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Lefschetz number and degree of a self-map(*)

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RÉSUMÉ. — Soit X un CW complexe fini connexe. Deux self-applications de X induisant la même application en homologie ont même nombre de Lefschetz et même degré. Pour les espaces elliptiques 1-connexes et les espaces formels, des formules relient le nombre de Lefschetz au polynôme caractéristique des applications induites en homotopie.

ABSTRACT. — Let X be a connected finite CW complex. Two self-maps of X inducing the same maps on the homotopy groups have same Lefschetz number and same degree. For simply connected elliptic spaces and formal spaces there are formulae connecting the Lefschetz number and the characteristic polynomials of the induced maps on $\pi_n(X) \otimes \mathbb{Q}$.

1. Introduction

Let X be a connected topological space that has the homotopy type of a finite CW complex, and let f be a self-map. We denote by $H_n(f)$ the linear map induced by f in rational homology. The number $\lambda_f = \sum_i (-1)^i \operatorname{tr} H_i(f)$ is called the Lefschetz number of f. It is well known that two homotopic self-maps f and g have the same Lefschetz number.

We here prove the following result.

THEOREM 1.1.— *Suppose $H_*(\Omega f; \mathbb{Q}) = H_*(\Omega g; \mathbb{Q})$, then the Lefschetz numbers of f and g are equal, $\lambda_f = \lambda_g$. Moreover, if X is a manifold then* $\deg(f) = \deg(g)$.

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This results suggests the following problem: Is it possible to deduce the number λ_f from the sequence of linear maps $\pi_n(f) \otimes \mathbb{Q}$, when the space is 1-connected?

In the general case the answer is negative. For instance the spaces

$$X = P^2(\mathbb{C}) \times (S^3 \vee S^2) \quad \text{and} \quad Y = S^2 \times (S^5 \vee S^2 \vee S^4)$$

have the same rational homotopy groups but not the same Euler–Poincaré characteristic $\chi(X) = 3$, $\chi(Y) = 4$. Moreover the spaces X and Y appear as total spaces of homotopy fibrations with the same homotopy fibre $Z = \bigvee_{n \geq 4} S^n \vee (\bigvee_{n \geq 6} S^n)$ and with basis X' and Y' finite CW complexes with the same Euler–Poincaré characteristic:

$$X' = P^2(\mathbb{C}) \times S^3 \times S^2\,, \quad Y' = S^2 \times S^5 \times S^2\,.$$

Recall that a finite simply connected CW complex is *elliptic* if

$$\sum_n \dim \pi_n(X) \otimes \mathbb{Q} < \infty\,.$$

In the elliptic case a result of Halperin [8] shows that the Lefschetz number of a self-map f can be deduced from the induced maps in homotopy.

THEOREM 1.2 [8].— *Let f be a self-map. Denote by A_n the matrix representing the linear map $\pi_n(f) \otimes \mathbb{Q}$ in some basis. We then have*

$$\lambda_f = \lim_{t \to 1} \prod_n \frac{\det(1 - t^{2n+2} A_{2n+1})}{\det(1 - t^{2n} A_{2n})}\,.$$

An elliptic space is a Poincaré duality space. Our next result is a formula giving the degree of f in terms of the induced maps in rational homotopy.

THEOREM 1.3.— *Denote by λ_i the eigenvalues of $\pi_{\mathrm{odd}}(f) \otimes \mathbb{Q}$ and by μ_j the eigenvalues of $\pi_{\mathrm{even}}(f) \otimes \mathbb{Q}$, then*

$$\deg(f) = \frac{\prod_j \mu_j}{\prod_i \lambda_i}\,.$$

2. Bar construction

Let A be an augmented graded differential $\mathbb{Q}$-algebra,

$$A = \bigoplus_{n \geq 0} A_n\,, \quad d : A_n \to A_{n-1}\,, \quad \varepsilon : A \to \mathbb{Q}\,.$$

