ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

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Annales de la faculté des sciences de Toulouse 6e série, tome 5, nº 3 (1996), p. 521-544

<http://www.numdam.org/item?id=AFST_1996_6_5_3_521_0>

Annales de la Faculté des Sciences de Toulouse Vol. V, n° 3, 1996

A hitting time for Lévy processes, with application to dams and branching processes[*]

ANTHONY G. PAKES[1]

RÉSUMÉ. — Nous obtenons de façon simple la densité du processus de Lévy à spectre positif pour le temps d'atteinte de zéro. Nous discutons de la relation avec les preuves antérieures, ainsi qu'avec celles de la théorie de la fluctuation en temps continu.

Le temps de vidange d'un barrage rempli par un subordinateur en est un cas particulier, pour lequel nous établissons une identité nouvelle. On utilise cette identité pour déduire une expression simple de la mesure canonique du phénomène régénérateur de vidange.

On obtient plusieurs théorèmes limites donnant des analogues continus de résultats connus sur les lois de Lagrange.

ABSTRACT. — A simple derivation is presented for the density of the zero-hitting time of a spectrally positive Lévy process. It is discussed in relation to existing proofs, and the result itself is discussed in relation to continuous time fluctuation theory.

The time to emptiness of a dam fed by a subordinator is a particular case. For this, a new identity is given, and it is used to derive a simple expression for the density of the canonical measure of the regenerative phenomenon of emptiness.

Several limit theorems are derived giving continuous analogues of known results for Lagrange laws.

KEY-WORDS : Lévy process, Subordinator, Hitting time density, dams, Branching processes, Exponential families, Limit theorems.

(*) Reçu le 24 mai 1994

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1. introduction

In this partly expository paper we consider (L_t), a conservative, spectrally positive Lévy process with $L_0 = z > 0$, that is, a real-valued process with stationary and independent increments whose paths are right-continuous with left hand limits. See Fristedt [12] or Skorohod [35].

Let $P_z(\cdot) = P(\cdot \mid L_0 = z)$ and $E_z(\cdot) = E(\cdot \mid L_0 = z)$. We are interested in the hitting time $T = \inf\{t \mid L_t = 0\}$. Of interest in its own right, this random variable occurs in several models of applied probability. To describe these, we need some notation.

The process (L_t) is determined by its cumulant function (cf) ψ: For $\theta \geq 0$ and all z,

$$E_z\left(\exp\left(-\theta(L_t - L_0)\right)\right) = \exp\left(-t\psi(\theta)\right) \tag{1.1}$$

where

$$\psi(\theta) = D\theta - \frac{v\theta^2}{2} + \int_{(0,1]} (1 - e^{-\theta x} - \theta x) n(\mathrm{d}x) + \int_{(1,\infty)} (1 - e^{-\theta x}) n(\mathrm{d}x) ,$$

and n is the (Lévy) measure, that is, a positive measure concentrating its mass on $(0, \infty)$ and satisfying $\int_0^\infty (1 \wedge x^2) n(\mathrm{d}x) < \infty$. The constant $D \in \mathbb{R}$ is the rate of deterministic drift and $v \geq 0$ is the variance of the Gaussian component. A corresponding decomposition is

$$L_t - L_0 = Dt + \sqrt{v}\, B_t + J_t ,$$

where (B_t) is a standard Brownian motion, and (J_t) is a Lévy process which jumps only to the right and $J_0 = 0$. This component is nondecreasing, i.e. it is a subordinator (Bingham [2] and Fristedt [12]), iff

Condition (S)

$$\int_0^1 1 \wedge x n(\mathrm{d}x) < \infty .$$

We assume that $\psi(\theta) \to -\infty$ as $\theta \to \infty$. This occurs if $P_z(L_t < 0) > 0$ for any, and hence all, $t > 0$. A sufficient condition is that $D > 0$, $v = 0$, and condition (S) do not occur together.

The continuous-state branching (CB) process controlled by ψ is defined as follows (Bingham [2]). Let $j(t) = \int_0^t \mathrm{d}v/L_v$ when $t < T$, $= \infty$ when $t \geq T$. Since $j(\,\cdot\,)$ is continuous and nondecreasing it has a generalized inverse $\tau(t) = \inf\{u \mid j(u) > t\}$. The randomly time changed process $Z_t = L_{\tau(t)}$ is a CB process. For this process the analogue of the total progeny is

$$J = \int_0^\infty Z_t \,\mathrm{d}t$$

and the change of variable $\tau = \tau(t)$ easily yields

$$J = \int_0^T L_\tau \,\mathrm{d}t = T\,.$$

The fundamental results about the law of T are as follows.

THEOREM 1.1.— *Suppose $\psi(\theta) \to -\infty$ as $\theta \to \infty$ and $z \geq 0$.*

(a) For each $s > 0$ there is a unique positive solution $\theta = \eta(s)$ of

$$\psi(\theta) = -s\,, \tag{1.2}$$

and

$$E_z(e^{-sT}) = e^{-z\eta(s)}\,. \tag{1.3}$$

(b) Let $K_L(x,t) = P_0(L_t \leq x)$, $x \in \mathbb{R}$. Then

$$G(t \mid z) = P_z(T \leq t) = -z\frac{\partial}{\partial z}\int_0^t K_L(-z,\tau)\frac{\mathrm{d}\tau}{\tau}\,. \tag{1.4}$$

(c) If $K_L(\,\cdot\,,t)$ has a density $k_L(\,\cdot\,,t)$, then $G(\,\cdot \mid z)$ has a density

$$g(t \mid z) = \frac{z}{t}\,k_L(-z,t)\,. \tag{1.5}$$

Although these results are by no means new, some published proofs are incomplete and we discuss this in Sections 2 and 3. A proof of the following integral version of (1.4),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^z G(t \mid v)\,\mathrm{d}v = t^{-1}\int_0^z yK_L(-\mathrm{d}y,t)\,, \tag{1.6}$$

is given by Gihman and Skorohod [14, p. 313] and, more recently, by Skorohod [35, p. 221], using factorization methods. These accounts are in terms of spectrally negative processes.

