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# Hardy Spaces on Compact Lie Groups(*)

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**RÉSUMÉ.** — Nous démontrons que l'espace de Hardy atomique $H_a^p(G)$ d'un groupe de Lie connexe compact semi-simple peut être caractérisé par des opérateurs maximaux associés aux noyaux de Poisson et de la chaleur si $0 < p \leq 1$, par l'opérateur maximal de Bochner-Riesz $S_*^\delta$ pourvu que $\delta > n/p - (n+1)/2$ où $n = \dim_{\mathbb{R}}(G)$, et, si $p = 1$, par des noyaux non différentiables satisfaisant une certaine propriété de Dini.

**ABSTRACT.** — We prove that the atomic Hardy space $H_a^p(G)$ of a connected semisimple compact Lie group $G$ can be characterized for $0 < p \leq 1$ by maximal functions based on Poisson and heat kernels, by the maximal Bochner-Riesz operator $S_*^\delta$ for $\delta > n/p - (n+1)/2$ where $n = \dim_{\mathbb{R}}(G)$, and, for $p = 1$, by certain nonsmooth kernels satisfying a Dini condition.

**KEY-WORDS** : Hardy space, atoms, maximal operators, Poisson kernel, heat kernel, Bochner-Riesz kernel, Dini condition

**AMS Classification** : 43A77, 42B30

Atomic decompositions of Hardy spaces of real functions on Euclidean spaces first arose in the work of R. Coifman [6] and R. Latter [13]. An abstract theory of atomic Hardy spaces was later developed by R. Coifman and G. Weiss [7] in the context of spaces of homogeneous type. These spaces include Euclidean spaces and compact Lie groups but do not in general have the structure on which to base a theory of Hardy space defined by maximal functions. It was noted by R. Coifman, Y. Meyer and G. Weiss in [7] and by A. Uchiyama in [18] that when a space of homogeneous type admits a certain family of kernels, a maximal function based Hardy space can be defined and shown to be equivalent to atomic Hardy space. Although the

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kernels in question are well-suited to an argument of L. Carleson [3], they are not necessarily intrinsic to any additional geometry (such as Riemannian structure) that a space of homogeneous type may possess. For example, compact Riemannian manifolds have Laplace–Beltrami operators which give rise via Poisson kernels to maximal function based Hardy spaces. In such cases it is of interest to obtain the atomic decomposition of Hardy spaces defined by maximal functions as was done for spheres by L. Colzani [8]. Moreover, the atomic Hardy spaces in [7] and [18] are, of necessity because of the more general structure, defined by duality. Where polynomials are available, such as is the case with compact Lie groups, it is desirable to have a direct definition of atoms in analogy with those in [7] and [13]. In this paper we establish the equivalence of the Hardy spaces on compact semisimple Lie groups that arise from their homogeneous type structure with those that arise from their Riemannian structure. We also characterize Hardy spaces by several standard maximal functions. In doing so we also set up a frawework which will be used elsewhere to generalize a number of familiar Euclidean results concerning Littlewood–Paley theory and Lipschitz spaces.

Let $G$ be a connected semisimple compact Lie group with invariant distance $d$ and Laplace–Beltrami $\Delta$ induced from the Killing form. The operators $\partial/\partial t - \Delta$ and $\partial^2/\partial t^2 - \Delta$ on $G^+ = G \times (0, \infty)$ give rise to heat and Poisson kernels $W_t(x)$ and $P_t(x)$ respectively. In Section 1, we describe some relationships among $P_t$, $W_t$, and the Bochner–Riesz kernel $S_R^\delta$ $(\delta \geq 0)$. The geometry reflected in these kernels is most easily seen via a metric $\widetilde{d}$, obtained in Section 1 from the Lie algebra $\mathfrak{g}$ of $G$, which is equivalent to $d$. It is this Riemannian geometry on which the analysis that follows is based.

In Section 2, we study the (maximal) Hardy space $H^p(G)$ of all distributions $f$ satisfying

$$P^+ f(x) = \sup_{t>0} \left| P_t * f(x) \right| \in L^p(G) \quad (0 < p \leq 1).$$

Using the Poisson kernel, we define other maximal functions $P_\varepsilon^* f$, $P_\varepsilon^+ f$, $P_{M,\varepsilon}^{**} f$ and the grand maximal function $f^*$. Then we prove that $f$ is in $H^p(G)$ if and only if one (and hence all) of these maximal functions is in $L^p(G)$ $(0 < p \leq 1)$. The proofs in this chapter are broadly based along standard arguments, but are technically more difficult since the Poisson kernel on $G$ is considerably less tractable than its $\mathbb{R}^n$ and sphere $\Sigma^n$ equivalents. Careful estimation of the kernels involves a combination of previous analysis of Cowling, Mantero, and Ricci [9] and J. L. Clerc [5].

Ellipticity is used to obtain estimates that are obtained in the Euclidean setting by harmonicity.

Section 3 deals with the atomic characterization of $H^p(G)$. We have chosen to work with the atomic Hardy space $H_a^p(G)$ introduced by J. L. Clerc in [5]. This atomization, by making use of the availability of polynomials on compact groups, resembles the Euclidean atomization more closely than those in [7] and [18], avoids the use of Lipschitz spaces and duality arguments, and is eminently suitable for analysis as was demonstrated in [5]. Unfortunately, it is considerably more difficult to work with polynomials on $G$ than on $\mathbb{R}^n$ or $\Sigma^n$. We first use geometric arguments to study the classical group $U(n)$ of unitary isometries. A result from approximation theory is then used to prove that any distribution in $H^p\big(U(n)\big)$ has an atomic decomposition. Next, unitary embedding, a well-known consequence of the Peter–Weyl theorem, allows us to transfer this result to $G$, yielding $H^p(G) = H_a^p(G)$. It should be noted at this point, that both atomic and maximal function Hardy space appearing here are defined differently from those in [18].

In Section 4, we define maximal functions by the heat kernel instead of the Poisson kernel; these give rise to $H^p$-spaces which we prove coincide with those defined by the Poisson kernel. We also investigate an analogue of a theorem of Fefferman and Stein pertaining to $C^\infty$-functions $\varphi$ on $\mathbb{R}^\ell$ that can be used to construct a kernel $\varphi_t(x) = t^{-n}\varphi(x/t)$ that characterizes $H_p(\mathbb{R}^\ell)$. Here, the obstacle to obtaining a simple analogue comes from the lack of dilation of $\mathbb{R}^+$ on $G$. We also give, this time generalizing work of Y. Han, a Dini condition sufficient for a (non-smooth) central function on $G$ to characterize $H^1(G)$. In the case of the Bochner–Riesz kernel $S_R^\delta(x)$, characterization of $H^p(G)$, is shown to hold for $\delta > n/p - (n+1)/2$ $(n = \dim_{\mathbb{R}}(G))$