The *bar construction* on A is the coalgebra $\big(B(A), D\big) = \big(T(s\overline{A}), D\big)$ with $s\overline{A} = \ker \varepsilon$ the augmentation ideal, and $D := D_1 + D_2$. The differentials D_i are given by:

$$D_1\big[sa_1 \mid \ldots \mid sa_k\big] := -\sum_{i=1}^{k} (-1)^{\varepsilon_i} \big[sa_1 \mid \ldots \mid sda_i \mid \ldots \mid sa_k\big]$$

and

$$D_2\big[sa_1 \mid \ldots \mid sa_k\big] := \sum_{i=2}^{k} (-1)^{\varepsilon_i} \big[sa_1 \mid \ldots \mid sa_{i-1}a_i \mid \ldots \mid sa_k\big]$$

with $\varepsilon_i = \sum_{j<i} |sa_j|$.

By filtering $T(s\overline{A})$ by the coalgebras $T^{\leq k}(s\overline{A})$, we obtain a spectral sequence satisfying

$$(E^1, d_1) \cong \big(B\big(H_*(A), D_2\big)\big)\,, \quad E^2 \cong H_*\left(B\big(H_*(A), D_2\big)\right)$$

and converging to $H_*\big(B(A)\big)$ [3]. One interesting property of the bar construction is the following theorem of J.- C. Moore [11].

THEOREM [11].— *If X is connected, then there exists a natural quasi-isomorphism of differential graded coalgebras*

$$C_*(X) \xrightarrow{\cong} B\big(C_*(\Omega X)\big)\,.$$

When applied to $A = C_*(\Omega X)$, the bar construction gives thus a spectral sequence

$$H_*\left(B\big(H_*(\Omega X)\big)\right) \Longrightarrow H_*\left(B\big(C_*(\Omega X)\big)\right) \cong H_*(X)\,.$$

This will be the main ingredient in the proof of Theorem 1.1.

THEOREM 2.1

a) Let X be a connected space, and f, g be endomorphisms of X inducing the same map on the rational homotopy groups, $\pi_n(f)\otimes\mathbb{Q} = \pi_n(g)\otimes\mathbb{Q}$, for $n \geq 2$, and on the fundamental group, then for $n \geq 0$, we have

$$\operatorname{tr} H_n(f;\mathbb{Q}) = \operatorname{tr} H_n(g;\mathbb{Q}) .$$

b) If $\sum_i \dim H^i(X,\mathbb{R}) < \infty$, then $\lambda_f = \lambda_g$.

c) If X is a manifold, then $\deg(f) = \deg(g)$.

Proof.— Denote by $\widetilde{X}$ the universal cover of X. For any field k, the fibration

$$\widetilde{X} \longrightarrow X \longrightarrow K\big(\pi_1(X),1\big)$$

induces an exact sequence of Hopf algebras [4]:

$$k \longrightarrow H_*(\Omega\widetilde{X};k) \longrightarrow H_*(\Omega X;k) \longrightarrow k\big[\pi_1(X)\big] \longrightarrow k .$$

When $k = \mathbb{Q}$, the Milnor–Moore theorem [12] implies that

$$H_*(\Omega\widetilde{X};\mathbb{Q}) \cong \mathcal{U}\pi_*(\Omega\widetilde{X}) \otimes \mathbb{Q} \cong \mathcal{U}\pi_{\geq 1}(\Omega X) \otimes \mathbb{Q} ,$$

where $\mathcal{U}$ means the enveloping algebra functor. It clearly follows that f and g induce the same maps on $H_*(\Omega X;\mathbb{Q})$.