While preparing lectures on storage theory, the author read Borovkov [6] and as a consequence found an elementary real variable approach to Theorem 1.1. Its simplicity seems to be worth recording. Also, it is opportune to do this in view of recent interest in natural exponential families (NEF's); see Seshadri [33]. Recently Letac and Mora [24, p. 20], cited the later proof of Borovkov [7] for its completeness and accessibility. Hence it is worth recording an even simpler version together with commentary on related literature and results.

The connection with NEF's is that the hitting time law, which we denote by $\mathcal{L}(T)$, is the second element of the inverse pair $\{\mathcal{L}(L_1)\,,\,\mathcal{L}(T)\}$; see Letac and Mora [24]. We meet this again in Section 4.

An important special case is where $v = 0$, $D < 0$, and condition (S) holds. Then we can write

$$\psi(\theta) = -r\theta + \int_0^\infty (1 - e^{-\theta x}) n(\mathrm{d}x) \tag{1.7}$$

where

$$r = -D + \int_{0+}^1 x n(\mathrm{d}x)\,.$$

The integral in (1.7), denoted by $i(\theta)$, is the cumulant function of a subordinator (I_t). A common interpretation is in terms of a store, or dam, which receives an inflow I_t during $[0\,,t\,]$ and the content is withdrawn at rate r, when it is available. See Kingman [23] for discussion of the precise specification of the content process, denoted here by (C_t). Thus $L_t = I_t - rt$ is the net inflow during $[0\,,t\,]$ and T is the time to first emptiness of the store. Another interpretation in terms of Jinira processes is mentioned in Section 3. There we also discuss Theorem 1 for these contexts. We prove an identity ((3.8) below) and use it to derive a simple expression for the canonical measure of the regenerative phenomenon $\mathcal{C} = \{t \mid C_t = 0\,,\ C_0 = 0\}$ (Prabhu and Rubinovitch [30]).

Comparison of (3.4) below with the discrete density of the zero hitting time of a left-continuous (or skip-free) random walk on $\mathbb{N}_+$ (see Wendel [39] and Pakes and Speed [28] for references) suggests thinking of the density (3.4) as a continuous basic Lagrange law. Johnson *et al.* [19] is a general reference, and Devroye [10] is a sound representative of the voluminous and very repetitious literature on Lagrange laws. Aspects of this connection are discussed in Section 4.

2. Proof of Theorem 1

(a) Clearly $\psi(0)=0$,

$$\psi''(\theta) = -v - \int_0^\infty x^2\, e^{-\theta x} n(\mathrm{d}x) < 0\,,$$

and $\psi(\,\cdot\,)$ is continuous. The assertion about (1.1) follows from graphical considerations. Since (L_t) is skip-free to the left, for $0 < x < z$ we have, writing T_z for T when $L_0 = z$,

$$T_z = T_{z-x} + T'_x\,.$$

The components on the right hand side are independent and T'_x has the same law as T_x. In other words $(T_x : z \geq 0)$ is a subordinator. Hence (1.2) follows for some positive, increasing, and $C^\infty(\mathbb{R}_+)$ function $\eta(\,\cdot\,)$.

Next, observe that T is a stopping time with respect to the natural filtration $(\mathcal{F}_t)$ of (L_t), and

$$M_t = \exp\bigl(-\theta(L_t - L_0) + t\psi(\theta)\bigr)$$

defines a positive martingale with respect to $(\mathcal{F}_t)$. Applying Doob's optional stopping theorem for the stopping times $n\wedge T$, then letting $n \uparrow \infty$ and noting that $L_T = 0$, gives $E_z(M_T) = 1$, i.e.

$$E_z(e^{T\psi(\theta)}) = e^{-\theta z}\,.$$

It follows that $\eta(\,\cdot\,)$ indeed is as asserted in (a). This argument, included here for completeness, is due to Bingham [1, p. 721]. For the record, we mention that $\eta(\,\cdot\,)$ has the representation

$$\eta(s) = as + \int_0^{\infty+} (1 - e^{-st})\mu(\mathrm{d}t) \tag{2.1}$$

and $\int_0^\infty 1 \wedge t\mu(\mathrm{d}t) < \infty$.

(b) A first passage decomposition and the strong Markov property for (L_t) yield

$$K_L(-z,t) = \int_0^t K_L(0, t-\tau) G(\mathrm{d}\tau \mid z)\,. \tag{2.2}$$

Letting $\widehat{K}_L(z,s) = \int_0^\infty K_L(z,t)\, e^{-st}\, \mathrm{d}t$, $z \in \mathbb{R}$, (1.1) and (2.2) yield

$$\widehat{K}_L(-z,s) = \widehat{K}_L(0,s)\, e^{-z\eta(s)}\,, \quad z \geq 0\,. \tag{2.3}$$

But

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-\theta x} \widehat{K}_L(\mathrm{d}x, s) &= \int_0^\infty e^{-st} E_z\left(\exp(-\theta(L_t - L_0))\right) \mathrm{d}t \\ &= \frac{1}{s+\psi(\theta)} \\ &= -\frac{\widehat{K}_L(0,s)\eta(s)}{\theta - \eta(s)} + \int_0^\infty e^{-\theta x} \widehat{K}_L(\mathrm{d}x, s)\,,\end{aligned}$$

where the last line results from using (2.3) to evaluate the left hand side. The second term on the right hand side is finite for $\theta > 0$, and $0 < \left(s + \psi(\theta)\right)^{-1} < \infty$ when $0 \leq \theta < \eta(s)$. Hence, by multiplying throughout by $\theta - \eta(s)$ and letting $\theta \to \eta(s)$, l'Hospital's rule gives

$$\widehat{K}_L(0,s) = -\frac{1}{\eta(s)\psi'\left(\eta(s)\right)} = \frac{\eta'(s)}{\eta(s)}\,; \tag{2.4}$$

the last term comes from differentiating (1.2).

