of the function $f(z)$. We assume (without losing generality) that $a_1^{(k)} \neq 0$, $\forall\ k = 1, 2, \ldots$ We use the same notations as above. Let, as before, $s_{kj} = H_k \cap H_j$ and let $r_{kj} = e^{v_3(s_{kj})}$.

We assume further that the polydisk $U_{kj} = U_{r_{kj}}(s_{kj})$ contains no points of the set $\Lambda \setminus (H_k \cup H_j)$, i.e.

$$U_{kj} \cap \Lambda \setminus (H_k \cup H_j) = \emptyset \tag{4}$$

We denote also by $\widetilde{U}_{kj}$ the polydisk $U_{r_{kj}/16}(s_{kj})$, and the union of all $\widetilde{U}_{kj}$ by U.

Now we are able to state our theorem on extension with a majorant of general type.

THEOREM 4.— *Let*

$$\log\left|f(z)\right| \leq v_1(z)\,, \quad z \in \mathbb{C}^2 \tag{5}$$

$$\log\left|\frac{\partial f(z)}{\partial z_1}\right| \geq -v_2(z)\,, \quad z \in \Lambda \setminus U\,. \tag{6}$$

Then for each $\varphi \in A(\Lambda)$ satisfying

$$\log\left|\varphi(z)\right| \leq u(z)\,, \quad z \in \Lambda\,, \tag{7}$$

there exists such $\Phi \in A(\mathbb{C}^2)$ that

i) $\Phi(z) = \varphi(z)$, $\forall\ z \in \Lambda$,

ii) $\log\left|\Phi(z)\right| \leq u^{[3]}(z) + 16\, v^{[3]}(z) + 3\log(1 + |z|^2) + C$, $z \in \mathbb{C}^2$.

The paper is organized as follows. First we prove our theorem 4 on extension with estimates of general type. From this theorem we deduce the sufficient analytic conditions for interpolation in the class $[\rho, h(z)]$ (theorem 2). Theorem 1 is just a particular case of theorem 2. Finally, we prove that our geometric conditions (theorem 3) imply that the condition (A) of theorem 1 holds.

Proof of theorem 4

We prove some preliminary statements first.

Introduce the following denotions:

$$\begin{aligned} &z = (\lambda, w) \in \mathbb{C}^2, \\ &U_{r,R}(z^0) = U_R(z^0) \setminus \overline{U_r(z^0)}, \\ &\Omega_\varepsilon(K) = \{z : \operatorname{dist}(z, K) \geq \varepsilon\}, \end{aligned}$$

where K is a set in $\mathbb{C}^2$ and dist is the euclidean metric.

LEMMA 1.— *Let an entire function $f(z)$ vanish on hyperplanes $H_1 = \{\lambda = \zeta w\}$ and $H_2 = \{\lambda = -\zeta w\}$ and have no other zeros in the polydisk $U_R(0)$, $R < 1$. Let, further, f satisfy (5) and*

$$\log|\nabla f(z)| \geq -v_2(z), \quad z \in (H_1 \cup H_2) \cap U_{R/8,R}(0). \tag{8}$$

Then

$$R \geq 2\, e^{-v^{[R]}(0)}; \tag{9}$$

$$|\zeta| \in [A^{-1}, A] \quad \textit{with} \quad A = \frac{R}{2} \exp\big(v^{[R]}(0)\big) \tag{10}$$

and

$$|f(\lambda, w)| \geq C_1 \varepsilon^2\, e^{-v_2(\zeta w, w) - 14\, v^{[R]}(0)}, \tag{11}$$

when $(\lambda, w) \in \Omega_\varepsilon(H_1 \cup H_2) \cap U_{R/8,R}(0)$, $\varepsilon \leq (1/4)\exp\big(-v^{[R]}(0)\big)$.

The lemma actually states that if on two zero hyperplanes we have an estimate for $\nabla f(z)$ from below outside a neighborhood of their intersection, then, first, the neighborhood is not too small, second, the angle between these hyperplanes is not too small, and third, outside some ε-neighborhood of these hyperplanes the function $f(z)$ can be estimated from below.

Proof.— It is enough to suppose that $|\zeta| \le 1$, and estimate $|\zeta|$ from below.

Denote $r = R/8$. Fix w, $|w| = r$ and consider the function $\varphi_w(\lambda) = f(\lambda, w)$. In the disk $|\lambda| \le r$ the function $\varphi_w(\lambda)$ has zeros at points $\pm\zeta w$ and has no other zeros in $|\lambda| \le R$. Since $\nabla f(z)$ at $z \in H_j$ is perpendicular to H_j, we get from (8):

$$\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\varphi_w(\pm\zeta w)\right| \ge \left(1+|\lambda|^2\right)^{-\frac{1}{2}} \exp\bigl(-v_2(\pm\zeta w, w)\bigr) \ge \frac{1}{2}\,\exp\bigl(-v_2(\pm\zeta w, w)\bigr)\,.$$

According to lemma 3 of [BT1], it is possible then to estimate the distance between the zeros from below:

$$2|\zeta w| \ge \left|\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\varphi_w(\pm\zeta w)\right| \cdot \left(\sup_{|\lambda+\zeta w|\le 1} |\varphi_w(\lambda)|\right)^{-1} \ge \frac{1}{2}\,\exp\bigl(-v(\zeta w, w)\bigr)\,.$$

Hence

$$2\cdot\frac{R}{8} \ge 2|\zeta w| \ge \frac{1}{2}\,\exp\bigl(-v(\zeta w, w)\bigr)$$

and

$$|\zeta| \ge \frac{e^{-v(\zeta w, w)}}{4r} \ge \frac{e^{-v^{[R]}(0)}}{4r} \ge \frac{2e^{-v^{[R]}(0)}}{r} = A\,,$$

and the first two statements are proved.

To prove the third statement we consider for fixed w, $r \le |w| \le R$, the function

$$\psi_w(\lambda) = \frac{\varphi_w\bigl(\zeta w + 2|\zeta w|\lambda\bigr)}{\bigl(\mathrm{d}/\mathrm{d}\lambda\bigr)\varphi_w(\zeta w)\cdot 2|\zeta w|\lambda}\,.$$

We have $\psi_w(0) = 1$, $M_{\psi_w}(1) \le 4\,e^{2v(\zeta w, \omega)}$, $\psi_w(\lambda) \ne 0$ for $|\lambda| < 1$.

Applying to the function ψ_w the Caratheodori inequality, we get

$$\bigl|\psi_w(\lambda)\bigr| \ge \frac{1}{16}\,e^{-4v(\zeta w, w)}\,, \quad |\lambda| \le \frac{1}{2}\,.$$

Taking $\lambda = -(1/2)\cdot\bigl(\zeta w/|\zeta w|\bigr)$ in the last inequality and using the previous estimates, by the definition of ψ_w we obtain

$$\bigl|\varphi_w(0)\bigr| \ge \frac{1}{32}\,e^{-v_2(\zeta w, w) - 4v(\zeta w, w)}\,.$$

Consider now the function

$$g_w(\lambda) = \frac{\varphi_w(\lambda)\zeta^2 w^2}{\varphi_w(0)(\zeta^2 w^2 - \lambda^2)} \quad \text{for} \quad |\omega| \in \left[r\,,\, \frac{R}{2}\right]\,,\ |\lambda| < R\,.$$

We have $g_w(0) = 1$, $M_{g_w}(R) \le 32 \cdot (1/3)\, e^{5v^{[R]}(0)}$ since
$|\zeta^2 w^2| \le |w|^2 \le \dfrac{R^2}{4}$ implies $\left|\dfrac{\zeta^2 w^2}{\zeta^2 w^2 - R^2}\right| \le \dfrac{R^2/4}{R^2 - R^2/4} = \dfrac{1}{3}$;
$g_w(\lambda) \ne 0$ for $|\lambda| < R$.