The two main techniques of Hardy theory on $\mathbb{R}^n$, dilation of space and Fourier transformation of functions, are largely unavailable in compact Lie groups. Some basic ideas on $\mathbb{R}^n$ are adaptable but are technically more difficult to execute; other ideas do not transfer at all and must be replaced with new ones. Another difficulty arises in the handling of polynomials which are necessarily more cumbersome than in the Euclidean setting. Our expositional strategy is to omit proofs which are obtainable in purely routine fashion from similar Euclidean proofs, but to give essentially full details where new techniques are needed. We have found in the literature occasional laxity in distinguishing between these two types of proof and have taken pains not to use the phrase "by a standard argument" unless justified. On the other hand, the analysis undertaken in the two papers [5] and [9] is so essential to our program, and so difficult to briefly recapitulate, that an attempt to allow our work to be read independently would have resulted in substantial and needless lengthening; where appropriate, we have utilized constructs from those papers, identically notated in our work, with no more than a reference to the source. Occasionally, we have taken the path that best allows for further investigation of the harmonic analysis of compact Lie groups; the second named author will pursue those matters in a series of papers elsewhere. In particular, in part because of this ulterior motivation and in part to fully utilize the available geometric structure, we have not directly compared the maximal Hardy spaces defined here with those defined by Uchiyama, which would have obviously been another way to proceed. A preliminary manuscript, differing from the present paper only in detail, was circulated in 1990; a number of matters that arose in the interim delayed publication until now.

## 1. Notation and Basic Material

Let $G$ be a connected compact semisimple Lie group of dimension $n$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{t}$ the Lie algebra of a fixed maximal torus $T$ in $G$ of dimension $\ell$. Let $A$ be a system of positive roots for $(\mathfrak{g}, \mathfrak{t})$; then $A$ has $(n-\ell)/2$ elements. Let

$$\rho = \frac{1}{2} \sum_{\alpha \in A} \alpha .$$

Let $\mathfrak{t}^+$ denote the open Weyl chamber of $\mathfrak{t}$ associated with the root ordering. The regular elements $\mathfrak{t}_r$ of $\mathfrak{t}$ consist of those $H \in \mathfrak{t}$ for which $\alpha(H) \notin 2\pi i\mathbb{Z}$ $(\alpha \in A)$. Let $\mathcal{A}$ be a connected component of $\mathfrak{t}_r$ contained in $\mathfrak{t}^+$ such that $0$ is $\mathrm{cl}(\mathcal{A})$, the topological closure of $\mathcal{A}$. Then $\mathcal{A}$ is a fundamental domain for the exponential map up to conjugacy in the sense that every element of $G$ is conjugate to a unique member of $\exp\big(\mathrm{cl}(\mathcal{A})\big)$. We let $\mathcal{H} : G \to \mathrm{cl}(\mathcal{A})$ be the resulting map.

Let $|\cdot|$ be the norm on $\mathfrak{g}$ induced by the negative of the Killing form $B$ on $\mathfrak{g}^{\mathbb{C}}$, the complexification of $\mathfrak{g}$. Then $|\cdot|$ induces a bi-invariant metric $d$ on $G$. Futhermore, since $B\big|_{\mathfrak{t}^{\mathbb{C}} \times \mathfrak{t}^{\mathbb{C}}}$ is nondegenerate, given $\lambda$ in $\hom_{\mathbb{C}}(\mathfrak{t}^{\mathbb{C}}, \mathbb{C})$ there is a unique element $H_\lambda$ of $\mathfrak{t}^{\mathbb{C}}$ such that $\lambda(H) = B(H, H_\lambda)$ for each $H \in \mathfrak{t}^{\mathbb{C}}$. We let $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the inner product and norm transferred from $\mathfrak{t}$ to $\hom_{\mathbb{C}}(\mathfrak{t}, i\mathbb{R})$ by means of this canonical isomorphism. Let $N = \{H \in \mathfrak{t} \mid \exp H = 1\}$. The weight lattice $P$ is defined by

$$P = \big\{\lambda \in \mathfrak{t} \mid \lambda(X) \in 2\pi\mathbb{Z}\,,\ X \in N\big\}$$

with dominant weights defined by

$$\Lambda = \big\{\lambda \in P \mid \langle\lambda\,,\,\alpha\rangle \geq 0\,,\ \alpha \in A\big\}\,.$$

Then $\Lambda$ provides a full set of parameters for the equivalence classes of irreducible representations of $G$: for $\lambda \in \Lambda$ the representation $U_\lambda$ has dimension

$$d_\lambda = \prod_{\alpha\in A} \frac{\langle\lambda + \rho\,,\,\alpha\rangle}{\langle\rho\,,\,\alpha\rangle} \tag{1.1}$$

and its associated character is

$$\chi_\lambda(x) = \frac{\sum_{s\in W} \varepsilon(s)\, e^{i\langle s(\lambda+\rho)\,,X\rangle}}{\varepsilon(s)\, e^{\langle s\rho\,,X\rangle}}\,, \quad x = \exp X\,,\ X \in \mathfrak{t} \tag{1.2}$$

where $W$ is the Weyl group, $\varepsilon(s)$ is the signature of $s \in W$, and $\langle\nu\,,\,H\rangle = \nu(H)$ for $H \in \mathfrak{t}$ and $\nu$ in $\hom(\mathfrak{t}, \mathbb{C})$.

For a positive constant $\gamma$ and $x_0$ in $G$ define the cone $\Gamma_\gamma(x_0)$ in $G^+ = G \times \mathbb{R}^+$ with base point $x_0$ by

$$\Gamma_\gamma(x_0) = \big\{(x,t) \in G^+ \mid d(x,x_0) < \gamma t\big\}\,.$$

We will also have occasion to consider the truncated cone $\Gamma_{\gamma,\varepsilon_0}(x_0) = \Gamma_\gamma(x_0) \cap \big(G \times (0\,,\,\varepsilon_0]\big)$. It is convenient to have another distance function $\widetilde{d}$ defined by

$$\widetilde{d}(x,y) = \sup_{h\in G} \big|\mathcal{H}(xh) - \mathcal{H}(yh)\big|\,, \quad x\,,\ y \in G\,. \tag{1.3}$$

**Proposition 1.1.**— *The metric $\widetilde{d}$ is equivalent to $d$.*

*Proof.*— We will show that there exists an $\varepsilon_0$ on $(0,1)$ such that $\varepsilon_0 d \leq \widetilde{d} \leq d$. In fact it is known [5] that $\big|\mathcal{H}(x) - \mathcal{H}(y)\big| \leq d(x,y)$ $(x,\ y \in G)$, whence $\widetilde{d} \leq d$. Taking $h = y^{-1}$ in (1.3), it is enough to show the existence of $\varepsilon_0$ depending only on $G$ such that $\varepsilon_0 d(x,y) \leq \big|\mathcal{H}(xy^{-1})\big|$ or, equivalently, $\varepsilon_0 d(g,e) \leq \big|\mathcal{H}(g)\big|$ for all $g$ in $G$. Indeed, if $\delta > 0$ is small enough so that $\exp : \mathfrak{g} \to G$ is a diffeomorphism for $|X| < \delta$, then $d(x,e) = |\mathcal{H}(x)|$ whenever $|\mathcal{H}(x)| < \delta$. Otherwise, $|\mathcal{H}(x)| \geq \delta D^{-1} d(x,e)$. where $D = \max_{g \in G} d(g,1)$ is the diameter of the group. Thus, $\varepsilon_0 = \delta D^{-1}$ will do. □

