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On the discrete Godbillon–Vey invariant and Dehn surgery on geodesic flows(*)

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RÉSUMÉ. — On donne une méthode pour construire des surfaces sectionnelles pour le flot géodésique d'une surface fermée à courbure négative, et dont l'application de premier retour est semi-conjuguée à un difféomorphisme torique Anosov. Cette construction est utilisée pour montrer que l'invariant discret de Godbillon-Vey [GS] n'est pas un invariant topologique, même quand on le restreint à $PL_+^\lambda(\mathrm{S}^1)$. Une variation de cette construction permet de construire des exemples de flots d'Anosov transitif liés à [BL].

ABSTRACT. — We give a method to construct surfaces of section for the geodisic flow on a negatively curved closed surface, with first return map semiconjugate to a toral Anosov diffeomorphism. This construction is used to prove that the discrete Godbillon - Vey invariant [GS] is not a topological invariant, even when restricted to $PL_+^\lambda(\mathrm{S}^1)$. A variation on that construction produces examples of transitive Anosow flows related to [BL].

Introduction

Let $\mathrm{PL}_+(\mathrm{S}^1)$ be the group of orientation preserving piecewise linear homeomorphisms of the circle $\mathrm{S}^1 = \mathbb{R}/\mathbb{Z}$. The *discrete Godbillon–Vey class* [GS] is the class $\overline{\mathrm{GV}} \in H^2(\mathrm{PL}_+(\mathrm{S}^1), \mathbb{R})$ represented by the 2-cocycle

$$(f,g) \mapsto \frac{1}{2}\sum_{x\in\mathrm{S}^1} \det\begin{pmatrix} \log g'(x+) & \log(f\circ g)'(x+) \\ \log\big(g'(x+)/g'(x-)\big) & \log\big((f\circ g)'(x+)/(f\circ g)'(x-)\big)\end{pmatrix}$$

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where, if $h \in \mathrm{PL}_+(\mathbf{S}^1)$, $h'(x+)\big(h'(x-)\big)$ denotes the right (left) derivative of h at the point $x \in \mathbf{S}^1$.

If $\Phi : G \to \mathrm{PL}_+(\mathbf{S}^1)$ is a representation of a group G into $\mathrm{PL}_+(\mathbf{S}^1)$, we can set

$$\overline{\mathrm{GV}}(\Phi) = \Phi^*(\overline{\mathrm{GV}}) \in H^2(G, \mathbb{R}),$$

where $\Phi^* : H^*\big(\mathrm{PL}_+(\mathbf{S}^1), \mathbb{R}\big) \to H^*(G, \mathbb{R})$ is the map induced by Φ. In particular, if G is the fundamental group of a closed oriented surface Σ_g of positive genus g, then we can identify $\overline{\mathrm{GV}}(\Phi)$ with the real number obtained from the canonical isomorphisms $H^2\big(\pi_1(\Sigma_g), \mathbb{R}\big) \simeq H^2(\Sigma_g, \mathbb{R}) \simeq \mathbb{R}$. This real number is the *discrete Godbillon–Vey invariant* of the representation $\Phi : \pi_1(\Sigma_g) \to \mathrm{PL}_+(\mathbf{S}^1)$.

Let $\lambda \in \mathbb{R}^+$ and let $\mathrm{PL}_+^\lambda(\mathbf{S}^1)$ be the subgroup of $\mathrm{PL}_+(\mathbf{S}^1)$ composed by homeomorphisms h such that $h'(x) \in \{\lambda^n \mid n \in \mathbb{Z}\}$ at every differentiable point. The discrete Godbillon–Vey invariant of a representation $\Phi : \pi_1(\Sigma_g) \to \mathrm{PL}_+^\lambda(\mathbf{S}^1)$ is then an integral multiple of $(1/2)\big(\log\lambda\big)^2$. In [Ghy] (see also [Ha2], [HM]), E. Ghys constructs two representations

$$\Phi_j : \pi_1(\Sigma_g) \to \mathrm{PL}_+^{\lambda_j}(\mathbf{S}^1), \quad j = 1, 2,$$

which are topologically conjugate but which have different Godbillon–Vey invariant; the proof of this second property is based on the fact that λ_1 and λ_2 are different so that

$$\frac{\big(\log\lambda_1\big)^2}{\big(\log\lambda_2\big)^2} \notin \mathbb{Q}.$$