With the help of the Carathedori inequality we get

$$\left|g_w(\lambda)\right| \ge 9 \cdot 32^{-2} \exp\!\left(-10\, v^{[R]}(0)\right), \quad |\lambda| \le \frac{R}{2}\,.$$

Hence (we use the estimate of $\left|\varphi_w(0)\right|$ and the fact that $|\zeta w| < 1$)

$$\begin{aligned}\left|f(\lambda, w)\right| &= \varphi_w(\lambda) = g_w(\lambda)\zeta^{-2} w^{-2} \varphi_w(0)(\zeta^2 w^2 - \lambda^2) \ge \\ &\ge 9 \cdot 2^{-10}\, e^{-10\, v^{[R]}(0)} \cdot 2^{-5}\, e^{-v_2(\zeta w, w) - 4\, v^{[R]}(0)} \cdot |\zeta^2 w^2 - \lambda^2| = \\ &= C_1\, e^{-v_2(\zeta w, w) - 14\, v^{[R]}(0)} \cdot |\zeta^2 w^2 - \lambda^2|\end{aligned}$$

when $|w| \in [\,r\,,\, R/2\,]$, $|\lambda| \le R/2$. If, additionally, $(\lambda, w) \in \Omega_\varepsilon(H_1 \cup H_2)$ then

$$\left|f(\lambda, w)\right| \ge C_1 \varepsilon^2\, e^{-v_2(\zeta w, w) - 14\, v^{[R]}(0)}\,. \tag{12}$$

Note that if $\varepsilon \le (1/4) e^{-v^{[R]}(0)}$, we have $\varepsilon \le r|\zeta|$ and so the set $B = \big\{(\lambda, w) : |\lambda| \in [\,r\,,\, R/2\,]\,,\ |w| = r\big\}$ is contained in $\Omega_\varepsilon(H_1 \cup H_2)$.

Hence (12) holds for $(\lambda, w) \in B$. Since the function $h_\lambda(w) = f(\lambda, w)$ for fixed $\lambda \in [\,r\,,\, R/2\,]$ has no zeros in $\big\{|w| < r\big\}$, the estimate (12) holds also inside this disk. Thus, for $(\lambda, w) \in \Omega_\varepsilon(H_1 \cup H_2) \cap U_{r, R/2}$ we obtain (11), and the lemma is proved.

LEMMA 2.— *Let $H_1 = \{\lambda = \zeta w\}$ and $H_2 = \{\lambda = -\zeta w\}$ be two hyperplanes with ζ satisfying (10) and let $\varphi \in A(H_1 \cup H_2)$ satisfy*

$$\log\left|\varphi(z)\right| \le u(z)\,, \quad z \in H_1 \cup H_2\,.$$

Then there exists such a function $P \in A(\mathbb{C}^2)$ that $P(z) = \varphi(z)$, $z \in H_1 \cup H_2$ and

$$M_P(R) \le \left(1 + R\, e^{v^{[R]}(0)}\right) e^{u^{[R]}(0)}\,. \tag{13}$$

Proof. — We may assume again that $|\zeta| \leq 1$.

Put

$$\psi(w) = \frac{\varphi(\zeta w, w) - \varphi(-\zeta w, w)}{2\zeta w}.$$

Obviously, $\psi(w)$ is an entire function with

$$M_\psi(R) \leq \frac{1}{R|\zeta|}\, e^{u^{[R]}(0)} \leq \frac{1}{2}\, e^{u^{[R]}(0)+v^{[R]}(0)}.$$

We now define $P(z)$ by

$$P(\lambda, w) = \varphi(\zeta w, w) + (\lambda - \zeta w)\psi(w).$$

It is evident, that P is entire, $P(z) = \varphi(z)$ for $z \in H_1 \cup H_2$. The required estimate (13) follows now from the estimate of $M_\psi(R)$.

The lemma is proved.

Proof of theorem 4

We use Hörmander's scheme. First a C^∞-function $h(z)$ solving the interpolation problem and having appropriate estimates of $\overline{\partial}$-derivatives is constructed. The construction consists of two steps. First we define a function $g(z)$ outside some neighborhoods of the points s_{kj} and give some estimates. Then we " paste " to g a function which solves the interpolation problem in these neighborhoods and check the estimates of the obtained function. The final step, as always, is based on solution of $\overline{\partial}$-problem with bounds.

Denote $\widetilde{\Lambda} = \Lambda \setminus U$; $\Lambda_w = \{\lambda : (\lambda, w) \in \Lambda\}$; $\widetilde{\Lambda}_w = \{\lambda : (\lambda, w) \in \widetilde{\Lambda}\}$.

Let $\lambda_0 \in \widehat{\Lambda}_w$. Then according to (6) we have

$$\log\left|\frac{\partial f}{\partial \lambda}(\lambda_0, w)\right| \geq -v_2(\lambda_0, w).$$

Using the mentioned lemma from [BT1] we obtain

$$\operatorname{dist}(\lambda_0\,,\, \Lambda_w \setminus \{\lambda_0\}) \geq \left|\frac{\partial f}{\partial \lambda}(\lambda_0, w)\right| \left(\sup\left\{|f(\lambda, w)| : |\lambda - \lambda_0| \leq 1\right\}\right)^{-1} \geq$$
$$\geq e^{-v(\lambda_0, w)}.$$

Set $r_{\lambda_0}(w) = e^{-v(\lambda_0, w)}$,

$$\psi_w(\lambda) = \frac{f\big(\lambda_0 + r_{\lambda_0}(w)\lambda\,,\, w\big)}{\dfrac{\partial f}{\partial \lambda}(\lambda_0, w)\lambda r_{\lambda_0}(w)}\,.$$

It is easy to see that $\psi_w(0) = 1$, $\psi_w(\lambda) \neq 0$ for $|\lambda| < 1$ and $M_{\psi_w}(1) \leq e^{2v(\lambda_0, w)}$. By the Caratheodori inequality for $|\lambda| \leq 1/2$ we have

$$\log\big|\psi_w(\lambda)\big| \geq -4v(\lambda_0, w)\,.$$

Hence

$$\log\big|f\big(\lambda_0 + r_{\lambda_0}(w)\lambda\,,\, w\big)\big| \geq -v_2(\lambda_0, w) - 5v(\lambda_0, w) + \log|\lambda|\,. \tag{14}$$

Let now $\chi(t)$ be a C^∞-function on $\mathbb{R}^+$, $\big|\chi(t)\big| \leq 1$, $\forall\, t$, $\chi(t) = 1$ for $t \leq 1/4$, $\chi(t) = 0$ for $t \geq 1/2$ and $C = \sup_t\big|\chi'(t)\big|$.

