ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

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Annales de la faculté des sciences de Toulouse 6e série, tome S5 (1996), p. 195-215

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Annales de la Faculté des Sciences de Toulouse n° spécial Stieltjes, 1996

Compact Jacobi matrices: from Stieltjes to Krein and $M(a, b)$

WALTER VAN ASSCHE[1]

RÉSUMÉ. — Dans une note à la fin de son article *Recherches sur les fractions continues*, Stieltjes donne la condition nécessessaire et suffisante pour qu'une fraction continue soit représentée par un fonction méromorphe dans tout le plan complexe. Ce résultat est lié à l'étude des opérateurs de Jacobi compacts. On présente le développement moderne des idées de Stieltjes en accordant une attention particulière aux résultats de M. G. Krein. On décrit ensuite la classe $M(a, b)$ des perturbations de l'opérateur constant de Jacobi par un opérateur compact, qui trouve son origine dans le travail de Blumenthal en 1889.

ABSTRACT. — In a note at the end of his paper *Recherches sur les fractions continues*, Stieltjes gave a necessary and sufficient condition when a continued fraction is represented by a meromorphic function. This result is related to the study of compact Jacobi matrices. We indicate how this notion was developped and used since Stieltjes, with special attention to the results by M. G. Krein. We also pay attention to the perturbation of a constant Jacobi matrix by a compact Jacobi matrix, work which basically started with Blumenthal in 1889 and which now is known as the theory for the class $M(a, b)$.

MOTS-CLÉS : Opérateurs de Jacobi compacts, polynômes orthogonaux, perturbations compacts, théorie spectral

KEY-WORDS : Compact Jacobi matrices, orthogonal polynomials, compact perturbations, spectral theory

AMS Classification : 42C05 40A15 47A10

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1. A theorem of Stieltjes

Stieltjes' research in *Recherches sur les fractions continues* [25] deals with continued fractions of the form

$$\cfrac{1}{\alpha_1 z + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 z + \cdots + \cfrac{1}{\alpha_{2n} + \cfrac{1}{\alpha_{2n+1} z + \ddots}}}}},$$

where the coefficients α_k are real and positive. Such a continued fraction is nowadays known as an S-fraction, where the S stands for Stieltjes. By setting $b_0 = 1/\alpha_1$ and $b_n = 1/(\alpha_n \alpha_{n+1})$ for $n \geq 1$, and by the change of variable $z = 1/t$, this continued fraction can be written as

$$\cfrac{b_0}{z + \cfrac{b_1}{1 + \cfrac{b_2}{z + \cfrac{b_3}{1 + \cfrac{b_4}{z + \ddots}}}}} = \cfrac{b_0}{1 + \cfrac{b_1 t}{1 + \cfrac{b_2 t}{1 + \cfrac{b_3 t}{1 + \cfrac{b_4 t}{1 + \ddots}}}}}, \tag{1.1}$$

where $b_k > 0$, which results from the positivity of the α_k. Finally we can "contract" this fraction by using repeatedly the identity

$$z + \frac{a}{1 + b/c} = z + a - \frac{ab}{b + c},$$

and the original S-fraction then changes to

$$\cfrac{\lambda_0}{z + a_1 - \cfrac{\lambda_1}{z + a_2 - \cfrac{\lambda_2}{z + a_3 - \cfrac{\lambda_3}{z + a_4 - \ddots}}}}, \tag{1.2}$$

with

$$a_n = b_{2n-2} + b_{2n-1}, \quad \lambda_n = b_{2n} b_{2n-1}. \tag{1.3}$$

Such a continued fraction is known as a J-fraction, where the letter J stands for Jacobi. This J-fraction and the original S-fraction are "nearly" equivalent in the sense that the n-th convergent of the J-fraction is identical to the $2n$-th convergent of the S-fraction.

During his work in [25], in particular the sections § 68-69, Stieltjes shows that the convergents of (1.1) are given by

$$\frac{P_n(z)}{Q_n(z)} = b_0 t \, \frac{U_n(t)}{V_n(t)},$$

where U_n and V_n are polynomials, and the convergence of the series $\sum_{k=1}^{\infty} b_k$ is necessary and sufficient for the convergence

$$\lim_{n\to\infty} U_n(t) = u(t), \quad \lim_{n\to\infty} V_n(t) = v(t), \tag{1.4}$$

for every $t \in \mathbb{C}$, uniformly on compact sets. The functions u and v are thus both entire functions as they are uniform limits of polynomials. Hence the continued fraction (1.1) converges to

$$\lim_{n\to\infty} \frac{P_n(z)}{Q_n(z)} = \frac{1}{\alpha_1 z} \frac{u(1/z)}{v(1/z)} = F(z)$$

and the function F is meromorphic in the complex t-plane and meromorphic in the complex z-plane without the origin. Furthermore the zeros of U_n and V_n are all real (and negative) and they interlace (nowadays a well known property for orthogonal polynomials, observed a century ago by Stieltjes), hence F has infinitely many poles in the z-plane, which accumulate at zero. Stieltjes then writes this function as

$$F(z) = \frac{s_0}{\alpha_1 z} + \frac{1}{\alpha_1} \sum_{k=1}^{\infty} \frac{s_k}{z + r_k}, \tag{1.5}$$

where $\sum_{k=0}^{\infty} s_k = 1$ and $s_k > 0$ for every $k > 0$ ($s_0 \geq 0$), and then uses the Stieltjes integral (which he introduced precisely for such purposes) to write it as

$$F(z) = \int_0^{\infty} \frac{\mathrm{d}\Phi(u)}{z + u},$$

where Φ is a (discrete) distribution function with jumps of size s_k/α_1 at the points r_k ($k > 0$), and also at the origin if $s_0 > 0$. So Stieltjes has proved the following result in [25, § 68-69]:

THEOREM 1.1.— *Suppose that $b_k > 0$ for $k \in \mathbb{N}$. Then*

$$\sum_{k=1}^{\infty} b_k < \infty \tag{1.6}$$

is a necessary and sufficient condition in order that the continued fraction (1.1) converges to

$$F(z) = \int_0^\infty \frac{\mathrm{d}\Phi(u)}{z+u} = \frac{1}{\alpha_1 z} \frac{u(1/z)}{v(1/z)},$$

where u and v are entire functions and F is meromorphic for $z \in \mathbb{C} \setminus \{0\}$.

In fact the condition (1.6) gives the *separate convergence* (1.4) of the numerator and denominator of the convergents of the continued fraction and allows to write F as the ratio of two entire functions in $\mathbb{C} \setminus \{0\}$. Such separate convergence of numerator and denominator of a continued fraction was proved earlier by Sleshinskii [24] in 1888, who showed that separate convergence holds for the continued fraction (1.1) whenever $\sum_{n=0}^{\infty} |b_n| < \infty$. Apparently Stieltjes did not know Sleshinskii's result, and in fact Sleshinskii's paper seemed to have been unnoticed for a long time until Thron gave due credit to him in his survey on separate convergence [27].

In a note at the end of his paper [25], Stieltjes wants to find all the cases such that the continued fraction (1.1) converges to a function F meromorphic for $t \in \mathbb{C}$ (or $z \in \mathbb{C}\setminus\{0\}$). and not only those for which one has the separate convergence (1.4) as in the case when (1.6) holds. Stieltjes still assumes the b_k to be positive. In the note he proves the following extension of Theorem 1.1:

THEOREM 1.2.— *Suppose that $b_k > 0$ for $k \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} b_n = 0 \tag{1.7}$$

is a necessary and sufficient condition in order that the continued fraction (1.1) converges to

$$F(z) = \int_0^\infty \frac{\mathrm{d}\Phi(u)}{z+u}$$

where F is meromorphic for $z \in \mathbb{C} \setminus \{0\}$.

He proves the necessity of the condition in section 3 of the note, and the sufficiency in section 4. Obviously condition (1.6) implies (1.7), but the latter condition is weaker.

The condition (1.7) given in Theorem 1.2 is also sufficient in the case where the coefficients b_n are allowed to be complex. This result was proved by Van Vleck (see Section 6). But for complex b_n condition (1.7) is no longer necessary, as was shown by Wall [32], [33].

2. Compact Jacobi operators

For the J-fraction (1.2) the condition (1.7) is equivalent to

$$\lim_{n\to\infty} a_n = 0, \qquad \lim_{n\to\infty} \lambda_n = 0. \tag{2.1}$$

Furthermore, the stronger condition (1.6) is equivalent to

$$\sum_{n=0}^{\infty} (a_{n+1} + \sqrt{\lambda_n}) < \infty. \tag{2.2}$$

Hence, Stieltjes' results in terms of the J-fraction (1.2) show that the J-fraction converges to a meromorphic function F in $\mathbb{C}\setminus\{0\}$ if and only if (2.1) holds, and this meromorphic function is given by $(\alpha_1 z)^{-1}u(1/z)/v(1/z)$, with u and v entire functions, if and only if (2.2) holds. The convergence holds uniformly on compact sets of the complex plane excluding the poles of F, which accumulate at the origin. In Stieltjes' analysis he always worked with S-fractions for which $b_n > 0$ for all n, which gives certain restrictions on the coefficients a_n and λ_n of the J-fraction, but in fact the results also hold for general real a_n and positive λ_n.

With the coefficients of the J-fraction (1.2) one can construct an infinite tridiagonal *Jacobi matrix*

$$J = \begin{pmatrix} -a_1 & \sqrt{\lambda_1} & 0 & 0 & \cdots \\ \sqrt{\lambda_1} & -a_2 & \sqrt{\lambda_2} & 0 & \cdots \\ 0 & \sqrt{\lambda_2} & -a_3 & \sqrt{\lambda_3} & \cdots \\ 0 & 0 & \sqrt{\lambda_3} & \ddots & \ddots \\ 0 & 0 & 0 & \ddots & \end{pmatrix}.$$

With this infinite matrix we associate an operator, which we also call J, acting on the Hilbert space ℓ_2 of square summable sequences. If the coefficients a_n and λ_n are bounded, then this operator is a self-adjoint and bounded operator. which we call the *Jacobi operator*. In order to find eigenvalues and eigenvectors, one needs to solve systems of the form $Ju = xu$, where $u \in \ell_2$ and x is an eigenvalue, which, when it exists, will be real due to the self-adjointness. This readily leads to a three-term recurrence relation

$$xu_n = \sqrt{\lambda_n} u_{n-1} - a_{n+1} u_n + \sqrt{\lambda_{n+1}} u_{n+1}, \qquad n \geq 0, \tag{2.3}$$

