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Discrete groups, Mumford curves and Theta functions(*)

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RÉSUMÉ. — Un groupe discret Γ, défini sur un corps complet par rapport à une valuation non archimédienne, induit une courbe X. Les fonctions thêta pour Γ sont à la base de la construction analytique de la variété de Jacobi de X. À l'aide des courants sur des arbres et des graphes et la de cohomologie de Γ, on développe une théorie des fonctions thêta. Un grand nombre de groupes discrets est calculé en connection avec des courbes de Shimura et des groupes de quaternions. On donne la relation entre un problème d'Abhyankar et les courbes modulaires de Drinfeld.

ABSTRACT. — A discrete group Γ given over some complete non archimedean valued field defines a curve X. The theta functions for Γ provide an analytic construction for the Jacobian variety of X. A theory of theta functions is developed with the help of currents on trees and graphs and the cohomology for Γ. In connection with Shimura curves and quaternions a great variety of discrete groups is explicity calculed. The relation between a problem of Abhyankar and Drinfeld's modular curves is given.

1. Introduction

This paper is an updated version of the notes of a lecture [P] given in 1981/82. In [GP], Chapter 6, there is an explicit construction of the Jacobian variety of a Mumford curve using theta functions for the corresponding Schottky group Γ given as certain infinite products. The lecture gave a translation of these formulas into currents and cohomology for the group Γ. This approach is precisely what E.U. Gekeler and M. Reversat need in their present work [GR] on the Jacobians of Drinfeld modular curves.

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The text of [P] is not easily found and the present article can be seen as the first publication of its content. The opportunity of writing this paper is used to extend the theory of theta functions to more general groups Γ than Schottky groups. In working this out, some new things came to light and some parts of [GP] are extended:

- Let $\Omega \subset \mathbb{P}^1$ denote the set of ordinary points of a discontinouos group Γ and suppose that $X := \Gamma\backslash\Omega$ is an (affine or projective) algebraic curve with compactification $\widehat{X}$. Then $\widehat{X}$ is a Mumford curve and the map $\Omega \to X$ *factors over the universal analytic covering of* $\widehat{X}$. Let Γ_{tors} denote the subgroup of Γ generated by the elements of finite order. Then $\Gamma/\Gamma_{\text{tors}}$ is the Schottky group defining $\widehat{X}$.
- The connection between the discrete groups of quaternions [GP, Chap. 9] and certain Shimura curves is given. A detailed study is made of the groups related with the Hamilton quaternions.
- There are many examples of discrete subgroups Γ of $\mathrm{PGl}(2, \mathbb{Q}_p)$ with as set of ordinary points the p-adic "upper half plane" :

$$\Omega := \mathbb{P}^1_{\mathbb{Q}_p} - \mathbb{P}^1(\mathbb{Q}_p)$$

  such that the quotient space $\Gamma\backslash\Omega$ is an *affine* rational curve.
- A conjecture of S.S. Abhyankar ([A1], [A2]) states that every quasi-p group (*i.e.* a finite group which is generated by its p-Sylov subgroups) is the Galois group of an unramified covering of the affine line in characteristic p. Using Drinfeld's modular curve we will construct examples of such Galois groups.

2. Invertible functions on Ω, currents on T

The field K is supposed to be complete with respect to a non-archimedean valuation. Let $\mathcal{L}$ denote a non empty compact subset of $\mathbb{P}^1(K)$ and let $\Omega = \mathbb{P}^1_K - \mathcal{L}$ denote the corresponding open analytic subset of $\mathbb{P}^1_K$. In the study of $O(\Omega)^*$, the group of invertible holomorphic functions on Ω, one uses an analytic reduction $r : \Omega \to \overline{\Omega}$ with respect to some pure covering of Ω by affinoid subsets.

In formal algebraic terms, one wants to construct a formal scheme $\widetilde{\Omega}$ over the valuation ring K^0 of K with "generic fibre" Ω and special fibre $\widetilde{\Omega} \otimes k$ equal to $\overline{\Omega}$ (k denotes the residue field of K). More details on the terminology can be found in [GP], [PV].

The scheme $\overline{\Omega}$ over k is locally of finite type. Every (irreducible) component of $\overline{\Omega}$ is a projective line over k and the intersection of two components is either empty or equal to one ordinary double point of $\overline{\Omega}$. Define the graph T by: the vertices t of T are the components $\overline{\Omega}_t$ of $\overline{\Omega}$ and $\{t_1, t_2\}$ is an edge if $\overline{\Omega}_{t_1} \cup \overline{\Omega}_{t_2} \neq \emptyset$. In fact, T turns out to be a tree.

The *construction of such a reduction* starts by choosing a compact subset $\mathcal{M}$ of $\mathbf{P}^1(K)$ containing $\mathcal{L}$ such that $\mathcal{L}$ is the set of limit points of $\mathcal{M}$. In case $\mathcal{L}$ is perfect (*i.e.* $\mathcal{L}$ has no isolated points) one can take $\mathcal{M} = \mathcal{L}$. The *tree* $T = T(\mathcal{M})$ is constructed in a combinatorical way out of $\mathcal{M}$ (see also [GP, p. 11]) as follows.

Let $\mathcal{M}^{(3)}$ denote the set of ordered triples a of distinct points of $\mathcal{M}$. We will define an equivalence relation $\sim$ on $\mathcal{M}^{(3)}$ and the set of vertices of T is given as the set of equivalence classes $[a] \in \mathcal{M}^{(3)}/_{\sim}$.

The standard reduction $R : \mathbb{P}^1_K \to \mathbb{P}^1_k$ is given by $(x_0 : x_1) \mapsto (\overline{x}_0 : \overline{x}_1)$ where the point $(x_0 : x_1)$ is normalized by $\max(|x_0|, |x_1|) = 1$ and where the bars denote the residues in the residue field k. For a triple $a = (a_0, a_1, a_\infty) \in \mathcal{M}^{(3)}$ we denote by γ_a the unique automorphism of $\mathbf{P}^1_K$ with $\gamma_a(a_i) = i$ for $i = 0, 1, \infty$. The reduction $R_a := R\gamma_a : \mathbb{P}^1_K \to \mathbb{P}^1_k$ maps a_0, a_1, a_∞ to $0, 1, \infty \in \mathbb{P}^1_k$. For $a, b \in \mathcal{M}^{(3)}$ we consider a combination of the two reductions namely $R_{a,b} = (R_a, R_b) : \mathbb{P}^1_K \to \mathbb{P}^1_k \times \mathbb{P}^1_k$. There are three cases.

1. The image of $R_{a,b}$ is one projective line. In this case we call a and b equivalent. The equivalence class of a is denoted by $[a]$.
2. The image of $R_{a,b}$ consist of two projective lines intersecting in one point p and this point does not lie in $R_{a,b}(\mathcal{M})$. In this case we call $\{[a], [b]\}$ an edge of T.
3. As before the image consist of two projective lines intersecting in one point p, but p lies in $R_{a,b}(\mathcal{M})$. In this case $[a] \neq [b]$ and $\{[a], [b]\}$ is not an edge of T.

One can show that the above defines a locally finite tree T. The set of (equivalence classes of) half lines in T is bijectively mapped to the set $\mathcal{L}$.

For any edge $e = \{[a], [b]\}$ the image $E := R_{a,b}(\mathcal{L})$ is a finite set and the complement $\Omega(e)$ of $R_{a,b}^{-1}(E)$ in $\mathbf{P}^1_K$ is an affinoid subset of Ω. For a vertex $[a]$ the image $A := R_a(\mathcal{L})$ is finite and the complement $\Omega([a])$ of $R_a^{-1}(A)$ in $\mathbb{P}^1_K$ is again an affinoid subset of Ω. Now the family $\{\Omega(e)\}$, where e runs in the set of edges of T, is a pure affinoid covering of Ω. Further $\Omega(e_1) \cap \Omega(e_2)$ is empty if e_1 and e_2 have no common vertex and the intersection is $\Omega([a])$

if $[a]$ is the common vertex of the two edges. The reduction r that we are looking for is the reduction with respect to this pure covering of Ω. We note in passing that each $\Omega(e)$ and $\Omega([a])$ can also be seen as an affine formal scheme over the valuation ring K^0 of K. If $[a]$ is a vertex of e then $\Omega([a])$ is an open formal subscheme of $\Omega(e)$ and the tree above prescribes the glueing of the affine pieces to a formal scheme $\widetilde{\Omega}$. Finally, the space $\overline{\Omega}$ consist of projective lines over k, one line for each vertex of T, and with its intersection pattern given by the edges of T.

Changing from one compact set $\mathcal{M}$ to a bigger one $\mathcal{M}_*$ (again with $\mathcal{L}$ as its set of limit points) corresponds with a subdivision of the tree $T(\mathcal{M})$. All reductions of Ω are obtained in this way. Indeed, let an analytic reduction $r : \Omega \to \overline{\Omega}$ be given. Take for every irreducible component L of $\overline{\Omega}$ a point $p_L \in \Omega$ with image $r(p_L)$ lying only on this component L. Then the set $\mathcal{M} := \mathcal{L} \cup \{p_L\}_{\text{all } L}$ is a compact set with limit points $\mathcal{L}$ and it can be seen that $\mathcal{M}$ induces the given reduction r.

A special case is the following: Let the field K be locally compact, then $\mathbb{P}^1(K)$ is compact. The space $\Omega := \mathbb{P}^1_K - \mathbb{P}^1(K)$ is sometimes called the "upper half space over K". The corresponding tree T has a canonical identification with the *Bruhat-Tits building of* $\mathrm{PSl}(2, K)$. The definition of this building goes as follows: Let K^0 denote the valuation ring of K and let V denote a vector space over K of dimension 2. The vertices of T are the equivalence classes $[M]$ of the rank two, free K^0-submodules of V. Two modules M_1, M_2 are equivalent if there exists a $\lambda \in K$ with $M_1 = \lambda M_2$. An edge of T is a couple $\{[M_1], [M_2]\}$ such that $M_1 \supset M_2$ and M_1/M_2 is isomorphic to the residue field of K.

Let G be any *locally finite graph*. We will use the following notations:

- $\overrightarrow{e}$ denotes an oriented edge; its origin and endpoint are written as $\overrightarrow{e}(0)$ and $\overrightarrow{e}(1)$; the opposite of $\overrightarrow{e}$ is denoted by $-\overrightarrow{e}$.
- a *current* (with values in $\mathbb{Z}$) on G is a map μ of the oriented edges of G into $\mathbb{Z}$ such that $\mu(-\overrightarrow{e}) = -\mu(\overrightarrow{e})$ and such that for every vertex a of G one has $\sum_{\overrightarrow{e}(0)=a} \mu(\overrightarrow{e}) = 0$.
- the *group of all currents on* G is denoted by $C(G)$.

THEOREM 2.1

There exists an exact sequence

$$0 \to K^* \to O(\Omega)^* \to C(T) \to 0\,.$$

Proof.— (Another proof of the statement can be found in [FP, pp. 47, 175, 176].) $O(\Omega)$ denotes the algebra of holomorphic functions on Ω and $O(\Omega)^*$ is the group of invertible holomorphic functions on Ω. We will first describe the map $f \mapsto \mu_f\,;\, O(\Omega)^* \to C(T)$.

Let t_0 be a vertex of T and let $(t_0, t_1), \ldots, (t_0, t_s)$ denote all oriented edges of T with origin t_0. They correspond to the double points $d_1, \ldots, d_s$ of $\overline{\Omega}$ situated on $\overline{\Omega}_{t_0}$. Now $\Omega^*_{t_0} := r^{-1}(\overline{\Omega}_{t_0} - \{d_1, \ldots, d_s\})$ is an affinoid subspace of Ω. Any $f \in O(\Omega)^*$ can be multiplied by a constant in K^* such that its supremum norm on $\Omega^*_{t_0}$ equals 1. The reduction $\overline{f}$ of f is then a regular function on $\overline{\Omega}_{t_0} - \{d_1, \ldots, d_s\}$ and extends uniquely to a rational function on the projective line $\overline{\Omega}_{t_0}$. The current μ_f is defined by $\mu_f((t_0, t_1)) = \mathrm{ord}_{d_1}(\overline{f})$. It is indeed a current since $\mu_f((t_1, t_0)) = -\mu_f((t_0, t_1))$ is easily seen to hold and $\sum_{i=1,\ldots,s} \mathrm{ord}_{d_i}(\overline{f}) = 0$. More explicitly, one can identify $\Omega^*_{t_0}$ with the affinoid subset

$$\{z \in K \mid |z| \leq 1 \text{ and } |z - a_2| = \cdots = |z - a_s| = 1\}$$

where $a_2, \ldots, a_s \in K$ have absolute value 1 and such that their residues in the residue field of K are distinct. The invertible function f can be written as $f = (z - a_2)^{n_2} \cdots (z - a_s)^{n_s} \lambda(1 + g)$ where $\lambda \neq 0$ is a constant and g is a function with supremum norm less than 1. Let the points $\infty, a_2, \ldots, a_s$ correspond to the double points $d_1, \ldots, d_s$ then $\mu_f((t_0, t_i))$ equals n_i for $i \neq 1$ and equals $-n_2 - \cdots - n_s$ for $i = 1$.

The proof starts with recalling (see [GP, p. 145, corol. 2.5]) that any bounded holomorphic function on Ω is constant. Let $f \in O(\Omega)^*$ satisfy $\mu_f = 0$. Then f is a function with constant absolute value on Ω. So $f \in K^*$. The surjectivity of the map $f \mapsto \mu_f$ is more difficult. We will use the sheaves: O^o = holomorphic functions of norm ≤ 1; O^{oo} = holomorphic functions with norm < 1; O^{o*} = the invertible elements of O^o; K^*O^{o*} = the holomorphic fuctions with constant absolute value ($\neq 0$); and their direct images under the reduction map $r : \Omega \to \overline{\Omega}$.