We set

$$g(\lambda, w) = \begin{cases} \chi\big(|\lambda - \lambda_0|\, e^{v(\lambda_0, w)}\big)\varphi(\lambda_0, w) & \text{if } \exists\ \lambda_0 \in \widetilde{\lambda}_w : \\ & \qquad |\lambda - \lambda_0| \leq (1/2) r_{\lambda_0}(w) \\ 0 & \text{in the opposite case.} \end{cases}$$

Note that $g \in C^\infty(\mathbb{C}^2 \setminus U)$ and that

$$\begin{aligned} &\overline{\partial} g(\lambda, w) \neq 0 \Leftrightarrow \\ &\qquad \Leftrightarrow (\lambda, w) \in A \overset{\text{def}}{=} \left\{(\lambda, w) : \exists\ \lambda_0 \in \widetilde{\Lambda}_w : |\lambda - \lambda_0| \cdot r_{\lambda_0}^{-1}(w) \in \left[\frac{1}{4}\,, \frac{1}{2}\right]\right\}. \end{aligned}$$

In view of (6) and (14) the equation $f(\lambda, w) = 0$ can be solved with respect to λ on the set $\big\{|\lambda - \lambda_0| e^{-v(\lambda_0, w)} < 1/2\,;\, |w - w_0| < \varepsilon\big\}$ for ε small enough. The corresponding function $\lambda(w)$ is holomorphic in the disk $\big\{|w - w_0| < \varepsilon\big\}$ and so $\overline{\partial}\lambda(w) = 0$ in this disk.

Besides that, since

$$0 \equiv \frac{\mathrm{d} f\big(\lambda(w), w\big)}{\mathrm{d} w} = \frac{\mathrm{d}\lambda}{\mathrm{d} w}\,\frac{\mathrm{d} f}{\mathrm{d}\lambda}\,\big(\lambda(w), w\big) + \frac{\mathrm{d} f}{\mathrm{d} w}\big(\lambda(w), w\big)\,,$$

we have

$$\left|\frac{\mathrm{d}\lambda}{\mathrm{d}w}\right| = \frac{\left|\dfrac{\mathrm{d}f}{\mathrm{d}w}(\lambda(w), w)\right|}{\left|\dfrac{\mathrm{d}f}{\mathrm{d}\lambda}(\lambda(w), w)\right|} \le e^{v(\lambda(w),w)} \tag{15}$$

(we have used here (6) and the fact that $\left|(\mathrm{d}f/\mathrm{d}w)(\lambda(w), w)\right| \le e^{v_1^{[1]}(\lambda,w)}$ by (5)).

Let us estimate now $\left|\overline{\partial} g(\lambda, w)\right|$ for $(\lambda, w) \in A$. We have

$$\begin{aligned}
|\overline{\partial} g| = |\overline{\partial}\chi \cdot \varphi| &= |\chi'| \left| \frac{\partial}{\partial \overline{\lambda}} \left(|\lambda - \lambda_0|\, e^{v(\lambda_0,w)}\right) + \frac{\partial}{\partial \overline{w}} \left(|\lambda - \lambda_0|\, e^{v(\lambda_0,w)}\right)\right| |\varphi| \\
&\le |\chi'|\,|\varphi| \left[\frac{1}{2}\, e^{v(\lambda_0,w)} + \frac{1}{2}\, e^{v(\lambda_0,w)} \left|\frac{\mathrm{d}\lambda_0(w)}{\mathrm{d}w}\right| + \right. \\
&\qquad \left. + \, e^{v(\lambda_0,w)} |\lambda - \lambda_0(w)|\,|\overline{\partial} v(\lambda_0, w)| \left(1 + \left|\frac{\mathrm{d}\lambda_0(w)}{\mathrm{d}w}\right|\right)\right] \\
&\le C \exp\left[u(\lambda_0, w) + \log\left|\overline{\partial} v(\lambda_0, w)\right| + \log\left|\frac{\mathrm{d}\lambda_0(w)}{\mathrm{d}w}\right|\right] \\
&\le C \exp\left[u(\lambda_0, w) + 2v(\lambda_0, w)\right]
\end{aligned}$$

(the last inequality holds by (3) and (15)).

Note that for $(\lambda, w) \in A$ the estimate (14) implies

$$\left|f(\lambda, w)\right| \ge \frac{1}{4} \exp\left(-v_2(\lambda_0, w) - 5v(\lambda_0, w)\right) \tag{16}$$

and hence

$$\left|\frac{\overline{\partial} g}{f}(\lambda, w)\right| \le \frac{C}{4}\, e^{(u+v_2)(\lambda_0,w)+7v(\lambda_0,w)} \le \frac{C}{4}\, e^{(u+v_2)^{[1/2]}(\lambda,w)+7v^{[1/2]}(\lambda,w)}.$$

Since $\overline{\partial} g = 0$ outside A we come to

$$\left|\frac{\overline{\partial} g}{f}(z)\right| \le \frac{C}{4}\, e^{(u+v_2)^{[1/2]}(z)+7v^{[1/2]}(z)}, \quad z \in \mathbb{C}^2 \setminus U. \tag{17}$$

We pass to the second step of our construction. We would like to use lemma 2 for the extension of φ to the neighborhoods of s_{kj}. Choose the coordinates (z_1', z_2') in the neighborhood of s_{kj} (which will become the origin for a while) so that the equations defining H_j and H_k have the form

$\{z_1' = \zeta z_2'\}$ and $\{z_1' = -\zeta z_2'\}$. Denote by $U_{kj}^{(\alpha)}$ the polydisk $U_{r_{kj}/\alpha}(s_{kj})$ (in old coordinates) and by $V_{kj}^{(\alpha)}$ the similar polydisk in new coordinates. We obviously have then

$$U_{kj}^{(2\alpha)} \subset V_{kj}^{(\sqrt{2}\alpha)} \subset U_{kj}^{(\alpha)}\,, \quad \forall\, \alpha > 0\,.$$

Note that (6) implies the estimate

$$\left|\nabla f(z' + s_{kj})\right| \geq \exp\bigl(-v_2(z' + s_{kj})\bigr)\,,$$

which holds for $z' + s_{kj} \in (H_k \cup H_j) \setminus V_{kj}^{8\sqrt{2}}$. Hence, in view of lemma 1, $v_3 \geq v$ and also (10) takes place. Thus the conditions of lemma 2 hold. According to this lemma, there exists an entire function $P_{kj}(z)$ which extends φ from H_j and H_k and satisfies (in view of (13) with $R = r_{kj}/\sqrt{2}$) the estimates

$$\left|P_{kj}(z)\right| \leq e^{u^{[1/\sqrt{2}]}(s_{kj})}\bigl(1 + r_{kj}\, e^{u^{[1/\sqrt{2}]}(s_{kj})}\bigr)\,, \quad z \in U_{kj}^{(2)}\,.$$

Hence

$$\left|P_{kj}(z)\right| \leq 2\, e^{u^{[1]}(z) + v^{[1]}(z)}\,, \quad z \in U_{kj}\,. \tag{18}$$