Consequently (2.3) takes the form

$$\int_0^\infty K_L(-z,t)\, e^{-st}\, \mathrm{d}t = \frac{\eta'(s)}{\eta(s)}\, e^{-z\eta(s)} = \eta'(s) \int_z^\infty e^{-v\eta(s)}\, \mathrm{d}v\,.$$

Integrating with respect to s over (s, ∞) and changing the order of integration, which is permissible by Fubini's Theorem and the measurability of (L_t), we obtain

$$\begin{aligned}\int_0^\infty t^{-1} K_L(-z,t)\, e^{-st}\, \mathrm{d}t &= \int_0^\infty e^{-st}\, \mathrm{d}t \int_0^t \tau^{-1} K_L(-z,\tau)\, \mathrm{d}\tau \\ &= \int_z^\infty v^{-1}\, e^{-v\eta(s)}\, \mathrm{d}v \\ &= \int_z^\infty v^{-1} \int_0^\infty e^{-st} G(\mathrm{d}t \mid v)\, \mathrm{d}v \\ &= \int_0^\infty e^{-st}\, \mathrm{d}t \int_z^\infty v^{-1} G(t \mid v)\, \mathrm{d}v\,.\end{aligned}$$

The uniqueness theorem for Laplace–Stieltjes transforms (LST's) now yields

$$\int_0^t \tau^{-1} K_L(-z,\tau)\,\mathrm{d}\tau = \int_z^\infty v^{-1} G(t \mid v)\,\mathrm{d}v\,. \tag{2.5}$$

Since the right hand side is absolutely continuous, (1.4) follows.

Suppose now that the density $k_L(x,t)$ of $K_L(x,t)$ exists and is measurable. The left hand side of (2.5) is

$$\int_0^t \tau^{-1} \int_z^\infty k_L(-x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau = \int_z^\infty \int_0^t \tau^{-1} k_L(-x,\tau)\,\mathrm{d}\tau\,\mathrm{d}x$$

whence

$$G(t \mid z) = z \int_0^t \tau^{-1} k_L(-x,\tau)\,\mathrm{d}\tau$$

and (1.5) follows.

For $v > 0$, Borovkov [6] proves the following version of (1.4),

$$\int_z^\infty v^{-1} G(t \mid v)\,\mathrm{d}v = \int_0^t K_L(-z,\tau)\frac{\mathrm{d}\tau}{\tau}\,, \tag{2.6}$$

which he understands to be equivalent to (1.5) in a generalized (but undefined) sence. Subsequently he (Borovkov [7, p. 66]) avoids both of these. Both of his presentations are set in the complex plane.

Most proofs of (1.4) and (1.5) from the 1960's use some form of limiting argument. This occurs by imbedding the laws of (L_t) in an indexed family, proving (1.4) and (1.5) for members of the indexed family, and then proceeding to the limit. For instance, Bingham [1] quotes (1.4) from Zolotarev [41]. In an earlier paper, Zolotarev [40] gives an intricate analytical proof of (1.5) when $n\{[x\,,\infty)\}$ is regularly varying at the origin with index $\varepsilon \in (0\,,2]$. In the later paper he introduces a family $\{n_\varepsilon\}$ of such Lévy measures which converges weakly to n as $\varepsilon \to 0$. Then (2.6) is satisfied by the corresponding DF's $G_\varepsilon(\,\cdot \mid v)$ and $K_{L,\varepsilon}(\,\cdot\,,t)$, and hence generally by letting $\varepsilon \to 0$, but his argument is incomplete at this point. Also, he assumes $E(L_t) \le 0$, though this seems unnecessary for his proof.

Using a "compensation" method, Keilson [21] derives (1.5) for the case $v > 0$ and $n(\mathbb{R}_+) < \infty$, that is, (J_t) is a compound Poisson process. He refers to the case $v = 0$, without showing how to attain it.

We end this section by explaining the fundamental nature of (1.3) for continuous time fluctuation theory. Suppose that $v \geq 0$ and that the average drift rate

$$m = E_0(L_1) = D + \int_{1+}^{\infty} xn(\mathrm{d}x)$$

satisfies $-\infty < m < 0$. Then $M = \sup_{t>0} L_t$ has a non-defective law whose LST is

$$E_0\big(\exp(-sM)\big) = -\frac{ms}{\psi(s)}\,.$$

This was first derived by Zolotarev [41]; see Takács [38, p. 47], Bingham [1, p. 725], and Prabhu [29, p. 78] for different proofs. Harrison [15] gives an elementary proof which makes essential use of (1.3). He also gives a simple proof of the familiar result that M has the same law as the limiting content C_∞ of the (generalized) dam process.

Let F be the distribution function (DF) of C_∞ and suppose the dam is modified to have a finite capacity γ. Then the limiting content of this finite dam has the DF $F(x)/F(\gamma)$. In addition, this also is the probability that a dam with initial content x empties before it overflows. Both results were found by Takács [38, chap. 6, for example]. Rogers [31] gives an elementary proof of the second result. See Bingham [3] for a summary of Takács' contribution to these topics.

3. When condition (S) holds

Throughout this section we assume condition (S), $v = 0$, and $r > 0$ in (1.7). This gives rise as follows to the Jirina process (Z_n), a discrete time $\mathbb{R}_+$ — valued branching process — see Pakes [27] and his references. In (1.7) suppose $r \leq 0$ and set $a = -r$. Then (L_t) is a subordinator with positive or zero drift. Discrete time n counts successive generations. At time n, Z_n measures the amount of "mass" present, a unit of which contributes to the next generation an amount equal in law to L_1. Disjoint subsets of n-th generation mass contribute independently to the next generation. Thus (Z_n) is a Markov chain whose transition kernel $\mathcal{K}(a, A)$, $z \in \mathbb{R}_+$ and $A \subset \mathbb{R}_+$, is determined by

$$\int_{\mathbb{R}_+} e^{-\theta x} \mathcal{K}(z, \mathrm{d}x) = e^{-z\psi(\theta)}\,.$$

Let $Z_0 = z$ and $S(z) = \sum_{n\geq 0} Z_n$ be the total mass ever existing. Then (Kallenberg [20, p. 21]) $\big(S(z) : z \geq 0\big)$ is a subordinator whose cf, $\eta(s)$, is the unique solution for θ of

$$\theta = s + \psi(\theta)\,. \tag{3.1}$$

This means that $S(z)$ is equal in law to the zero hitting time of $(L_t - t)$, $L_0 = z$. Note that $S(z) = \infty$, if a $a \geq 1$.

To reformulate Theorem 1.1 under condition (S), let $K_I(\,\cdot\,, t)$ be the DF of I_t and $k_I(\,\cdot\,, t)$ be its density, when this exists. Both are supported in $\mathbb{R}_+$. Since $K_L(x, t) = K_I(x + rt,\, t)$, the following result is an immediate corollary of Theorem 1.1.