Let the sheaf $\mathcal{Q}$ on $\overline{\Omega}$ be defined by the exact sequence:

$$0 \to r_*(K^*O^{o*}) \to r_*O^* \to \mathcal{Q} \to 0\,. \tag{1}$$

From the above it is seen that $\mathcal{Q}$ is a skyscraper sheaf with as non zero stalks $\mathbb{Z}$ at the double points of $\overline{\Omega}$. Let $A(T)$ denote the group of maps μ from the oriented edges of T with values in $\mathbb{Z}$ satisfying $\mu(-\overrightarrow{e}) = -\mu(\overrightarrow{e})$. One identifies $A(T)$ with $\mathcal{Q}(\overline{\Omega})$ by choosing an orientation for every double point of $\overline{\Omega}$. From (1) one obtains the following exact cohomology sequence on $\overline{\Omega}$:

$$0 \to K^* \to O(\Omega)^* \to A(T) \to H^1(r_*K^*O^{o*}) \to H^1(r_*O^*) \to 0\,. \qquad (2)$$

The group $H^1(r_*O^*)$ is trivial as can be seen as follows. Every line bundle on any (connected) affinoid subset $A \subset \mathbb{P}^1_K$ is trivial since $O(A)$ is a principal ideal domain. Hence $H^1(\Omega, O^*) = H^1(r_*O^*)$. Further, write Ω as the union of connected affinoid subsets Ω_n such that $\Omega_n \Subset \Omega_{n+1}$ for all n. An easy argument with approximation of functions (see [FP, p. 43, (1.8.6)]) shows that $H^1(\Omega, O^*) = 0$.

Let $|K^*|$ denote the constant sheaf on $\overline{\Omega}$ with the group $|K^*|$ as stalks. Since T is a tree, $H^1(\overline{\Omega}, |K^*|) = 0$. The following exact sequence:

$$0 \to r_*O^{o*} \to r_*K^*O^{o*} \to |K^*| \to 0 \qquad (3)$$

shows now that $H^1(r_*O^{o*}) = H^1(r_*K^*O^{o*})$. Next consider the exact sequence:

$$0 \to r_*(1+O^{oo}) \to r_*O^{o*} \to O^*_{\overline{\Omega}} \to 0\,. \qquad (4)$$

One can show that $H^1(r_*(1+O^{oo})) = 0$ with the same method which proved that $H^1(r_*O^*) = 0$. This implies that $H^1(r_*O^{o*}) = H^1(O^*_{\overline{\Omega}})$. Let $H(T)$ denote the group of integer valued maps on the set of vertices of T. Then one can identify $H(T)$ with $H^1(O^*_{\overline{\Omega}})$ in the following natural way. $H^1(O^*_{\overline{\Omega}})$ describes the group of isomorphy classes of line bundles on $\overline{\Omega}$. Define $H^1(O^*_{\overline{\Omega}}) \to H(T)$ by $L \mapsto L|_{\overline{\Omega}_t} \in \mathrm{Pic}(\overline{\Omega}_t) = \mathbb{Z}$. One easily verifies that the map $A(T) \to H(T)$ derived from sequence (2) and the identification $H^1(r_*K^*O^{o*}) = H(T)$ has the form $\mu \mapsto \left(t \mapsto \sum_{\vec{e},\, \vec{e}(0)=t} \mu(\vec{e})\right)$. This proves the theorem. □

Examples 2.1.1

(a) If $\mathcal{L} = \{a_1, a_2, \ldots, a_n\}$ is a finite set then the tree has precisely n ends and $C(T) = \mathbb{Z}^{n-1}$. Suppose for convenience that $\infty \in \Omega$, then $O(\Omega)^*$ consisits of the functions $\lambda \prod (z-a_i)^{l_i}$ with $\sum l_i = 0$ and $\lambda \in K^*$.

(b) $C(T)$ can be identified with the group of finite additive measures m on $\mathcal{L}$ with values in $\mathbb{Z}$ and with $m(\mathcal{L}) = 0$. For any oriented edge $\vec{e}$ one considers the open and closed subset $\mathcal{L}(\vec{e})$= the set of all half lines starting at $\vec{e}(0)$ and containing $\vec{e}$. These sets form a basis for the topology of $\mathcal{L}$. Given a current μ, one defines a measure m on $\mathcal{L}$ by $m(\mathcal{L}(\vec{e})) = \mu(\vec{e})$. This is easily seen to be a bijection between the two groups defined above.

(c) Let a measure m as above be given. Suppose for convenience that $\infty \notin \mathcal{L}$. Choose a $\pi \in K$ with $0 < |\pi| < 1$. For any $n \geq 1$ one considers a decomposition of $\mathcal{L}$ into disjoint open and closed subsets $\mathcal{L}_i$ $(1 \leq i \leq s)$ each of diameter $\leq |\pi|^n$. Take a point $a_i \in \mathcal{L}_i$ for each i. Put $F_n = \prod (z - a_i)^{m(\mathcal{L}_i)}$. Then one can show that $F = \lim F_n$ exists and that $F \in O(\Omega)^*$ has an image in $C(T)$ corresponding to the measure m. A combination of (b) and (c) provides an elementary proof of 2.1.

COROLLARY 2.1.2.— *Let $\Omega^1(\Omega)$ denote the space of holomorphic 1-forms on Ω and let $C(T,K)$ denote the group of currents on T with values in K. Suppose that the characteristic of K is zero. Then there is an exact sequence:*

$$0 \to O(\Omega) \xrightarrow{\mathrm{d}} \Omega^1(\Omega) \to C(T,K) \to 0\,.$$

Proof.— Suppose for convenience that $\infty \in \mathcal{L}$. Let ω denote a holomorphic 1-form on Ω. The measure m corresponding to ω can be described as follows. Let U be a compact and open subset of $\mathcal{L}$. There exist finitely many disjoint disks $B_1, \ldots, B_n$ such that their union contains $\mathcal{L}$ and such that $U = (B_1 \cup \cdots \cup B_s) \cap \mathcal{L}$. Then

$$m(U) := \sum_{i=1}^{s} \mathrm{res}_{\partial B_i}(\omega)$$

A ringdomain such as ∂B_i is isomorphic to $\{z \in K \mid |z| = 1\}$. For a holomorphic 1-form ω on such a ringdomain one can write a Laurent series $\omega = \sum_{n \in \mathbf{Z}} a_n z^n \, \mathrm{d}z$. The coefficient a_{-1} is by definition the residue of ω with respect to the ringdomain. This term depends only on the chosen orientation of ∂B and does not depend on the choice of the variable. The non trivial part of the statement is that every measure on $\mathcal{L}$ with values in K and total measure 0 is the image of a holomorphic 1-form. This can be done in an elementary way as in 2.1.1 part (c). □

PROPOSITION 2.2

The following sequence is exact:

$$0 \to O(\Omega)^* \to M(\Omega)^* \to \mathrm{Div}(\Omega) \to 0$$

and

$$H^1(\Omega, O^*) = H^1(\Omega, M^*) = H^1(\Omega, \mathrm{Div}) = 0\,.$$

Proof.— As usual M, M^*, Div denote the sheaves of meromorphic functions, invertible meromorphic functions and divisors. We note that for any admissible open U the group $\mathrm{Div}(U)$ consists of the $\mathbb{Z}$-valued functions on U with "discrete" support. In this context $S \subset U$ is called discrete if the intersection of S with any affinoid subset of U is finite. The sheaf Div is a skyscraper sheaf and has trivial cohomology. We have proved above that $H^1(\Omega, O^*) = 0$ and so, the proposition follows from the exact sequence of sheaves on Ω:

$$0 \to O^* \to M^* \to \mathrm{Div} \to 0\,.$$

One can make the above explicit as follows. Suppose for convenience that $\infty \in \Omega$ and choose for every $\lambda \in \Omega$, $\lambda \neq \infty$ a point $\mathrm{pr}(\lambda) \in \mathcal{L}$ such that

$$\big|\lambda - \mathrm{pr}(\lambda)\big| = \min\big\{|\lambda - \mu| \mid \mu \in \mathcal{L}\big\}\,.$$

Further let $\mathrm{pr}(\infty)$ be any fixed point of $\mathcal{L}$. Then one can verify that for any divisor $D = \sum n_i x_i$ (with discrete support) the infinite product

$$\prod \left(\frac{z - x_i}{z - \mathrm{pr}(x_i)}\right)^{n_i}$$

converges and defines an element of $M(\Omega)^*$ with divisor D. □

3. Discrete groups Γ and their quotients

3.1. Discontinuous groups

Let $\Gamma \subset \mathrm{PGl}(2, K)$ be an infinite subgroup. A point $p \in \mathbb{P}^1_K$ is called a *limit point* of Γ if there exists a point $q \in \mathbb{P}^1_K$ and a sequence of distinct elements γ_n of Γ such that $\lim \gamma_n(q) = p$. The set of all limit points is denoted by $\mathcal{L}$. The group Γ is called *discontinuous* (*or discrete*) if $\mathcal{L} \neq \mathbb{P}^1_K$ and if for every point p the closure of its orbit $\overline{\Gamma p}$ is compact.

If the field K happens to be a local field (*i.e.* K is locally compact) then this condition on $\overline{\Gamma p}$ is superfluous. Moreover, for a local field K, the group Γ is discontinuous if and only if Γ is discrete as a topological subgroup of $\mathrm{PGl}(2, K)$.

Let Γ be discrete with set of limit points $\mathcal{L}$. Then $\mathcal{L}$ is compact and $\Omega = \mathbf{P}^1_K - \mathcal{L}$ is of the type studied in section 2. One can choose a compact and Γ-invariant set $\mathcal{M}$ with $\mathcal{L}$ as its set of limit points. If $\mathcal{L}$ is a perfect set one can take $\mathcal{M} = \mathcal{L}$. If $\mathcal{L}$ is not perfect (in that case $\mathcal{L}$ consists of at most two points and the group is called elementary) one can take $\mathcal{M} = \mathcal{L} \cup \bigcup_{i=1,\ldots,s} \Gamma p_i$ where p_i are points in Ω. Then Γ acts discontinuously on Ω and this induces an action on $\overline{\Omega}$ and T.

As we will show later the quotient $X := \Gamma \backslash \Omega$ is a 1-dimensional, connected, non-singular analytic space over K. We are interested in the case that X is an algebraic curve over K. There are two important examples of this:

- Γ is a finitely generated group and hence a finite extension of a Schottky group and the curve X is projective (see [GP, Chapter 1, (3.1) and Chapter 3, (2.2)]);
- $K = \mathbf{F}_q((t^{-1}))$ and Γ is a subgroup of finite index of the group $\mathrm{PGl}(2, \mathbb{F}_q[t])$ acting on $\Omega = \mathbb{P}^1_K - \mathbb{P}^1(K)$. In this case X is an affine curve. For certain congruence subgroups Γ (and with a more general field K) the corresponding curve is called *Drinfeld modular curve* (see [G2]).

More general, a connected, one-dimensional, non singular analytic space X is called *quasi algebraic* if there exists a projective non singular and connected curve $\widehat{X}$ and a compact subset $\mathcal{N}$ of $\widehat{X}$ (for the topology on $\widehat{X}$ induced by the topology of the field K) such that X is isomorphic to $\widehat{X} - \mathcal{N}$.

One might hope that every quotient X is quasi algebraic. There are however more complicated quotients as shows the example 4.1.

We will assume that Γ acts without *inversions* on the tree T (*i.e.* $\gamma \overrightarrow{e} \neq -\overrightarrow{e}$ for every $\gamma \in \Gamma$ and edge $\overrightarrow{e}$ of T). In the case that there are inversions one can change the reduction of Ω and T by enlarging the compact and Γ-invariant set $\mathcal{M}$ such that there are no inversions in the new situation. This condition assures us that the quotient $\Gamma \backslash T$ is a (locally finite) graph. Since Γ is discontinuous the stabilizer of any vertex and any edge is a finite group.

According to [S2, pp. 75, 76], the quotient is also a *graph of (finite) groups*. The groups attached to vertices and edges are the stabilizers of the corresponding vertices and edges of T. Further Γ is canonical isomorphic to the *fundamental group of this graph of groups*. Let Γ_{tors} denote the subgroup of Γ generated by the set of torsion elements (*i.e.* the elements of finite order) of Γ. This subgroup of Γ is normal and according to [S2, p. 77, cor. 1], the group $\Gamma/\Gamma_{\mathrm{tors}}$ is isomorphic to the (ordinary) fundamental group of the graph $\Gamma\backslash T$. The topological fundamental group of any connected (locally finite) graph (finite or not) is a free group. Hence we conclude that $\Gamma/\Gamma_{\mathrm{tors}}$ is a free group. Let g, $0 \leq g \leq \infty$, denote the rank of this group. Note that g is equal to the dimension of the vector space $\Gamma_{ab} \otimes \mathbb{Q}$. For convenience we call g the *rank* of the group Γ.

3.2. Definition and some properties of X

Let $\pi : \Omega \to X$ denote the canonical map. The Grothendieck topology of X is given by: a subset $A \subset X$ is admissible if $\pi^{-1}(A)$ is admissible. Similarly, one defines admissible coverings of admissible subsets. For any affinoid subset $B \subset \Omega$ the set $A := \pi(B)$ is admissible since $\pi^{-1}(A) = \bigcup_{\gamma\in\Gamma} \gamma B$ is admissible according to the discreteness of the group Γ. For any sheaf of abelian groups $\mathcal{T}$ on Ω with a Γ-*action* σ (*i.e.* a collection of isomorphisms $\sigma(\gamma) : \gamma^*\mathcal{T} \to \mathcal{T}$ such that $\sigma(\gamma_1\gamma_2) = \gamma_2^*(\sigma(\gamma_1)\sigma(\gamma_2)$ for all $\gamma_1, \gamma_2 \in \Gamma$), one defines a sheaf $\mathcal{S} = \pi_*^\Gamma \mathcal{T}$ (of abelian groups) on X by $\mathcal{S}(A) = \mathcal{T}(\pi^{-1}A)^\Gamma$. The functor π_*^Γ is a left exact functor. In particular the structure sheaf O_X on X is defined as $\pi_*^\Gamma O_\Omega$. Take a small enough connected affinoid (for instance the affinoids occuring in the pure covering of Ω) subset $B \subset \Omega$ with $A := \pi(B)$. Then $\bigcup_{\gamma\in\Gamma} \gamma B$ has some connected component C with as stabilizer the finite subgroup Γ_0 of Γ. As is easily seen $\pi^{-1}(A) = \bigcup_{\gamma\in\Gamma/\Gamma_0} \gamma C$ and $O_X(A) = O_\Omega(C)^{\Gamma_0}$. The latter algebra is an affinoid algebra since it is the ring of invariants of an affinoid algebra under the action of a finite group. Moreover A is isomorphic to an affinoid subset of $\mathbb{P}^1_K$ since $\Gamma_0\backslash\mathbb{P}^1_K$ is isomorphic to $\mathbb{P}^1_K$. This proves that X is a 1-dimensional, connected, non-singular analytic space over K, locally for the Grothendieck topology isomorphic to $\mathbb{P}^1_K$.