In order to " paste " the functions $P_{kj}(z)$ to the function $g(z)$ we set

$$\chi_{kj}(\lambda, w) = \chi\left(\frac{\left|\lambda - (s_{kj})_1\right|}{r_{kj}/2}\right)\chi\left(\frac{\left|w - (s_{kj})_2\right|}{r_{kj}/2}\right)\,.$$

We have then

$$\chi_{kj}(\lambda, w) = \begin{cases} 1, & \text{if } (\lambda, w) \in U_{kj}^{(8)}; \\ 0, & \text{if } (\lambda, w) \notin \bigcup_{k,j} U_{kj}^{(4)}, \end{cases}$$

which shows that the function $\bigl(1 - \chi_{kj}(z)\bigr) \cdot g(z)$ is correctly defined in $\mathbb{C}^2$. Set now

$$h(z) = \begin{cases} \chi_{kj}(z) P_{kj}(z) + \bigl(1 - \chi_{kj}(z)\bigr) \cdot g(z)\,, & z \in U_{kj}^{(4)}; \\ g(z)\,, & z \in \mathbb{C}^2 \setminus U_{kj}^{(4)}. \end{cases}$$

We estimate $\left|\overline{\partial} h(z) \cdot (f(z))^{-1}\right|$. For $z \notin U_{kj}^{(4)}$ we have $\overline{\partial} h = \overline{\partial} g$, and $\overline{\partial} h = 0$ for $z \in U_{kj}^{(8)}$.

Let $z \in U_{kj}^{(4)} \setminus U_{kj}^{(8)}$. Then

$$\overline{\partial} h = \overline{\partial} \chi_{kj} \cdot (P_{kj} - g) + \overline{\partial} g \cdot (1 - \chi_{kj}),$$

and hence

$$\left|\overline{\partial} h\right| = \left|\overline{\partial} \chi_{kj}\right| \cdot \left|P_{kj} - g\right| + \left|\overline{\partial} g\right|,$$

$$\left|\frac{\overline{\partial} h}{f}\right| = \left|\overline{\partial} \chi_{kj}\right| \cdot \frac{\left|P_{kj} - g\right|}{|f|} + \frac{\left|\overline{\partial} g\right|}{|f|}.$$

First estimate $\left|\overline{\partial} \chi_{kj}\right|$:

$$\left|\overline{\partial} \chi_{kj}(z)\right| = \left|\chi_{kj}'\right| \left(\frac{1}{4}\, e^{v(s_{kj})} + \frac{1}{4}\, e^{v(s_{kj})}\right) \leq \frac{1}{2}\, e^{v(s_{kj})}.$$

Put $A_{kj} = A \cap \left(U_{kj}^{(4)} \setminus U_{kj}^{(8)}\right)$ and note that for $z \in A_{kj}$ we have an estimate of $\left|f(z)\right|$ from below, and so, using (16) and (18) we get

$$\begin{aligned} \frac{\left|P_{kj}(z) - g(z)\right|}{\left|f(z)\right|} &\leq \frac{\left|P_{kj}(z)\right| + \left|g(z)\right|}{\left|f(z)\right|} \leq \\ &\leq 16\, e^{(u^{[1]} + v_2)^{[1/2]}(z) + 6 v^{[1]}(z)}, \quad z \in A_{kj}. \end{aligned} \tag{19}$$

Since on the set

$$\left\{(\lambda, w) : \exists\ \lambda_0 \in \widetilde{\Lambda}_w : |\lambda - \lambda_0| r_{\lambda_0}^{-1}(w) < \frac{1}{4}\right\} \cap \left(U_{kj}^{(4)} \setminus U_{kj}^{8}\right)$$

the function $\left(P_{kj}(z) - g(z)\right)/f(z)$ is holomorphic and on the boundary we have (19), by the maximum principle, we get

$$\left|\frac{P_{kj}(z) - g(z)}{f(z)}\right| \leq C\, e^{(u^{[1]} + v_2)^{[1]}(z) + 6 v^{[1]}(z)},$$

$$z \in E_{kj} \stackrel{\text{def}}{=} \left\{(\lambda, w) : \exists\ \lambda_0 \in \widetilde{\Lambda}_w : |\lambda - \lambda_0| \cdot r_{\lambda_0}^{-1}(w) \leq \frac{1}{2}\right\} \cap \left(U_{kj}^{(4)} \setminus U_{kj}^{8}\right).$$

It remains to estimate $\left|(P_{kj} - g)/f\right|$ on the set $G_{kj} \stackrel{\text{def}}{=} U_{kj}^{(4)} \setminus \left(U_{kj}^{8} \cup E_{kj}\right)$. Note that on G_{kj} our function $g \equiv 0$, so that we have to estimate $\left|P_{kj}/f\right|$.

We use lemma 1. Introduce the same coordinates as before, when we constructed P_{kj}. It was already noted that $U_{kj} \supset V_{kj}^{(\sqrt{2})}$; $U_{kj}^{(16)} \subset V_{kj}^{(8\sqrt{2})}$. Set $R = r_{kj}/\sqrt{2}$, $\varepsilon = (1/4)\, e^{-v^{[1]}(s_{kj})}$.

Now we find ourselves within the conditions of lemma 1, which implies that on the set $V'_{kj} \stackrel{\text{def}}{=} V_{kj}^{(\sqrt{2})} \setminus V_{kj}^{(8\sqrt{2})}$ we have the estimate (11). Observe that $V'_{kj} \supset U_{kj}^{(4)} \setminus U_{kj}^{(8)}$. Hence for $z \in U_{kj}^{(4)} \setminus U_{kj}^{(8)}$, $\operatorname{dist}(z\,,\, H_j \cup H_k) \geq \varepsilon$, we get from (11):

$$\left|f(\lambda, w)\right| \geq \frac{1}{16} C_1 \exp\left[-v_2^{[1]}(s_{kj}) - 14\, v^{[1]}(s_{kj})\right], \tag{20}$$

$$\left|\frac{P_{kj}(z)}{f(z)}\right| \leq C_2 \exp\Big[\big(u^{[1]} + v_2^{[1]}\big)^{[1]}(z) + 15\, v^{[1]}(z)\Big]\,. \tag{21}$$

The estimate (20) in view of the maximum principle and (16) holds also for those points of G_{kj}, the distance from which to $H_j \cup H_k$ is less than ε. Hence (21) holds as well. So for $z \in U_{kj}^{(4)} \setminus U_{kj}^{(8)}$ (17) and (21) imply

$$\left|\frac{\overline{\partial} h(z)}{f(z)}\right| \leq C_3 \exp\left[\big(u^{[1]} + v_2^{[1]}\big)^{[1]}(z) + 16\, v^{[2]}(z)\right]\,. \tag{22}$$

We construct the function Φ in the form $\Phi(z) = h(z) - \beta(z) f(z)$, where

$$\overline{\partial}\beta = \frac{\overline{\partial} h(z)}{f(z)} \equiv \alpha\,. \tag{23}$$

Set

$$\psi(z) = \big(u^{[1]} + v_2^{[1]}\big)^{[1]}(z) + 16\, v^{[2]}(z) + 2\log\big(1 + |z|^2\big)\,.$$

Evidently, $\psi \in \mathrm{PSH}(\mathbb{C}^2)$. From (22) it follows that

$$\int_{\mathbb{C}^2} |\alpha|^2\, e^{-2\psi}\, \mathrm{d}\omega_2 < \infty\,.$$

By Hömander's theorem in this case the equation (23) has such a solution β that

$$\int_{\mathbb{C}^2} |\beta|^2\, e^{-2\psi - 2\log(1+|z|^2)}\, \mathrm{d}\omega_2 < \infty\,.$$

It is easy to show now that

$$\int_{\mathbb{C}^2} |\Phi|^2\, e^{-2\psi - 2v_1 - 2\log(1+|z|^2)}\, \mathrm{d}\omega_2 < \infty$$

which implies the desired estimate. Now, since $h(z) = \varphi(z)$ for $z \in \Lambda$, we have $\Phi(z) = \varphi(z)$ for $z \in \Lambda$. This completes the proof.