On the other hand f and g induce maps:

$$C_*(\Omega X) \xrightarrow{C_*(\Omega f)} C_*(\Omega X) , \quad C_*(\Omega X) \xrightarrow{C_*(\Omega g)} C_*(\Omega X)$$

and therefore morphims of spectral sequences

$$E^i \xrightarrow{E^i(f)} E^i , \quad E^i \xrightarrow{E^i(g)} E^i .$$

At the E^1 level, these are the morphisms induced by the bar construction:

$$B\big(H_*(\Omega X;\mathbb{Q})\big) \xrightarrow{B(H_*(\Omega f))} B\big(H_*(\Omega X;\mathbb{Q})\big)$$

$$B\big(H_*(\Omega X;\mathbb{Q})\big) \xrightarrow{B(H_*(\Omega g))} B\big(H_*(\Omega X;\mathbb{Q})\big) .$$

This implies that $E^i(f) = E^i(g)$ for all $i \geq 1$ and thus $E^\infty(f) = E^\infty(g)$.

Recal now that $E^\infty(X)$ is the graded vector space associated to $H_*(X)$ for some filtration. It follows that for $n \geq 0$, we have

$$\operatorname{tr} H_n(f;\mathbb{Q}) = \operatorname{tr} H_n(g;\mathbb{Q})\,. \square$$

Example.— Let $X = S^2 \vee (S^1 \times S^1)$ and f, g be the maps

$$f : X \xrightarrow{f_1} S^2 \xrightarrow{i} X\,, \quad g : X \xrightarrow{g_1} S^2 \xrightarrow{i} X$$

where

$$f_1 := \mathrm{id} \vee h \quad \text{et} \quad g_1 := \mathrm{id} \vee q\,.$$

Here h consists to collapse $S^1 \times S^1$ into a point and q is the canonical projection $q : S^1 \times S^1 \to S^1 \wedge S^1 \cong S^2$. Clearly $\pi_*(f) = \pi_*(g)$. However, f and g are not homotopic because the maps h and q are not. Nevertheless, by Theorem 2.1, $\lambda_f = \lambda_g$

COROLLARY 2.1.— *A self-map of a simply connected compact manifold or of a simply connected finite* CW *complex that induces zero on the rational homotopy groups has a fixed point.*

3. Degree of a self-map of an elliptic space

In this section we prove Theorem 1.3.

THEOREM 3.1.— *Let X be an elliptic space and f be a self-map. Denote by λ_i the eigenvalues of $\pi_{\mathrm{odd}}(f) \otimes \mathbb{Q}$ and by μ_j the eigenvalues of $\pi_{\mathrm{even}}(f) \otimes \mathbb{Q}$, then*

$$\deg(f) = \frac{\prod_j \mu_j}{\prod_i \lambda_i}\,.$$

We prove the theorem by induction on the dimension of $\pi_*(X) \otimes \mathbb{Q}$ by using the theory of Sullivan minimal models [7]. When $\pi_*(X) \otimes \mathbb{Q} = 1$, a Sullivan minimal model of X is given by $\big(\bigwedge u, 0\big)$, with u of odd degree. Denote by g a model of f. We have $g(u) = \mu \cdot u$ and $\deg(f) = \mu$.

We now suppose the theorem has been proved for self-maps of elliptic spaces Z such that $\dim \pi_*(Z) \otimes \mathbb{Q} < n$ and we suppose that f is a self-map of an elliptic space X with $\dim \pi_*(X) \otimes \mathbb{Q} = n$. We denote by $(\bigwedge Z, d)$ the Sullivan minimal model of X and by $g : \bigwedge Z \to \bigwedge Z$ a minimal model for f. We denote also by r the minimum degree p with $\pi_p(X) \otimes \mathbb{Q} \neq 0$; we have $Z = Z^{\geq r}$ and $Z^r \neq 0$. We have to consider separately the case r is odd and the case r is even.

When r is odd we form the KS extension (see [7] for the definition)

$$\left(\bigwedge Z^r, 0\right) \longrightarrow \left(\bigwedge Z, d\right) \longrightarrow \left(\bigwedge Z^{>r}, d\right).$$

The naturality of the Serre spectral sequence implies that

$$\deg g = \deg g\restriction_{\bigwedge Z^r} \cdot \deg \overline{g},$$

where $\overline{g}$ denotes the projection of g on $\bigwedge Z^{>r}$, $\overline{g} : \bigwedge Z^{>r} \to \bigwedge Z^{>r}$. We obtain the result by induction.