THEOREM 3.1.— *Let T be the time to first emptiness of a dam with a subordinator inflow process (I_t) and withdrawal rate r. The cf $\eta(s)$ of T is the unique positive solution for θ of*

$$r\theta - i(\theta) = s\,. \tag{3.2}$$

When $C_0 = z$,

$$G(t \mid z) = -z\frac{\partial}{\partial z}\int_0^t K_I(r\tau - z,\, \tau)\frac{\mathrm{d}\tau}{\tau} \tag{3.3}$$

and when $k_I(\,\cdot\,, t)$ exists,

$$g(t \mid z) = \frac{z}{t}\, k_I(rt - z,\, t)\,. \tag{3.4}$$

Comparison of (3.1) and (3.2) reveals that the total mass $S(z)$ of a Jirina process has the same law as the first emptiness time provided that in (3.2) we replace r by $1 + r$.

The formula (3.4) has a long history. Kendall [22] observed that (3.4) solves the integral equation

$$g(t \mid z) = \int_0^\infty g(t - z/r \mid u)k_I\Big(u, \frac{z}{r}\Big)\,\mathrm{d}u\,, \tag{3.5}$$

obtained using a familiar hitting time decompostion. He explicitly set aside the question of uniqueness of its solutions. Hasofer [16] observed that the most general solution has the form $\int_0^{rt-z} k_I(rt - uz, t)\,\mathrm{d}V(u)$ where $V(\,\cdot\,)$ is an arbitrary function of bounded variation. By computing the Laplace transform of the right hand side of (3.4) he shows this gives the solution that is sought. His proof requires that $E_0 I_1 < \infty$, a condition which excludes stable inflow processes, amongst others, and he imposes a further bounded variation condition (see his Theorem 3).

Zolotarev [40] (see his "Theorem") gives essentially (3.4) by a purely analytical argument having no explicit reference to Lévy processes. See also Corollary 5 of Zolotarev [41]. We should mention that Takács [37] uses the ballot theorem to derive a version of (1.6) under our present conditions. Also, see his later account Takács [38, p. 57]. Prabhu [29, p. 81], proved (3.4) using factorization methods.

We have mentioned above that some authors derive identities like (2.6) when $v > 0$ and then allow $v \to 0$. It is worth showing that this gambit has problems. For example, differentiating (2.3) with respect to z , then formally taking the derivative inside the integral defining $\widehat{K}_L(-z,t)$, and then letting $z \to 0$ and using (2.4) gives (Borovkov [6, eq. (7)])

$$\widehat{k}_L(0,s) = \eta'(s)\,. \tag{3.6}$$

This is legitimate when $v > 0$ because then $k_L(\,\cdot\,,t)$ is bounded and C^∞. But this relation is not generally valid when $v = 0$. The left hand side of (3.6) $\to 0$ as $s \to \infty$ for any $v \geq 0$. When $v > 0$, and/or condition (S) fails, the law $\mathcal{L}(T)$ has zero as its first point of increase (for any $z > 0$). This occurs because (L_t) can diffuse back to the origin within arbitrarily short intervals. But under the conditions imposed in this section, movement back to the origin is governed by the linear drift term whence the first point of increase of $\mathcal{L}(T)$ is z/r, and then $\eta'(s) \to 1/r$ as $s \to \infty$. It seems in this case, then, that (3.6) should be replaced by

$$\widehat{k}_L(0,s) = \eta'(s) - r^{-1}\,, \tag{3.7}$$

and direct computation with the stable(1/2) inflow process supports this conjecture.

Using an argument similar to that leading to (2.4) shows that (3.7) from follows from

$$k_L(-z,t) = r^{-1}g(t \mid z) + \int_0^t k_L(0,t-\tau)g(\tau \mid z)\,\mathrm{d}\tau\,. \tag{3.8}$$

Essentially this identity, but lacking the first term on the right, occurs as (4) in Borovkov [6], and it is asserted as above (with $r = 1$) in Gani and Prabhu [13, eq. (6.3)].

We prove (3.8) directly as follows. Observe that

$$k_L(-z,t)\,\mathrm{d}z = P_z\big(L_t \in (0,\mathrm{d}z)\big)\,.$$

The argument of $P_z(\,\cdot\,)$ is achieved by

(i) first hitting the origin at τ $(0<\tau<t)$ followed by motion back to zero at t, and the union of such events accounts for the integral at (3.8);

(ii) first hitting the origin near time t.

This has probability $g(t \mid z)\,\mathrm{d}t$. But the increment in (L_t) as it crosses the origin at t is

$$\mathrm{d}z = L_t - L_{t-\mathrm{d}t} + r\,\mathrm{d}t + \mathrm{o}(\mathrm{d}t) = r\,\mathrm{d}t + \mathrm{o}(\mathrm{d}t)$$

where we have used Fubini's differentiation theorem, or Theorem 1 of Shtatland [34]. Hence this contribution is $g(t \mid z)\,\mathrm{d}z/r$, and (3.8) follows.

The Lévy measure μ of $\mathcal{L}(T)$ has a positive atom at infinity $\mu(\{\infty\}) = \eta \equiv \eta(0+)$, iff $m>0$, and then

$$P_z(T=\infty) = e^{-z\eta}\,.$$

When k_I exists, integration of (3.7) gives the explicit representation (*cf.* (2.1))

$$\eta(s) = \frac{s}{r} + \frac{\mu(\{\infty\})}{r} + \int_0^\infty e^{-st} k_L(0,t)\,\frac{\mathrm{d}t}{t}\,.$$

In particular μ has the density

$$\mu(\mathrm{d}t) = k_I(rt,t)\,\frac{\mathrm{d}t}{t}$$

on $(0,\infty)$. This seems to be new and it can be used as follows.