Let $X_1, X_2, \ldots, X_n$ be an orthonormal basis of $\mathfrak{g}$. Form the Casimir operator $\Delta = \sum_{i=1}^{n} X_i^2$. Then $\Delta$ is an elliptic bi-invariant operator on $G$ which is independent of the choice of orthonormal basis of $\mathfrak{g}$. The solution of the heat equation

$$\Delta\varphi(x,t) = \frac{\partial \varphi}{\partial t}(x,t), \quad \varphi(x,0^+) = f(x) \quad \text{on } G^+ = G \times \mathbb{R}^+$$

for $f \in L^1(G)$ is given by $\varphi(x,t) = W_t * f(x)$ where

$$W_t(x) = \sum_{\lambda \in \Lambda} e^{-t(\|\lambda+\rho\|^2 - \|\rho\|^2)} d_\lambda \chi_\lambda(x), \quad x \in T \tag{1.4}$$

is the Gauss-Weierstrass kernel. Here and throughout this paper, unspecified measures on $G$ and $T$ are bi-invariant Haar measures normalized to have total mass one; Lebesgue spaces, in particular, are defined with respect to these measures and $|E|$ denotes the Haar measure of a subset $E$ of $G$ except for the Weyl group $W$, in which case $|\cdot|$ denotes cardinality. The solution of the Poisson equation

$$\Delta\varphi(x,t) = \frac{\partial \varphi^2}{\partial t^2}(x,t), \quad \varphi(x,0^+) = f(x) \quad \text{for } f \in L^1(G)$$

is given by $\varphi(x,t) = P_t * f(x)$ where

$$P_t(x) = \sum_{\lambda \in \Lambda} e^{-t(\|\lambda+\rho\|^2 - \|\rho\|^2)^{1/2}} d_\lambda \chi_\lambda(x), \quad x \in T \tag{1.5}$$

is the Poisson kernel. We will also consider the Bochner–Riesz kernel:

$$S_t^\delta(x) = \sum_{\lambda \in \Lambda} \left(1 - \frac{\|\lambda+\rho\|^2}{t^{-2}}\right)_+^\delta d_\lambda \chi_\lambda(x), \quad \delta > 0, \ x \in T. \tag{1.6}$$

These three kernels are central functions on $G$ and are determined by their restrictions to $T$ as given in (1.4), (1.5) and (1.6).

Let $\varphi_\lambda(x) = \sqrt{d_\lambda}\, U_\lambda(x)$ $(x \in G)$ and put $\varphi_\lambda(f) = \int_G f(x)\varphi_\lambda(x)^{-1}\,\mathrm{d}x$ for $f \in L^1(G)$. Then $f$ has Fourier expansion

$$\sum_{\lambda\in\Lambda} d_\lambda \chi_\lambda * f(x) = \sum_{\lambda\in\Lambda} \operatorname{tr}\big(\varphi_\lambda(f)\varphi_\lambda(x)\big) \tag{1.7}$$

and in particular

$$S_\lambda^\delta * f(x) = \sum_{\lambda\in\Lambda} \Big(1 - \|t(\lambda+\rho)\|^2\Big)_+^\delta \operatorname{tr}\big(\varphi_\lambda(f)\varphi_\lambda(x)\big) \tag{1.8}$$

There are relationships among these three kernels that we shall make use of. The first we list is Bochner's subordination formula [14]:

$$\begin{aligned} P_t(x) &= \pi^{-1/2} \int_0^\infty u^{-1/2}\, e^{-u}\, W_{t^2/4u}(x)\,\mathrm{d}u \\ &= \frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-(t^2/4u)}\, u^{-1/2} W_u(x)\,\mathrm{d}u\,. \end{aligned} \tag{1.9}$$

In order to describe the relationship between $W_t$ and $S_t^\delta$ we will briefly introduce classes of functions discussed more fully in [16]. Let $\mathrm{AC}_{\mathrm{loc}}(0,\infty)$ be the space of local absolutely continuous functions. Let $\mathbb{C}[0\,,\,\infty]$ denote the space of uniformly continuous functions $e(t)$ on $[0\,,\infty)$ such that $e(\infty) = \lim_{t\to\infty} e(t)$ exists. Let $\mathrm{BV}_1$ denote the class of functions on $[0\,,\,\infty)$ of bounded variation. We define $\mathrm{BV}_{1+\delta}$ for $\delta \geq 0$ as follows. Set $\alpha = \delta - [\delta]$. To begin with, suppose $\delta \in \mathbb{N}^+$, in which case $\alpha = 0$ and define

$$\begin{aligned} \mathrm{BV}_{1+\delta} = \Big\{ e \in \mathbb{C}[0\,,\,\infty] \mid e^{(\beta)} \in \mathrm{AC}_{\mathrm{loc}}(0,\infty)\,,\ \alpha \leq \beta \leq \delta - 1\,, \\ e^{(\delta)} \in \mathrm{BV}_{\mathrm{loc}}\,,\ \|e\|_{\mathrm{BV}_{1+\delta}} < \infty \Big\} \end{aligned}$$

where

$$\|e\|_{\mathrm{BV}_{1+\delta}} = \Gamma(\delta+1)^{-1} \int_0^\infty t^\delta \big|\,\mathrm{d}\, e^{(\delta)}(t)\big| + \lim_{t\to\infty} |e(t)|\,.$$

For $\delta$ non integral, in which case $0 < \alpha < 1$, define

$$(I_\xi^{1-\alpha}\, e)(t) = \Gamma(1-\alpha)^{-1} \int_t^\xi (s-t)^{-\alpha}\, e(s)\,\mathrm{d}s\,, \quad 0 \leq t < \xi$$

and set

$$e^{(\alpha)}(t) = \lim_{\xi\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} (I_\xi^{1-\alpha}\, e)(t)\,.$$

For $k = 0, 1, 2, \ldots, [\delta]$ set

$$e^{(k+\alpha)}(t) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\, e^{(\alpha)}(t)\,.$$