The aim of this note is to prove that even restricted to $\mathrm{PL}_+^\lambda(\mathbf{S}^1)$ the discrete Godbillon–Vey invariant is not a topological invariant, at least for some λ.

THEOREM .— *There exist $\lambda \in \mathbb{R}^+$ and two representations*

$$\Phi_j : \pi_1(\Sigma_{409}) \to \mathrm{PL}_+^\lambda(\mathbf{S}^1), \quad j = 1, 2$$

which are topologically conjugate and which have different discrete Godbillon–Vey invariant.

The proof of this result is strongly inspired from [Ghy]: our representations Φ_j are topologically conjugate to the representation (into $\mathrm{Diff}_+^\infty(\mathbf{S}^1)$) corresponding to the stable foliation of the geodesic flow on Σ_{409} w.r.t. to an hyperbolic metric. The main step is the observation that the geodesic flow on an hyperbolic surface Σ_g, $g \geq 2$, admits two essentially different surfaces of section of genus one: the one introduced by Birkhoff ([Fri], [Ghy]) and another one which we describe in Section 2. We give a description of these surfaces of section which is a little different from that explained in [Fri]; this will lead to the digression of Section 3, where we construct transitive Anosov flows which admit a transverse torus which is not a global cross section (compare [BL] for other examples).

1. The surface of section of Birkhoff

Let Σ_g denote a closed oriented surface of genus $g \geq 2$ equipped with any hyperbolic metric. Let $\phi_t : T_1\Sigma_g \to T_1\Sigma_g$ denote the (Anosov) geodesic flow on the unitary tangent bundle of Σ_g. The topological equivalence class of ϕ_t does not depend on the chosen hyperbolic metric.

Let $\{\gamma_j\}_{j=1}^{2g+2}$ be the collection of simple closed oriented geodesics on Σ_g shown in the following picture:

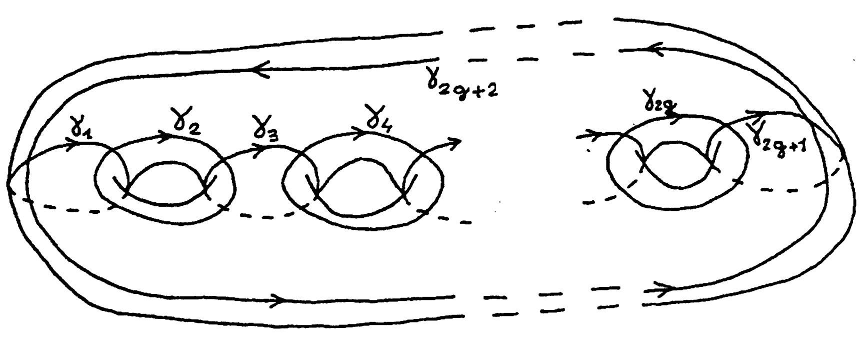

Fig. 1

The geodesics $\{\pm\gamma_j\}_{j=1}^{2g+2}$ ($+\gamma_j = \gamma_j$, $-\gamma_j = \gamma_j$ with reversed orientation) lift to $T_1\Sigma_g$ to $4g+4$ closed orbits of the flow ϕ_t, which we will denote by $\{\Gamma_j^\pm\}_{j=1}^{2g+2}$ ($\Gamma_j^\pm$ projects onto $\pm\gamma_j$). These closed orbits are boundary of a *surface of section* S for ϕ_t: S is an embedded surface in $T_1\Sigma_g$, with $\partial S = \bigcup_{j=1}^{2g+2} \Gamma_j^\pm$, $\int S$ is transverse to ϕ_t and every orbit of ϕ_t intersects S in a uniformly bounded time. This surface S can be described in two ways.