Analytic conditions of interpolation

We prove theorem 2 now. Note that the conditions of theorem 1 contain the estimate of the derivative in some certain direction. However, with the help of the following lemma, for functions of finite order this condition may be formulated invariantly, with (6) replaced by an estimate of $\log\left|\nabla f(z)\right|$.

LEMMA 3.— *Let* $\{a^{(k)}\}_1^\infty$ *be a sequence of points in* $\mathbb{C}^N$, *and* $\{\gamma_k\}_1^\infty$ *be a sequence of positive numbers,*

$$\sum_{k=1}^{\infty} \gamma_k^2 < \frac{1}{N-1}.$$

Then there exists a vector $\tau \in \mathbb{C}^N$, $|\tau| = 1$, *that*

$$\left|\left\langle \tau, \frac{a^{(k)}}{|a^{(k)}|} \right\rangle\right| \geq \gamma_k, \quad \forall\ k = 1, 2, \ldots$$

Proof.— Let S_1 be the unit sphere in $\mathbb{C}^N$ and σ be the normalized surface measure on S_1. The following equality takes place for a summable function φ of one variable (see [Rud], p. 23):

$$\forall\ \eta \in S_1 : \quad \int_{S_1} \varphi\big(\langle \zeta, \eta \rangle\big)\, \mathrm{d}\sigma(\zeta) = \frac{N-1}{\pi} \int_0^1 r(1-r^2)^{N-2}\, \mathrm{d}r \int_0^{2\pi} \varphi(re^{i\theta})\, \mathrm{d}\theta .$$

Fix $k \in \mathbb{N}$ and consider a function

$$\varphi_k(\lambda) = \varphi_k(re^{i\theta}) = \begin{cases} 0, & r < \gamma_k; \\ 1, & r \geq \gamma_k. \end{cases}$$

Set $\eta_k = a^{(k)}/|a^{(k)}|$. We have then

$$\begin{aligned} \sigma\left\{\zeta \in S_1 : \left|\langle \zeta, \eta_k \rangle\right| \geq \gamma_k\right\} &= \int_{S_1} \varphi_k\big(\langle \zeta, \eta_k \rangle\big)\, \mathrm{d}\sigma(\zeta) \\ &= \frac{N-1}{\pi} \cdot 2\pi \int_{\gamma_k}^1 r(1-r^2)^{N-2}\, \mathrm{d}r \\ &= (1-\gamma_k^2)^{N-1} \geq 1 - (N-1)\gamma_k^2 . \end{aligned}$$

Thus $\sigma\left\{\zeta \in S_1 : \left|\langle\zeta\,,\,\eta_k\rangle\right| < \gamma_k\right\} \le (N-1)\gamma_k^2$. Therefore

$$\sigma \bigcup_{k=1}^{\infty} \left\{\zeta \in S_1 : \left|\langle\zeta\,,\,\eta_k\rangle\right| < \gamma_k\right\} \le (N-1)\sum_{k=1}^{\infty}\gamma_k^2 < 1\,,$$

and there exists $\tau \in S_1 : \tau \notin \bigcup_{k=1}^{\infty} \left\{\zeta \in S_1 : \left|\langle\zeta\,,\,\eta_k\rangle\right| < \gamma_k\right\}$.

The lemma is proved.

We are able to prove our theorem on analytic conditions of interpolation in the class $\left[\,\rho\,,\, h(z)\,\right]$.

Proof of theorem 2

As it was mentioned already, we are going to use our theorem 4. First of all we show that the estimate (A) holds with $\nabla f(z)$ replaced by $\partial f(z)/\partial\tau$, where τ is some vector on S_1. To do this we note that due to the fact that $f(z)$ has order ρ, the series $\sum_{k=1}^{\infty} |a^{(k)}|^{-\lambda}$ converges for each $\lambda > \rho$ (see [Pa3]). Choose the numbers $\gamma_k = \exp\left(-C|a^{(k)}|^{3\rho/4}\right)$, where the constant C is chosen so that $\sum_{k=1}^{\infty}\gamma_k^2 < 1/(N-1)$. Such a choice is possible since for $C \ge 1$

$$\gamma_k^2 = \exp\left(-C|a^{(k)}|^{3\rho/2}\right) < \exp\left(-C\log|a^{(k)}|^{3\rho/2}\right) = |a^{(k)}|^{-3C\rho/2}\,.$$

By lemma 3, there exists such a vector $\tau \in S_1$ that

$$\left|\left\langle \tau\,,\, \frac{a^{(k)}}{|a^{(k)}|} \right\rangle\right| \ge \gamma_k\,, \quad \forall\ k = 1,\, 2,\, \ldots$$

Hence for $z \in H_k$ we have

$$\left|\frac{\partial f(z)}{\partial \tau}\right| = \left|\left\langle \nabla f(z)\,,\, \tau\right\rangle\right| = \left|\nabla f(z)\right| \cdot \left|\left\langle \frac{a^{(k)}}{|a^{(k)}|}\,,\, \tau \right\rangle\right| \ge \gamma_k\left|\nabla f(z)\right|,$$

and therefore

$$\log\left|\frac{\partial f(z)}{\partial \tau}\right| \ge \log\left|\nabla f(z)\right| + \log\gamma_k = \log\left|\nabla f(z)\right| - C\left|a^{(k)}\right|^{3\rho/4} \ge$$
$$\ge \log\left|\nabla f(z)\right| - C\left|z\right|^{3\rho/4}.$$

In view of (A) there exists such negligible $w(z)$, that

$$\log\left|\nabla f(z)\right| \geq L_f(z) - w(z)\,, \quad z \in \Lambda \setminus U\,.$$

Putting $\widetilde{w}(z) = w(z) - C|z|^{3\rho/4}$, we get

$$\log\left|\frac{\partial f(z)}{\partial \tau}\right| \geq L_f(z) - \widetilde{w}(z)\,, \quad \forall\, z \in \Lambda \setminus U\,, \tag{24}$$

with $\widetilde{w}$ negligible.

By a well-known lemma due to A. Martineau (see [Ro3], p. 323) each locally bounded negligible function has a nonnegative radial negligible plurisubharmonic majorant. Since in what follows we are going to use negligible functions only in estimates from above, we may assume that each such function is plurisubharmonic.