With these interpretations of $e^{(\alpha)}, \ldots, e^{(\delta)}$, we define $\mathrm{BV}_{1+\delta}$ as above. Also, define

$$\mathrm{BV}^b_{\delta+1} =$$
$$= \left\{ e \in \mathrm{BV}_{\delta+1} \;\middle|\; \|e\|_{\mathrm{BV}^b_{\delta+1}} = \Gamma\big(\delta+b+1\big)^{-1} \int_0^\infty t^{\delta+b} \big|\,\mathrm{d}\, e^{(b)}(t)\big| < \infty \right\}$$

for $\delta \geq 0$ and $b > 0$. Finally, we consider the normalized subclass

$$\widetilde{\mathrm{BV}}^b_{\delta+1} = \big\{ e \in \mathrm{BV}^b_{\delta+1} \mid e(0) = 1\,,\ e(\infty) = 0 \big\}\,. \tag{1.10}$$

For any $e$ in this class it is known [16] that

$$e(t) = c \int_t^\infty (s-t)^\delta\, \mathrm{d}e^{(\delta)}(s)\,, \quad 0 \leq t\,. \tag{1.11}$$

For each $f \in L^1(G)$ and $e \in \widetilde{\mathrm{BV}}^b_{\delta+1}$ we define the $e$-summation of $f$ by

$$e_t * f(x) = \sum_{\lambda\in\Lambda} e\Big(\|\lambda+\rho\|^2\Big)\, d_\lambda \chi_\lambda * f(x)\,, \quad (x,t) \in G^+\,. \tag{1.12}$$

Then, by (1.1), (1.11) and Fubini's theorem,

$$\begin{aligned}
e_t * f(x) &= \\
&= C \int_G f(y^{-1}x) \sum_{\lambda\in\Lambda} e\Big(t\|\lambda+\rho\|^2\Big) \left( \prod_{\alpha\in A} \langle \lambda+\rho\,,\, \alpha\rangle \right) \chi_\lambda(y)\,\mathrm{d}y \\
&= C \int_G f(y^{-1}x) \sum_{\lambda\in\Lambda} \left( \prod_{\alpha\in A} \langle \lambda+\rho\,,\, \alpha\rangle \right) \times \\
&\quad \times \left( \int_{t\|\lambda+\rho\|^2}^\infty \Big(s - t\|\lambda+\rho\|^2\Big)^\delta\, \mathrm{d}e^{(\delta)}(s) \right) \chi_\lambda(y)\,\mathrm{d}y \\
&= C \int_G f(y^{-1}x) \int_0^\infty s^\delta \sum_{\substack{t\|\lambda+\rho\|^2 \leq s \\ \lambda\in\Lambda}} \prod_{\alpha\in A} \langle \lambda+\rho\,,\, \alpha\rangle \times \\
&\quad \times \left(1 - \frac{t}{s}\|\lambda+\rho\|^2\right)^\delta\, \mathrm{d}e^{(\delta)}(s) \chi_\lambda(y)\,\mathrm{d}y
\end{aligned}$$

whence

$$e_t * f(x) = C \int_0^\infty s^\delta S^\delta_{t/s} * f(x)\, \mathrm{d}e^{(\delta)}(s)\,. \tag{1.13}$$

It is easy to verify that $e(t) = e^{-t}$ belongs to $\widetilde{\mathrm{BV}}^b_{\delta+t}$ for all $\delta \geq 0$ and $b > 0$. With this choice of $e(t)$, (1.13) gives for $f \in L^1(G)$:

$$W_t * f(g) = C\, e^{t\|\rho\|^2} \int_0^\infty s^\delta\, e^{-s} (S^\delta_{t/s} * f)(x)\, \mathrm{d}s \tag{1.14}$$

or

$$W_t = C\, e^{t\|\rho\|^2} \int_0^\infty s^\delta\, e^{-s} S^\delta_{t/s}\, \mathrm{d}s\,. \tag{1.15}$$

Another important formula, the details of which may be found in [9], is obtained by Poisson summation:

$$W_t(x) = C \frac{e^{t\|\rho\|^2} t^{-n/2}}{D(x)} \sum_{Z \in N} \left( \prod_{\alpha \in A} \alpha(X + Z) \right) e^{-\|X+Z\|^2/4t} \tag{1.16}$$

where $x = \exp X \in T$ and $D(x)$ is the denominator in (1.2). Also to be found in [9] is a useful approximation of the heat kernel in a Sobolev norm. To wit, choose a radial function $\eta \in C^\infty(\mathfrak{t})$ that is supported in a neighbourhood of 0 whose translates by distinct elements of $N$ are disjoint and that is identically one in a smaller neighbourhood of 0. Let $K_t(x)$ be the central function on $G$ defined on $T$ by

$$K_t(x) = e^{2t\|\rho\|^2} t^{-\ell/2} \sum_{H \in N} \eta(X + H)\, e^{-\|X+H\|^2/4t}\,, \quad x = \exp X\,,\ X \in \mathfrak{t}\,. \tag{1.17}$$

For $s \in \mathbb{N}^+$ let $H^{2,s}(G)$ denote the Sobolev space consisting of functions $f$ with derivatives of order up to $s$ in $L^2(G)$ and with finite Sobolev norm

$$\|f\|_{2,s} = \left( \sum_{\lambda \in \Lambda} d_\lambda \|\varphi_\lambda(f)\|^2 \|\lambda + \rho\|^{2s} \right)^{1/2}\,.$$

In [9] it is shown that for each pair $(s, N)$ of positive integers and each $t > 0$ there is an integer $q = q(s, N)$ and a bi-invariant differential operator $\Delta_{q,t}$ of degree $q$ with coefficients whose dependence on $t$ involves only powers of $t$ up to $q$ such that

$$\|W_t - \Delta_{q,t} K_t\|_{2,s} = \mathrm{O}(t^N) \quad \text{as } t \to 0\,. \tag{1.18}$$

Let $\mathcal{S}(G)$ be the class of infinitely differentiable functions on $G$. For any $f$ in $\mathcal{S}(G)$, we have $f(x) = \sum_{\lambda\in\Lambda} \operatorname{tr}(\varphi_\lambda(f)\varphi_\lambda(x))$, and by the Peter–Weyl theorem, $\|f\|^2_{L^2(G)} = \sum_{\lambda\in\Lambda}\|\varphi_\lambda(f)\|^2$ where the norm of a finite dimensional operator $A$ on a complex Hilbert space is taken to be the Hilbert–Schmidt norm $\|A\| = (\operatorname{tr} AA^*)^{1/2}$. The following proposition is the analogue for semisimple compact connected Lie groups of a standard torus result which is often used in the theory of multiple Fourier series and which is in fact used in the proof of its analogue below.