Description 1 ([Fri]).— $\Sigma_g \setminus \bigcup_{j=1}^{2g+2} \gamma_j$ is composed by four $(2g+2)$-gons. Take two of these $(2g+2)$-gons, such that their closures intersect only in correspondence of their vertices, and fill them with two foliations by strictly convex circles with a centre-type singularity. Then S is the closure of the set of unitary tangent vectors which are tangent to these convex circles. It is easy to see that S is a torus with $4g+4$ holes.

Description 2.— Each γ_j has a natural coorientation, induced by the orientations of Σ_g and γ_j. For every $j = 1, \ldots, 2g+2$ let C_j be the set of unitary tangent vectors at points of γ_j which are non-negative w.r. to the coorientation of γ_j. Each C_j is a closed cylinder, bounded by Γ_j^+ and Γ_j^-; it intersects C_{j-1} and C_{j+1} along segments and does not intersect the other C_i's; $\int C_j$ is transverse to ϕ_t. A surgery along $C_j \cap C_{j+1}$, as described in [Fri], produces an embedded surface S', bounded by $U_{j=1}^{2g+2}\Gamma_j^{\pm}$, diffeomorphic to the torus with $4g+4$ holes. This S' is a surface of section for ϕ_t.

The two surfaces S and S' are isotopic *rel(boundary)*, the isotopy being realized be the flow ϕ_t itself. The flow ϕ_t induces on S (or S') a first return map f, which is semiconjugate to an hyperbolic toral automorphism ([Ghy], [Ha1]). The conjugacy class of the automorphism has been calculated by N. Hashiguchi (see also [Chr] for a particular case and [Ghy] for the computation of the trace).

PROPOSITION 1 ([Ha1]).— *The first return map f defined by ϕ_t on the surface of section S is topologically semiconjugate to the hyperbolic toral automorphism defined by*

$$A_g = \begin{pmatrix} 2g^2-1 & 2g^2+2g \\ 2g^2-2g & 2g^2-1 \end{pmatrix}.$$

Let $\mathcal{F}^s$ be the stable foliation of ϕ_t. It is transverse to the fibres of $T_1\Sigma_g \to \Sigma_g$ and hence it corresponds to a representation

$$\Psi_g : \pi_1(\Sigma_g) \to \mathrm{Diff}_+^\infty(\mathrm{S}^1).$$

Let $\lambda_g \in (1, +\infty)$ be the largest eigenvalue of A_g. The above proposition, or more simply the computation of λ_g done in [Ghy], has the following consequence.

COROLLARY 1 ([Ghy], [Ha2]).— *The representation Ψ_g is topologically conjugate to a representation*

$$\Phi_g : \pi_1(\Sigma_g) \to \mathrm{PL}_+^{\lambda_g}(\mathrm{S}^1).$$

The explicit form of Φ_g can be found in [Ha2].

2. Another surface of section

Let $\{\mu_j\}_{j=1}^{2g+1}$ be the collection of simple closed oriented geodesics on Σ_g shown in the following picture.

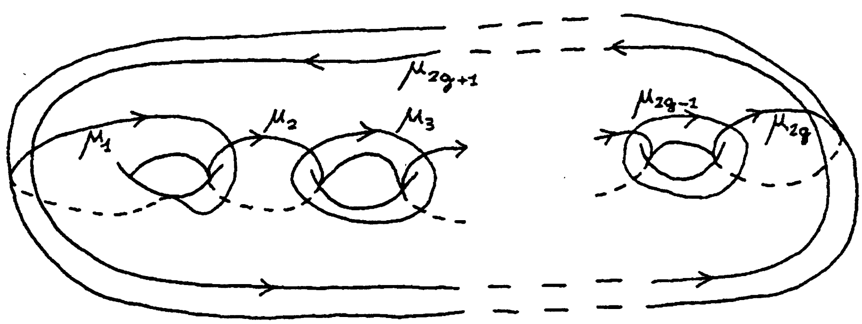

Fig. 2

(We have replaced γ_1 and γ_2 with a single geodesic μ_1, freely homotopic to the product of γ_1 and γ_2.)