where $u_{-1} = 0$. The solution when $u_0 = 1$ is such that $u_n = p_n(x)$ is a polynomial of degree n in the variable x and this is precisely the denominator polynomial for the n-th convergent of the J-fraction. Another solution, with $u_0 = 0$ and $u_1 = 1$ gives a polynomial $u_n = p_{n-1}^{(1)}(x)$ of degree $n-1$, and this is the numerator polynomial for the n-th convergent of the J-fraction. Applying the spectral theorem to the Jacobi operator J shows that there is a positive measure μ on the real line such that J is unitarily isomorphic to the multiplication operator M acting on $L_2(\mu)$ in such a way that the unit vector $e_0 = (1, 0, 0, \ldots) \in \ell_2$ (which is a cyclic vector) is mapped to the constant function $x \mapsto 1$, and $J^n e_0$ is mapped to the monomial $x \mapsto x^n$. A simple verification, using the three-term recurrence relation, shows that the unitary isomorphism also maps the n-th unit vector

$$e_n = (\underbrace{0, 0, \ldots,}_{n \text{ zeros}} 1, 0, 0, \ldots) \in \ell_2$$

to the polynomial p_n, and since $\langle e_n, e_m \rangle = \delta_{m,n}$ in the Hilbert space ℓ_2, this implies that $\langle p_n, p_m \rangle = \int p_n(x) p_m(x)\, d\mu(x) = \delta_{m,n}$ in the Hilbert space $L_2(\mu)$, showing that we are dealing with orthogonal polynomials. For more regarding this connection between spectral theory and orthogonal polynomials, see e.g., [9], [20], [10, § XII.10, pp. 1275-1276, [26, pp. 530-614]). Unfortunately, the spectral theorem (and the Riesz representation theorem) came decades after Stieltjes so that Stieltjes was not using the terminology of orthogonal polynomials, even though he clearly was aware of this peculiar orthogonality property of the denominator polynomials, as can be seen from section 11 in [25].

The spectrum of the operator J corresponds to the support of the *spectral measure* μ. This spectrum is real since J is self-adjoint. The measure μ in general consists of an absolutely continuous part, a singular continuous part and an atomic (or discrete) part, and the supports of these three parts correspond to the absolutely continuous spectrum, the singular continuous spectrum and the point spectrum. The point spectrum is the closure of the set of eigenvalues of J, and thus all $x \in \mathbb{R}$ for which $\sum_{n=0}^{\infty} p_n^2(x) < \infty$ are in the point spectrum. Moreover, one can show that

$$\mu(\{x\}) = \left(\sum_{n=0}^{\infty} p_n^2(x) \right)^{-1} .$$

In terms of the Jacobi operator J, Stieltjes' results can be formulated as follows. The spectral measure μ corresponds to the distribution function Φ in (1.5) and is purely atomic and the point spectrum has 0 as its only accumulation point if and only if (2.1) holds. When (2.2) holds, then the eigenvalues are the reciprocals of the zeros of the entire function v, which is obtained as the limit

$$\lim_{n\to\infty} x^{-n} p_n\left(\frac{1}{x}\right) = v(x) .$$

Since v is the uniform limit (on compacta) of a sequence of polynomials, it follows that v is an entire function. In fact, it is a canonical product completely determined by its zeros, and the order of this canonical product is less than or equal to one. Therefore the sum $\sum_{k=0}^{\infty} 1/|z_k|$ converges, where z_k are the zeros of the entire function v.

In modern terminology, the condition (2.1) implies that the Jacobi operator J is a *compact operator*. In general, a linear operator A acting on a Hilbert space $\mathcal{H}$ is called compact if it maps the unit ball in $\mathcal{H}$ onto a set whose closure is compact. In other words, A is compact if for every bounded sequence ψ_n $(n \in \mathbb{N})$ of elements in the Hilbert space $\mathcal{H}$, there is always a subsequence $A\psi_n$ $(n \in \Lambda \subset \mathbb{N})$ that converges. Compact operators are sometimes also known as completely continuous operators, but this terminology is not so much in use anymore. It is not hard to see that for an operator associated with a banded matrix A of bandwidth $2m+1$, i.e. $A_{i,j} = 0$ whenever $|i-j| > m$, for some fixed m, compactness is equivalent with the condition that $\lim_{n\to\infty} A_{n,n+k} = 0$ for every k with $-m \leq k \leq m$ [1, § 31, pp. 93-94]. Indeed, a diagonal matrix $(m = 0)$ is compact if and only if the entries on the diagonal tend to 0. A banded matrix is of the form $A = A_0 + \sum_{k=1}^{m} (V^* A_k + B_k V)$, where V is the shift operator and A_0, A_k, B_k $(k = 1, 2, \ldots, m)$ are diagonal operators, and since V is bounded and compact operators form a closed two-sided ideal in the set of bounded operators, this shows that A is compact if and only if each of the diagonal matrices A_0, A_k, B_k is compact. Hence a Jacobi operator is compact if and only if (2.1) holds. The simplest linear operators are, of course, operators acting on a finite dimensional Hilbert space, in which case we are dealing with matrices. Next in degree of difficulty are the compact operators, which can be considered as limits of finite dimensional matrices. Indeed, the structure of the spectrum of a compact operator is quite similar to the spectrum of a matrix since it is a pure point spectrum with only one accumulation point at the origin, a result known as the *Riesz–Schauder theorem.* This is in perfect agreement with Stieltjes' result (Theorem 1.2) and the poles of the meromorphic function F in fact correspond to the point spectrum (the eigenvalues) of the operator J. So, Stieltjes' theorem is an anticipation of the Riesz–Schauder theorem (proved by Schauder in 1930) regarding the spectrum of a compact operator, but restricted to tridiagonal operators. Similarly, Stieltjes' Theorem 1.1 is related to a subclass of the compact operators, namely those compact operators for which $\sum_{k=0}^{\infty} |x_k| < \infty$, where x_k are the eigenvalues of the operators. These operators are known as *trace class operators.* One can show that a banded operator A is trace class if $\sum_{n=0}^{\infty} \sum_{k=-m}^{m} |A_{n,n+k}| < \infty$, hence the condition (2.2) means that J is trace class, in which case the eigenvalues are in ℓ_1.