PROPOSITION 1.2.— *A function $f$ belongs to $\mathcal{S}(G)$ if and only if for every positive integer $m$ there exists a constant $C(m, f)$ such that*

$$\|\varphi_\lambda(f)\| \le \frac{C(m,f)}{(1+|\lambda|)^m}.$$

*Proof.*— If $f \in \mathcal{S}(G)$, then $\widehat{f}(y) = \int_G f(x)f(yx)\,\mathrm{d}x$ belongs to $\mathcal{S}(G)$ and

$$\begin{aligned}
\|\varphi_\lambda(f)\|^2 &= \operatorname{tr}(\varphi_\lambda(f)\varphi_\lambda(f)^*)\\
&= d_\lambda \int_G\int_G f(x)f(y)\operatorname{tr}(U_\lambda(x^{-1})U_\lambda(y))\,\mathrm{d}x\,\mathrm{d}y\\
&= d_\lambda \int_G\int_G f(x)f(y)\operatorname{tr}(U_\lambda(yx^{-1}))\,\mathrm{d}x\,\mathrm{d}y\\
&= d_\lambda \int_G \widehat{f}(y)\,\chi_\lambda(y)\,\mathrm{d}y\\
&= d_\lambda |W|^{-1}\int_T\int_G \widehat{f}(yty^{-1})\sum_{s\in W} e^{i\langle s(\lambda+\rho)\,,\,y\rangle}\overline{D(x)}\,\mathrm{d}y\,\mathrm{d}t
\end{aligned}$$

by the Weyl integration formula. Now the inner integral is infinitely differentiable on the $\ell$-torus $T$, whence $\|\varphi_\lambda(f)\| \le G(1+|\lambda|)^{-m}$ for some constant $C = C(m, f)$ by the well-known result of multiple Fourier series that we referred to.

Now suppose that for each $m > 0$ there exists a constant $C = C(m, f)$ such that $\|\varphi_\lambda(f)\| \le G(1+|\lambda|)^{-m}$. Clearly $f_\lambda = \operatorname{tr}\varphi_\lambda(f)\varphi_\lambda$ is a $C^\infty$-function on $G$. For each $Y \in \mathfrak{g}$ and for every matrix coefficient $\varphi_\lambda^{u,v}(x) = \langle\varphi_\lambda(x)u\,,\,v\rangle$ we have (as in [14, p. 43]):

$$\|Y\varphi_\lambda^{u,v}\|_\infty \le A\sum_{k=0}^{(n/2)+1}\|\Delta^k\varphi_\lambda^{u,v}\|_{L^2(G)} \le A|\lambda|^{n+1}.$$

Hence, $\sum_{\lambda\in\Lambda} \operatorname{tr}\big(\varphi_\lambda(f)\,Y\varphi_\lambda(x)\big)$ converges uniformly on $G$ and $Yf(x) = \sum_{\lambda\in\Lambda} \operatorname{tr}\big(\varphi_\lambda(f)\,Y\varphi_\lambda(x)\big)$ is continuous on $G$. By induction we obtain $Y^J f \in C(G)$ for all multi-indices $J = (j_1, j_2, \ldots, j_n)$ and for all $Y_1, Y_2, \ldots, Y_n \in \mathfrak{g}$, which proves that $f \in \mathcal{S}(G)$. □

A complete topology on $\mathcal{S}(G)$ can now be defined in the standard way by declaring the family

$$N_{\varepsilon,m} = \left\{ f \in \mathcal{S}(G) \;\middle|\; \sup_\lambda (1+|\lambda|)^m \|\varphi_\lambda(f)\| < \varepsilon \right\}, \quad m \in \mathbb{N}^+,\ \varepsilon > 0$$

to be a local base at 0. The subspace $\mathcal{S}'(G)$ of the space of formal Fourier series $\sum_{\lambda\in\Lambda} \operatorname{tr}\big(C_\lambda \varphi_\lambda(x)\big)$ (where each $C_\lambda$ is an operator in the representation space of $U_\lambda$) consists of Fourier series for which $\|C_\lambda\| = \mathrm{O}\big(|\lambda|^m\big)$ for some $m \in \mathbb{N}^+$. We topologize the space $\mathcal{S}'(G)$ of Schwartz distributions by defining the local base at 0:

$$N_{\varepsilon,m} = \left\{ \sum \operatorname{tr}(C_\lambda \varphi_\lambda) \in \mathcal{S}'(G) \;\middle|\; \sup_\lambda (1+|\lambda|)^{-m} \|C_\lambda\| < \varepsilon \right\},$$

$m \in \mathbb{N}^+$, $\varepsilon > 0$.

For $f = \sum_{\lambda\in\Lambda} \operatorname{tr}\big(C_\lambda \varphi_\lambda\big)$ and $\psi = \sum_{\lambda\in\Lambda} \operatorname{tr}\big(c_\lambda \varphi_\lambda\big)$ in $\mathcal{S}'(G)$, the convolution of $f$ and $\psi$ is defined by

$$(f * \psi)(x) = \sum_{\lambda\in\Lambda} \operatorname{tr}\big(C_\lambda c_\lambda \varphi_\lambda(x)\big).$$

The space of Schwartz distributions is closed under convolution and the Dirac distribution $\delta(x) = \sum_{\lambda\in\Lambda} \operatorname{tr} \varphi_\lambda(x)$ is an identity element for convolution. If $f \in \mathcal{S}'(G)$ and $\psi \in \mathcal{S}$, then

$$f * \psi(x) = \int_G f(y)\psi(y^{-1}x)\,\mathrm{d}y.$$

## 2. Maximal Functions

For any distribution $f$ in $\mathcal{S}'(G)$, $P_t * f$ is a measurable function on $G^+$. With the interpretation of $G$ as boundary of $G^+$, we have maximal functions associated with three different boundary approaches. The radial maximal function $P^+f$, the nontangential maximal function $P^*_\gamma f$, and the tangential maximal function $P^{**}_M f$ are defined for $x$ in $G$ by

$$P^+f(x) = \sup_{t>0} \left| P_t * f(x) \right| \tag{2.1}$$

$$P^*_\gamma f(x) = \sup_{(y,t)\in\Gamma_\gamma(x)} \left| P_t * f(y) \right| \tag{2.2}$$

$$P^{**}_M f(x) = \sup_{(y,t)\in G^+} \left| P_t * f(y) \right| \left( \frac{t}{d(x,y)+t} \right)^M . \tag{2.3}$$

When $\gamma = 1$ we write $P^*f$ for $P^*_1 f$. We also define local maximal functions for $\varepsilon_0 > 0$ as follows:

$$P^+_{\varepsilon_0} f(x) = \sup_{0<t\leq\varepsilon_0} \left| P_t * f(x) \right| ,$$

$$P^*_{\gamma,\varepsilon_0} f(x) = \sup_{\substack{(y,t)\in\Gamma_\gamma(x) \\ 0<t\leq\varepsilon_0}} \left| P_t * f(y) \right| , \quad P^*_{\varepsilon_0} f = P^*_{1,\varepsilon_0} f$$

and

$$P^{**}_{M,\varepsilon_0} f(x) = \sup_{(y,t)\in G\times(0,\varepsilon_0\,]} \left| P_t * f(y) \right| \left( \frac{t}{d(x,y)+t} \right)^M .$$