Let $\{M_j^{\pm}\}_{j=1}^{2g+1}$ be the closed orbits of ϕ_t corresponding to $\{\pm\mu_j\}_{j=1}^{2g+1}$. Then $\bigcup_{j=1}^{2g+1} M_j^{\pm}$ is the boundary of a surface of section T for ϕ_t, which, as before, can be constructed in two ways.

Description 1.— $\Sigma_g\backslash\bigcup_{j=1}^{2g+1}\mu_j$ is composed by two $(2g+1)$-gons and one $(4g+2)$-gon. The closures of the two $(2g+1)$-gons intersect only at vertices, and the closure of the $(4g+2)$-gon has selfintersections in correspondence of its vertices. We may repeat the construction of Section 1 starting either from foliations by convex circles on the two $(2g+1)$-gons, or from a foliation on the single $(4g+2)$-gon.

Description 2.— Same construction of Section 1, using cylinders over geodesics μ_j.

These two constructions produce surfaces of section which are isotopic *rel(boundary)* and which are diffeomorphic to a torus with $4g+2$ holes.

Working as in [Hal] (see also [Chr], [Ghy]) one obtains the following analogue of Proposition 1.

PROPOSITION 1'.— *The first return map h defined by ϕ_t on the surface on section T is topologically semiconjugate to the hyperbolic toral automorphism defined by*

$$B_g = \begin{pmatrix} 4g^2 - 2g - 1 & 2g^2 - 2g \\ 8g^2 - 2 & 4g^2 - 2g - 1 \end{pmatrix} .$$

Let $\nu_g \in (1, +\infty)$ be the largest eigenvalue of B_g.

COROLLARY 1'.— *The representation Ψ_g is topologically conjugate to a representation*

$$\Phi'_g : \pi_1(\Sigma_g) \to \mathrm{PL}_+^{\nu_g}(\mathrm{S}^1) .$$

3. Digression: Anosov flows with a transverse torus

Let $\{\eta_j\}_{j=1}^N$ be a collection of closed oriented (hence cooriented) geodesics on Σ_g, not necessarily simple, such that:

(i) $\bigcup_j \eta_j$ has no triple point,

(ii) every other geodesic on Σ_g intersects some η_j in the positive direction.

For example, $\{\eta_j\}_j$ can reduce to a single "long" closed geodesic, "well distributed" in Σ_g.

Then construction 2 of the previous two sections (construction 1 is not always available) produces a surface of section Ω for ϕ_t with boundary $\bigcup_{j=1}^N N_j^{\pm}$, where $N_j^{\pm}$ is the lift of $\pm\eta_j$.

It is not difficult to see that the stable foliation $\mathcal{F}^s$ induces on Ω a foliation $\mathcal{G}$ with $2k_j^{\pm}$ semisaddles on $N_j^{\pm}$, where $k_j^{\pm}$ is the number of positive intersections of $\pm\eta_j$ with $\bigcup_{i=1}^N \eta_i$. This permits the computation of the homeomorphism type of Ω, via Poincaré–Hopf formula.