3. Some orthogonal polynomials with compact Jacobi matrix

Stieltjes' theorems were rediscovered half a century later during the investigation of (modified) Lommel polynomials. First, H. M. Schwartz [23] considered continued fractions of the form (1.1) but allowed the b_k to be complex, and the more general J-fraction (1.2) with complex λ_k and a_k.

Later Dickinson [7], Dickinson, Pollak, and Wannier [8], and Goldberg [13] considered the polynomials $h_{n,\nu}$ satisfying the recurrence relation

$$h_{n+1,\nu}(x) = 2x(n+\nu)h_{n,\nu}(x) - h_{n-1,\nu}(x),$$

with initial conditions $h_{-1,\nu} = 0$ and $h_{0,\nu} = 1$. These polynomials appear in the study of Bessel functions and allow to express a Bessel function $J_{n+\nu}$ as a linear combination of two Bessel functions J_ν and $J_{\nu-1}$ as

$$J_{\nu+n}(x) = h_{n,\nu}(1/x)J_\nu(x) - h_{n-1,\nu+1}(1/x)J_{\nu-1}(x),$$

reducing the investigation of the asymptotic behaviour of Bessel functions with high index to the investigation of the polynomials $h_{n,\nu}$, which are known as Lommel polynomials. considering $p_n = \sqrt{(n+\nu)/\nu}\, h_n$, the three-term recurrence is of the form

$$xp_n(x) = \frac{1}{2\sqrt{(n+\nu)(n+\nu+1)}}\, p_{n+1}(x) + \frac{1}{2\sqrt{(n+\nu)(n+\nu-1)}}\, p_{n-1}(x),$$

which corresponds to a J-fraction and Jacobi operator with coefficients $a_n = 0$ and $\lambda_n = \left[4(n+\nu)(n+\nu-1)\right]^{-1}$. Clearly $\lim_{n\to\infty} \lambda_n = 0$ so that Stieltjes' Theorem 1.2 holds, and we can conclude that the Lommel polynomials are orthogonal with respect to an atomic measure with support a denumerable set with accumulation point at the origin. The spectrum of the Jacobi operator can be identified completely by investigating the asymptotic behaviour of the Lommel polynomials, and it turns out that the spectrum consists of the closure of the set $\{1/j_{k,\nu-1} : k \in \mathbb{Z}\}$, where $j_{k,\nu-1}$ are the zeros of the Bessel function $J_{\nu-1}$. These points indeed accumulate at the origin, but the origin itself is not an eigenvalue of the operator J. Note that Goldberg [13] observed that the analysis of Dickinson, Pollak, and Wannier [7], [8] was incomplete since they did not give any information whether or not the accumulation point 0 had positive mass. The Jacobi operator in this case is not trace class, since (2.2) is not valid. This is compatible with the asymptotic behaviour $j_{n,\nu} \sim \pi n$ for the zeros of the Bessel function.

For the Bessel functions there are several q-extensions, with corresponding Lommel polynomials. For the Jackson q-Bessel functions the q-Lommel polynomials were introduced by Ismail [14] who showed that these polynomials are orthogonal on a denumerable set similar as for the Lommel polynomials but involving the zeros of the Jackson q-Bessel functions. For the Hahn–Exton q-Bessel function the q-analogue of the Lommel polynomials turn out to be Laurent polynomials and in [16] it is shown that they obey orthogonality with respect to a moment functional acting on Laurent polynomials.

Other families of orthogonal polynomials with a compact Jacobi matrix include the Tricomi–Carlitz polynomials, for which the asymptotic behaviour was recently studied by Goh and Wimp [12]. These polynomials satisfy the three-term recurrence relation

$$(n+1)f_{n+1}(x) - (n+\alpha)xf_n(x) + f_{n-1}(x) = 0,$$

with $f_0 = 1$ and $f_{-1} = 0$. For the orthonormal polynomials $[n!(\alpha+n)/\alpha]^{1/2} f_n$ this gives $a_n = 0$ and $\lambda_n = n/[(n+\alpha)(n+\alpha-1)]$, so that $\lambda_n \to 0$ but the Jacobi operator is not trace class. The spectral measure now is supported on the set $\{\pm 1/\sqrt{k+\alpha} : k = 0, 1, 2, \ldots\}$, which is indeed a denumerable set with an accumulation point at the origin, and the elements are not summable. The Tricomi–Carlitz polynomials are also known as the Carlitz–Karlin–McGregor polynomials [3] because Karlin and McGregor showed that they turn out to be the orthogonal polynomials for the imbedded random walk of a queueing process with infinitely many servers and identical service time rates. There are a number of other examples of orthogonal polynomials arising from birth-and-death processes for which the Jacobi operator is compact. Van Doorn [29] showed that the orthogonal polynomials for a queueing process studied by B. Natvig in 1975, where potential customers are discouraged by queue length, are orthogonal on a denumerable set accumulating at a point. The birth-and-death process governing this queueing process has birth rates $\lambda_n = \lambda/(n+1)$ $(n \geq 0)$, which expresses that the rate of new customers decreases as the number n of customers in the queue increases, and death rates $\mu_0 = 0$ and $\mu_n = \mu$, which expresses that the service time does not depend on the queue length. The corresponding orthogonal polynomials then satisfy the three-term recurrence relation

$$-xQ_n(x) = \lambda_n Q_{n+1}(x) - (\lambda_n + \mu_n)Q_n(x) + \mu_n Q_{n-1}(x).$$