Prabhu and Rubinovitch [30] have shown that $r\mu$ is the canonical measure of $\mathcal{C}$, i.e. if $p(t) = P_0\ (t \in \mathcal{C})$ then

$$\int_0^\infty e^{-st} p(t)\,\mathrm{d}t = \left(s + r\int_{0+}^{\infty+} (1-e^{-st})\,\mu(\mathrm{d}t)\right)^{-1} = \frac{r}{\eta(s)}\,.$$

See Prabhu and Rubinovitch [30] for a thorough account of the properties of $\mathcal{C}$. They show also that

$$r\mu\big((t\,,\infty]\big) = \int_{0+}^{\infty+} \big(1 - G(t \mid z)\big) n(\mathrm{d}z)\,.$$

But since the integrand is equal to $1 - e^{-z\eta} + \int_t^\infty g(\tau \mid z)\,\mathrm{d}\tau$ we deduce that

$$\int_0^\infty g(t \mid z)\, n(\mathrm{d}z) = r k_I(rt, t)\,,$$

another new identity.

4. Asymptotic properties of $\mathcal{L}(t)$

As we mentioned in Section 1, the density (3.4) is very similar in functional form to the class of discrete laws known collectively as basic Lagrange laws. See, for example, (2′) in Pakes and Speed [28] and also the references cited there. However, in the absence of a continuous version of the Lagrange formula (for reversion of series), we do not recommend associating "Lagrange" with densities of the type (3.4).

Explicit examples can be written down using known examples of the infinitely divisible (infdiv) densities $k_I(x,t)$. Recall that the underlying law cannot be compound Poisson, thus eliminating from consideration many common infdiv laws.

Example 1.— For the gamma inflow process we have $i(\theta) = \log(1+\theta/\alpha)$, $\alpha > 0$, and

$$k_I(x,t) = \frac{\alpha\big(\alpha x\big)^{t-1}}{\Gamma(t)}\, e^{-\alpha x}\,, \quad x > 0\,.$$

Here r enters simply as a scaling constant so with no loss of flexibility we set $r = 1$ to get

$$g(t \mid z) = \frac{\alpha z\big(\alpha(t-z)\big)^{t-1}}{\Gamma(1+t)} \exp\big(-\alpha(t-z)\big)\,, \quad t \geq z\,. \tag{4.1}$$

This was first given by Kendall [22, p. 211] and subsequently has been attributed by Letac and Mora [24, p. 20] to P. Ressel. They appended a note recommending the descriptor "Kendall–Ressel". This family is analogous to the Borel–Tanner law (Moran [25, p. 101]) which, annoyingly, is called the Poisson-delta law by Johnson, Kotz and Kemp [19, eq. (3.119) on p. 143].

Many positive infdiv laws have infinite first moment, and then $\mathcal{L}(T)$ is defective. Non-defective hitting laws can be obtained be defining NEF's of subordinators as follows. This construction obviously is more generally applicable.

For each $c \geq 0$ denote by $(I_{c,t})$ a subordinator whose cf is $i(\theta + c) - i(c)$. Its density, assuming it exists, is

$$k_I(x,t\,;\,c) = k_I(x,t)\,e^{-cx+ti(c)}\,. \tag{4.2}$$

Letting $m_c = E(I_{c,1}) - r$ denote the mean drift rate, we see that $m_c \leq 0$ iff $c \geq \bar{c}$ where $\bar{c} = 0$ if $i'(0) \leq r$, and $\bar{c}$ is the unique positive solution of $i'(c) = r$ when $i'(0) > r$.

Now let $\widetilde{T}$ denote the zero hitting time of the process $(I_{c,t} - rt)$. Then for any $z > 0$, $\mathcal{L}(\widetilde{T})$ is non-defective if $c \geq \bar{c}$. Applying (3.4) to (4.2) shows that the density of $\widetilde{T}$ is

$$\widehat{g}(t \mid z) = g(t \mid z)\,e^{-t\gamma+cz}$$

where $\gamma = cr - i(c)$. From graphical considerations and (3.2), we can invert this relation as $c = \eta(\gamma)$ where $\gamma \geq \bar{\gamma} = i(\bar{c}) - r\bar{c}$. Note that $\bar{\gamma} = 0$, if $\bar{c} = 0$, < 0 otherwise. Hence

$$\widetilde{g}(t \mid z) = g(t \mid z)\,e^{-\gamma t + z\eta(\gamma)} \tag{4.3}$$

and

$$\int_0^\infty e^{-st}\widetilde{g}(t \mid z)\,\mathrm{d}t = \exp\big(-z\big(\eta(s+\gamma) - \eta(\gamma)\big)\big)\,.$$

These laws comprise a NEF of non-defective laws for $\gamma \geq \bar{\gamma}$. This construction induces families of inverse pairs of laws $\{\mathcal{L}(I_{c,1}), \mathcal{L}(\widetilde{T})\}$. See Letac and Mora [24, sect. 5] (they use "reciprocal" for "inverse"). We can allow $\gamma < 0$ when $\bar{c} > 0$; in fact $P_z(\widetilde{T} > t) = \mathrm{O}(e^{\bar{\gamma}t})$ in this case. Later we obtain an exact estimate of the upper tail of hitting time laws.

If (I_t) is a gamma process then so is $(I_{c,t})$ and hence the general form of (4.1) is preserved under the NEF transformation.

Example 2.— If (I_t) is a stable(α) subordinator, with cf $(A/\alpha)\theta^\alpha$, $0 < \alpha < 1$, then $I_{c,t}$ has a Hougaard [17] law $H(\alpha, At, c)$. This law is well-defined for $\alpha = 1$ when $c > 0$ and then $(I_{c,t})$ is a gamma process. Apart from one other special case (see below) the density $k(\,\cdot\,,\,\cdot\,;c)$ does not have an elementary closed form. Hougaard [17, p. 389] gives a series representation which induces a complicated series expression for $\widetilde{g}(t \mid z)$. In the general case we have

$$\bar{c} = \left(\frac{A}{r}\right)^{1/(1-\alpha)} \quad \text{and} \quad \bar{\gamma} = -r\bar{c}(\alpha^{-1} - 1)\,.$$

The case $\alpha = 1/2$ yields closed expressions. It is known, Seshadri [33], when $c > 0$ that $I_{c,t}$ has an inverse-Gaussian law. For convenience we set $r = 1$ and $a = A^2$. Then $\bar{c} = a = -\bar{\gamma}$ and

$$k_I(x,t) = y\sqrt{\frac{a}{\pi x^3}}\exp\left(-\frac{at^2}{x}\right).$$

Explicit computation of the cf of T yields the representation

$$\widetilde{T} = z + \Gamma(\mu z, \sigma)$$

where

$$\mu = \sqrt{\frac{2a}{\sigma}}\,, \quad \sigma = \gamma + a$$

and $\Gamma(\mu, \sigma)$ has the inverse Gaussian density

$$p_{\mu,\sigma}(x) = \frac{\mu}{\sqrt{2\pi x^3}}\exp\left(-\frac{(x-\mu)^2}{2\sigma x}\right), \quad x > 0\,.$$

The same approach can be applied to the Bessel function densities described by Feller [11, pp. 437-8].