In case X is a projective curve this means that X is a Mumford curve. In case X is an affine curve it follows that the corresponding projective curve $\widehat{X}$ is a Mumford curve. More general, if X happens to be quasi-algebraic then $X = \widehat{X} - \mathcal{N}$ and $\widehat{X}$ is again a Mumford curve (as one easily verifies).

Let M denote again the sheaf of meromorphic functions on an analytic space. Then $\pi_*^\Gamma M_\Omega = M_X$. Indeed, with the notations A, B, C as above, $M_X(A)$ is the quotient field of $O_X(A)$ and equals the field of invariants of the quotient field $M_\Omega(C)$ of $O_\Omega(C)$ under the action of the finite group Γ_0.

The sheaf of divisors Div on any analytic space is defined by $\mathrm{Div}(A)$ is the group of all $\mathbb{Z}$-valued functions on A with discrete support. In this context a set $S \subset A$ is called discrete if its intersection with every affinoid subset of A is finite. One easily sees that $\pi_*^\Gamma \operatorname{Div}_\Omega \supset \operatorname{Div}_X$ and equality holds if and only if Γ acts without fixed points on Ω.

The pure affinoid covering of Ω by the sets $\Omega(e), \Omega([a])$ gives rise to a pure covering of the quotient X and a corresponding reduction $r : X \to \overline{X}$. In fact $\overline{X} = \Gamma\backslash\overline{\Omega}$. The irreducible components of $\overline{X}$ are rational curves over the residue field k of K. The intersection of two components consist of a finite number of ordinary double points. A component can also have self-intersection, which means that the rational curve may have a finite number of ordinary double points. The intersection graph of the components of $\overline{X}$ coincides with $\Gamma\backslash T$. The following theorem gives an idea what the possibilities for the quotients X can be. *A surprising feature for groups of finite rank is that the construction of $X = \Gamma\backslash\Omega$ factors over the Schottky uniformization of the Mumford curve $\widehat{X}$.*

THEOREM 3.3

Let $X = \Gamma\backslash\Omega$ and let g denote the rank of Γ. Then:

(a) X is a projective algebraic curve if and only if Γ is finitely generated. The genus of X equals g.

(b) X is quasi-algebraic if and only if g is finite. The projective curve $\widehat{X}$ and the compact subset $\mathcal{N}$ with $X \cong \widehat{X} - \mathcal{N}$ are uniquely determined by X. Further $\widehat{X}$ is a Mumford curve of genus g.

(c) X is isomorphic to $\mathbb{P}^1_K - \mathcal{N}$ if and only if $g = 0$.

(d) Suppose that g is finite. Put $\Gamma_1 := \Gamma/\Gamma_{\mathrm{tors}}$ and $X_1 := \Gamma_{\mathrm{tors}}\backslash\Omega$. Then X_1 can be identified with $\mathbb{P}^1_K - \mathcal{M}$ for some compact set $\mathcal{M}$. The action of Γ_1 on X_1 extends to an action on $\mathbb{P}^1_K$. The set $\mathcal{M}$ is Γ_1 invariant and contains the set of limit points $\mathcal{L}_1$ of Γ_1. Put $\Omega_1 = \mathbb{P}^1_K - \mathcal{L}_1$. Then Γ_1 is a Schottky group and and the embedding of X into the projective curve $\widehat{X}$ is given by $X = \Gamma_1\backslash X_1 \subset \Gamma_1\backslash\Omega_1 =: \widehat{X}$.

Suppose that Γ is finitely generated then $X_1 = \Omega_1$ and $\mathcal{L}_1$ is empty if $g = 0$, $\mathcal{L}_1$ consists of two points if $g = 1$ and is an infinite, perfect set if $g > 1$.

(e) X is called algebraic if X is an algebraic curve (either affine or projective). The following statements are equivalent:

(1) X is algebraic.

(2) The graph $\Gamma\backslash T$ has finitely many ends.

Proof.— Part (a) is proved in [GP, pp. 20, 106]. We suppose now that the group Γ is not finitely generated.

Suppose that $g = 0$. Then it is easily seen that X satisfies the conditions of [GP, p. 145, (2.5)]. Hence X is isomorphic to $\mathbb{P}^1_K - \mathcal{N}$ for some compact subset $\mathcal{N}$ of $\mathbb{P}^1_K$.

Let now g be finite. Choose a finite part of the graph $\Gamma \backslash T$ containing all the cycles of the graph. The complement of this finite part consist of finitely many components which are trees. For every component we apply [GP, p. 144, (2.4)] and find that the space corresponding to the component has the form $A - \mathcal{N}_A$ where A is an affinoid subspace of $\mathbb{P}^1_K$ and where $\mathcal{N}_A$ is a compact set. Glueing the finitely many affinoids A to the affinoid space corresponding with the chosen finite part yields the complete, non singular, connected curve $\widehat{X}$ and the compact set $\mathcal{N}$ such that $X \cong \widehat{X} - \mathcal{N}$. The uniqueness of the construction is clear. Further the curve $\widehat{X}$ is by construction locally isomorphic to $\mathbb{P}^1_K$ and therefore a Mumford curve. Finally it is easily seen that the choosen finite part of $\Gamma \backslash T$ is essentially the graph of the reduction of $\widehat{X}$. Hence the genus of $\widehat{X}$ is g.

Suppose that g is infinite and that X can be seen as an analytic subspace of some projective non singular and connected curve Z. Take a positive integer n and a finite connected part of the graph $\Gamma \backslash T$ with Betti number $b \geq n$. The corresponding affinoid subspace A of X lies in Z. Using a suitable reduction of Z one can easily see that the genus of Z is at least b. This is a contradiction and we find that X can not be embedded into any projective curve. Hence we have proved (a), (b) and (c).

In part (d) we have to show that the action of Γ_1 extends to $\mathbb{P}^1_K$. This is proved in [GP, p. 152]. Hence Γ_1 is a finitely generated, discontinuous and torsion free subgroup of $\mathrm{PGl}(2, K)$. By definition ([GP, p. 6]) this is a Schottky group. The rest of the statement (d) is more or less obvious.

For part (e) we note that an end in a graph is (the equivalence class of) a subgraph isomorphic to a half line. We know that $X = \widehat{X} - \mathcal{M}$ for some compact set $\mathcal{M}$. It is easily seen that $\mathcal{M}$ is finite if and only if the graph of some reduction (and hence any reduction) of X has finitely many ends. From this (e) follows. This proves the theorem. $\square$

4. Examples of discrete groups

4.1. Examples with $g = \infty$

Let Γ be a Schottky group on more than one generator. Its commutator subgroup $\Gamma_1 = [\Gamma, \Gamma]$ is a free group on countably many generators and the set of limit points of Γ_1 is perfect and invariant under Γ. Hence this set coincides with the set of limit points of Γ. The quotient $\Gamma_1 \backslash \Omega$ is a one-dimensional analytic space of "infinite genus" and has no embedding in any projective curve (see also 6.5.2).

One might think that every discrete group of infinite rank lies in a discrete, finitely generated group. This is not the case as the following example indicates.

Let K denote a local field. A triple (a, b, q) with a, $b \in \mathbb{P}^1(K)$; $a \neq b$ and $q \in K$ with $0 < |q| < 1$ determines a hyperbolic element $\gamma \in \mathrm{PGl}(2, K)$ given by $\gamma\left({}^{z-a}/_{z-b}\right) = q\left({}^{z-a}/_{z-b}\right)$. One can easily construct a sequence of triples $\{(a_n, b_n, q_n)\}_{n \geq 1}$ such that the group Γ generated by the corresponding hyperbolic elements γ_n is discrete and free on those generators. Moreover the triples can be chosen in such a way that the smallest subring R of K such that $\gamma_n \in \mathrm{PGl}(2, R)$ for all n is not finitely generated over $\mathbb{Z}$. From the last property it follows that Γ does not lie in any finitely generated subgroup of $\mathrm{PGl}(2, K)$.

4.2. Shimura curves and groups of quaternions

Shimura curves can be defined, like the modular curves $X_0(N)$, by a moduli problem. We have to introduce some objects and notations.

4.2.1. Quaternian algebras

A *quaternion algebra D over a field k* is a simple algebra (*i.e.* there are no two-sided ideals different from $\{0\}$ and D) with center k and of dimension 4 over k. For a field k with $\mathrm{char}(k) \neq 2$, D has a basis $\{1, e_1, e_2, e_3\}$ over k such that the multiplication is given by the formulas:

$$e_1 e_2 = e_3;\ e_2 e_1 = -e_3;\ e_1^2 = a;\ e_2^2 = b;\ e_3^2 = -ab \quad \text{with } a,\ b \in k^*.$$

Examples

1. $k = \mathbb{R}$. There are two possibilities:

$$D = M(2 \times 2, \mathbb{R})$$

and $D = \mathbb{H}$ the skew field of *Hamilton quaternions*, given by

$$a = b = -1\,.$$

2. $k = \mathbb{Q}_p$. There are again two possibilities:

$$D = M(2 \times 2, \mathbb{Q}_p)$$

and $D =$ the skew field that one can describe as follows:

$$D = \mathbb{Q}_{p^2} 1 + \mathbb{Q}_{p^2} u$$

with multiplication rules $u^2 = p$; $\lambda u = u\mathrm{Fr}(\lambda)$ for $\lambda \in \mathbb{Q}_{p^2}$. One has written here $\mathbb{Q}_{p^2}$ for the unique non ramified extension of degree 2 of $\mathbb{Q}_p$ and Fr for the action of Frobenius on $\mathbb{Q}_{p^2}$.

3. $k = \mathbb{Q}$. Let D be a quaternion algebra over $\mathbb{Q}$ and let v be a place of $\mathbb{Q}$ (meaning that v is a prime number or that $v = \infty$). One says that D is *ramified* at v if $D \otimes \mathbb{Q}_v$ is a skew field. The number of ramified places is finite and even. Moreover, for every finite set S of places with an even cardinality there exists a unique quaternion algebra over $\mathbb{Q}$ with S as its set of ramified places.The *discriminant* of a quaternion algebra D over $\mathbb{Q}$ is given by the formula:

$$d = \pm \prod_{p \text{ ramified}} p$$

where the $+$ sign corresponds with "D is not ramified at ∞".

4. The discriminant of $M(2 \times 2, \mathbb{Q})$ is 1. The discriminant of the skew field $\mathbb{H}$ is -2.

Eichler orders

Let $D/\mathbb{Q}$ be a quaternion algebra with discriminant d and let $f \in \mathbb{Z}$, $f > 0$, $(d, f) = 1$. An *Eichler order* $\theta \subset D$ of level f is a subring of D, free of rank 4 as $\mathbb{Z}$-module, such that:

- for $p \mid d$, $\theta \otimes \mathbb{Z}_p$ is the unique maximal order $\mathbb{Z}_{p^2}[u]$ of the skew field $\mathbb{Q}_{p^2}[u]$;

- for $p \nmid d$, $\theta \otimes \mathbb{Z}_p$ is conjugated with

$$\left\{ \begin{pmatrix} a & b \\ fc & d \end{pmatrix} \,\middle|\, a,\, b,\, c,\, d \in \mathbb{Z}_p \right\} \subset M(2 \times 2, \mathbb{Q}_p) \cong D \otimes \mathbb{Q}_p \,.$$

We note that "$f = 1$" is equivalent to "the Eichler order is maximal". For $d > 0$, there exists only one conjugacy class of Eichler orders of level f in D. For $d < 0$, the Eichler orders of level f in D form a finite set of classes under conjugacy.

4.2.2. The moduli problem

Fix an Eichler order $\theta \subset D$ with discriminant $d > 1$ and level f. One considers abelian surfaces A with the following additional structure:

1. an injective ringhomomorphism $i : \theta \to \mathrm{End}(A)$;
2. a subgroup $B \subset A[f] = \ker(\cdot f : A \to A)$ such that B is a cyclic θ-module of order f;
3. the map $\theta \xrightarrow{i} \mathrm{End}(A) \to \mathrm{End}(\mathrm{Lie}\ A) = M(2 \times 2)$ has the property

$$\mathrm{Tr}\big(i(d) \mid \mathrm{Lie}\ A\big) = \mathrm{Tr}_{D/\mathbb{Q}}(d)\,.$$

   Here $\mathrm{Tr}_{D/\mathbb{Q}}$ denotes the reduced trace of D. The equality is meant to hold in the structure sheaf of the base scheme of A.

The data above define a functor on schemes over $\mathbb{Z}[1/f]$ and according to [D2] there is a coarse moduli scheme $S_{d,f}$ of finite type and projective over $\mathbb{Z}[1/f]$ connected with the functor. The curve $S_{d,f}$ is called the *Shimura curve*.

4.2.3. The uniformizations

Description of the complex Shimura curve $S_{d,f} \otimes \mathbb{C}$

The group $\theta^* \subset (D \otimes \mathbb{R})^* \cong \mathrm{GL}(2, \mathbb{R})$ acts on $\Omega = \mathbb{C} - \mathbb{R}$. The *complex Shimura curve* is isomorphic to $\theta^* \backslash \Omega$.

Since $d > 1$, the quotient is a compact and connected Riemann surface.