We observe that we have the following pointwise inequalities:

$$P^+f(x) \leq P^*_\gamma f(x) \leq CP^{**}_M f(x) , \quad P^+_{\varepsilon_0} f(x) \leq P^*_{\gamma,\varepsilon_0} f(x) \leq CP^{**}_{M,\varepsilon_0} f(x) \tag{2.4}$$

$$P^+_{\varepsilon_0} f(x) \leq P^+ f(x) , \quad P^*_{\gamma,\varepsilon_0} f(x) \leq P^*_\gamma f(x) , \quad P^{**}_{M,\varepsilon_0} f(x) \leq P^{**}_M f(x) . \tag{2.5}$$

The following two constructs play the same role in the current setting as they do in Euclidean space. For a measurable function $f$ on $G$, the Hardy–Littlewood maximal function $Mf$ is defined by

$$Mf(x) = \sup_{h>0} \left| B(x,h) \right|^{-1} \int_{B(x,h)} \left| f(y) \right| dy , \quad x \in G \tag{2.6}$$

where $B(x,h) = \{ y \in G \mid d(x,y) < h \}$. The distribution function $\lambda_f$ of a measurable function $f$ on $G$ is defined by

$$\lambda_f(\alpha) = \left| \left\{ x \in G \;\middle|\; \left| f(x) \right| > \alpha \right\} \right| , \quad \alpha \in [\,0\,,\infty) . \tag{2.7}$$

The weak $(1,1)$ property

$$\lambda_{Mf}(\alpha) \le \frac{C}{\alpha} \|f\|_{L^1(G)}, \quad \alpha > 0, \; f \in L^1(G),$$

although not universally valid on Riemannian manifolds, is well-known and easily obtained in the present setting.

For each $x \in G$ we choose $\tau$ small enough so that $\exp^{-1} \circ L_{x^{-1}} : B(x,\tau) \to \mathfrak{g}$ (where $L$ denotes left multiplication) gives a coordinate chart $(y_1, \ldots, y_n)$ where $x^{-1}y = \exp(y_1 X_1 + \cdots + y_n X_n)$ and $d(y,x)^2 \approx y_1^2 + \cdots + y_n^2$. Since $G$ is compact, $\tau$ may be uniformly chosen. We do not fix such a $\tau$ once and for all at this time but it is to be understood that any future reference to a parameter labeled $\tau$ entails that it is small enough that this condition is satisfied.

LEMMA .— *For any $t > 0$ and $\gamma \ge 1$, there exists a constant $C$ which does not depend on $t$ or $\gamma$ such that $|B(x, (\gamma+1)t)|^{-1}|B(x,t)| > C\gamma^{-n}$.*

*Proof.*— It is clear that the inequality is only at issue for $0 < t < D$ (where $D$ denotes the diameter of $G$) and by invariance we may take $x$ to be the identity element $e$. In the case that $(\gamma+1)t \le D$ it follows from Proposition 1.1 that we may find positive constants $\mu$ and $\nu$ such that

$$\begin{aligned} \left|B(e, (\gamma+1)t)\right| = \int_{B(e,(\gamma+1)t)} \mathrm{d}y &\le \int_{\substack{|H| \le \mu(\gamma+1)t \\ H \in \mathfrak{t}}} \prod_{\alpha \in A} |\alpha(H)| \, \mathrm{d}H \\ &\le C\gamma^n \int_{|H| \le t} \prod_{\alpha \in A} |\alpha(H)|^2 \, \mathrm{d}H \end{aligned}$$

and

$$|B(e,t)| \ge \int_{|H| \le \nu t} \prod_{\alpha \in A} |\alpha(H)|^2 \, \mathrm{d}H \ge C \int_{|H| \le t} \prod_{\alpha \in A} |\alpha(H)|^2 \, \mathrm{d}H$$

from which the lemma follows. In the remaining case, i.e. when $(\gamma+1)t \ge D$, we prove, as in the first case, that $|B(e,t)| \ge Ct^n \ge C\gamma^{-n}$ and $|B(x, (\gamma+1)t)| = |G| = 1$, whence the required estimate. □

PROPOSITION 2.1.— *If $0 < p < \infty$, $M > n/p$, and $0 < \varepsilon_0 < \tau$ there are constants $C_1$ and $C_2$ depending only on $G$, $p$, and $M$ such that*

$$\|P^{**}_{M,\varepsilon_0} f\|_{L^p(G)} \le C_1 \|P^*_{\varepsilon_0} f\|_{L^p(G)} \le C_2 \|P^+_{2\varepsilon_0} f\|_{L^p(G)}, \quad f \in \mathcal{S}'(G).$$

*Proof.*— First we prove

$$\|P^*_{\gamma,\varepsilon_0}f\|^p_{L^p(G)} \le C\gamma^n \|P^*_{\varepsilon_0}f\|^p_{L^p(G)} \quad \text{for } \gamma \ge 1\,. \tag{2.8}$$

We let

$$E_\alpha = \Big\{ y \in G \;\Big|\; \big|P^*_{\varepsilon_0}f(y)\big| > \alpha \Big\} \quad \text{and} \quad E^*_\alpha = \{ M\chi_{E_\alpha} > C_0\gamma^{-n} \}$$

where $C_0$ is the fixed geometric constant of the previous lemma. Then

$$|E^*_\alpha| \le C\gamma^n |E_\alpha| \tag{2.9}$$

since the Hardy–Littlewood operator is of weak type $(1,1)$. We will show that

$$\{P^*_{\gamma,\varepsilon_0}f > \alpha\} \subset E^*_\alpha \tag{2.10}$$

which together with (2.9) implies that

$$\big|\{P^*_{\gamma,\varepsilon_0}f > \alpha\}\big| \le C\gamma^n\,|E_\alpha| \tag{2.11}$$

We obtain (2.8) from (2.11) by the standard distribution function argument:

$$\begin{aligned}\|P^*_{\gamma,\varepsilon_0}f\|^p_{L^p(G)} &= p\int_0^\infty \alpha^{p-1}\big|\{P^*_{\gamma,\varepsilon_0}f > \alpha\}\big|\,\mathrm{d}\alpha \le \\ &\le C\gamma^n p\int_0^\infty \alpha^{p-1}|E_\alpha|\,\mathrm{d}\alpha = C\gamma^n\|P^*_{\varepsilon_0}f\|^p_{L^p(G)}\,.\end{aligned}$$

We prove (2.10) by working with the complements of the indicated sets. Fix $g \notin E^*_\alpha$. Let $(g_1,t) \in \Gamma_{\gamma,\varepsilon_0}(g)$. We claim that the ball $B(g_1,t)$ cannot be contained in $E_\alpha$. If it were, we would have $B(g_1,t) \subset E_\alpha \cap B\big(g\,,\,(\gamma+1)t\big)$ by the triangle inequality and therefore, by the lemma preceding this proposition,

$$\begin{aligned}M(\chi_{E_\alpha})(g) &\ge \frac{1}{\big|B\big(g\,,\,(\gamma+1)t\big)\big|}\int_{B(g\,,\,(\gamma+1)t)}\chi_{E_\alpha}(y)\,\mathrm{d}y \\ &\ge \frac{\big|E_\alpha \cap B\big(g\,,\,(\gamma+1)t\big)\big|}{\big|B\big(g\,,\,(\gamma+1)t\big)\big|} \ge \frac{\big|B(g_1,t)\big|}{\big|B\big(g\,,\,(\gamma+1)t\big)\big|} \ge C_0\gamma^{-n}\end{aligned}$$

which contradicts $g \notin E^*_\alpha$. Thus, for each $g \notin E^*_\alpha$ and $(g_1,t) \in \Gamma_{\gamma,\varepsilon_0}(g)$, we may pick a $g_0 \in B(g_1,t) \cap E^c_\alpha$. Then $\big|P_t * f(g_1)\big| \le \big|P^*_{\varepsilon_0}f(g_0)\big| \le \alpha$ and since $(g_1t) \in \Gamma_{\gamma,\varepsilon_0}(g)$ is arbitrary, $\big|P^*_{\gamma,\varepsilon_0}f(g)\big| \le \alpha$, which proves (2.10).