When r is even, we work over the field of complex numbers and we choose an eigenvector x of $Z^r : g(x) = \lambda x$. As x is a cocycle of even degree, a power of x has to be a coboundary: $[x^n] = 0$ for some n. We can take n large enough so that $n \cdot |x| > 1 + |t|$ for all nonzero homogeneous elements t in Z. We form the commutative differential graded algebra $(\bigwedge x \otimes \bigwedge y, d)$, $d(y) = x^m$. We extend g to $(\bigwedge x \otimes \bigwedge y, d)$ by putting $g(y) = \lambda^m y$ and we form the KS extension of elliptic spaces

$$\begin{aligned}\left(\bigwedge x \otimes \bigwedge y, d\right) &\longrightarrow \left(\bigwedge Z \otimes \bigwedge y, d\right)\\ &\longrightarrow \left(\bigwedge Z / (\overset{+}{\bigwedge} x \cdot \bigwedge Z), d\right) = \left(\bigwedge Y, d\right).\end{aligned}$$

By induction the formula for the degree is true for $(\bigwedge Y, d)$. The formula is trivially true for $(\bigwedge x \otimes \bigwedge y, d)$. By the Serre spectral sequence the formula is thus satisfied for $(\bigwedge Z \otimes \bigwedge y, d)$. A new application of the Serre spectral sequence applied to the KS extension

$$\left(\bigwedge Z, d\right) \longrightarrow \left(\bigwedge Z \otimes \bigwedge y, d\right) \longrightarrow \left(\bigwedge y, 0\right)$$

yields the result. □

4. Computation of the Lefschetz number

S. Halperin has shown [8] how the Lefschetz number of a self-map f of an elliptic space X can be deduced from the maps $\pi_n(f) \otimes \mathbb{Q}$

THEOREM 4.1 [8].— *Denote by A_n the matrix representing the linear map $\pi_n(f) \otimes \mathbb{Q}$ in some basis. We then have*

$$\lambda_f = \lim_{t \to 1} \prod_n \frac{\det(1 - t^{2n+2} A_{2n+1})}{\det(1 - t^{2n} A_{2n})} .$$

This formula has been used by G. Lupton and J. Oprea [10] to prove that powers of self-maps of a Lie group always have a fixed point.

Among all simply connected spaces a very interesting class of spaces is given by the formal spaces. A 1-connected finite CW complex is called *formal* if X and its cohomology algebra have the same minimal model. For instance 1-connected compact Kähler manifolds are formal [2]. It follows from a general construction of Halperin-Stasheff [9] that a formal space X admits a special minimal model $(\bigwedge Z, d)$ called its bigraded minimal model. This model is equipped with a bigradation $Z = \bigoplus_{p,q \geq 0} Z_p^q$ satisfying:

1) $d : Z_p^q \to (\bigwedge^{\geq 2} Z)_{p-1}^{q+1}$
2) $H^*(\bigwedge Z, d) = H_0^*(\bigwedge Z, d)$.

Moreover if f is a continuous map from X into X, then f admits a model $f' : (\bigwedge Z, d) \to (\bigwedge Z, d)$ satisfying

$$f'(Z_p^q) \subset \left(\bigwedge Z\right)_{\leq p}^q .$$

We denote by ϕ_p^q the matrix representing the projection θ_p^q of f' on Z_p^q:

$$\theta_p^q : Z_p^q \longrightarrow Z_p^q .$$

We then have the following formula.

PROPOSITION 4.1.— *With the previous notations*

$$\prod_{p,q} \left[\det\left(1 - (-t)^{p+q} \phi_p^q\right)\right]^{(-1)^{q+1}} = \sum_n \operatorname{tr} H^n(f; \mathbb{Q}) t^n .$$

This formula enables the computation of the right hand side in terms of the characteristic polynomials of the matrices ϕ_p^q for $p+q \leq \dim X$.

Theorem 4.1 and Propositions 4.1 are in fact particular cases of a more general formula obtained for spaces equipped with a weight decomposition.