Description of the p-adic Shimura curve $S_{d,f} \otimes \mathbb{Q}_p$ for $p \mid d$

Let D' denote the quaternion algebra over $\mathbb{Q}$ obtained from D by *exchanging p and ∞*. This means that the discriminant of D' is $d' = -d/p$. Let $\theta' \subset D'$

be an order of level f (the conjugacy class of this order is no longer unique). Define $\Gamma \subset \mathrm{GL}(2,\mathbb{Q}_p)$ by:

$$\Gamma = \theta'\left[\frac{1}{p}\right]^* \subset (D'\otimes\mathbb{Q}_p)^* \cong \mathrm{GL}(2,\mathbb{Q}_p)\,.$$

The approximation theorem for algebraic groups over $\mathbb{Q}$ implies that Γ is discrete and co-compact. In particular, the set of ordinary points of Γ is $\Omega = \mathbb{P}^1_{\mathbb{Q}_p} - \mathbb{P}^1(\mathbb{Q}_p)$. The group Γ acts with inversions on the tree T of Ω. Put $\Gamma_+ = \{\gamma\in\Gamma \mid v_p(\det\gamma) \equiv 0 \bmod 2\}$, where v_p denotes the additive valuation of $\mathbb{Q}_p$.

Drinfeld's theorem implies

$$S_{d,f}\otimes\mathbb{Q}_{p^2} \cong (\Gamma_+\backslash\Omega)\otimes\mathbb{Q}_{p^2}\,.$$

We have to be more precise and to explain the Frobenius twist occuring in the statement. For this purpose we introduce the following notations:

- $\mathbb{Z}_{p^2} = W(\mathbb{F}_{p^2})$ = the ring of integers of $\mathbb{Q}_{p^2}$;
- $\mathbb{Z}_{p^\infty} = W(\mathbb{F}_{p^\infty})$ = the completion of the maximal unramified extension of $\mathbb{Z}_p$;
- Fr denotes the Frobenius action on $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_{p^\infty}$; further Ω is seen as formal scheme over $\mathbb{Z}_p$ and $\Omega\widehat{\otimes}\mathbb{Z}_{p^\infty}$ is a scalar extension of Ω;
- the group Γ acts on $\Omega\widehat{\otimes}\mathbb{Z}_{p^\infty}$ by the formula:

  $$\gamma(\omega\otimes\lambda) = \big([\gamma]\omega\big)\otimes \mathrm{Fr}^{n(\gamma)}(\lambda)$$

  in which $\omega\in\Omega$, $\lambda\in\mathbb{Z}_{p^\infty}$, $[\gamma]$ = the image of γ in $\mathrm{PGL}(2,\mathbb{Q}_p)$ and $n(\gamma) = v_p(\det\,\gamma)$.

A precise version of Drinfeld's theorem is [D2], [Ri], [JL]

THEOREM . — *The formal $\mathbb{Z}_p$-schemes $S_{d,f}\otimes\mathbb{Z}_p$ and $\Gamma\backslash(\Omega\widehat{\otimes}\mathbb{Z}_{p^\infty})$ are isomorphic.*

We observe that the matrix $p\cdot\mathrm{Id}$ belongs to Γ_+ and so

$$\Gamma_+\backslash\big(\Omega\widehat{\otimes}\mathbb{Z}_{p^\infty}\big) = \big([\Gamma_+]\backslash\Omega\big)\otimes\mathbb{Z}_{p^2}\,,$$

where $[\Gamma_+] = \Gamma_+/\{p^n\cdot\mathrm{Id}\mid n\in\mathbb{Z}\}$ is the image of Γ_+ in $\mathrm{PGL}(2,\mathbb{Q}_p)$. The group $\Gamma/\Gamma_+ = \{1,\omega\}$ and ω acts on $[\Gamma_+]\backslash\Omega$ by $[\omega]$ and on $\mathbb{Z}_{p^2}$ by Fr. Hence $S_{d,f}\otimes\mathbb{Z}_p$ is a *Frobenius twist* of $[\Gamma_+]\backslash\Omega$ defined by

$$\{1,\omega\}\backslash\big([\Gamma_+]\backslash\Omega\big)\otimes\mathbb{Z}_{p^2}\,.$$

In particular

$$S_{d,f} \otimes \mathbb{Z}_{p^2} \cong \left([\Gamma_+]\backslash\Omega\right) \otimes \mathbb{Z}_{p^2} .$$

Description of the graph $\Gamma_+\backslash T$ in terms of the arithmetic of θ'

The graph $\Gamma_+\backslash T$ is the dual graph of the reduction $S_{d,f} \otimes \mathbb{F}_{p^2}$ of $S_{d,f}$ and also the dual graph of the reduction $\Gamma_+\backslash\Omega \otimes \mathbb{F}_p$ of $\Gamma_+\backslash\Omega$ seen as formal scheme over $\mathbb{Z}_p$. The tree T is the Bruhat-Tits building of the group $\mathrm{PSl}(2, \mathbb{Q}_p)$. The isomorphism $(D' \otimes \mathbb{Q}_p)^* \cong \mathrm{GL}(2, \mathbb{Q}_p)$ can be used to give a new description of T: the vertices of T are the classes of normalized left θ'-ideals of D'. A left θ'-ideal M is called *normalized* if $M \otimes \mathbb{Z}_l = \theta' \otimes \mathbb{Z}_l$ for every prime number $l \neq p$. Two normalized ideals M_1, M_2 are equivalent if $M_1 = p^m M_2$ for some m. Further $\{[M_1], [M_2]\}$ is an edge if the order of M_1/M_2 is p^2. The group $\Gamma = \left(\theta'[1/p]\right)^*$ acts on normalized ideals by multiplication on the right. The set $(\Gamma\backslash T)_0$, the vertices of $(\Gamma\backslash T)$, is a finite set and its cardinal is the class number h of θ'. Then $(\Gamma_+\backslash T)_0 = \{X_1, \ldots, X_h, X'_1, \ldots, X'_h\}$ and ω operates by $\omega(X_i) = X'_i$; $\omega^2 = id$. Every edge $y \in (\Gamma_+\backslash T)_1$ has a certain length $l(y) \geq 1$. Let a denote the double point corresponding with y. Then the local ring of a has the form $\mathbb{Z}_{p^\infty}[\![X, Y]\!]/(XY - p^{l(y)})$. One easily sees that the order of the stabilizer of any edge of T, with image y, in the group Γ_+ is equal to $l(y)$. There are formulas of Eichler for the numbers h and $l(y)$.

4.2.4. $S_{2p,1}$ and Hamilton quaternions

For the choice $d = 2p$ with $p > 2$ a prime number and $f = 1$, the algebra D' is $\mathbb{H}$, the skew field of Hamilton quaternions over $\mathbb{Q}$. The class number of $\mathbb{H}$ is 1 and its maximal order (unique up to conjugation) will be denoted by H. A base of this order as $\mathbb{Z}$-module is $\{\rho = 1/2\,(1+e_1+e_2+e_3),\ e_1,\ e_2,\ e_3\}$. For any commutative ring with unit R we write $H(R)$ for $H \otimes R$. The group Γ, defined above, is $H\left(\mathbb{Z}[1/p]\right)^*$. This group has been studied in [GP, ch. 9] in some detail. At the time [GP] was written the authors were not aware of the connection with the Shimura curves.

Let us start by investigating the tree T. A (fractional) left H-ideal M in $H(\mathbb{Q})$ is normalized if and only if $M \otimes \mathbb{Z}[1/p] = H\left(\mathbb{Z}[1/p]\right)$. Every left ideal in H is principal (see [HW]) and it follows that every class of normalized left ideals has uniquely the form $[Hu]$ where $u \in H$ is an element with norm a power of p and such that u is not divisible by p. Such an element u lies in Γ and as a consequence Γ has only one orbit on the vertices of T and $[\Gamma_+]$ has two orbits. The edges of T starting in $[H]$ are the pairs $\{[H], [Hu]\}$ where u

runs in the set of elements of H with norm p. The quotients H/Hu are the non trivial left ideals of the algebra $H/pH = H(\mathbb{F}_p) \cong M(2 \times 2, \mathbb{F}_p)$. Their number is of course $p+1$ and this is in accordance with $r_4(p) = 8(p+1)$; $\#H^* = 24$ and an easy calculation like [HW, Theorem 371]. The stabilizer of $[H]$ in Γ or Γ_+ is the group H^*. The edges of $[\Gamma_+]\backslash T$ are the orbits of the $p+1$ edges above under the action of H^* by right multiplication. In order to understand this action we introduce a curve $\mathbf{P}$ over $\mathbb{Z}$ canonically attached to the situation.

$\mathbf{P}$ is the set of isotropic lines in the $\mathbb{Z}$-module V generated by $\{e_1, e_2, e_3\}$ (this is the orthoplement of the vector $1 \in H$), with respect to the quadratic form induced by H. Hence $\mathbf{P}$ is the subspace of $\mathbb{P}^2$ given by the homogeneous equation $x_1^2+x_2^2+x_3^2 = 0$. The curve $\mathbf{P}_{\mathbb{Q}_p}$ is the projective line over $\mathbb{Q}_p$ since $\mathbf{P}$ has points with coordinates in in $\mathbb{F}_p$. The groups $[\Gamma]$ and $[\Gamma_+]$ act by conjugation on V. Let $\mathrm{SO}(3)$ denote the affine algebraic group, defined over $\mathbb{Z}$ of the orthogonal automorphisms of V with determinant 1 and let H^*/Z denote the quotient of H^* by its centralizer Z, seen as an affine algebraic group over $\mathbb{Z}$.

An explicit calculation of the map between the affine rings of the two affine group schemes over $\mathbb{Z}$ proves that the morphism $H^*/Z \to \mathrm{SO}(3)$, defined by the action of H^* on V, is an isomorphism over $\mathbb{Z}[1/2]$. In particular $H^*/Z(\mathbb{Z}[1/2p])$ is equal to $\mathrm{SO}(3, \mathbb{Z}[1/2p])$. Further $H^*/Z(\mathbb{Z}) \cong A_4$ is a subgroup of index 2 in $\mathrm{SO}(3, \mathbb{Z}) \cong S_4$ and $H^*/Z(\mathbb{Z}[1/2])$ is mapped bijectively to $\mathrm{SO}(3, \mathbb{Z}) = \mathrm{SO}(3, \mathbb{Z}[1/2])$. Hence $\Gamma/\{\pm p^m\} = H^*/Z(\mathbb{Z}[1/p])$ has index 2 in $\mathrm{SO}(3, \mathbb{Z}[1/p]) = \mathrm{SO}(3, \mathbb{Z}[1/2p])$.

The groups $[\Gamma]$ and $[\Gamma_+]$ act on $\mathbf{P}_{\mathbb{Q}_p}$. The space of ordinary points for this action is $\Omega = \mathbf{P}_{\mathbb{Q}_p} - \mathbf{P}(\mathbb{Q}_p)$. Every edge $\{[H], [Hu]\}$ as above induces the isotropic line $(Hu/Hp) \cap V \otimes \mathbb{F}_p$. One finds in this way a bijection between the edges above and the points of $\mathbf{P}(\mathbb{F}_p)$. The group H^* or better the quotient $H^*/\{\pm 1\} \cong A_4$ acts on V and $V \otimes \mathbb{F}_p$ as the subgroup of index two of the group S_4 of all symmetries with determinant 1 of the cube. The structure of the orbits of the action of A_4 on $\mathbf{P}(\mathbb{F}_p)$ depends on p mod 12. The genus g of the curves $S_{2p,1}$ and $[\Gamma_+]\backslash\Omega$ is equal to the number of orbits minus one. For $p = 3$ there is only one orbit and $g = 0$. For $p > 3$ we tabulate g and the number of edges y with $l(y) = 2$, 3. The stable reduction of the curve $[\Gamma_+]\backslash\Omega$ mod p has two components, both isomorphic to the projective line over $\mathbb{F}_p$. The two components meet in $g+1$ double points; the numbers l have the same meaning for the local equation at the double points as in 4.2.3. The table also gives $[\Gamma_+]\backslash T$ as graph of groups and according to [S2] gives the structure of the group $[\Gamma_+]$, since the stabilizers of the two vertices are isomorphic to A_4. The curve $S_{2p,1} \otimes \mathbb{F}_{p^2}$ has a very similar structure.

p mod 12	$l=2$	$l=3$	$g([\Gamma_+])$
1	1	2	$(p+11)/12$
5	1	0	$(p-5)/12$
7	0	2	$(p+5)/12$
11	0	0	$(p-11)/12$

We note that the curve $[\Gamma_+]\backslash\Omega$ has features in commun with $X_0(p)$. The table of the reduction mod p of the latter curve is:

p mod 12	$l=2$	$l=3$	$g_0(p)$
1	0	0	$(p-13)/12$
5	0	1	$(p-5)/12$
7	1	0	$(p-7)/12$
11	1	1	$(p+1)/12$

4.2.5. A zoo of quaternion groups

It is interesting to consider other groups of quaternions. The group $\Gamma/\{\pm p^n\}$ is called Λ in [GP, ch. 9]. For the genus of the corresponding curve (or the rank of Λ) there is the formula:

$$\frac{p+1}{24} - \frac{r_3(p)}{48} + \frac{\delta_4(p)}{4} + \frac{2\delta_3(p)}{3},$$

where $\delta_k(p) = 1$ for $p \equiv 1$ mod k and otherwise 0 (and $k = 3$, 4). Further $r_3(p)$ denotes the number of representations of p as sum of three squares. This formula can be obtained as follows. The genus of $\Lambda\backslash\Omega$ is equal to the genus of its reduction $(\Lambda\backslash\Omega)\otimes\mathbb{F}_p = \Lambda\backslash(\Omega\otimes\mathbb{F}_p)$. The latter space has only one irreducible component because Λ acts transitively on (the vertices of T) = (the irreducible components of $\Omega\otimes\mathbb{F}_p$). Further, a double point of $\Omega\otimes\mathbb{F}_p$ disappears in $\Lambda\backslash(\Omega\otimes\mathbb{F}_p)$ if and only if the corresponding edge of T admits an inversion in Λ. In particular, the edge $\{[H],[Hu]\}$ corresponds to a disappearing double point if and only if u can be chosen such that

$u^2 = -p$. This amounts to $u = x_1e_1 + x_2e_2 + x_3e_3$ with x_1, x_2, $x_3 \in \mathbb{Z}$; $\sum x_i^2 = p$ and the first non zero x_i is positive. This explains the first part $(p+1)/24 - r_3(p)/48$ of the formula. The last part follows from an examination of what happens for the fixed points of the elements of A_4. We note that the degree two map $[\Gamma_+]\backslash\Omega \to \Lambda\backslash\Omega$ is ramified in $r_3(p)/12$ points. For the first 50 primes we give the values of the rank of Λ. This table extends and corrects [GP, p. 268].

$$(3,0),\ (5,0),\ (7,1),\ (11,0),\ (13,1),\ (17,0),\ (19,1),\ (23,1),$$
$$(29,0),\ (31,2),\ (37,2),\ (41,0),\ (43,2),\ (47,2),\ (53,1),\ (59,1),$$
$$(61,2),\ (67,3),\ (71,3),\ (73,3),\ (79,4),\ (83,2),\ (89,1),\ (97,4),$$
$$(101,1),\ (103,5),\ (107,3),\ (109,4),\ (113,3),\ (127,6),\ (131,3),$$
$$(137,4),\ (139,5),\ (149,3),\ (151,7),\ (157,6),\ (163,7),\ (167,7),$$
$$(173,4),\ (179,5),\ (181,6),\ (191,8),\ (193,8),\ (197,6),$$
$$(199,9),\ (211,8),\ (223,10),\ (227,7),\ (229,8)\,.$$

The group Λ lies as normal subgroup of index 2 in the group

$$\widetilde{\Lambda} := H\big(\mathbb{Z}[1/2p]\big)^*/\{\pm p^m\}\,.$$

This group acts with inversions on T and its subgroup $\widetilde{\Lambda}_+ = \{\lambda \in \widetilde{\Lambda} \mid v_p(\det\lambda) \equiv 0 \bmod 2\}$ of index 2 acts without inversions on T. The quotient graph $\widetilde{\Lambda}_+\backslash T$ has two vertices. The edges of the quotient graph are the orbits of $\mathbf{P}(\mathbb{F}_p)$ under the action of the stabilizer of one of the vertices. This stabilizer is the group $H\big(\mathbb{Z}[1/2]\big)^*/\{\pm 2^n\}$. The last group is isomorphic with S_4 and acts on the space $V \otimes \mathbb{F}_p$ as the group of symmetries with determinant 1 of the cube. From this description one can calculate the data of the quotient graph and the genus of the corresponding curve. For $p > 3$ we have tabulated the results (see next page).