We can now establish the first required inequality, without regard to the size of $\varepsilon_0$:

$$\left\|P^{**}_{M,\varepsilon_0} f\right\|_{L^p(G)} \leq C\left\|P^{*}_{\varepsilon_0} f\right\|_{L^p(G)}, \quad M > \frac{n}{p}, \; \varepsilon_0 > 0. \tag{2.12}$$

For fixed $(y,t) \in G \times (0\,,\,\varepsilon_0]$ and for any $x \in G$, fix a $\gamma = 2^m$ $(m \in \mathbb{N}^+)$ such that $d(x,y) \leq \gamma t$. Then

$$\begin{aligned}\left|P_t * f(y)\right| \left(\frac{t}{dy,x+t}\right)^M &\leq \left|P_t * f(y)\right| \left(\frac{d(y,x)}{t}\right)^{-M} \\ &\leq \gamma^{-M}\left|P_t * f(y)\right| \leq \gamma^{-M}\left|P^{*}_{\gamma,\varepsilon_0} f(x)\right|\end{aligned}$$

whence

$$\left|P^{**}_{M,\varepsilon_0} f(x)\right| \leq \sup_{m \in \mathbb{N}^+} \left|2^{-mM} P^{*}_{2^m,\varepsilon_0} f(x)\right|, \quad x \in G.$$

Therefore, by (2.8)

$$\begin{aligned}\left\|P^{**}_{M,\varepsilon_0} f\right\|^p_{L^p(G)} &\leq \sum_{m=1}^{\infty} \int_G 2^{-Mmp}\left|P^{*}_{2^m,\varepsilon_0} f\right|^p \mathrm{d}g \\ &\leq \sum_{m=1}^{\infty} 2^{-Mmp+mn} \int_G \left|P^{*}_{\varepsilon_0} f\right|^p \mathrm{d}g \\ &\leq C\left\|P^{*}_{\varepsilon_0} f\right\|^p_{L^p(G)}\end{aligned}$$

for $M > n/p$ where $C = C(M,p,n)$.

We now prove the remaining required inequality:

$$\left\|P^{*}_{\varepsilon_0} f\right\|_{L^p(G)} \leq C\left\|P^{+}_{2\varepsilon_0} f\right\|_{L^p(G)} \quad 0 < \varepsilon_0 < \tau\,, \; 0 < p < \infty. \tag{2.13}$$

Since the Hardy–Littlewood maximal operator is bounded on $L^2(G)$ it suffices to show that for $\varepsilon_1$ small enough

$$\left|P^{*}_{\varepsilon_1/4} f(x)\right|^{p/2} \leq CM\left(P^{+}_{\varepsilon_1/2} f\right)^{p/2}(x), \quad x \in G\,, \; 0 < p < \infty. \tag{2.14}$$

Applying Theorem 1 of [17] (together with the remark following that theorem) to the function $P_t * f$ on $G^+$ (which is a solution of the elliptic operator $\Delta + \partial^2/\partial t^2$) we have for $t < \varepsilon_1/4$ and $d(y', y) < t$

$$\begin{aligned}
|P_t * f(y')|^{p/2} &\le C t^{-(n+1)} \int_{\{d(z,y)^2+(s-t)^2 \le t^2\}} |P_s * f(z)|^{p/2}\, \mathrm{d}z\, \mathrm{d}s \\
&\le C t^{-(n+1)} \int_0^{2t} \int_{B(y,t)} |P_s * f(z)|^{p/2}\, \mathrm{d}z\, \mathrm{d}s \\
&\le C t^{-(n+1)} \int_0^{2t} \int_{B(y,t)} |P^+_{\varepsilon_1/2} f(z)|^{p/2}\, \mathrm{d}z\, \mathrm{d}s \\
&\le C t^{-n} \int_{B(y,t)} |P^+_{\varepsilon_1/2} f(z)|^{p/2}\, \mathrm{d}z \le C M\big(P^+_{\varepsilon_1/2} f\big)^{p/2}(y)
\end{aligned}$$

and (2.13) holds for $\tau = \varepsilon_{1/4}$. □

PROPOSITION 2.2.— *If $0 < p < \infty$ and $M > n/p$, then*

$$\|P^{**}_M f\|_{L^p(G)} \le C \|P^* f\|_{L^p(G)}, \quad f \in \mathcal{S}'(G).$$

Since the proof of Proposition 2.2 only uses ideas found in the proof of Proposition 2.1 it is omitted.

We now define another maximal function associated with distributions on $G$ which is more flexible in some way than those discussed above. For $N \in \mathbb{N}^+$ and $x \in G$ we define the subclass $K_N(x)$ of $\mathcal{S}(G)$ as all those $\varphi \in \mathcal{S}(G)$ satisfying:

i) $\operatorname{Supp}\varphi \subset B(x, h)$;

ii) $\displaystyle\sup_{t,x} \left| \frac{\mathrm{d}^N}{\mathrm{d}t^N} (P_t * \varphi)(x) \right| \le h^{-N-n}$;

iii) $\|\varphi\|_\infty \le h^{-n}$ for some $h > 0$.

We note that $h = D$, the diameter of $G$, is permitted.

For distributions $f$ in $\mathcal{S}'(G)$ and $\varphi$ in $K_N(x)$ we use the pairing $\langle f, \varphi\rangle = \int_G f(g)\overline{\varphi(g)}\, \mathrm{d}g$. The grand maximal function $f^*$ is defined by

$$f^*(x) = \sup\Big\{ |\langle f, \varphi\rangle| \;\Big|\; \varphi \in K_N(x) \Big\}, \quad f \in \mathcal{S}'(G) \tag{2.16}$$

in analogy with the $C$. Fefferman-Stein grand maximal function for $\mathbb{R}^n$ [11]. Of course, the dependence on $N$ is concealed by the notation. The restriction of $N$ to even integers below is made only to simplify computations and is not essential.