In particular, the function $l(z) = \left|L_f(z) + \left(-L_f(z)\right)^{[1]}\right|$ is also negligible and by the remark above there exists such negligible plurisubharmonic function $\widetilde{w}_1$ that $\widehat{w}_1 \geq l(z)$.

Now, in view of the conditions of our theorem, continuity of $L_f(z)$ and the well-known property of the radial indicator (see [Ro3], p. 287, d)) we get

$$\log\left|f(z)\right| \leq L(z) + w_2(z)\,, \quad \forall\, z \in \mathbb{C}^2$$

with some negligible plurisubharmonic $w_2(z) \geq 0$ and also

$$\log\left|\varphi(z)\right| \leq h(z) + w_3(z)\,, \quad \forall\, z \in \Lambda$$

with some negligible plurisubharmonic $w_3(z) \geq 0$.

Put $\widetilde{w}_2(z) = \max\{\widetilde{w}_1(z), w_2(z), w_3(z)\}$, $u = h(z) + \widetilde{w}_2(z)$, $v_1 = L_f(z) + \widetilde{w}_2$, $v_2 = -L_f(z) + \widetilde{w}_2(z)$.

By theorem 4 we can construct the required function $\Phi(z)$ with the estimate

$$\log\left|\Phi(z)\right| \leq u^{[3]}(z) + 16\, v^{[3]}(z) + C\log\left(2 + |z|^2\right).$$

It is easy to see that two last terms are negligible. Hence the inequality implies the desired estimate $L_\Phi(z) \leq h(z)$.

The theorem is proved.

Geometrical conditions of interpolation

Now we establish sufficient conditions for interpolation of geometric type. Introduce the following denotions. For a function $f(z)$ with zero set Λ of the form (1) we set

$$f_k(z) = \frac{f(z)}{\left(|a^{(k)}| - \langle z\,,\, a^{(k)}/|a^{(k)}|\rangle\right)}\,.$$

It is easy to check (for example, with the help of Hadamard factorisation representation for entire functions with plane zeros, [Pa3], theorem 2), that

$$\left|\nabla f(z)\right| = \left|f_k(z)\right| \quad \text{for } z \in H_k\,.$$

Hence we will estimate $|f_k(z)|$ instead of $|\nabla f(z)|$. Remind that for a point $z \in \mathbb{C}^N$ we denote the number of $H_k \in \Lambda$ intersecting the ball $B_t(z)$ by $n_z(t)$, that $\widetilde{n}_z(t) = \left[n_z(t) - 1\right]^+$ and, finally, $\Phi_z(\alpha) = |z|^{-\rho}\,\widetilde{n}_z(\alpha|z|)$. Let $\delta \in (0,1)$ be fixed. Consider the functions

$$f_z^\delta(w) = \prod_{\left|\,|a^{(k)}| - \langle z, \frac{a^{(k)}}{|a^{(k)}|}\rangle\right| < \delta|z|} \frac{|a^{(k)}| - \langle z + w\,,\, a^{(k)}/|a^{(k)}|\rangle}{(1+\delta)|z|}\,,$$

$$\widetilde{f}_z^\delta(w) = \prod_{0 < \left|\,|a^{(k)}| - \langle z, \frac{a^{(k)}}{|a^{(k)}|}\rangle\right| < \delta|z|} \frac{|a^{(k)}| - \langle z + w\,,\, a^{(k)}/|a^{(k)}|\rangle}{(1+\delta)|z|}\,.$$

Note that for $|w| < \delta|z|$ we have

$$\log\left|f_z^\delta(w)\right| < 0\,. \tag{25}$$

We will need two auxilliary statements.

LEMMA 4.— *The following equality is true:*

$$\left|\log\left|f_z^\delta(0)\right|\right| = |z|^\rho\left(\Phi_z(\delta)\log\left(1 + \frac{1}{\delta}\right) + \int_0^\delta \Phi_z(\alpha)\frac{\mathrm{d}\alpha}{\alpha}\right).$$

Proof.— Note, that the value

$$\left| |a^{(k)}| - \left\langle z\,,\ \frac{a^{(k)}}{|a^{(k)}|} \right\rangle \right|$$

is exactly the distance from z to H_k. Hence

$$\left|\log|f_z^\delta(0)|\right| = -\sum \left\{ \log \left| |a^{(k)}| - \left\langle z\,,\ \frac{a^{(k)}}{|a^{(k)}|} \right\rangle \right| - \log\big((1+\delta)|z|\big) \right\}$$

$$= -\int_0^{\delta|z|} \log t \, \mathrm{d}\widetilde{n}_z(t) + \widetilde{n}_z\big(\delta|z|\big) \log\big((1+\delta)|z|\big)\,.$$

Integrating the last expression by parts gives the required equality.

We remark that for $\delta \in (0,1)$ we have

$$\int_0^\delta \frac{\Phi_z(\alpha)}{\alpha}\, \mathrm{d}\alpha \ge \int_{\delta^2}^\delta \frac{\Phi_z(\alpha)}{\alpha}\, \mathrm{d}\alpha \ge \Phi_z(\delta^2) \log \frac{1}{\delta}\,,$$

and hence by (G2)

$$\sup_{z \in \Lambda \setminus U} \Phi_z(\delta) \log \frac{1}{\delta} \xrightarrow[\delta \to 0]{} 0\,. \tag{26}$$

The next statement gives some information on the structure of the exceptional set E of a function of completely regular growth. It might be interesting apart from interpolation problems.

LEMMA 5.— *Let $E \subset \mathbb{C}^N$ be a C_0^0-set and let $\delta > 0$. Then there exists a number $R_0 > 0$ such that for each $z^0 \in \mathbb{C}^N \setminus B_{R_0}(0)$ there exists a complex line l through z^0 and a circumference γ, centered at z^0, having radius less than $\delta|z^0|$ and lying on l, such that $\gamma \cap E = \emptyset$.*

The proof of this lemma will be given later. Assuming that the assertion of the lemma 5 holds, we show that conditions (G1)-(G2) imply (A) for the canonical product $f(z)$, associated with our set of hyperplanes Λ. As we have mentioned already, (G1) implies that f is of completely regular growth with continuous indicator $h(z)$. Thus the following lemma will complete the proof of theorem 3.

LEMMA 6.— *Let $f(z)$ be a function of completely regular growth with hyperplane set of zeros Λ. Let this function satisfy (G2) with some set $U \subset \mathbb{C}^N$, such that $\widetilde{n}_z(0) = 0$ for $z \in U$. Then (A) holds.*

Proof.— Fix $\varepsilon > 0$. For $z \in H_k \setminus U$ put $q_z^\delta(w) = f(z+w)/f_z^\delta(w)$. The functions $f_z^\delta(w)$ and $\widetilde{f}_z^\delta(w)$ differ in one term. Hence we have

$$q_z^\delta(w) = \frac{f_k(z+w)}{\widetilde{f}_z^\delta(w) \cdot (1+\delta)|z|}\,.$$

Observe that function $q_z^\delta(w)$ does not vanish in $B_{\delta|z|}(0)$. Thus $\log\left|q_z^\delta(w)\right|$ is pluriharmonic in w in $B_{\delta|z|}(0)$. Estimate $\log\left|q_z^\delta(0)\right|$.