A space X is said to have a *weight decomposition* if X admits a minimal model $(\bigwedge Z, d)$ where Z is given a bigradation $Z = \bigoplus_{p,q\geq 0} Z_p^q$ satisfying

$$d : Z_p^q \to \left(\bigwedge Z\right)_{p-1}^{q+1}.$$

PROPOSITION 4.2.— *If X is equipped with a weight decomposition, we have:*

$$\prod_{p,q} \left[\det\left(1 - (-t)^{p+q}\phi_p^q\right)\right]^{(-1)^{q+1}} = \sum_n \left(\sum_{p+q=n} (-1)^p \operatorname{tr} H_p^q(f)\right) t^n .$$

Proof.— The Euler formula for the subcomplexes

$$0 \longrightarrow \left(\bigwedge Z\right)_p^0 \longrightarrow \left(\bigwedge Z\right)_{p-1}^1 \longrightarrow \left(\bigwedge Z\right)_{p-2}^2 \longrightarrow \cdots$$
$$\longrightarrow \left(\bigwedge Z\right)_0^p \longrightarrow 0$$

gives the formula

$$\sum_i (-1)^{p-i} \operatorname{tr}\left(\bigwedge f'\right)_{p-i}^i = \sum_i (-1)^{p-i} \operatorname{tr} H_{p-i}^i(f) .$$

We then take the sum of these formulae and obtain:

$$\sum_n \left(\sum_i (-1)^{n-i} \operatorname{tr}\left(\bigwedge f'\right)_{n-i}^i\right) t^n = \sum_n \left(\sum_i (-1)^{n-i} \operatorname{tr} H_{n-i}^i(f)\right) t^n .$$

This gives the result. □

A formal space admits a weight decomposition with cohomology concentrated in lower degree 0. Proposition 4.1 follows thus from Proposition 4.2.

Proof of Theorem 4.2.— Let X be an elliptic space, $(\bigwedge Z, d)$ its minimal model and $g : (\bigwedge Z, d) \to (\bigwedge Z, d)$ the map induced by f. We denote by d_σ the pure differential associated to d [6]. This differential is defined by:

$$\begin{cases} d_\sigma(Z^{\text{even}}) = 0 \\ d_\sigma(Z^{\text{odd}}) \subset (\bigwedge Z^{\text{even}}) \\ (d - d_\sigma)(Z^{\text{odd}}) \subset (\bigwedge^+ Z^{\text{odd}}) \otimes (\bigwedge Z^{\text{even}}) . \end{cases}$$

By [6], $H^*(\bigwedge Z, d_\sigma)$ is finite dimensional. It is then easy to see that g induces an endomorphism g_σ of the differential graded algebra $(\bigwedge Z, d_\sigma)$. It follows that

$$\lambda_{g_\sigma} = \lambda_g .$$

As $(\bigwedge Z, d_\sigma)$ is equiped with a weigth decomposition satisfying $Z^{\text{odd}} = Z_1$ et $Z^{\text{even}} = Z_0$. Theorem 4.2 follows from Proposition 4.2. □

An other case where λ_f can be deduced from the characteristic polynomials associated to the maps $\pi_n(f) \otimes \mathbb{Q}$ is the case when the Lie algebra $\pi_*(\Omega X) \otimes \mathbb{Q}$ has a finite dimensional cohomology:

$$\dim \operatorname{Tor}_*^{H_*(\Omega X;\mathbb{Q})}(\mathbb{Q}, \mathbb{Q}) < \infty .$$

In this case, for a self-map f, we have the formula

$$\Big(\sum_{p,q} (-1)^p t^q \operatorname{tr} \operatorname{Tor}_{p,q}^{H_*(\Omega f;\mathbb{Q})}(\mathbb{Q}, \mathbb{Q})\Big)\Big(\sum_n \operatorname{tr} H_n(\Omega f; \mathbb{Q}) t^n\Big) = 1 .$$