The calculation of the rank $g(\widetilde{\Lambda})$ of $\widetilde{\Lambda}$ can in principle be done with the method that gave the rank of Λ. The formula reads:

$$g(\widetilde{\Lambda}) = \frac{p+1}{48} - \frac{r_3(p) + r_3(2p)}{96} + \frac{\delta_2(p)}{4} + \frac{\delta_3(p)}{3} + \frac{3\delta_4(p)}{8}$$

p mod 24	$l=2$	$l=3$	$l=4$	$g(\widetilde{\Lambda}_+)$
1	1	1	1	$(p+23)/24$
5	0	0	1	$(p-5)/24$
7	0	1	0	$(p-7)/24$
11	1	0	0	$(p-11)/24$
13	0	1	1	$(p+11)/24$
17	1	0	1	$(p+7)/24$
19	1	1	0	$(p+5)/24$
23	0	0	0	$(p-23)/24$

where $\delta_2(p) = 1$ if $p \equiv 1, 3 \bmod 8$ and otherwise $=0$. We note that the degree two map $\widetilde{\Lambda}_+\backslash\Omega \to \widetilde{\Lambda}\backslash\Omega$ is ramified in $\big(r_3(p) + r_3(2p)\big)/24$ points. Another calculation is the following. As we have seen before $\widetilde{\Lambda} \cong \mathrm{SO}\big(3, \mathbb{Z}[1/2p]\big)$. The congruence subgroup

$$\widetilde{\Lambda}(3) := \big\{\lambda \in \widetilde{\Lambda} \mid \lambda \equiv 1 \in \mathrm{SO}(3, \mathbb{F}_3)\big\}$$

has no elements of finite order since a matrix of the form $Id + 3A$ and $A \neq 0$ can not have finite order. The map $\widetilde{\Lambda} \to \mathrm{SO}(3, \mathbb{F}_3)$ is surjective since $S_4 = H\big(\mathbb{Z}[1/2]\big)^*/\{\pm 2^m\} \to \mathrm{SO}(3, \mathbb{F}_3)$ is an isomorphism. It follows that $\widetilde{\Lambda}(3)$ acts transitively on the vertices of T. Hence $\widetilde{\Lambda}(3)$ is a free group of rank $(p+1)/2$. The group S_4 acts by conjugation on $\widetilde{\Lambda}(3)_{ab} \otimes \mathbb{Q}$ and the rank of $\widetilde{\Lambda}$ is equal to the number of trivial representations present in the representation above. For the rank of $\widetilde{\Lambda}$ one finds the following algorithm : the number of solutions in integers q, x_1, x_2, x_3 with $q > 0$; $0 < x_1 < x_2 < x_3$ of

$$q^2 + 9(x_1^2 + x_2^2 + x_3^2) = \delta p$$

where $\delta = 2, 4$ for $p \equiv 2, 1 \bmod 3$. For the first 50 primes, the non zero values of the rank of $\widetilde{\Lambda}$ are:

$$(71,1),\ (79,1),\ (107,1),\ (109,1),\ (113,1),\ (127,1),\ (131,1),$$
$$(137,1),\ (139,1),\ (149,1),\ (151,2),\ (163,1),\ (167,2),\ (173,1),$$
$$(179,2),\ (181,1),\ (191,3),\ (193,2),\ (197,2),\ (199,2),\ (211,3),$$
$$(223,1),\ (227,2),\ (229,1)\,.$$

4.2.6. Strange animals

Let Δ denote any discontinuous group acting on Ω. The ramification index e_q of $\Omega \to \Delta\backslash\Omega$ at a point $q \in \Delta\backslash\Omega$ is the order of the stabilizer of any $Q \in \Omega$ with image q. The point q is called ramified if $e_q > 1$. For the groups $[\Gamma_+]$, $\Lambda = [\Gamma]$, $\tilde{\Lambda}_+$, $\tilde{\Lambda}$ a calculation shows that the number of ramified points is even.

No ramification occurs only for $[\Gamma_+]$ with $p \equiv 1 \bmod 12$ and for $\tilde{\Lambda}_+$ with $p \equiv 1 \bmod 24$. In these cases the rank of the group must be at least 2, since otherwise impossible unramified coverings of the rational or elliptic curve $\Delta\backslash\Omega$ would occur.

Two ramification points occur for $[\Gamma_+]$ with $p \equiv 7 \bmod 12$, for $\tilde{\Lambda}_+$ with $p \equiv 13, 17, 19 \bmod 24$ and for $\Lambda = [\Gamma]$ with $p \equiv 7 \bmod 24$. Again the rank of these groups must be different from 0. Rank 1 occurs for $[\Gamma_+]$ with $p = 7$, for $\tilde{\Lambda}_+$ with $p = 13, 17, 19$ and for $\Lambda = [\Gamma]$ with $p \equiv 7 \bmod 24$ or $p = 13, 37$. Let Δ be a rank one example with only two ramification points for the map $\Omega \to \Delta\backslash\Omega = E$ with E some elliptic curve. The algebraic fundamental group of $E - \{a, b\}$ is the (pro-finite) completion of a free group on three generators, this group maps surjectively to the completion of Δ.

More *surprising* is the application of (3.3) to groups Δ as above having rank 1 (and any number of ramified points). *The quotient of the "upper half plane" Ω by Δ_{tors} is isomorphic to $\mathbb{P}^1_{\mathbb{Q}_p} - \{0, \infty\}$.*

We note that $\Omega \to \mathbb{P}^1_{\mathbb{Q}_p} - \{0, \infty\}$ is ramified above the points of at least two $\Delta/\Delta_{\text{tors}} = \langle q\rangle$-orbits. Here $q \in K^*$ with $0 < |q| < 1$ acts on $\mathbb{P}^1_{\mathbb{Q}_p} - \{0, \infty\}$ by multiplication. For a subgroup N of finite index in Δ_{tors} the quotient $N\backslash(\mathbb{P}^1_{\mathbb{Q}_p} - \{0, \infty\})$ is (in contrast with the complex modular situation) in general not an affine curve. As an example one can take for N the kernel of the map $\Delta_{\text{tors}} \to \mathrm{SO}(3, \mathbb{F}_3)$. The group N has no torsion and can not be finitely generated since Δ_{tors} is not finitely generated. Hence N is a free group on countably many generators and the quotient of $\mathbb{P}^1_{\mathbb{Q}_p} - \{0, \infty\}$ by N is a curve of "infinite genus". It seems unlikely that a more faithful p-adic analogue of the $\mathrm{PSl}(2, \mathbb{Z})$ action on the complex upper half space exists.

4.3. Abhyankar's conjecture

For a function field over a finite field and a given place of the function field one can construct Drinfeld modules of rank 2 and a Drinfeld modular curve. For the general situation we refer to [G1]. We will restrict our attention to the function field $K = \mathbb{F}_q(t)$ with q a power of the prime p and with "$t = \infty$" as choosen place.

The completion of this field is the field of formal Laurent series $\widehat{K} = \mathbb{F}_q((t^{-1}))$. The upper half space Ω is again defined as $\mathbb{P}^1_{\widehat{K}} - \mathbb{P}^1(\widehat{K})$. We consider the group $\Gamma := \mathrm{Gl}(2, \mathbb{F}_q[t])$ acting as a discontinuous group on Ω. This group acts without inversions on the tree of Ω and according to J.-P. Serre [S2, p. 118], the quotient graph is a half line. Using **3.3** one sees that $\Gamma\backslash\Omega$ is isomorphic to $\mathbf{A}^1_{\widehat{K}}$. The same holds for the group $\Gamma(1) := \mathrm{Sl}(2, \mathbb{F}_q[t])$; an explicite calculation of the quotient is made in [GP, ch. 10]. Let f denote a polynomial in $\mathbf{F}_q[t]$ and let $\Gamma(f)$ denote the congruence subgroup defined by $\Gamma(f) = \ker\big(\Gamma(1) \to \mathrm{PSl}(2, \mathbb{F}_q[t]/(f))\big)$. The compactification $X\big(\Gamma(f)\big)$ of the quotient $\Gamma(f)\backslash\Omega$ is a *Drinfeld modular curve*.

Because subgroups of finite index of $\mathrm{Sl}(2, \mathbb{F}_q[t])$ will give examples for the conjecture of S.S. Abhyankar, we give some results concerning Γ. By [S2, pp. 118, 121], one has

$$\mathrm{Gl}(2, k[t]) = \mathrm{Gl}(2, k) *_{B(k)} B(k[t])$$

and a similar statement holds for $\mathrm{Sl}(2, \cdot)$.

Using this one finds for $q \neq 2$ that the commutator subgroup of Γ is $\Gamma(1) := \mathrm{Sl}(2, \mathbb{F}_q[t])$. For $q = 2$ one has that $\Gamma_{ab} = \mathbf{F}_2[t]$. It can be shown that $\Omega/[\Gamma, \Gamma]$ is isomorphic to the affine line over $\mathbf{F}_q((t^{-1}))$ and that the canonical map to Ω/Γ is the division by a lattice $\mathbf{F}_q[t]\tau$.

Further $\Gamma(1)$ is equal to its commutator subgroup except for $q = 2$, 3. In the case of $q = 3$ one has that $\Gamma(1)_{ab} = \mathbf{F}_3[t]$ and it can be shown that the canonical map $\Omega/\big[\Gamma(1), \Gamma(1)\big] = \mathbf{A}^1 \to \Omega/\Gamma(1) = \mathbf{A}^1$ is division by a $\mathbb{F}_3[t]$-lattice of rank one.

From now on we assume that the characteristic is not 2 and that $q > 3$. For convenience we write also $\Gamma(1)$ for the group $\mathrm{PSl}(2, \mathbb{F}_q[t])$. The set of points of Ω with a non trivial stabilizer form one orbit $\Gamma(1)\omega_0$, where ω_0 is a point in $\mathbf{F}_{q^2} - \mathbb{F}_q$. The stabilizer of ω_0 is a cyclic group of order $(q+1)/2$. The map $f : \Omega \to \Gamma(1)\backslash\Omega \cong \mathbf{A}^1$ can be normalized such that $f(\omega_0) = 0$. Hence f is only ramified above 0 and the points above 0 have cyclic ramification of order $(q+1)/2$. For a subgroup $\Delta \subset \Gamma(1)$ of finite index, which contains no elliptic elements of $\Gamma(1)$, the canonical map of the affine curves

$$\Delta\backslash\Omega \to \Gamma(1)\backslash\Omega = \mathbf{A}^1$$

has cyclic ramification of order $(q+1)/2$ at the points above 0 and there is wild ramification above ∞. In order to construct examples for

ABHYANKAR'S CONJECTURE

Every quasi p-group (i.e. a finite group generated by its p-Sylov subgroups) is the Galois group of an unramified covering of the affine line in characteristic p (over some algebraically closed field).

We have to remove the cyclic ramification above. This is done by an analytic version of Abhyankar's lemma.

Define the analytic space Ω^* to be the normalisation of

$$\left\{(\omega,\lambda) \in \Omega \times \mathbf{A}^1 \mid \lambda^{(q+1)/2} = f(\omega)\right\}.$$

Since f has a zero of order $(q+1)/2$ at each point of the orbit $\Gamma(1)\omega_0$ the map $g : \Omega^* \to \Omega$ is unramified of degree $(q+1)/2$ and g becomes a cyclic covering after a quadratic unramified field extension of $\widehat{K}$. An easy calculation with the reductions modulo (t^{-1}) shows that Ω^* is connected and that the irreducible components of its reduction have positive genus. Hence Ω^* is a curve of "infinite genus" in the terminology of 4.1. The action of $\Gamma(1)$ on Ω lifts in an obvious way to Ω^*. The morphism

$$\Gamma(1)\backslash\Omega^* \to \Gamma(1)\backslash\Omega$$

has degree $(q+1)/2$ and is ramified only above 0 (and above ∞). It follows that $\Gamma(1)\backslash\Omega^*$ is isomorphic to $\mathbf{A}^1$ and we have obtained the following result.