In particular, Ω has genus one and the first return map is topologically semiconjugate to an hyperbolic toral automorphism if and only if $k_j^{\pm} = 1$ for every j. This means that every η_j is simple and that (modulo reordering) $\eta_j \cap \eta_i = \emptyset$ for $|j - i| \geq 2$, η_{j+1} (η_{j-1}) intersects η_j in exactly one point and in negative (positive) direction. This, in turn, implies that $\{\eta_j\}_j$ coincides with one of the collections $\{\gamma_j\}_{j=1}^{2g+2}$ or $\{\mu_j\}_{j=1}^{2g+1}$ previously examinated (remark that condition (ii) implies that $\Sigma_g \setminus \bigcup_j \eta_j$ is simply connected).

For this reason, we don't know how to construct other surfaces of section with first return map semiconjugate to a hyperbolic toral automorphism, different from the two constructed above.

Because first return maps associated to surfaces of section are semiconjugate to pseudo Anosov diffeomorphism [Fri], our construction gives very concrete examples of such diffeomorphisms. The geodesics which form the boundary of the surface of section will correspond to fixed points of the pseudo Anosov diffeomorphism, and a geodesic which intersects $\bigcup_j \eta_j$ k times in the positive direction will correspond to a periodic point of period k. For instance, if $\{\eta_j\}_j$ is a single long closed geodesic, then the corresponding first return map will be semiconjugate to a pseudo Anosov diffeomorphism on a surface of large genus whose singular set is composed by only two (fixed) points. A "generic" choice of the geodesic will ensure that such a diffeomorphism is not a branched covering of a toral automorphism.

Let now the collection $\{\eta_j\}_{j=1}^N$ of closed oriented geodesics satisfy:

(i) $\bigcup_j \eta_j$ is connected and has no triple point,

(ii) $k_j^{\pm} = 1$ for every j,

but doesn't satisfy the condition that every other geodesic intersects some η_j in the positive direction. Then a Dehn surgery along $\{\eta_j\}_{j=1}^N$ [Fri], [Goo] produces a transitive Anosov flow $\psi_t : M \to M$ which admits a transverse torus T but which is not the suspension of a toral Anosov automorphism (compare [BL]). For example, $\{\eta_j\}_j$ can be a single closed geodesic with exactly one point of self-intersection, or it can be a "piece" of the collection of Birkhoff's section (fig. 3).

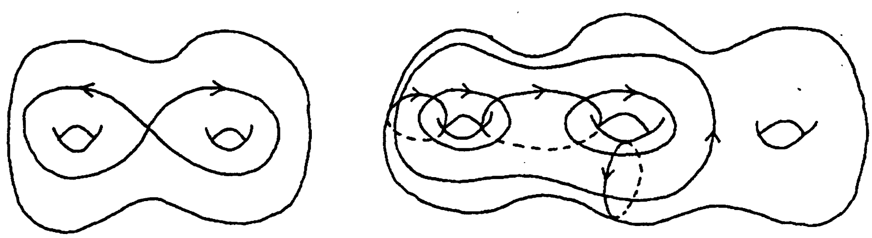

Fig. 3

Remark that, in general, the maximal ψ_t-invariant set in $M \setminus T$ will be large, *i.e.* not reduced to a finite collection of closed orbits (this is a difference with [BL]). In the second example above, this maximal invariant set is isomorphic to the set of geodesics (lifted in $T_1\Sigma_g$) which are completely contained in an handle of Σ_g (the handle to the right in the picture). If

$\gamma \subset \Sigma_g$ is the unoriented geodesic which separates this handle from the region of Σ_g containing $\bigcup_j \eta_j$ and if $T_\gamma \subset T_1\Sigma_g$ is its preimage in $T_1\Sigma_g$, then after the Dehn surgery T_γ will correspond to a torus $S \subset M$ (disjoint from T) with the properties:

(a) S contains two closed orbits γ_1 and γ_2 of ψ_t and is transverse to ψ_t outside $\gamma_1 \cup \gamma_2$;

(b) S separates M in two pieces, one of which is a "piece" of geodesic flow (or the geodesic incomplete flow on a holed torus with negative curvature and geodesic boundary), the other one is diffeomorphic to the complement of a solid torus embedded in a torus bundle over the circle (more precisely, the torus bundle whose monodromy is A_2) and disjoint from a fiber (*e.g.* T) of that bundle.