Therefore, denoting by $Q(t)$ the rational function $\sum_n \operatorname{tr} H_n(\Omega f; \mathbb{Q}) t^n$, we have

$$\sum_{p,q} (-1)^{p+q} \operatorname{tr} \operatorname{Tor}_{p,q}^{H_*(\Omega f;\mathbb{Q})}(\mathbb{Q}, \mathbb{Q}) = \lim_{t \to -1} \frac{1}{Q(t)} .$$

On the other hand, there is a natural Milnor–Moore spectral sequence

$$E_2^{p,q} = \operatorname{Tor}_{p,q}^{H_*(\Omega f;\mathbb{Q})}(\mathbb{Q}, \mathbb{Q}) \Longrightarrow H_{p+q}(X; \mathbb{Q}) .$$

We deduce the following result.

Proposition 4.3.— *If* $\dim \operatorname{Tor}_*^{H_*(\Omega f;\mathbb{Q})}(\mathbb{Q}, \mathbb{Q}) < \infty$, *then*

$$\lambda_f = \lim_{t \to -1} \frac{1}{Q(t)} ,$$

where $Q(t)$ *denotes the Poincaré series* $\sum_n \operatorname{tr} H_n(\Omega f; \mathbb{Q}) t^n$.

Example.— Let X be a formal space. For each $\mu \in \mathbb{Q} \setminus \{0\}$, the map

$$\overline{\mu} : H^*(X) \longrightarrow H^*(X)\,, \quad x \longmapsto \overline{\mu}(x) := \mu^{|x|} \cdot x$$

can be realized by an automorphism $\widetilde{\mu}$ of X. Denote by $(\bigwedge Z, d)$ the bigraded model of X. The restriction of the map $\widetilde{\mu}$ at Z_p^q consists in the multiplication by μ^{p+q}. We remark that the formula given by Proposition 4.1 is a generalization of the formula given by Halperin and Stasheff [9] in the case $f = \mathrm{id}$:

$$\prod_{p,q} \left[1 - (-t)^{p+q}\right]^{(-1)^{q+1} \dim Z_p^q} = \sum_n \dim H^n(f; \mathbb{Q}) t^n\,.$$

The left hand side of the formula has a nonzero radius of convergence R. We now replace t by $-\mu$ and we take $\mu < R$. The infinite product

$$\prod_{p,q} \left[1 - (\mu)^{p+q}\right]^{(-1)^{q+1} \dim Z_p^q}$$

converges and we thus have this following corollary.

COROLLARY 4.1.— *If* $0 < \mu < R$, *then*

$$\lambda_{\overline{\mu}} = \prod_{p,q} \left[1 - (\mu)^{p+q}\right]^{(-1)^{q+1} \dim Z_p^q}\,.$$

5. Self-maps with finite image in homotopy

Let X be a simply connected space and let f be a self-map such that the image of $\pi_*(f) \otimes \mathbb{Q}$ in finite dimensional. In this case, we have the following result.

PROPOSITION 5.1.— *The space X admits as retract an elliptic space Y with $\pi_*(Y) \otimes \mathbb{Q}$-isomorphic to the subspace of $\pi_*(X) \otimes \mathbb{Q}$ generated by the eigenspaces of nonzero eigenvalue.*

COROLLARY 5.1.— *Let X be a simply connected finite* CW *complex and f be a self-map satisfying* $\dim \operatorname{Im} \pi_*(f) \otimes \mathbb{Q} < \infty$. *Denoting by r the multiplicity of* 1 *as eigenvalue of $\pi_{\text{even}}(f) \otimes \mathbb{Q}$ and by s the multiplicity of* 1 *as eigenvalue of $\pi_{\text{odd}}(f) \otimes \mathbb{Q}$, we have:*

(1) $s \geq r$,

(2) if $s > r$, *then* $\lambda_f = 0$,

(3) if $s = r$, *then*

$$\lambda_f = \frac{\prod_{\lambda_i \neq 0,1}(1-\lambda_i)}{\prod_{\mu_i \neq 0,1}(1-\mu_i)},$$

where the λ_i are the eigenvalues of $\pi_{\text{odd}}(f) \otimes \mathbb{Q}$ and the μ_i the eigenvalues of $\pi_{\text{even}}(f) \otimes \mathbb{Q}$.