PROPOSITION 4.3.1.— *For every subgroup $\Delta \subset \Gamma(1)$ of finite index, the morphism*

$$\Delta\backslash\Omega^* \to \Gamma(1)\backslash\Omega^* \cong \mathbf{A}^1_{\widehat{K}} \quad \textit{is unramified}.$$

Examples and Remarks 4.3.2

Choose $\Delta = \Gamma(f)$. The proposition gives a Galois covering of the affine line over $\widehat{K}$ with group $\mathrm{PSl}(2, \mathbb{F}_q[t]/(f))$ by a (potentially) cyclic covering of $X(\Gamma(f))$ of degree $(q+1)/2$. In particular for $f = t$ one finds $\mathrm{PSl}(2, \mathbb{F}_q)$ as Galois group and the genus of the corresponding curve is $((q-1)/2)^2$. Looking at the reduction modulo (t^{-1}) one also finds the same group as Galois group of a covering of the affine line over $\mathbb{F}_q$. This covering could have been obtained more easily by applying Abhyankar's lemma (in order to remove the tame ramification) to the map

$$\mathbb{P}^1_{\mathbb{F}_q} \to \mathrm{PSl}(2, \mathbb{F}_q) \backslash \mathbb{P}^1_{\mathbb{F}_q} \cong \mathbf{P}^1_{\mathbb{F}_q} .$$

Comparing the above with the quotient of $\mathbf{P}^1_{\mathbb{F}_q}$ by a Borel subgroup of $\mathrm{PSl}(2, \mathbb{F}_q)$ one finds the covering of degree $q+1$ of the projective line by itself given by $x \mapsto ((x^{q+1}+1)/x)$. Apparently, $\mathrm{PSl}(2, \mathbb{F}_q)$ is the Galois group of this equation.

5. Cohomology for a discrete group Γ

THEOREM 5.1

(a) Let $\mathcal{T}$ be a sheaf on Ω provided with a Γ-action. Suppose that $\mathcal{T}$ has trivial cohomology on Ω. Then the canonical map $\alpha : H^1(X, \pi_*^\Gamma \mathcal{T}) \to H^1(\Gamma, \mathcal{T}(\Omega))$ is injective.

(b) The map α is an isomorphism if X has an admissible covering by connected affinoids A such that for a connected component B of $\pi^{-1}A$ with (finite) stabilizer Γ_0 the group $H^1(\Gamma_0, \mathcal{T}(B))$ is 0.

(c) The condition in (b) are satisfied in the following cases:

1) Γ is a Schottky group (and all $\mathcal{T}$).

2) $\mathcal{T}(U)$ is a $\mathbb{Q}$-module for all U.

Proof. — Let $\xi \in H^1(X, \pi_*^\Gamma \mathcal{T})$ be given as a 1-cocycle $(\xi_{i,j})$ of a covering of X by affinoids $\{A_i\}$. Then, after refining the covering $\{A_i\}$, there are elements $f_i \in \mathcal{T}(\pi^{-1}A_i)$ such that $f_i - f_j = \xi_{i,j}$ on $\pi^{-1}(A_i \cap A_j)$. For $\gamma \in \Gamma$ one can glue the sections $\gamma f_i - f_i$ to an element $c(\gamma) \in \mathcal{T}(\Omega)$ and clearly $\gamma \mapsto c(\gamma)$ is a 1-cocycle for $H^1(\Gamma, \mathcal{T}(\Omega))$. If this cocycle happens to be trivial then one easily sees that $\xi = 0$. This proves the first statement.

Let $c : \gamma \mapsto c(\gamma)$ represent an element of $H^1(\Gamma, \mathcal{T}(\Omega))$. Choose a covering of X by connected affinoids $\{A_i\}$ with the property of the statement. Write $\pi^{-1}(A_i)$ as a disjoint union $\bigcup_{\gamma\in\Gamma/\Gamma_i} \gamma B_i$ where B_i is a connected affinoid set with finite stabilizer Γ_i. Then we consider the image c_i of c in $H^1(\Gamma, \mathcal{T}(\pi^{-1}A_i))$. The Γ module $\mathcal{T}(\pi^{-1}A_i)$ is induced by the Γ_i module $\mathcal{T}(B_i)$. According to [S1, exercice p. 125], one has that $H^1(\Gamma, \mathcal{T}(\pi^{-1}A_i))$ is isomorphic to $H^1(\Gamma_i, \mathcal{T}(B_i))$. The latter group is supposed to be 0 and this yields elements $f_i \in \mathcal{T}(\pi^{-1}A_i)$ satisfying $c(\gamma) = \gamma f_i - f_i$ on $\pi^{-1}A_i$. The elements $f_i - f_j \in \mathcal{T}(\pi^{-1}(A_i \cap A_j))$ are Γ-invariant. The image ξ of the 1-cocycle $(f_i - f_j)$ in $H^1(X, \pi_*^\Gamma \mathcal{T})$ is the element which is mapped onto c.

Finally, a finite group acting on a $\mathbb{Q}$ vector space has trivial cohomology according to [S1, proposition 4, p. 138].

COROLLARY 5.2

Suppose that the quotient X is quasi algebraic. Then the genus of $\widehat{X}$ is equal to the dimension of the $\mathbb{Q}$-vector space $\Gamma_{ab} \otimes \mathbb{Q}$.

Proof.— The statement is already proven in (3.3). We will now give another proof using Γ cohomology.

Apply theorem (5.1) to the sheaf $\mathcal{T} = \mathbb{Q}_\Omega$, the constant sheaf on Ω with stalk $\mathbb{Q}$ and obvious Γ-action. Then $\pi_*^\Gamma \mathcal{T} = \mathbb{Q}_X$ and so $H^1(X, \mathbb{Q}) = \mathrm{Hom}(\Gamma, \mathbb{Q})$. By $\widehat{X}$ we denote the curve satisfying $X = \widehat{X} - \mathcal{L}$. As remarked before, $\widehat{X}$ is a Mumford curve of some genus g, and as a consequence $\widehat{X}$ has a uniformization by a Schottky group free of rank g. Applying the theorem for this uniformization and the constant sheaf $\mathbb{Q}$ one finds $g = \dim H^1(\widehat{X}, \mathbb{Q})$. Using the fact that any subset $A - \mathcal{M}$ of $\mathbf{P}^1_K$, where A is affinoid and $\mathcal{M}$ is compact has trivial cohomology for any constant sheaf one finds $H^1(\widehat{X}, \mathbb{Q}) = H^1(X, \mathbb{Q})$. This proves the statement. □

Remark.— —The result above has been proved in [GP, ch. 8, section 4] for finitely generated discontinuous groups. For other groups, U.E. Gekeler [G2] has given a proof.

COROLLARY 5.3

$H^1(X, M_X^*) \to H^1(\Gamma, M(\Omega)^*)$ is an isomorphism.

Further $H^1(X, M_X^*) = 0$ for every quasi algebraic curve X.

Proof.— Let B denote a connected component of $\pi^{-1}A$ with finite stabilizer Γ_0 where A is any connected affinoid of X. Then Γ_0 is the Galois group of the field extension $M(A) \to M(B)$. Hilbert 90 proves then $H^1(\Gamma_0, M(B)^*) = 0$. Apply now theorem (5.1) for the first statement.

Suppose that X is quasi algebraic. From the exact sequence of sheaves

$$0 \to O^* \to M^* \to \mathrm{Div} \to 0$$

one obtains a surjective map $H^1(X, O^*) \to H^1(X, M^*)$. Hence the latter group will be 0 if we can show that every line bundle L on X has a non trivial meromorphic section. We think that this is the case for any connected, non singular, one-dimensional analytic space. The proof for a quasi analytic X is very easy. Let the embedding $X \subset \widehat{X}$ be given. The line bundle L on X extends to a line bundle $\widehat{L}$ on $\widehat{X}$. Indeed, any line bundle on an analytic set of the form $A - \mathcal{M}$, with A an affinoid subset of the projective line and $\mathcal{M}$ compact, is trivial and hence extends to A.

By GAGA the line bundle $\widehat{L}$ is algebraic and has a non zero rational section above $\widehat{X}$. The restriction of this section to X is a non zero meromorphic section of L.

Definition 5.4

A (meromorphic) automorphic function for Γ is an element $f \in M(\Omega)^*$ such that $(\gamma^* f)f^{-1} = c(\gamma) \in K^*$ holds for every $\gamma \in \Gamma$. The homomorphism $\gamma \mapsto c(\gamma)$ is called the automorphy factor. The element f is called a theta functionfor Γ if $f \in O(\Omega)^*$.

Corollary 5.5

Suppose that $X = \Gamma \backslash \Omega$ is quasi algebraic. For every homomorphism $c \in \mathrm{Hom}(\Gamma, K^*)$ there exists a automorphic function f with automorphy factor c. This f is unique upto an element of $M(X)^*$.

Proof.— — Define the group $\mathcal{Q}$ by the exact sequence

$$0 \to K^* \to M(\Omega)^* \to \mathcal{Q} \to 0.$$

After taking invariants under Γ one finds the exact sequence:

$$M(X)^* \to \mathcal{Q}^\Gamma \to \mathrm{Hom}(\Gamma, K^*) \to H^1(\Gamma, M(\Omega)^*) \ldots$$

The statement follows now from corollary 5.3 and the observation that $\mathcal{Q}^\Gamma$ is equal to the group of automorphic functions modulo the constant functions.

6. Theta functions for a Schottky group

In this section we assume that the group Γ *is a Schottky group of rank* g. For convenience we take K to be algebraically closed. This enables us to identify any reduced algebraic (or analytic) variety over K with its set of points over K.

Let $\gamma_1, \ldots, \gamma_g$ denote free generators for the group. As in 3.1 one defines $\mathcal{L}$, Ω, T and $X := \Gamma\backslash\Omega$. The curve X is a Mumford curve of genus g and $G := \Gamma\backslash T$ is a finite graph with Betti number g. Cohomology for such a group can be calculated as follows: Let M denote a Γ-module. One forms the complex

$$d : M \to M^g \quad \text{with } d(m) = (m - \gamma_1(m), \ldots, m - \gamma_g(m)) \text{ for } m \in M\,.$$

Then $M^\Gamma = H^0(\Gamma, M) = \ker(d)$ and $H^1(\Gamma, M) = \operatorname{coker}(d)$ and

$$H^i(\Gamma, M) = 0 \quad \text{for} \quad i \geq 2\,.$$

The sequence of 2.1 induces an exact sequence

$$0 \to C(T)^\Gamma \to \operatorname{Hom}(\Gamma, K^*) \to H^1\big(\Gamma, O(\Omega)^*\big) \to H^1\big(\Gamma, C(T)\big) \to 0\,.$$

The term $H^1\big(\Gamma, O(\Omega)^*\big)$ equals $H^1(X, O_X^*)$ according to 5.1. The elements of the last group represent (equivalence classes of) analytic line bundles on X. By GAGA analytic line bundles are algebraic and hence the group is equal to $\operatorname{Pic}(X)$.

Let $\Theta(\Gamma)$ *denote the group of theta functions for the group* Γ. Then by definition, one has the following exact sequence:

$$0 \to K^* \to \Theta(\Gamma) \to C(T)^\Gamma \to 0\,.$$

For any locally finite graph G we denote by $A(G)$ the group of functions μ on the oriented edges of G with values in $\mathbb{Z}$, satisfying $\mu(-\overrightarrow{e}) = -\mu(\overrightarrow{e})$. Let $H(G)$ denote the group of all functions on the vertices of G with values in $\mathbb{Z}$ and let $d : A(G) \to H(G)$ be given by $d(\mu)(a) = \sum_{\overrightarrow{e}(0)=a} \mu(\overrightarrow{e})$. Clearly $\ker(d) = C(G)$, the group of currents on G.

LEMMA 6.1

Let G be a finite, connected graph. There exists an exact sequence

$$0 \to C(G) \to A(G) \xrightarrow{d} H(G) \xrightarrow{\phi} \mathbb{Z} \to 0$$

and there exists a canonical isomorphism $\pi_1(G)_{ab} \xrightarrow{\sim} C(G)$.

Proof.— The lemma is well known. The map ϕ is given by $\phi(f) = \sum_{a\in G} f(a)$. The non trivial part is to prove that ker $\phi \subset$ im d. Let G^* denote a maximal subtree of G, then

$$0 \to A(G^*) \to H(G^*) \to \mathbb{Z} \to 0$$

is exact; $A(G^*)$ can be seen as subgroup of $A(G)$; $A(G)/A(G^*) \cong \mathbb{Z}^g$ where g is the number of edges of G not belonging to G^* and $H(G^*) = H(G)$. This proves the first statement of the lemma.

Let γ be a closed path in G. One can describe γ by a sequence of oriented edges $\{\overrightarrow{e}_1, \ldots, U\overrightarrow{e}_n\}$ satisfying $\overrightarrow{e}_i(1) = \overrightarrow{e}_{i+1}(0)$ for $i = 1, \ldots, n-1$ and where $\overrightarrow{e}_n(1) = \overrightarrow{e}_1(0)$. One associates to γ the current μ_γ given by:

$$\mu_\gamma(\overrightarrow{e}) = \sum_{i=1}^{n} \{\overrightarrow{e}_i, \overrightarrow{e}\}$$

where

$$\{\overrightarrow{e}, \overrightarrow{f}\} = 1, -1, 0 \quad \text{for } \overrightarrow{e} = \overrightarrow{f}, \overrightarrow{e} = -\overrightarrow{f}, \overrightarrow{e} \neq \pm\overrightarrow{f}\,.$$

One easily sees that μ_γ is a current only depending on the homology class of γ and that $\mu_{\gamma_1\gamma_2} = \mu_{\gamma_1} + \mu_{\gamma_2}$. In order to see that the resulting map $\pi_1(G)_{ab} \to C(G)$ is an isomorphism, one uses again the maximal subtree G^*. Let $\{\overrightarrow{e}_1, \ldots, \overrightarrow{e}_g, -\overrightarrow{e}_1, \ldots, -\overrightarrow{e}_g\}$ denote the set of oriented edges belonging to G and not to G^*. For every $\overrightarrow{e}$ in this set, one constructs the closed path $\gamma(\overrightarrow{e}) = \{\overrightarrow{e}, \overrightarrow{f}_1, \ldots, \overrightarrow{f}_s\}$ where $\{\overrightarrow{f}_1, \ldots, \overrightarrow{f}_s\}$ is the shortest path in G^* from $\overrightarrow{e}(1)$ to $\overrightarrow{e}(0)$. Then $\{\gamma(\overrightarrow{e}_1), \ldots, \gamma(\overrightarrow{e}_g)\}$ is a free basis of $\pi_1(G)_{ab}$ and the $\mu_i = \mu_{\gamma(\overrightarrow{e}_i)}$ have the property $\mu_i(\overrightarrow{e}_j) = \delta_{i,j}$. Let μ be any current of G. Then $\widetilde{\mu} = \mu - \sum_{i=1}^{g} \mu(\overrightarrow{e}_i)\mu_i$ satisfies $\widetilde{\mu}(\overrightarrow{e}_i) = 0$ for all i. Hence $\widetilde{\mu}$ is also a current for G^* and so $\widetilde{\mu} = 0$. This shows that $\pi_1(G)_{ab} \cong C(G)$.