Hence that example is a sort of connected sum between two classical examples of Anosov flows, or it is a suspension of a toral automorphism (A_2) to which a handle (a piece of geodesic flow) has been glued. It would be interesting to show the existence of an operation of "glueing a handle" (glueing a piece of geodesic flow) to any transitive Anosov flow.

Finally, let us remark that, taking $\{\eta_j\}_j$ such that $\bigcup_j \eta_j$ has ℓ connected components instead being connected, we obtain examples of transitive Anosov flows with ℓ pairwise non isotopic transverse tori.

4. Proof of the theorem

Recall that λ_g (resp. ν_g) is the largest eigenvalue of A_g (resp. B_g). A direct computation shows that $\lambda_{35} = \nu_{25}$, hence we set

$$\lambda = \lambda_{35} = \nu_{25}\,.$$

The surface Σ_{409} covers both Σ_{35} and Σ_{25}. The multiplicity of a covering $\Sigma_{409} \xrightarrow{i} \Sigma_{35}$ is 12, whereas the multiplicity of a covering $\Sigma_{409} \xrightarrow{j} \Sigma_{25}$ is 17.

Let $\Phi_{35} : \pi_1(\Sigma_{35}) \to \mathbf{PL}_+^\lambda(\mathbf{S}^1)$ be the representation of corollary 1 and

$$\Phi_0 : \pi_1(\Sigma_{409}) \to \mathbf{PL}_+^\lambda(\mathbf{S}^1)$$

its lift defined by $i : \Sigma_{409} \to \Sigma_{35}$. Clearly, Φ_0 is topologically conjugate to Ψ_{409}.

Let $\Phi'_{25} : \pi_1(\Sigma_{25}) \to \mathrm{PL}^{\lambda}_{+}(\mathrm{S}^1)$ be the representation of corollary 1' and

$$\Phi'_0 : \pi_1(\Sigma_{409}) \to \mathrm{PL}^{\lambda}_{+}(\mathrm{S}^1)$$

its lift defined by $j : \Sigma_{409} \to \Sigma_{25}$. Also, Φ'_0 is topologically conjugate to Ψ_{409} and hence to Φ_0.

The discrete Godbillon–Vey invariant of Φ_g has been evaluated by Hashiguchi [Ha2]:

$$\overline{\mathrm{GV}}(\Phi_g) = -(4g+4)(\log \lambda_g)^2 .$$

It follows that

$$\overline{\mathrm{GV}}(\Phi_0) = 12 \cdot (-4 \cdot 35 - 4)(\log \lambda)^2 = -1728(\log \lambda)^2 .$$

The discrete Godbillon–Vey invariant of Φ'_g is an integral multiple of $(1/2)(\log \nu_g)^2$, and so

$$\overline{\mathrm{GV}}(\Phi'_0) = 17 \cdot \frac{n}{2} (\log \lambda)^2 , \quad n \in \mathbb{Z} .$$

Because $(1728 \cdot 2)/17$ is not an integer, we necessarily have

$$\overline{\mathrm{GV}}(\Phi_0) \neq \overline{\mathrm{GV}}(\Phi'_0)$$

and this completes the proof.

Remark.— The geodesic flow on Σ_g is obtained from the suspension of $B_g : \mathbb{T}^2 \to \mathbb{T}^2$ through $(1,1)$-Dehn surgeries along $4g+2$ closed orbits of period 1; the result of Hashiguchi (see the comments at the end of the introduction of [Ha2]) suggests $\overline{\mathrm{GV}}(\Phi'_g) = -(4g+2)(\log \nu_g)^2$ and consequently $\overline{\mathrm{GV}}(\Phi'_0) = -1734 (\log \lambda)^2$.

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