Proof of the proposition 5.1.— Denote by $\left(\bigwedge Z, d\right)$ the minimal model of X, by g a minimal model for f and by φ the linear part of g. We decompose Z as the direct sum of the eigenspaces V_λ associated to φ. We write $Z = V \oplus W$, with

$$V = \bigoplus_{\lambda \neq 0} V_\lambda \quad \text{and} \quad W = V_0 .$$

We will show by induction on the degree that we can modify V and W in order to have $d(V) \subset \bigwedge V$, $g(V) \subset \bigwedge V$, $d(W) \subset \bigwedge^+ W \otimes \bigwedge V$ and $g(W) \subset \bigwedge^+ W \otimes \bigwedge V$. We take homogeneous basis v_n of V and w_n of W satisfying $|v_n| > |v_{n-1}|$ and $\varphi(v_n) - \lambda_n v_n \in \bigwedge(v_1, \ldots, v_{n-1})$, and a similar formula for the elements w_n.

Suppose the properties are satisfied for $v_1, \ldots, v_{n-1}$ and $w_1, \ldots, v_{m-1}$. We then have

$$\begin{cases} d(v_n) = \beta_n + \gamma_n & \begin{cases} \beta_n \in \bigwedge V \\ \gamma_n \in \bigwedge V \otimes \bigwedge^+ W \end{cases} \\ g(v_n) = \lambda_n v_n + u_n + \delta_n & \begin{cases} u_n \in \bigwedge^{\geq 2} V \otimes (v_1, \ldots, v_{n-1}) \\ \delta_n \in \bigwedge V \otimes \bigwedge^+ W . \end{cases} \end{cases}$$

There exists integers r and s with $g^{\circ^r}(\gamma_n) = 0$ and $g^{\circ^s}(\delta_n) = 0$. We choose an integer p greater than r and s, and we replace v_n by $g^{\circ^p}(v_n)$. We have

$$\begin{cases} d(g^{\circ^p}(v_n)) = g^{\circ^p}(\beta_n) \in \bigwedge V \\ g(g^{\circ^p}(v_n)) - \lambda_n v_n \in \bigwedge^{\geq 2} V \oplus (v_1, \ldots, v_{n-1}) . \end{cases}$$

We now consider the element w_n. Suppose

$$d(w_n) = \alpha_m + \beta_m \quad \text{with} \quad \alpha_m \in \overset{+}{\bigwedge} W \otimes \bigwedge V \text{ and } \beta_m \in \bigwedge V\,.$$

Then $g^{\circ^s}(\beta_n)$ is a coboundary for some integer s. By the next lemma the element β_n is therefore also a coboundary, $\beta_m = d(\gamma_m)$. Now by definition of W, γ_m is a decomposable element. We can therefore replace w_m by $\omega_m - \gamma_m$ in order to have $d(w_m) \in \bigwedge^+ W \otimes \bigwedge V$.

We now write

$$g(w_m) = \mu_m + \nu_m \quad \text{with} \quad \mu_m \in \overset{\geq 2}{\bigwedge} V \text{ and } \nu_m \in \overset{+}{\bigwedge} W \otimes \bigwedge V\,.$$

Of course by induction hypothesis, $g^{\circ^s}(\mu_m)$ is a cocycle, so that μ_m is also a cocycle. We replace w_m by $w_m - g^{-1}(\mu_m)$, where g^{-1} denotes the inverse of the function $g : \left(\bigwedge V\right)^t \to \left(\bigwedge V\right)^t$, with $t = \deg(w_m)$. □

LEMMA 5.1.— *Let E be a finite type vector space over the complex numbers, let $f : E \to E$ be an isomorphism, and let S be a graded sub vector space of E invariant for f. Suppose f°^s} belongs to S for some element x and some integer s, then the element x belongs to S.*

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