LEMMA 6.2

Let G be an infinite (locally finite) tree and let E denote the collection of the ends of G. This set has a natural topology as 0-dimensional compact space. The group of currents $C(G)$ on G can be identified as the group of finite additive measures m on E with values in $\mathbb{Z}$ and satisfying $m(E) = 0$. The following sequence is exact:

$$0 \to C(G) \to A(G) \xrightarrow{d} H(G) \to 0 .$$

Proof.— For the surjectivity of d one needs a combinatorical argument. Let $V \in H(G)$, choose a vertex a of G. We try to find $f \in A(G)$ with $df = V$. An edge e is called finite if one of the two components of the graph $G - \{e\}$ is finite. Consider the $\overrightarrow{e}$ with $\overrightarrow{e}(0) = a$. For a finite $\overrightarrow{e}$ the values of f on the finite component corresponding to e is uniquely determined. For the infinite $\overrightarrow{e}$ with $\overrightarrow{e}(0) = a$ we choose values $f(\overrightarrow{e})$ such that at $df(a) = V(a)$. Now we proceed with the same method applied to the vertices $\overrightarrow{e}(1)$, where $\overrightarrow{e}$ are the infinite edges with $\overrightarrow{e}(0) = a$. This proces gives the desired f. The second part of the statement is analoguous to 2.1.1.

LEMMA 6.3

(1) $C(T)^\Gamma = C(\Gamma\backslash T)$ and is canonical isomorphic to Γ_{ab}.

(2) $H^1(\Gamma, C(T)) = \mathbb{Z}$ and the map $H^1(\Gamma, O(\Omega)^*) \to H^1(\Gamma, C(T))$ coincides with the degree map $\mathrm{Pic}(X) \to \mathbb{Z}$.

(3) There is a canonical exact sequence:

$$0 \to \Gamma_{ab} \xrightarrow{c} \mathrm{Hom}(\Gamma_{ab}, K^*) \to \mathrm{Pic}^0(X) \to 0 .$$

Proof.— (1) follows form 6.1. For (2) we consider the exact sequence of Γ modules:

$$0 \to C(T) \to A(T) \to H(T) \to 0 .$$

The modules $A(T)$ and $H(T)$ are induced Γ modules and the cohomology sequence reads:

$$0 \to C(\Gamma\backslash T) \to A(\Gamma\backslash T) \to H(\Gamma\backslash T) \to H^1(\Gamma, C(T)) \to 0 .$$

According to 6.1 the last group is isomorphic to $\mathbb{Z}$. The last part of (2) is easily verified. Part (3) follows from (1) and (2) applied to the sequence:

$$0 \to C(T)^\Gamma \to \mathrm{Hom}(\Gamma, K^*) \to H^1(\Gamma, O(\Omega)^*) \to H^1(\Gamma, C(T)) \to 0 .$$

THEOREM 6.4

Let $\langle \cdot , \cdot \rangle$ denote the bilinear form on Γ_{ab} with values in K^* derived from c in 6.3.3.

(1) $\langle \cdot , \cdot \rangle$ is symmetric.

(2) $(\cdot , \cdot) := -\log\left|\langle \cdot , \cdot \rangle\right|$ is a real bilinear, symmetric and positive definite form.

(3) The isomorphism of the groups

$$\mathrm{Hom}(\Gamma_{ab}, K^*)/c(\Gamma_{ab}) \to \mathrm{Pic}^0(X)$$

is an isomorphism of analytic varieties.

Proof.— (1) Let v, $w \in \Gamma$ with images $\overline{v}$, $\overline{w} \in \Gamma_{ab}$ and let $u_w \in \Theta(\Gamma)$ denote an element which has the same image as w in $C(\Gamma\backslash T)$. Then $\langle \overline{w}, \overline{v}\rangle = v^*(u_w)(u_w)^{-1}$. In [GP, p. 190], the symmetry of this expression is proved by using an explicit formula for u_w. Suppose that $\infty \in \Omega$ then

$$u_w = \prod_{\gamma\in\Gamma} \frac{z - \gamma w(a)}{z - \gamma(a)}$$

where $a \in \Omega$ is a point not lying in the Γ-orbit of ∞. It is not difficult to see that u_w is indeed the element of $\Theta(\Gamma)$ with image $\overline{w} \in C(\Gamma\backslash T)$ normalized by $u_w(\infty) = 1$. By considering u_w and similar expressions as functions of two variables z, a, the symmetry comes out. We do not know how to translate this trick in terms of Γ-cohomology.

(2) The form $(\cdot , \cdot)$ will be given explicitely in terms of the graph $\Gamma\backslash T$ and a function h on its edges, given by $h(e) = -\log|\pi_e|$, where π_e is defined as the element appearing in the local equation $(xy - \pi_e)$ above the double point corresponding to e.

Let $\mathrm{Pot}(T)$ denote the group of potentials on T, *i.e.* the real functions V on the vertices of T such that the expression $dV \in A(T)$, given by the formula

$$dV(\overrightarrow{e}) = \frac{V(\overrightarrow{e}(1)) - V(\overrightarrow{e}(0))}{-\log|\pi_e|},$$

is actually a current on T. The exact sequence of Γ-modules 2.1 has a morphism to the exact sequence of Γ-modules:

$$0 \to \mathbb{R} \to \mathrm{Pot}(T) \to C(T) \to 0$$

given by taking $-\log|\cdot|$. This shows that $(\cdot,\cdot)$ is derived from the cohomology sequence:

$$0 \to C(\Gamma\backslash T) \to \mathrm{Hom}(\Gamma, \mathbb{R}) \to H^1(\Gamma, \mathrm{Pot}(T)) \to \cdots$$

Let μ denote a current of T and let p_0 denote a fixed vertex of T. A potential V with $dV = \mu$ can easily be given by integration as follows: let q denote a vertex and let $\{\vec{e}_1, \ldots, \vec{e}_n\}$ denote the shortest path from p_0 to q. Then

$$V(q) = \sum_{i=1}^{n} h(e_i)\mu(\vec{e}_i).$$

For $\gamma \in \Gamma$ we denote by V_γ the potential with dV_γ is equal to the current $\mu_\gamma \in C(T)$ corresponding to γ. One finds:

$$(\gamma_1, \gamma_2) = V_{\gamma_1}(\gamma_2 p_0) - V_{\gamma_1}(p_0) = \sum_{i=1}^{n} h(e_i)\mu_{\gamma_1}(\vec{e}_i)$$

where $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is the shortest path from p_0 to $\gamma_2 p_0$. Since μ_{γ_1} is Γ-invariant, one can consider this formula as an expression in $\Gamma\backslash T$ and replace p_0 and $\vec{e}_i$ by their images in $\Gamma\backslash T$. Hence:

$$(\gamma_1, \gamma_2) = \sum_{i=1}^{n} h(e_i)\mu_{\gamma_1}(e_i) = \sum_{i=1}^{n}\sum_{j=1}^{n} h(e_i)\{\vec{e}_i, \vec{f}_j\}$$

where $\{\vec{f}_1, \ldots, \vec{f}_n\}$ is the path in $\Gamma\backslash T$ corresponding to γ_1. Moreover

$$\sum_{\vec{e}\in\Gamma\backslash T} \{\vec{e}_i, \vec{e}\}\{\vec{f}_j, \vec{e}\} = 2\{\vec{e}_i, \vec{f}_j\}.$$

This gives finally the formula:

$$(\gamma_1, \gamma_2) = \frac{1}{2} \sum_{\vec{e}\in\Gamma\backslash T} \mu_{\gamma_1}(\vec{e})\mu_{\gamma_2}(\vec{e})h(e)$$

and proves (2).

(3) can be proved as follows. Let $\mathcal{T}$ denote the algebraic torus over K with character group Γ_{ab}. On $\mathcal{T}$ there is a canonical analytic structure. One considers the line bundle

$$\mathbf{A}^1 \times \mathcal{T} \times \Omega \to \mathcal{T} \times \Omega$$

with Γ-action given by $\gamma(a,t,\omega) = \left(\overline{\gamma}(t)a, t, \gamma(\omega)\right)$, where $\overline{\gamma}$ denotes the image of γ in Γ_{ab}. Dividing by the Γ-action one finds a line bundle $L \to \mathcal{T} \times X$. Using the universal property of the Poincaré bundle $\mathcal{P}$ on $\mathrm{Pic}^0(X) \times X$ (this property remains valid in the category of analytic spaces) one finds an analytic morphism $\mathcal{T} \to \mathrm{Pic}^0(X)$. This morphism coincides with the map on the K valued points $\mathrm{Hom}(\Gamma_{ab}, K^*) \to \mathrm{Pic}^0(X)$ of **6.3.3**. The morphism is surjective and its kernel $\Lambda := c(\Gamma_{ab})$ is discrete in $\mathcal{T}$ since the form is positive definite. Therefore the quotient $\mathcal{T}/\Lambda$ has a natural structure as analytic variety. This variety is then isomorphic to $\mathrm{Pic}^0(X)$. We note that the symmetry of $\langle\,\cdot\,,\,\cdot\,\rangle$ and the fact that $(\,\cdot\,,\,\cdot\,)$ is positive definite also proves that the analytic torus $\mathcal{T}/\Lambda$ is an abelian variety (see [FP, ch. 6]). In our situation this is obvious.

Remarks 6.5

(1) In the case of a field K with a discrete valuation, it is natural to normalize $-\log|\cdot|$ to be the additive valuation of K. The edges e of the graph $\Gamma\backslash T$ are then given a weight $h(e) \in \mathbb{N}$ and one finds in **6.4** certain positive definite quadratic forms over the integers. One can ask if all positive definite quadratic forms occur in this way. For $g = 1, 2$ (and probably 3) one finds all such forms. For $g > 3$ there are more positive definite quadratic forms than those constructed with Schottky groups. They do occur for principally polarized abelian varieties over K having a multiplicative reduction. Hence the question above is related to the non surjectivity of the Torelli map for $g > 3$. This theme is worked out in [G].

(2) We use again the notations Γ, Ω, X, $\mathcal{T}$ as above. Fix a point $\omega_0 \in \Omega$ and let $x_0 \in X$ be its image. Normalize the theta functions u_γ by $u_\gamma(\omega_0) = 1$. The map $\Omega \xrightarrow{\phi} \mathcal{T}$, is given by $\phi(\omega) = \left\{\overline{\gamma} \mapsto u_\gamma(\omega)\right\}$. After dividing out the action of Γ one finds the usual immersion $X \to \mathrm{Pic}^0(X)$ which sends x_0 to $0 \in \mathrm{Pic}^0(X)$. The preimage of $X \subset \mathrm{Pic}^0(X)$ in $\mathcal{T}$ is an analytic covering of X with group Γ_{ab}. This preimage coincides with the image of the map ϕ. From this we draw the conclusion that ϕ induces a closed immersion of the curve with "infinite genus" (see 4.1) $[\Gamma,\Gamma]\backslash\Omega$ into $\mathcal{T}$.

7. Theta functions for general groups

It is again convenient to worke over an algebraically closed field K. Let Γ denote a discontinuous group. We will assume that its rank g is finite and that Γ is not a finite group. We will use the following notations.

- The set of limit points of Γ is the set $\mathcal{L}$; $\Omega = \mathbb{P}^1 - \mathcal{L}$; T is the tree $T(\mathcal{L}_*)$, where $\mathcal{L}_*$ is a compact, Γ-invariant set having $\mathcal{L}$ as set of limit points choosen such that Γ acts without inversions on T; Y is the graph $\Gamma\backslash T$; $X = \Gamma\backslash\Omega$; $\Theta(\Gamma)$ is the group of theta functions for Γ *i.e.* the invertible functions f on Ω with $(\gamma^* f)(f)^{-1} \in K^*$ for all $\gamma \in \Gamma$.
- Γ_{tors} is the normal subgroup of Γ generated by the elements of finite order; according to **3.3** $\Omega_{\text{tors}} := \Gamma_{\text{tors}}\backslash\Omega \cong \mathbb{P}^1 - \mathcal{M}$ and $T_{\text{tors}} := \Gamma_{\text{tors}}\backslash T$ is a tree for $\mathbb{P}^1 - \mathcal{M}$ corresponding to some Γ-invariant set $\mathcal{M}_*$ having $\mathcal{M}$ as its set of limit points.
- $\Gamma_1 = \Gamma/\Gamma_{\text{tors}}$ is a Schottky group (if $g > 0$) according to (**3.3**); the set of limit points of Γ_1 is $\mathcal{L}_1$; this set is contained in $\mathcal{M}$ and $\Omega_1 := \mathbb{P}^1 - \mathcal{L}_1 \supset \mathbb{P}^1 - \mathcal{M}$; the tree T_1 is the tree $T(\mathcal{L}_{1,*})$, where $\mathcal{L}_{1,*}$ is choosen to be a Γ-invariant set contained in $\mathcal{M}_*$ and having $\mathcal{L}_1$ as its set of limit points. $\widehat{X} := \Gamma_1\backslash\Omega_1$ and according to **3.3** $X \subset \widehat{X}$ and the complement of X in $\widehat{X}$ is a compact set; further $\Theta(\Gamma_1)$ is the group of theta functions (defined on Ω_1) for the group Γ_1. Let Y_1 denote the graph $\Gamma_1\backslash T_1$.