Since $f(z)$ has completely regular growth, by [Az], p. 165, outside a ball $B_{R_\varepsilon}(0)$ with R_ε large enough and outside a C_0^0-set E_ε it holds

$$|z|^{-\rho}\log\left|f(z)\right| > L_f\Big(\frac{z}{|z|}\Big) - \varepsilon\,.$$

Hence for such ω that $z + w \notin E_\varepsilon$ and $|z+w| > R_\varepsilon$, in view of (25) we have

$$|z+w|^{-\rho}\log\left|q_z^\delta(w)\right| > L_f\left(\frac{z+w}{|z+w|}\right) - \varepsilon\,, \delta < 1\,,\ |w| < \delta|z|\,.$$

In view of continuity of $L_f(z)$ on the unit sphere, one can take δ_1 small enough, so that the following implication is true:

$$\begin{aligned}\{|w| < \delta_1|z|\,, z+w \notin E_\varepsilon,\ |z+w| > R^\varepsilon\} \Rightarrow \\ |z|^{-\rho}\log\left|q_z^\delta(w)\right| > L_f\left(\frac{z}{|z|}\right) - 2\varepsilon\,. \qquad (27)\end{aligned}$$

We apply our lemma 5 now with $E = E_\varepsilon$, $z^0 = z$ and $\delta = \delta_1$. It follows, that for $|z| > R_0$ a circumference lying outside E_ε exists in each ball $B_{\delta_1|z|}(z)$. If $|z| > \max(2R_\varepsilon, R_0)$, $|w| < \delta_1|z|$, then, in view of pluriharmonicity of $\log\left|q_z^\delta(w)\right|$ (for $\delta \le \delta_1$) in $B_{\delta_1|z|}(0)$, by the minimum principle for harmonic functions, (27) holds also for $w = 0$. Hence

$$|z|^{-\rho}\log\left|q_z^\delta(0)\right| > L_f\left(\frac{z}{|z|}\right) - 2\varepsilon\,. \qquad (28)$$

Now, by lemma 4, we have

$$\left|\log\left|f_z^\delta(0)\right|\right| = |z|^\rho\left(\Phi_z(\delta)\log\Big(1+\frac{1}{\delta}\Big) + \int_0^\delta \Phi_z(\alpha)\frac{\mathrm{d}\alpha}{\alpha}\right)\,. \qquad (29)$$

Since $\log|f_z(z)| = \log|q_z^\delta(0)| + \log|f_z^\delta(0)| + \log((1+\delta)|z|)$, by (G2), (26), (28) and (29) taking δ small enough we obtain for $z \in H_k \setminus U$, $|z| > R_1(\varepsilon)$:

$$|z|^{-\rho}\log|f_z(z)| \geq L\Big(\frac{z}{|z|}\Big) - 3\varepsilon\,.$$

Thus lemma 4 and theorem 3 are proved.

Some properties of C_0^0-sets in $\mathbb{C}^N$

We conclude the paper by proving lemma 5 on the structure of C_0^0-sets in $\mathbb{C}^N$.

Assume the converse, i.e. for each complex line l through z^0, there does not exist a circumference γ lying outside E. We will show that this contradicts E being a C_0^0-set.

Let $E_{z^0} = E \cap B_{\delta|z^0|}(z^0)$ and let $\mathcal{B} = \{b_1, b_2, \ldots\}$ be some covering of E_{z^0} by balls $b_j = B_{r_j}(z^{(j)})$. We are going to estimate the value $\sum r_j^{2N-3/2}$.

For $\zeta \in \mathbb{C}^N$, $|\zeta| = 1$, denote by L_ζ the $(2N-1)$-dimensional real hyperplane through z^0 with normal vector ζ. Thus a one-to-one correspondence is established between the unit sphere S_1 in $\mathbb{C}^N$ and the set of $(2N-1)$-dimensional hyperplanes through z^0.

Let $w \in \mathbb{C}^N$. A *circular projection of a point $w \in L_\zeta$* is defined as the points of intersection of the circumference $\{z^0 + e^{i\theta}w,\ \theta \in [0\,,\,2\pi)\}$ with L_ζ (there are exactly two such points, unless w lies in the $(N-1)$-dimensional complex hyperplane $M_\zeta \subset L_\zeta$; in the latter case we assume that the circular projection of w is $\pm w$).
A *circular projection of a set $D \subset \mathbb{C}^N$* onto L_ζ, $\text{c.p.}_{L_\zeta}(D)$, is a union of circular projections of all points in D.

Denote $\delta|z^0|$ by R. It is no loss of generality to assume that the radii of b_j do not exceed $R/4$. Denote by B_{z^0} the set $B_R(z^0) \setminus B_{R/2}(z^0)$. Since, by our assumption, each circumference of radius $r \in (R/2, R)$ contains at least one point of E_{z^0}, it follows that

$$\text{c.p.}_{L_\zeta}(E_{z^0} \cap B_{z^0}) = L_\zeta \cap B_{z^0}\,, \quad \forall\, \zeta \in S_1\,. \tag{30}$$

Let m_{2N-1} be the Lebesgue measure in $\mathbb{R}^{2N-1}$. Consider the quantity

$$J = \sum_{\substack{b_j \in \mathcal{B} \\ z^{(j)} \notin B_{R/4}(z^0)}} \int_{S_1} m_{2N-1}\big(\text{c.p.}_{L_\zeta}(b_j)\big)\, d\sigma(\zeta)\,,$$

where σ is the normalized surface measure on S_1. Property (30) implies a lower bound for J:

$$J = \int_{S_1} m_{2N-1}\big(\text{c.p.}_{L_\zeta}(E \cap B_{z^0})\big)\, d\sigma(\zeta) \geq c_1(N) R^{2N-1} . \tag{31}$$

Now estimate J from above. In order to do this, we represent each integral in the sum defining J in the form

$$\left\{ \int_{\{\zeta\,:\,\text{dist}(L_\zeta, z^{(j)}) \leq \rho_j\}} + \int_{\{\zeta\,:\,\text{dist}(L_\zeta, z^{(j)}) > \rho_j\}} \right\} m_{2N-1}\big(\text{c.p.}_{L_\zeta}(b_j)\big)\, d\sigma(\zeta) ,$$

We estimate each integral separately. First note, that the set $\Omega = \{z^0 + e^{i\theta} b_j ,\, \theta \in [\,0\,, 2\pi\,)\}$ is a body of rotation in $\mathbb{C}^N$, which is obtained by rotating the center of b_j around z^0 along the circumference of radius $|z^{(j)} - z^0| < {}^{5R}/_{4}$. Since $(2N-1)$-dimensional area of the section of this body by L_ζ does not exceed the $(2N-1)$-dimensional area of its surface, we get:

$$m_{2N-1}\big(\text{c.p.}_{L_\zeta}(b_j)\big) \leq c_2(N) r_j^{2N-2} |z^{(j)} - z^0| .$$