The orthonormal polynomials q_n then satisfy

$$xq_n(x) = \sqrt{\lambda_n \mu_{n+1}}\, q_{n+1}(x) + (\lambda_n + \mu_n)\, q_n(x) + \sqrt{\lambda_{n-1}\mu_n}\, q_{n-1}(x),$$

and since

$$\lim_{n\to\infty} \lambda_{n-1}\mu_n = 0, \qquad \lim_{n\to\infty} \lambda_n + \mu_n = \mu,$$

it follows that these polynomials correspond to a Jacobi matrix J which can be written as $J = \mu I + J_p$, where J_p is a compact operator. Hence the orthogonality measure is denumerable with only one accumulation point at μ. Van Doorn gives a complete description of the support of the measure. Chihara and Ismail [6] studied these polynomials in more detail and showed that the point μ is not a mass point of the orthogonality measure, even though it is an accumulation point of mass points. Chihara and Ismail also study the queueing process with birth and death rates

$$\lambda_n = \frac{\lambda}{n+a}, \quad \mu_{n+1} = \frac{\mu(n+1)}{n+a}, \qquad n \geq 0,$$

for which the Jacobi operator is again of the form $J = \mu + J_p$ with J_p a compact operator. The case $a = 1$ corresponds to the situation studied by Natvig and van Doorn. Another way to model a queueing process where potential customers are discouraged by queue length is to take

$$\lambda_n = \nu q^n, \quad \mu_n = \mu(1 - q^n), \qquad 0 < q < 1,$$

in which case the decrease is exponential. The corresponding orthogonal polynomials turn out to be q-polynomials of Al-Salam and Carlitz [5, § 10, p. 195].

The orthogonal polynomials U_n associated with the Rogers–Ramanujan continued fraction [2]

$$U_{n+1}(x) = x(1 + aq^n)U_n(x) - bq^{n-1}U_{n-1}(x), \qquad 0 < q < 1,$$

have a compact Jacobi operator, which in addition belongs to the trace class. Several orthogonal polynomials of basic hypergeometric type (q-polynomials) have a Jacobi matrix which is a compact operator that belongs to the trace class, so that Stieltjes' Theorem 1.1 can be used to find the orthogonality relation for these polynomials. Often this orthogonality relation can be written in terms of the q-integral

$$\int_0^b f(t)\, \mathrm{d}_q t = b(1 - q) \sum_{n=0}^{\infty} f(bq^n) q^n ,$$

and for $a < 0 < b$ this q-integral is given by

$$\int_a^b f(t)\, \mathrm{d}_q t = \int_0^b f(t)\, \mathrm{d}_q t + \int_0^{-a} f(-t)\, \mathrm{d}_q t ,$$

so that the support of the measure is the geometric lattice $\{aq^k, bq^k , k = 0, 1, 2, \ldots\}$ which is denumerable and has 0 as the only accumulation point. The orthogonal polynomials of this type are the *big q-Jacobi polynomials*, the *big q-Laguerre polynomials*, the *little q-Jacobi polynomials*, the *little q-Laguerre polynomials* (also known as the *Wall polynomials* [5, § 11 on p. 198]), the *alternative q-Charlier polynomials*, and the *Al-Salam–Carlitz polynomials*, which we already mentioned earlier. These polynomials, with references to the literature, can be found in [17].

4. Krein's theorem

The most interesting extension of Stieltjes' Theorem 1.2 on compact Jacobi operators was made by M. G. Krein [18]. He considered operators of the form $g(J)$, where J is a Jacobi operator and g a polynomial. It is not so hard to see that the matrix for the operator $g(J)$ is banded and symmetric, and when J is a bounded operator, then $g(J)$ is also bounded. The bandwidth of $g(J)$ is $2m+1$ when g is a polynomial of degree m. In [18], Krein first shows that a banded operator A with matrix $(a_{i,j})_{i,j\geq 0}$ is compact if and only if $\lim_{i,j\to\infty} a_{i,j} = 0$. But his main result is:

THEOREM 4.1 (Krein).— *In order that the spectrum of J consists of a bounded set with accumulation points in $\{x_1, x_2, \ldots, x_m\}$, it is necessary and sufficient that J is a bounded operator and $g(J)$ is a compact operator, where $g(x) = (x-x_1)(x-x_2)\cdots(x-x_m)$.*

The polynomial g of lowest degree for which $g(J)$ is a compact operator is known as the *minimal polynomial*, and the zeros of the minimal polynomial correspond exactly to the accumulation points of the spectrum of J. Krein explicitly refers to Stieltjes' work, which is a special case where the minimal polynomial is the identity $g : x \mapsto x$ and the spectrum is a compact subset of $(-\infty, 0]$ or $[0, \infty)$ if we make a reflection through the origin. Krein mentions a remark by N. I. Akhiezer that, by changing Stieltjes' reasoning somewhat, one may by his method obtain the result for one accumulation point without the restriction that the spectrum is on the positive (or negative) real axis. However, Krein finds it improbable that the result for $m > 1$ accumulation points could be proved by Stieltjes' method.