The aim of this section is to describe the uniformization of the Jacobian variety $\text{Jac}(\widehat{X}) = \text{Pic}^0(\widehat{X})$ *of* $\widehat{X}$ *in terms of the theta functions for* Γ. We note that $\Theta(\Gamma_1) \subset \Theta(\Gamma)$ and that, according to section 6, this uniformization is given by the exact sequence:

$$0 \to K^* \to \Theta(\Gamma_1) \to \text{Hom}(\Gamma_1, K^*) \to \text{Pic}^0(\widehat{X}) \to 0\,.$$

Further $\overline{\Gamma} := (\Gamma_1)_{ab} = \Gamma_{ab}/\{\text{its torsion subgroup}\}$ and the symmetric bilinear form $\overline{\Gamma} \times \overline{\Gamma} \to K^*$ is the essential part of the uniformization. In locating $\Theta(\Gamma_1)$ as subgroup of $\Theta(\Gamma)$ there are two obstructions: the possible torsion in Γ and the possibility that X is not a complete curve. The first obstruction is responsible for subgroups of finite index, the second obstruction is overcome by the method of [GR] of considering currents (*et cetera*) with compact support.

The natural injection $O(\Omega_{\text{tors}})^* \to O(\Omega)^*$ induces an injection

$$C(T_{\text{tors}}) \to C(T).$$

This leads to defining natural maps $A(T_{\text{tors}}) \to A(T)$; $H(T_{\text{tors}}) \to H(T)$ as follows.

Let $m(e)$ denote the order of the stabilizer in Γ_{tors} of any edge $\overrightarrow{f}$ of T lying above the edge $\overrightarrow{e}$ of T_{tors}. And let $m(a)$ denote the order of the stabilizer of any vertex b of T lying above the vertex a of T_{tors}. For $g \in A(T_{\text{tors}})$, the image $G \in A(T)$ is given by $G(\overrightarrow{f}) = m(e)g(\overrightarrow{e})$ where $\overrightarrow{e} \in T_{\text{tors}}$ is the image of $\overrightarrow{f}$. The map $H(T_{\text{tors}}) \to H(T)$ is defined in a similar way. For $d \in H(T_{\text{tors}})$, the image $D \in H(T)$ is given by $D(b) = m(a)d(a)$ where b has image a in T_{tors}.

One obtains in this way an injection of the exact sequence $0 \to C(T_{\text{tors}}) \to A(T_{\text{tors}}) \to H(T_{\text{tors}})$ into the exact sequence $0 \to C(T) \to A(T) \to H(T)$. The exact sequence $0 \to C(Y) \to A(Y) \to H(Y)$ is obtained from the first sequence by taking the invariants under the action of the group Γ_1. As a consequence $0 \to C(Y) \to A(Y) \to H(Y)$ is mapped injectively in the exact sequence $0 \to C(T)^{\Gamma} \to A(T)^{\Gamma} \to H(T)^{\Gamma}$. In particular $C(Y)$ is a submodule of $C(T)^{\Gamma}$.

Let $C(Y)_c$, $A(Y)_c$, $H(Y)_c$ denote the subgroups of elements with finite support in the corresponding groups. The homology groups of the graph Y with coefficients in $\mathbb{Z}$ is given by the exact sequence:

$$0 \to H_1(Y, \mathbb{Z}) \to A(Y)_c \to H(Y)_c \to H_0(Y, \mathbb{Z}) \to 0.$$

Of course $H_0(Y, \mathbb{Z}) = \mathbb{Z}$ and $C(Y)_c = H_1(Y, \mathbb{Z}) = \overline{\Gamma} \cong \mathbb{Z}^g$ since Y is obtained from the tree $T_{\text{tors}} = \Gamma_{\text{tors}} \backslash T$ by dividing out the free action of the group Γ_1.

Let Y_c denote the finite subgraph of Y supporting all the currents $C(Y)_c$. By definition $C(Y)_c = C(Y_c)$ and one can identify the exact sequence $0 \to C(Y)_c \to A(Y)_c \to H(Y)_c$ with the sequence $0 \to C(Y_c) \to A(Y_c) \to H(Y_c)$.

Let $C(T)_c^{\Gamma}$ denote the subgroup of $C(T)^{\Gamma}$ consisting of the currents which have a finite support modulo the action of Γ. Similarly, one defines

$$A(T)_c^{\Gamma} \subset A(T)^{\Gamma} \quad \text{and} \quad H(T)_c^{\Gamma} \subset H(T)^{\Gamma}.$$

One can identify $A(T)_c^{\Gamma}$ with $A(Y_c)' :=$ the group of alternating functions f on the oriented edges of Y_c such that $f(\overrightarrow{e}) \in ({}^1/_{m(e)})\mathbb{Z}$ for every edge. In the same way, $H(T)_c^\Gamma$ can be identified with $H(Y_c)' :=$ the group of functions f on the vertices of Y_c such that $f(a) \in ({}^1/m(a))\mathbb{Z}$ for every vertex a. Put $A(Y_c)'' := A(Y_c)'/A(Y_c) =$ the group of alternating functions f on the edges of Y_c such that $f(\vec{e}) \in ({}^1/m(e))\mathbb{Z}/\mathbb{Z}$ and similarly $H(Y_c)'' := H(Y_c)'/H(Y_c)$. There is an obvious boundary map $d'' : A(Y_c)'' \to H(Y_c)''$ and an exact sequence of complexes:

$$\begin{aligned} 0 \to \{A(Y_c) \to H(Y_c)\} \to \{A(Y_c)' \to H(Y_c)'\} \\ \to \{A(Y_c)'' \to H(Y_c)''\} \to 0\,. \end{aligned}$$

This induces an exact sequence

$$0 \to C(Y_c) \to C(T)_c^\Gamma \to \ker d'' \to \mathbb{Z}\,.$$

PROPOSITION 7.1

(1) The graph Y_c is a subdivision of the graph Y_1.

(2) The image of $\Theta(\Gamma_1)/K^*$ is the subgroup of finite index oo $C(Y)_c = C(Y_c)$ of $C(T)_c^\Gamma \subset \Theta(\Gamma)/K^*$.

(3) The quotient $C(T)_c^\Gamma/C(Y_c)$ is equal to $\ker d''$.

Proof.— First some generalities. Let $\mathcal{M}_1 \subset \mathcal{M}_2$ denote compact subsets of $\mathbf{P}^1$. Then there is a canonical injective map from the set of vertices of $T(\mathcal{M}_1)$ to the set of vertices of $T(\mathcal{M}_2)$. An edge of the first tree need not be an edge of the second tree. Let S denote the subtree of the second tree spanned by the vertices of the first tree. One has: a vertex s of $T(\mathcal{M}_2)$ lies in S if and only if s lies on the shortest path between two vertices of $T(\mathcal{M}_1)$. Hence S is a subdivision of $T(\mathcal{M}_1)$. Further $S = T(\mathcal{M}_2)$ if and only if the sets $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same set of limit points.

We apply this to $\mathcal{L}_{1,*} \subset \mathcal{M}_*$ and we divide by the action of Γ_1. Then the subdivision $\Gamma_1\backslash S$ of Y_1 is a subgraph of Y, this subgraph is the support of $C(Y)_c$ and hence equal to Y_c. This proves (1).

We note further that $\Theta(\Gamma_1)/K^* = C(Y_1)$ (according to 6.3) and so $C(Y_1) = C(Y_c)$. The rest of the proposition follows easily with the help of the exact sequences above.

Remarks 7.2

The map $\overline{\Gamma} \to \Theta(\Gamma)$ can be made explicit by using infinite products. Take $w \in \Gamma$ and assume for the sake of the formula that $\infty \in \Omega$ and let $a \in \Omega$ be a point not lying in the Γ-orbit of ∞. Then the expression

$$u_w := \prod_{\gamma\in\Gamma} \frac{z-\gamma w a}{z-\gamma a}$$

is easily seen to belong to $O(\Omega)^*$, to be independent of the choice of a and to have as current the image of w in $C(T)_c^{\Gamma}$. In particular u_w does only depend on the image $\overline{w}$ of w in $\overline{\Gamma}$ and we can write $u_{\overline{w}} = u_w$. The map $\overline{w} \mapsto u_{\overline{w}}$ is the desired explicit map.

COROLLARY 7.3

(1) $f \in \Theta(\Gamma)$ belongs to $\Theta(\Gamma_1)$ if and only if the current of f belongs to $C(T)_c^{\Gamma}$ and the automorphy factor of f is trivial on the torsion elements of Γ_{ab}.

(2) There is an injective map $C(T)_c^{\Gamma}/C(Y_c) \to \mathbf{Hom}((\Gamma_{ab})_{\mathrm{tors}}, K^*)$.

(3) If the characteristic of K is $p \neq 0$ then the finite group $C(T)_c^{\Gamma}/C(Y_c)$ has no p-torsion.

Proof.— The "only if" part of (1) is trivial. Let f satisfy the conditions above. Since f is Γ_{tors}-invariant, f can be seen as an invertible function on $\Gamma_{\mathrm{tors}}\backslash\Omega = \mathbb{P}^1 - \mathcal{M}$. It follows that the current μ of f on T satisfies $\mu(\vec{\tilde{e}}) \in m(e)\mathbb{Z}$ (where $\vec{\tilde{e}}$ is mapped onto $\vec{e} \in T_{\mathrm{tors}}$). Hence $\mu \in C(Y_c)$ and $f \in \Theta(\Gamma_1)$. The other statements follow at once from (1).

COROLLARY 7.4

(1) The injective maps

$$\mathbf{Div}(X) \to \mathbf{Div}(\Omega)^{\Gamma} \quad \textit{and} \quad H^1(X, O_X) \to H^1(\Gamma, O(\Omega)^*)$$

have a cokernel isomorphic to $\prod_{x\in X} \mathbb{Z}/\mathbb{Z}e_x$, where e_x denotes the order of the stabilizer of any point $\omega \in \Omega$ with image x.

(2) If X is an algebraic curve then the analytic cohomology group $H^1_{\mathrm{an}}(X, O_X^*)$ coincides with the algebraic cohomology group

$$H^1_{\mathrm{alg}}(X, O_X^*).$$

Further $O_{\mathrm{an}}(X)^* = O_{\mathrm{alg}}(X)^*$.

Proof.— (1) Consider the exact sequence of Γ-modules:

$$0 \to O(\Omega)^* \to M(\Omega)^* \to \mathrm{Div}(\Omega) \to 0$$

and its cohomology sequence, using 5.3:

$$0 \to O(X)^* \to M(X)^* \to \mathrm{Div}(\Omega)^\Gamma \to H^1\big(\Gamma, O(\Omega)^*\big) \to 0\,.$$

The injective map

$$\mathrm{Div}(X) \to \mathrm{Div}(\Omega)^\Gamma$$

has clearly cokernel $\prod_{x\in X} \mathbb{Z}/\mathbb{Z}e_x$. Comparing this with the exact sequence

$$0 \to O(X)^* \to M(X)^* \to \mathrm{Div}(X) \to H^1(X, O_X^*) \to 0$$

one finds that the injective map $H^1(X, O_X^*) \to H^1\big(\Gamma, O(\Omega)^*\big)$ has the same cokernel.

(2) If X happens to be complete, this is of course a consequence of GAGA. Suppose now that X is an affine curve with compactification $\widehat{X}$. The map $H^1_{\mathrm{alg}}(X, O_X^*) \to H^1_{\mathrm{an}}(X, O_X^*)$ is surjective since any analytic line bundle on X has an extension to an analytic, and hence algebraic, line bundle on $\widehat{X}$. Suppose that the (algebraic) line bundle L on $\widehat{X}$ is trivial as analytic line bundle on X. Then L has a nowhere vanishing analytic section f above X. Take a point $q \in \widehat{X} - X$, a local parameter z at q and a local generator e of L at q. Then $f = (\sum_{n=-\infty}^{\infty} a_n z^n)e$ and this infinite sum converges and has no zeros for $0 < |z| \le \epsilon$. It follows that the infinite sum can only have finitely many negative terms. Hence the analytic section f extends to a meromorphic analytic section of L on all of $\widehat{X}$. Such a section is by GAGA a rational section of L with its poles and zeros in $\widehat{X} - X$. Hence the restriction to X of L as algebraic line bundle is also trivial. The second statement of (2) has a similar proof.

Remarks and Examples 7.5

(1) Suppose that X is an affine algebraic curve. Comparing the Γ_1-cohomology of the two exact sequences

$$0 \to K^* \to O(\Omega_{\mathrm{tors}})^* \to C(T_{\mathrm{tors}}) \to 0$$

and

$$0 \to K^* \to O(\Omega_1)^* \to C(T_1) \to 0$$

one finds an exact sequence

$$0 \to O(X)^*/K^* \to C(Y)/C(Y_1) \to \ker\big(\mathrm{Pic}^0(\widehat{X}) \\ \to H^1(X, O_X^*)\big) \to 0\,.$$

This sequence is easily seen to be identical with

$$0 \to O(X)^*/K^* \to \mathrm{Div}_0(\widehat{X} - X) \\ \to \mathrm{Div}_0(\widehat{X} - X)/\sim \to 0$$

Further the group $C(Y)/C(Y_1)$ is a subgroup (possibly of finite index) of $\Theta(\Gamma)/\Theta(\Gamma_1)$. An interesting question is whether the group $\mathrm{Div}_0(\widehat{X} - X)/\sim$ is finite for groups of arithmetic type. In [G2] it is shown that this group is finite for any congruence subgroup of $PSl(2, \mathbb{F}_q[t])$.

(2) Put $\Delta = [\Gamma_+]$ and $p = 13$ (see 4.2.4). Then $X = \widehat{X}$, all $e_x = 1$, $\Delta_{ab} = \mathbb{Z}^2 \oplus \mathbb{Z}/3\mathbb{Z}$ and $C(T)^\Delta/C(Y) = \ker d'' \cong \mathbb{Z}/3\mathbb{Z}$. Apparently, the map $C(T)^\Delta/C(Y) \to \mathrm{Hom}\big((\Delta_{ab})_{\mathrm{tors}}, K^*\big)$ of 7.3.2 is here a bijection.

(3) Put $\Delta = [\Gamma_+]$ and $p = 23$. In this case Δ_{tors} is an interesting group. It is the amalgam of countably many copies of A_4 and the quotient $\Delta_{\mathrm{tors}}\backslash\Omega = \mathbb{P}^1 - \{0, \infty\}$.

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