It is easy to see that the σ-measure of the set $\{\zeta : \text{dist}(L_\zeta, z^{(j)}) \leq \rho_j\}$ does not exceed $c_3(N)\rho_j/|z^{(j)} - z^0|$. Hence

$$\int_{\{\zeta\,:\,\text{dist}(L_\zeta, z^{(j)}) \leq \rho_j\}} m_{2N-1}\big(\text{c.p.}_{L_\zeta}(b_j)\big)\, d\sigma(\zeta) \leq c_4(N) r_j^{2N-2} \rho_j .$$

The estimate of the second integral is more complicated. For simplicity we assume that $L_\zeta = \{z \in \mathbb{C}^N : \text{Im}\, z_N = 0\} + z^0$. For $z \in b_j$ estimate the difference $|\arg z_N - \arg z_N^{(j)}|$. We have

$$|z - z^{(j)}| \leq r_j \Rightarrow |z_N - z_N^{(j)}| \leq r_j \quad ; \quad \text{dist}(L_\zeta, z^{(j)}) = |\,\text{Im}\, z_N^{(j)}| \geq \rho_j .$$

It isn't hard to see, that then

$$|\arg z_N - \arg z_N^{(j)}| \leq \tan|\arg z_N - \arg z_N^{(j)}| \leq \frac{r_j}{\rho_j} .$$

Hence in this case the intersection $L_\zeta \cap \Omega$ is contained in the set $\widetilde{\Omega} = \{z^0 + e^{i\theta} b_j ,\, |\theta - \theta_0| \leq {}^{r_j}/_{\rho_j}\}$ with $\theta_0 = -\arg z_N^{(j)}$. The $(2N-1)$-dimensional surface area of $\widetilde{\Omega}$ does not exceed

$$c_5(N) r_j^{2N-1} \cdot \frac{|z^{(j)} - z^0|}{\rho_j} + c_6(N) r_j^{2N-1} ,$$

and we have

$$m_{2N-1}\big(\text{c.p.}_{L_\zeta}(b_j)\big) \le c_7(N) r_j^{2N-1} \cdot \frac{|z^{(j)} - z^0|}{\rho_j} \le \frac{5}{4}\, c_7(N) r_j^{2N-1} \frac{R}{\rho_j}\,.$$

Hence

$$\int_{\{\zeta\,:\,\mathrm{dist}(L_\zeta, z^{(j)}) > \rho_j\}} m_{2N-1}\big(\text{c.p.}_{L_\zeta}(b_j)\big)\, \mathrm{d}\sigma(\zeta) \le c_8(N) r_j^{2N-1} \frac{R}{\rho_j}\,.$$

Choosing $\rho_j = \sqrt{R r_j}$, we obtain the estimate

$$\int_{S_1} m_{2N-1}\big(\text{c.p.}_{L_\zeta}(b_j)\big)\, \mathrm{d}\sigma(\zeta) \le c_9(N) R^{1/2} r_j^{2N-3/2}\,.$$

This implies

$$J \le c_{10}(N) \sum_{\substack{b_j \in \mathcal{B} \\ R/4 < |z^{(j)} - z^0| < 5R/4}} r_j^{2N-3/2} R^{1/2}\,.$$

Comparing with (31) one gets

$$\sum_{b_j \in \mathcal{B}} r_j^{2N-3/2} \ge c_{11}(N) R^{2N-3/2}\,.$$

Hence, for arbitrary covering $\widetilde{\mathcal{B}}$ of $E \cap B_{(1+\delta)|z^0|}(0)$ one has

$$\big((1+\delta)|z^0|\big)^{-(2N-2+1/2)} \sum_{b_j \in \widetilde{\mathcal{B}}} r_j^{2N-2+1/2} \ge c_{12}(N)\,, \quad \forall\, z^0 \in \mathbb{C}^N\,.$$

This contradicts E being a $C_0^{1/2}$-set. The lemma is proved.

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References

[Az] AZARIN (V.S.) .— *On asymptotic behavior of subharmonic functions of finite order*, Mat. Sb., **108** (1979) pp. 147-167.

[BT1] BERENSTEIN (C.A.) and TAYLOR (B.A.) .— *Interpolation problems in* $\mathbb{C}^N$ *with applications to harmonic analysis*, J. Anal. Math., **38** (1980) pp. 188-254.

[BT2] BERENSTEIN (C.A.) and TAYLOR (B.A.) .— *On the geometry of interpolating varieties*, Lect. Notes in Math., Springer-Verlag, **919** (1982) pp. 1-25.

[Gr] GRISHIN (A.F.) .— *On sets of regular growth of entire functions*, I, Theor. Funk., Funk. Anal. i Ih Pril., **40** (1983) pp 36-47.

[GrRu] GRISHIN (A.F.) and RUSSAKOVSKII (A.M.) .— *Free interpolation by entire functions*, Theor. Funk., Funk. Anal i Ih Pril., **44** (1985) pp. 32-42.

[Gru] GRUMAN (L.) .— *The regularity of growth of entire functions whose zeros are hyperplanes*, Ark. Mat., **10** (1972) pp. 23-31.

[Hö] HÖRMANDER (L.) .— *An introduction to complex analysis in several variables*, North Holland (1973).

[LGru] LELONG (P.) and GRUMAN (L.) .— *Entire functions of several complex variables*, Springer-Verlag (1989).

[Pa1] PAPUSH (D.E.) .— *On interpolation on discrete sets in* $\mathbb{C}^N$, Theor. Funk., Funk. Anal. i Ih Pril., **55** (1991) pp. 113-124.

[Pa2] PAPUSH (D.E.) .— *Entire functions of several variables with a regular set of " plane " zeros*, Sib. Math. J., **32** (1991) pp. 120-130.

[Pa3] PAPUSH (D.E.) .— *On the growth of entire functions with " plane " zeros*, Theor. Funk., Funk. Anal. i Ih Pril., **48** (1987) pp. 117-125.

[Ro1] RONKIN (L.I.) .— *On extension with estimates of functions holomorphic on a zero set of a polynomial*, Theor. Funk., Funk. Anal. i Ih Pril., **36** (1981) pp. 89-103.

[Ro2] RONKIN (L.I.) .— *On extention with estimates of functions holomorphic on a zero set of a pseudopolynomial*, Sib. Math. J., **24** (1983) pp. 150-163.

[Ro3] RONKIN (L.I.) .— *Introduction to the theory of entire functions of several variables*, Transl. of Math. Monographs, V. 44, AMS (1974).

[Ro4] RONKIN (L.I.) .— *Entire functions*, Encyclopaedia of Mathematical Sciences. Several Complex Variables III, Springer-Verlag (1990).

[RORu] RONKIN (L.I.) and RUSSAKOVSKII (A.M.) .— *On extension with estimates of functions holomorphic on an algebraic set*, Ann. Pol. Math., **46** (1985) pp. 403-431.

[Ru] RUSSAKOVSKII (A.M.) .— *Extension with estimates from analytic sets of codimension 1*, Theor. Funk., Funk. Anal. i Ih Pril., **51** (1989) pp. 88-94.

[Rud] RUDIN (W.) .— *Function theory in the unit ball of* $\mathbb{C}^N$, Springer-Verlag, New York (1980).