We will now establish some limit theorems related to the upper tail of $\mathcal{L}(T)$. Expressed in general terms, the first of these can be extracted from results of Bingham [1]. But here we give more specific results which, for the subordinator case, will parallel results for Lagrange laws obtained by Pakes and Speed [28]. Of course, some of these results are known, but seem not easily accessible.

Suppose that $n_2 = \int x^2 n(\mathrm{d}x) < \infty$. Then in the general case

$$m = E_0(L_1) = D + \int_1^\infty x n(\mathrm{d}x)\,,$$

and when condition (S) holds we have, as above, $m = n_1 - r$, where $n_1 = \int_0^\infty x n(\mathrm{d}x)$. When $m < 0$ some computation from (1.2) yields

$$\mu_T = E_1(T) = -m^{-1} \quad \text{and} \quad \sigma_T^2 = \mathrm{var}_1(T) = -m^{-3}(v + n_2)\,.$$

The central limit theorem yields

$$\frac{T - \mu_T z}{\sigma_T \sqrt{z}} \Longrightarrow \mathcal{N}(0,1) \text{ as } z \to \infty\,.$$

For comparison, see Theorem 4 of Pakes and Speed [28].

Suppose now that $m = 0$ and

$$\psi(\theta) = -\frac{1}{2}\,\theta^\delta L\Big(\frac{1}{\theta}\Big) \tag{4.4}$$

where $1 < \delta \leq 2$ and $L(\,\cdot\,)$ is positive and slowly varying at infinity (SV). This is equivalent to $x^\delta n(x^{-1}, \infty)$ being SV. If second order moments are finite then $L(x) \to v + n_2$.

For the subordinator case, if $F(x) = P_0(I_1 \leq x)$, we have $1 - F(x) \sim n(x, \infty)$, see Bingham, Goldie and Teugels [4, p. 341]. When $\delta < 2$, (4.4) is equivalent to

$$\int_0^x y^2 F(\mathrm{d}y) \sim \frac{x^{2-\delta} L(x)}{\Gamma(3-\delta)}\,, \quad 1 < \delta < 2\,,$$

and when $\delta = 2$ this truncated second moment is SV.

It follows immediately (Feller [11, p. 574]) that $\big((I_t - rt)/a_t\big)$ converges weakly to a spectrally positive stable(δ) law. More specifically, choose $a_t > 0$ so

$$t\psi\Big(\frac{\theta}{a_t}\Big) \longrightarrow -\frac{\theta^\delta}{2}\,. \tag{4.5}$$

Then

$$t = \frac{a_t^\delta}{L(a_t)} \quad \text{and} \quad a_t = t^{1/\delta} M(t)$$

where M is SV, and

$$\frac{L_t}{a_t} \Longrightarrow S_\delta \quad \text{where } E(e^{-\theta S_\delta}) = \exp\Big(-\frac{\theta^\delta}{2}\Big)\,.$$

If $p_\delta(x)$ denotes the density of S_δ, then

$$p_\delta(0) = \frac{2^{1/\delta}}{\delta \Gamma(1 - \delta^{-1})}\,. \tag{4.6}$$

The following result is analagous to Theorem 5 of Pakes and Speed [28].

THEOREM 4.1.— *Let $r = n_1$ and (4.4) hold, and $b_z = -1/\psi(1/z)$. As $z \to \infty$,*

$$\frac{T}{b_z} \Longrightarrow S_{1/\delta},$$

the positive stable law whose cf is $s^{1/\delta}$.

Proof.— From (1.2) and (4.4) we have $\eta(s) = s^{1/\delta}M(1/s)$, $s \to 0$, where M is SV. Hence the cf of T/b_z is

$$z\eta\left(-s\psi\left(\frac{1}{z}\right)\right) \sim zs^{1/\delta}\eta\left(-\psi\left(\frac{1}{z}\right)\right) = s^{1/\delta}. \square$$

Next, we obtain some local limit theorems for the hitting density under the following conditions. The first of these is related to the above NEF transformation of k_L.

Condition (M).— There is a unique $\overline{\theta} \in \mathbb{R}$ such that

$$\psi(\overline{\theta}) = \sup_{\theta} \psi(\theta).$$

In most cases $\overline{\theta}$ is the unique solution of $\psi'(\theta) = 0$, and such a $\overline{\theta} > 0$ exists when $m > 0$, and $\overline{\theta} = 0$ when $m = 0$. Nothing in general can be said when $m < 0$, apart from $\overline{\theta} \leq 0$.

Condition (B).— For some $t > 0$ the density $k_L(\cdot, t)$ exists and is bounded.

This condition is satisfied by the gamma and stable processes and, under mild regularity conditions, by any density which is a generalized gamma convolution; see Bondesson [5, p. 50].

A ratio limit theorem holds under conditions (M) and (B): For any x, $y \in \mathbb{R}$,

$$\lim_{t\to\infty} \frac{k_L(y, t+\tau)}{k_L(x,t)} = \exp\left((x-y)\overline{\theta} - \tau\psi(\overline{\theta})\right).$$

See Stone [36, p. 88]. The following result is an immediate inference from (1.5).

THEOREM 4.2.— *If conditions (B) and (M) hold then for r, x, $z > 0$,*

$$\lim_{t\to\infty} \frac{g(t+\tau \mid x)}{g(t \mid z)} = \frac{x}{z} \exp\big((z-x)\overline{\theta} - \tau\psi(\overline{\theta})\big)\,.$$

This can be compared with the ratio limit theorem for basic Lagrange laws given by Pakes and Speed [28, Theorem 1].