In terms of the corresponding orthogonal polynomials, Krein's theorem says that when $g(J)$ is a compact operator, then the polynomials will be orthogonal with respect to a discrete measure μ and the support of this measure has accumulation points at the zeros of g. It is not so difficult to prove that orthogonal polynomials can have at most one zero in an interval $[a, b]$ for which $\mu([a,b]) = 0$. This means that also the zeros of the orthogonal polynomials will cluster around these zeros of g.

In terms of the continued fraction (1.2) Krein's result implies that the continued fraction will converge to a function F which is meromorphic in $\mathbb{C}\setminus\{x_1, x_2, \ldots, x_m\}$ and the poles of this meromorphic function accumulate at the zeros of g.

Recently it has been shown [11] that Krein` theorem can be restated in terms of orthogonal matrix polynomials, where the polynomials have matrix coefficients with matrices from $\mathbb{R}^{m\times m}$. Orthogonal matrix polynomials satisfy a three-term recurrence relation with matrix coefficients, and with these matrix recurrence coefficients one can form a block Jacobi matrix, which defines a self-adjoint operator, but now one does not have a single cyclic vector, but a set of m cyclic vectors. Consequently, the spectrum is not simple and the spectral measure is a (positive definite) $m \times m$ matrix of measures $M = (\mu_{i,j})_{1\leq i,j\leq m}$. Starting with an ordinary Jacobi matrix, the matrix $g(J)$ is banded and can be considered as a block Jacobi matrix, where the subdiagonals are triangular matrices. If $g(J)$ is compact, then by the Riesz–Schauder theorem the spectrum $\sigma(g(J))$ of $g(J)$ has only one accumulation point at the origin, which means that the spectral matrix of measures is discrete and the support, which is the support of the trace measure $\sum_{j=1}^{m} \mu_{j,j}$, has only one accumulation point at the origin. The spectral matrix of measures for $g(J)$ is connected with the spectral measure for μ and in particular $\sigma(J) \subset g^{-1}(\sigma(g(J)))$, and since $\sigma(g(J))$ has only one accumulation point at 0, it follows that the spectrum $\sigma(J)$ of J has accumulation points at $g^{-1}(0)$, which are the zeros of g.

5. The class $M(a,b)$ and Blumenthal's theorem

Compact Jacobi operators have also shown to be of great use in studying orthogonal polynomials on an interval. In this section we will change notation and consider the three-term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x)\,, \tag{5.1}$$

so that $(-a_{n+1}, \sqrt{\lambda_n})$ in (2.3) corresponds to (b_n, a_n) in (5.1). This notation is more common nowadays. If we consider orthogonal polynomials satisfying a three-term recurrence relation with constant coefficients,

$$x\tilde{p}_n(x) = \frac{a}{2}\,\tilde{p}_{n+1}(x) + b\,\tilde{p}_n(x) + \frac{a}{2}\,\tilde{p}_{n-1}(x)\,,$$

with initial values $\tilde{p}_0 = 1$ and $\tilde{p}_{-1} = 0$, then these polynomials are given by

$$\tilde{p}_n(x) = U_n\left(\frac{x-b}{a}\right),$$

where the U_n are the *Chebyshev polynomials of the second kind*, defined as

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad x = \cos\theta.$$

For these polynomials the orthogonality relation is

$$\frac{2}{\pi}\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}\,\mathrm{d}x = \delta_{m,n},$$

which follows easily from the orthogonality of the trigonometric system $\{\sin k\theta, k = 1, 2, 3, \ldots\}$. Hence, by an affine transformation, the polynomials $\tilde{p}_n$ ($n \in \mathbb{N}$) obey the orthogonality conditions

$$\frac{2}{\pi a^2}\int_{b-a}^{b+a} \tilde{p}_n(x)\tilde{p}_m(x)\sqrt{a^2-(x-b)^2}\,\mathrm{d}x = \delta_{m,n}.$$

Hence these polynomials are orthogonal on the interval $[b-a, b+a]$ and they will serve as a comparison system for a large class of polynomials for which the essential support is $[b-a, b+a]$. A measure μ on the real line can always be decomposed as $\mu = \mu_{ac} + \mu_{sc} + \mu_d$, where μ_{ac} is absolutely continuous, μ_{sc} is singular and continuous, and μ_d is discrete (or atomic). The essential support of μ corresponds to the support of $\mu_{ac} + \mu_{sc}$ together with the accumulation points of the support of μ_d. Hence, if a measure has essential support equal to $[b-a, b+a]$ then μ can have mass points outside $[b-a, b+a]$, but the accumulation points should be on this interval. The Jacobi operator for $\tilde{p}_n$ has a matrix with constant values

$$J_0 = \begin{pmatrix} b & a/2 & 0 & 0 & \ldots \\ a/2 & b & a/2 & 0 & \ldots \\ 0 & a/2 & b & a/2 & \ldots \\ 0 & 0 & a/2 & \ddots & \ddots \\ 0 & 0 & 0 & \ddots & \end{pmatrix}.$$

If we perturb this operator by adding to it a compact Jacobi operator J_p, so that we obtain a Jacobi operator

$$J = \begin{pmatrix} b_0 & a_1 & 0 & 0 & \ldots \\ a_1 & b_1 & a_2 & 0 & \ldots \\ 0 & a_2 & b_2 & a_3 & \ldots \\ 0 & 0 & a_3 & \ddots & \ddots \\ 0 & 0 & 0 & \ddots & \end{pmatrix} = J_0 + J_p\,,$$

then the Jacobi operator J has entries for which

$$\lim_{n\to\infty} a_n = \frac{a}{2}, \qquad \lim_{n\to\infty} b_n = b, \tag{5.2}$$

and we say that J is a compact perturbation of J_0. There is a very nice result regarding compact perturbations of operators, which is quite useful in the analysis of orthogonal polynomials [20].

THEOREM 5.1 (H. Weyl).— *Suppose A is a bounded and self-adjoint operator and C is a compact operator, then $A+C$ and A have the same essential spectrum.*

Applied to our analysis of orthogonal polynomials, this means that the orthogonal polynomials corresponding with a Jacobi operator $J = J_0 + J_p$, where J_p is compact, have an essential spectrum on $[b-a, b+a]$, hence the orthogonality measure μ for these polynomials has support $[b-a, b+a] \cup E$, where E is at most denumerable with accumulation points only at $b \pm a$. Compact perturbations of the operator J_0 occur quite often, and in 1979 Paul Nevai [21] introduced the terminology $M(a,b)$ for the class of orthogonal polynomials for which (5.2) holds. The investigation of the class $M(a,b)$, however, goes back almost a century. The first to consider this class was O. Blumenthal, a student of Hilbert, whose Inaugural Dissertation [4] was devoted to this class. In his dissertation, Blumenthal proves the following result regarding the continued fraction (1.1).

THEOREM 5.2 (Blumenthal).— *Streben die Grössen b_n den endlichen von 0 verschiedenen limites:*

$$\lim b_{2n} = \ell, \qquad \lim b_{2n+1} = \ell_1 \tag{5.3}$$

zu, so liegen innerhalb des ganzen Intervalles

$$\left\{-(2\sqrt{\ell\ell_1}+\ell+\ell_1) \le z \le 2\sqrt{\ell\ell_1}-\ell-\ell_1\right\}$$

überall dicht Nullstellen der Funktionen-Reihe Q_{2n}, ausserhalb desselben nähern sich die Nullstellen mit wachsendem n einer endlichen Zahl von Grenzpunkten. (If the b_n converge to positive limits $b_{2n} \to \ell$, $b_{2n+1} \to \ell_1$, then the zeros of the sequence of functions Q_{2n} will be dense on the interval $\left[-(2\sqrt{\ell\ell_1}+\ell+\ell_1)\,,\, 2\sqrt{\ell\ell_1}-\ell-\ell_1\right]$, outside of which the zeros for increasing n will approach a finite number of limit points.)

In terms of the J-fraction, the convergence in (5.3) is equivalent with

$$\lim_{n\to\infty} b_n = -\ell - \ell_1 = b, \qquad \lim_{n\to\infty} a_n = \sqrt{\ell\ell_1} = \frac{a}{2},$$

which corresponds to the class $M(a,b)$, and Blumenthal's conclusion is that the zeros of the denominator polynomials (the orthogonal polynomials) are dense on the interval $[b-a, b+a]$ (the essential spectrum) and that outside this interval the zeros converge to a finite number of limit points. The latter statement, however, turns out to be incorrect, since outside the interval $[b-a, b+a]$ there can be a denumerable number of limit points of the zeros, which can only accumulate at the endpoints $b \pm a$, which means that outside $[b-a-\epsilon, b+a+\epsilon]$ there are a finite number of limit points, and this is true for every $\epsilon > 0$ (but not for $\epsilon = 0$). Except for this, Blumenthal's theorem is really a beautiful result and a nice complement to Stieltjes' Theorem 1.2 which deals with the special case $\ell = \ell_1 = 0$.

Blumenthal's proof of the theorem was based on a result by Poincaré [22] which describes the ratio asymptotic behaviour of the solution of a finite order linear recurrence relation when the coefficients in the recurrence relation are convergent.

THEOREM 5.3 (Poincaré). — *If in the recurrence relation*

$$y_{n+k} = \sum_{j=0}^{k-1} a_{j,n} y_{n+j}$$

the recurrence coefficients have limits

$$\lim_{n\to\infty} a_{j,n} = a_j, \qquad 0 \le j < k,$$

and if the roots ξ_i $(i = 1, 2, \ldots, k)$ of the characteristic equation

$$z^k = \sum_{j=0}^{k-1} a_j z^j$$

all have different modulus, then either $y_n = 0$ for all $n \ge n_0$ or there is a root ξ_ℓ of the characteristic equation such that

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \xi_\ell.$$

For a nice and comprehensible proof, see [19]. The case relevant for the class $M(a, b)$ corresponds to the second order recurrence relation (5.1) for orthogonal polynomials corresponding to a Jacobi matrix J, and (5.2) expresses the fact that the recurrence coefficients have limits. The characteristic equation then is

$$2xz = az^2 + 2bz + a$$

for which the roots are

$$\xi_1 = \frac{x - b + \sqrt{(x-b)^2 - a^2}}{a}, \quad \xi_2 = \frac{x - b - \sqrt{(x-b)^2 - a^2}}{a}.$$