We now transform to the NEF (or conjugate or associated) law whose density is

$$k_L(x,t\,;\overline{\theta}) = k_L(x,t)\exp\big(-\overline{\theta}x + t\psi(\overline{\theta})\big)\,. \tag{4.7}$$

Its cf is $\overline{\psi}(\theta) = \psi(\theta+\overline{\theta}) - \psi(\overline{\theta})$, $\theta \geq 0$. Then

$$g(t \mid z) = \frac{z}{t}\, k_L(-z,t\,;\overline{\theta}) \exp\left(-(z\overline{\theta} + t\psi(\overline{\theta}))\right)\,. \tag{4.8}$$

To go further we assume (*cf.* (4.4)) a new condition.

Condition (R).— $\psi'(\overline{\theta}) = 0$ and for some $1 < \delta \leq 2$,

$$\overline{\psi}(\theta) = -\frac{1}{2}\,\theta^{\delta} L\Big(\frac{1}{\theta}\Big)$$

where L is SV.

This says that the law defined by (4.7) with $t = 1$ is attracted to the above spectrally positive stable(δ) law. Let $\overline{n}_i = \int_0^\infty x^i\, e^{-\overline{\theta}x} n(\mathrm{d}x)$, $i = 1, 2$. condition (R) entails $\overline{n}_1 < \infty$ and $D - v\overline{\theta} + \overline{n}_1 = 0$. If also $\overline{n}_2 < \infty$ then $\delta = 2$ and $L(\infty) = v\overline{\theta} + \overline{n}_2$.

If a_t is chosen so (4.5) is satisfied for $\overline{\psi}$, $\overline{L}_t/a_t \Rightarrow S_\delta$, and a local limit theorem for densities, Ibragimov and Linnik [18, p. 126], takes the form

$$\lim_{t\to\infty} \sup_y \big|a_t k_L(a_t y, t\,;\overline{\theta}) - p_\delta(y)\big| = 0\,. \tag{4.9}$$

Choosing $y = -z/a_t$, (4.8) yields the foreshadowed result about the upper tail of $\mathcal{L}(T)$.

THEOREM 4.3.— *If conditions (M), (B) and (R) hold, then as* $t \to \infty$

$$g(t \mid z) \sim z\, e^{\overline{\theta} z - t\psi(\overline{\theta})} p_\delta(0) \frac{t^{-1-1/\delta}}{M(t)}$$

and $p_\delta(0)$ *is given by (4.6). If* $\overline{n}_2 < \infty$ *then*

$$g(t \mid z) \sim z\, e^{\overline{\theta} z - t\psi(\overline{\theta})} \left(2\pi V t^3\right)^{-1/2}$$

where $V = v\overline{\theta} + \overline{n}_2$.

Contrast this with Theorem 2 of Pakes and Speed [28]. The finite variance case is analagous to the much older result of Otter [26] for basic Lagrange laws.

Example 1 again.— Here $\overline{\theta} = r^{-1} - \alpha$, $\psi(\overline{\theta}) = \alpha r - 1 - \log(\alpha r)$ and $\overline{n}_2 = r^2$. When $r = 1$,

$$g(t \mid z) \sim z(2\pi)^{-1/2}\, e^{-(z+t)(\alpha-1)} \alpha^t t^{-3/2} .$$

This result is derived directly from (4.1) by Prabhu [29, p. 83], and it is strikingly similar to a corresponding result for the Borel–Tanner law; see Pakes and Speed [28, p. 748].

Example 2 again.— Here we take (I_t) as the Hougaard subordinator, so

$$\psi(\theta) = -r\theta + \left(\frac{A}{\delta}\right)\left((\theta + c)^\alpha - c^\alpha\right), \quad c \geq 0 \text{ and } 0 < \alpha < 1 .$$

Then

$$\overline{\theta} = \left(\frac{A}{r}\right)^{1/(1-\alpha)} - c ,$$

$$\psi(\overline{\theta}) = rc - \left(\frac{A}{\alpha}\right)c^\alpha + (\alpha^{-1} - 1)\left(\frac{A}{r}\right)^{1/(1-\alpha)} ,$$

$$\overline{n}_2 = (1-\alpha) r \left(\frac{r}{A}\right)^{1/(1-\alpha)} \quad \text{and} \quad \delta = 2 .$$

Condition (B) is satisfied as the Hougaard law is a NEF transformation of a positive stable law, and the latter has a bounded density. Hence Theorem 4.3 applies in this case.

Fix $x > 0$ and let $y = x^{-\delta}$ in (4.9). Choosing $z = x^{-1/\delta} a_t$ in (4.8) yields the following local limit result for the hitting laws.

THEOREM 4.4.— *Assume conditions (M), (B) and (R) hold, and that* t, $z \to \infty$ *in such a way that* $tz^{-\delta}L(z) \to x > 0$. *Then*

$$\sup_x \left| \frac{z^\delta}{L(z)} e^{-\overline{\theta} z - \psi(\overline{\theta})} g(t \mid z) - x^{-1-1/\delta} p_\delta(-x^{-1/\delta}) \right| \longrightarrow 0\,.$$

The corresponding result for basic Lagrange laws appears as Theorem 3 of Pakes and Speed [28]. When $\overline{n}_2 < \infty$ our result becomes

$$\sup_x \left| \frac{z^2}{V} e^{-\overline{\theta} z - t\psi(\overline{\theta})} g(t \mid z) - \frac{1}{\sqrt{2\pi x^3}} e^{-1/2x} \right| \longrightarrow 0\,.$$

Pakes and Speed [28] give some limit results for general Lagrange laws. In our context these amount to allowing L_0 to have a positive law which is parameterized by $\ell = E(L_0) < \infty$ and which satisfies $L_0 \to_p \infty$ as $\ell \to \infty$. If $\zeta(\,\cdot\,)$ is the cf of L_0 then the cf of T is $\zeta\big(\eta(s)\big)$. By considering the conditional characteristic function (CF), of $(T - \mu_T L_0)/\sigma_T\sqrt{L_0}$, given L_0, it is straightforward to prove the following result.

THEOREM 4.5.— *Let* $m < 0$, $n_2 < \infty$, *and* $w^2 = \mathrm{var}(L_0) < \infty$ *and define*

$$\mu = \ell\mu_T \quad \textit{and} \quad \sigma^2 = \ell\sigma_T^2 + w^2\mu_T^2\,,$$

the mean and variance of T. *If*

$$\frac{L_0}{\ell} \Longrightarrow 1 \quad \textit{and} \quad \frac{L_0 - \ell}{w} \Longrightarrow N(0,1)$$

as $\ell \to \infty$, *then*

$$\frac{T - \mu}{\sigma} \Longrightarrow N(0,1)\,.$$

Limit results for Lagrange laws, whose proofs rest on limit theorems for random sums, are given by Pakes and Speed [28]. They correct and generalize an earlier result of Consul and Shenton [8]. The basis of these results transfers in a simple manner to give the following limit results.