These two roots have equal modulus whenever $(x-b)^2 - a^2 \leq 0$, hence for $x \in [b-a, b+a]$, so that this simple observation already gives the important interval. Poincaré's theorem then shows that for $x \notin [b-a, b+a]$ the ratio $p_{n+1}(x)/p_n(x)$ converges to one of the two roots of the characteristic equation. Poincaré's theorem does not tell you which root, but in the case of orthogonal polynomials, we know that for x large enough the ratio $p_{n+1}(x)/p_n(x)$ behaves like $2x/a$ as $x \to \infty$, hence we need to choose the root with largest modulus whenever x is large enough. This asymptotic behaviour can then be used to obtain information about the set of limit points of the zeros of the orthogonal polynomials, which is how Blumenthal arrived at his results. For a contemporary approach, see [20].

Blumenthal's result thus deals with compact perturbations of Chebyshev polynomials. If more can be said of the (compact) perturbation operator $J - J_0$, then more can also be said of the spectral measure for the Jacobi operator J. If $J - J_0$ is a trace class operator, i.e.,

$$\sum_{k=1}^{\infty} \left(\left| a_k - \frac{a}{2} \right| + |b_k - b| \right) < \infty,$$

then there is a beautiful theorem by Kato and Rosenblum [15, Thm. 4.4 on p. 540] that tells something about the nature of the spectral measure on the essential spectrum [9].

THEOREM 5.4 (Kato–Rosenblum).— *Suppose A is a self-adjoint operator in a Hilbert space $\mathcal{H}$ and C is a trace class operator in $\mathcal{H}$ and that $A+C$ is self-adjoint. Then the absolutely continuous parts of A and $A+C$ are unitarily equivalent.*

The spectral measure for the operator J_0 is absolutely continuous on $(b-a, b+a)$, hence the Kato–Rosenblum theorem implies that the orthogonal polynomials corresponding to the Jacobi operator J are orthogonal with respect to a measure with an absolutely continuous part in $(b-a, b+a)$. The measure can still have a discrete part outside $(b-a, b+a)$. For even more information regarding this absolutely continuous part, one needs an even stronger condition such as [28]

$$\sum_{k=1}^{\infty} k \left(\left| a_k - \frac{a}{2} \right| + |b_k - b| \right) < \infty,$$

in which case $\mu'(x) = g(x)(x-b-a)^{\pm 1/2}(x-b+a)^{\pm 1/2}$, where g is continuous and strictly positive on $[b-a, b+a]$. Furthermore, in this case the number of mass points outside $[b-a, b+a]$ is finite and the endpoints $b \pm a$ are not mass points.

6. Van Vleck's results

The class $M(a, b)$ received a lot of attention the past two decades, starting with Nevai in [21] who introduced the terminology and obtained various results. See [28] and the references given there for a survey on the class $M(a, b)$. In the mean time it has become clear that this class has already been studied in detail almost a century ago by Blumenthal (see previous section), but also by Edward B. Van Vleck. He studied the class in terms of continued fractions, much in the spirit of Stieltjes who also studied the class of compact operators in terms of continued fractions. In [30] and [31] Van Vleck considers continued fractions as in (1.1) for which the coefficients converge. He does not require the restrictions $b_n > 0$ and allows the coefficients to be complex.

THEOREM 6.1 (Van Vleck). — *If in the continued fraction*

$$\cfrac{b_0}{1 + \cfrac{b_1 t}{1 + \cfrac{b_2 t}{1 + \cfrac{b_3 t}{1 + \cfrac{b_4 t}{1 + \ddots}}}}} \tag{6.1}$$

one has $\lim_{n\to\infty} b_n = b$, *then the continued fraction will converge in* $\mathbb{C}$ *except*

1) *along the whole or part of a rectilinear cut from* $-1/4b$ *to* ∞ *with an argument equal to that of the vector from the origin to* $-1/4b$,
2) *possibly at certain isolated points* $p_1, p_2, p_3, \ldots$

The limit of the continued fraction is holomorphic in $\mathbb{C}\backslash[-1/4b, \infty)$ *except at the points* $p_1, p_2, \ldots$ *which are poles.*

Van Vleck's proof is again based on Poincaré's theorem. Van Vleck actually shows that the exceptional points can have accumulation points on the cut. In case all the b_n are positive these exceptional points can only accumulate at the point $-1/4b$. Van Vleck also considers the corresponding J-fraction (1.2). This will also be a continued fraction with converging coefficients, and if $b_n \to b$, then obviously

$$a_n = b_{2n-2} + b_{2n-1} \to 2b, \quad \lambda_n = b_{2n}b_{2n-1} \to b^2.$$

The limit of the continued fraction (1.2) is equal to $F(1/z)$, where $F(t)$ is the limit of the continued fraction (6.1). Hence $F(1/z)$ is analytic in the complex plane cut along the segment $[-4b, 0]$, except at the points $1/p_1$, $1/p_2, \ldots$ which are poles. The cut $[-4b, 0]$ is indeed the essential spectrum, since this J-fraction is one that corresponds to the class $M(2b, -2b)$. Van Vleck also considers the limiting case $b \to 0$ and thus was able to generalize Stieltjes' Theorem 1.2 for complex coefficients, showing that (1.7) is a sufficient condition (but not necessary condition, see Wall [32], [33]) for a continued fraction to converge to a meromorphic function.

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