THEOREM 4.6.— *If $m < 0$, $n_2 < \infty$ and*

$$\frac{L_0}{\ell} \text{ converges weakly to a law whose DF is } H(\,\cdot\,), \tag{4.10}$$

then $(T-\mu)/\sigma_T\sqrt{\ell}$ converges weakly to a law whose characteristic function (CF) is $\int_0^\infty \exp(-xt^2/2)H(\mathrm{d}x)$.

This limit law is a variance mixture of normal laws, the structural properties of which are discussed by Rosinski [32]. A similar result in terms of a scale mixture of stable laws holds when $m = 0$.

THEOREM 4.7.— *If $m = 0$ and (4.4) and (4.10) hold, then T/b_ℓ converges weakly to a positive law whose LST is $\int_0^\infty \exp(-xs^{1/\delta})H(\mathrm{d}x)$.*

Example 3.— Let $a > 0$, $b \in \mathbb{R}$ and suppose L_0 has the density $h_\alpha(x) = x^{-1-b}\, e^{-\alpha x} 1_{[1,\alpha)}(x)$. If $b \leq 1$ then $\ell \to \infty$ as $a \to 0$, indeed

$$\ell \sim \frac{1-b}{a} \text{ if } b > 1 \quad \text{and} \quad \ell \sim \frac{1}{a} \log \frac{1}{a} \text{ if } b = 1\,.$$

When $b < 0$ further calculation shows that L_0/ℓ converges weakly to the gamma law whose LST is $\big(1+(1-b)^{-1}\theta\big)^b$. Then Theorems 4.5 and 4.6 are applicable provided their other hypotheses are satisfied. When $m < 0$ the CF of the limit law is $\big(1+t^2/2(1-b)\big)^b$, defining a symmetric Linnik law of index 2. This law can be represented in the obvious way as the difference of independent gamma variates with shape parameter $-b$. When $m = 0$ the limit law is a positive Linnik law. See Devroye [9] for references on this attribution.

Now suppose $b = 0$ and let $\lambda(\theta)$ be the LST of L_0. Computation shows for $0 < x < 1$ and $\theta > 0$ that $\lambda(\theta a^x) \to x$ as $a \to 0$. The limit is the LST of the measure $x\delta_0$. Consequently $P(L_0 a^x \leq 1) \to x$, or $\log L_0/\log a^{-1} \Rightarrow \mathcal{U}[0\,,1]$, the uniform law on $[0\,,1]$. So when $m < 0$, we infer that $\log T/\log a^{-1} \Rightarrow \mathcal{U}[0\,,1]$. Similarly, when $m = 0$ and (4.4) holds, we have

$$\frac{\delta^{-1}\log T + M(T)}{\log a^{-1}} \Longrightarrow \mathcal{U}[0\,,1]\,.$$

Finally, if $b > 0$ then $h_0(x)$ is a Pareto density. Hence as $a \to 0$, $\mathcal{L}(T)$ converges weakly to a Pareto scale mixture of the limiting normal and stable laws of Theorems 4.5 and 4.6, respectively.

The result which follows is a continuous analogue of the "Second Consul and Shenton theorem", as it is discussed by Pakes and Speed [28], giving an approximation for $\mathcal{L}(T)$ when L_0 is large and $m \approx 0$. Its proof, which is based on (1.2), is similar in strategy to that of Theorem 7 in Pakes and Speed [28], but simpler in detail.

As above we assume $\mathcal{L}(L_0)$ is parametrized by ℓ and satisfies (4.10). We shall regard the law of (L_t) as being parametrized by $m \in (-\infty, 0)$ through its cf. For such m let ψ_m denote the cf and suppose the resulting family of cf's satisfies:

$$\text{as } m \longrightarrow 0- \,,\ \psi_m(\theta) \longrightarrow \psi_*(\theta) \text{ a cf };\tag{4.11a}$$

$$\psi_*'(0) = 0 \quad \text{and} \quad V_* = -\psi_*''(0) > 0 \,;\tag{4.11b}$$

and

$$\begin{gathered}\text{using obvious notation, the family of measures}\\ \left\{x^2 K_L(\mathrm{d}x, 1\,;\, m) : m < 0\right\} \text{ is tight.}\end{gathered}\tag{4.11c}$$

If $V_m = -\psi_m''(0)$, which is finite, then

$$\psi_m(\theta) = m\theta - \frac{\left(V_m - R_m(\theta)\right)\theta^2}{2}\,.\tag{4.12}$$

Under assumption (4.11) we have $R_m(\theta) \to 0$ as m, $\theta \to 0$.

Let $\eta_m(s)$ be the cf of T induced by ψ_m. Then working from (4.12) and (1.2) it is easily seen that $\eta_m(m^2 s) \le -ms$, and hence that

$$\rho_m(s) = V_m - R_m\left(\eta_m(m^2 s)\right) \longrightarrow V_* \quad \text{as } m \to 0- \,.$$

Consequently

$$\eta_m(s) = \frac{m - m\sqrt{1 + 2s\rho_m(s)}}{\rho_m(s)} \sim -m\eta_*(s)\,,$$

where

$$\eta_*(s) = \frac{-1 + \sqrt{1 + 2V_* s}}{V_*}\,.$$

The next theorem follows easily. Let $\widehat{h}(\theta)$ be the LST of the DF $H(\,\cdot\,)$ in (4.10).

THEOREM 4.8.— *Suppose (4.10) and (4.11) are satisfied. If $\ell \to \infty$ and $m \to 0-$ in such a way that $\ell|m| \to \kappa > 0$, then m^2T converges in law to a limit whose LST is $\widehat{h}(\kappa\eta_*(s))$.*

If $H(\cdot)$ is degenerate, $L_0/\ell \Rightarrow$ const. > 0, then the limit law is inverse-Gaussian. This result supplements the limit-theorem derivations of this law discussed by Seshadri [33].

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