PROPOSITION 2.3.— *For $N \in 2\mathbb{N}^+$, there is a constant $C$ depending only on $N$ and $G$ such that*

$$P^+f(x) \leq Cf^*(x)\,, \quad f \in \mathcal{S}'(G)\,, \, x \in G \tag{2.17}$$

*Proof.*— Let $t_0 = \tau/2$ where $\tau$ is as in Proposition 2.1. Then

$$\begin{aligned} P^+f(x) &\leq \sup_{0<t\leq t_0} \left| \int_G f(y)\overline{P_t(yx^{-1})}\,\mathrm{d}y \right| + \sup_{t>t_0} \left| \int_G f(y)\overline{P_t(yx^{-1})}\,\mathrm{d}y \right| \\ &\leq P^+_{t_0}f(x) + Qf(x)\,. \end{aligned}$$

We estimate the maximal function $Qf$ first. It will be clear that $Qf(x) \leq Cf^*(x)$ with $C$ independent of $f$ and $x$ if we prove that there exists a constant $C$ independent of $t$ and $x$ such that

$$CP_t(yx^{-1}) \in K_N(x)\,, \quad t > t_0\,, \quad x \in G\,. \tag{2.18}$$

In fact, if $\widetilde{P}_t(y) = P_t(x^{-1}y)$,

$$P_s * \widetilde{P}_t(y) = P_{t+s}(x^{-1}y) = \sum_{\lambda\in\Lambda} e^{-(t+s)\left(\|\lambda+\rho\|^2-\|\rho\|^2\right)^{1/2}} d_\lambda\chi_\lambda(x^{-1}y)$$

whence

$$\left| \frac{\mathrm{d}^N}{\mathrm{d}s^N} P_{s+t}(x^{-1}y) \right| \leq C \sum_{\lambda\in\Lambda} e^{-(t+s)\|\lambda\|} \|\lambda\|^{2\xi+n-\ell} \quad \text{for } \xi > 2N\,.$$

It follows that

$$\sup_{y\in G\,,\, s>0} \left| \frac{\mathrm{d}^N}{\mathrm{d}s^N} P_s * \widetilde{P}_t(y) \right| \leq C$$

with $C$ dependent only on $t_0$. In the same way we obtain

$$\sup_{t\geq t_0} \|\widetilde{P}_t(y)\|_{L^\infty(G)} \leq C$$

which yields (2.18).

We turn to a similar estimate of $P_{t_0}^{+}f$. For each $x \in G$, $\exp^{-1} \circ L_{x^{-1}}$ (where $L$ denotes left multiplication) gives a coordinate chart for the ball $B(x, t_0)$. For a fixed $t$ in $(0, t_0)$, there exists a $j_0 \in \mathbb{N}$ such that $2^{j_0+1}t \leq \delta < 2^{j_0+2}t$. For $0 \leq j \leq j_0$ let

$$A_j(x) = \left\{ y \in G \mid 2^{j-1}t < d(x,y) < 2^{j+1}t \right\}.$$

Set

$$A(x) = \left\{ y \in G \mid d(x,y) > 2^{j_0-4}t \right\}.$$

Then $A_j(x) \subset B(x,\delta)$ for these $j$ and $d(y,x) > \varepsilon_0/16$ for $y \in A(x)$. Using the indicated chart we may choose a $C^\infty$ partition of unity $\left\{ \varphi_j (0 \leq j < j_0) \,, \varphi \right\}$ relative to the covering $\left\{ A_j(x) \right\}_j \cup \left\{ A(x) \right\}$ of $G$ with the property:

$$\left| (X^J \varphi_k)(x) \right| \leq C_J \left( 2^k t \right)^{-|J|}, \quad x \in G \,,\ J = (j_1, j_2, \ldots, j_n) \in \mathbb{N}^n$$

where $C_J$ does not depend on $k$. In terms of the modified kernel given by (1.17), for a fixed element $t$ of $(0, t_0)$ we have by (1.9):

$$\begin{aligned}
\widetilde{P}_t(y) &= P_t(x^{-1}y) \\
&= \frac{t}{2\sqrt{\pi}} \int_1^\infty e^{-t^2/4u} W_u(x^{-1}y) u^{-3/2}\, du + \\
&\quad + \frac{t}{2\sqrt{\pi}} \int_0^1 e^{-t^2/4u} \big( W_u(x^{-1}y) - \Delta_{q,u} K_u(x^{-1}y) \big) u^{-3/2}\, du \\
&\quad + \frac{t}{2\sqrt{\pi}} \int_0^1 e^{-t^2/4u} \Delta_{q,u} K_u(x^{-1}y)\, u^{-3/2}\, du
\end{aligned}$$

which we write as $P_t^{(1)}(y) + P_t^{(2)}(y) + P_t^{(3)}(y)$, noting that each term is in $\mathcal{S}(G)$. Now for $s > n/2$ we have by Sobolev's theorem

$$\begin{aligned}
\| P_t^{(1)} \|_\infty &\leq Ct \int_1^\infty e^{-t^2/4u} \| W_u \|_\infty u^{-3/2}\, du \\
&\leq Ct \int_1^\infty e^{-t^2/4u} \| W_u \|_{2,s} u^{-3/2}\, du \\
&\leq Ct \int_1^\infty e^{-t^2/4u} u^{-s-(n+1/2)-1}\, du \leq C\,.
\end{aligned}$$

Also

$$\| P_t^{(2)} \|_\infty \leq Ct \int_1^\infty e^{-t^2/4u} \| W_u \Delta_{q,u} K_u \|_{2s} u^{-3/2}\, du \leq C$$

by (1.18). For $N$ even and $i \in \{1,2\}$,

$$\frac{\mathrm{d}^N}{\mathrm{d}s^N} P_s * P_t^{(i)} = -\frac{\mathrm{d}^{N-2}}{\mathrm{d}s^{N-2}} \Delta(P_s * P_t^{(i)}) = -\frac{\mathrm{d}^{N-2}}{\mathrm{d}s^{N-2}} P_s * \Delta P_t^{(i)}$$
$$= P_s * \Delta^{N/2} P_t^{(i)},$$

the last equality resulting from iteration. The same argument used to estimate $\|P_t^{(i)}\|_\infty$ now yields $\|\Delta^{N/2}P_t^{(i)}\|_\infty \leq C$. Thus,

$$\left|\frac{\mathrm{d}^N}{\mathrm{d}s^N} P_s * P_t^{(i)}(y)\right| \leq \|\Delta^{N/2}P_t^{(i)}\|_\infty \|P_s\|_{L_1(G)} \leq C$$

where $C$ is independent of $y \in G$, $t < t_0$ and $s > 0$.

To recapitulate our progress, we have shown that

$$P^+ f(x) \leq C f^*(x) + \sup_{0<t\leq t_0} \left|P_t^{(3)} * f(x)\right|.$$

The task of showing that the last term is pointwise dominated by $f^*$ is most nettlesome. We write

$$P_t^{(3)}(y) = \sum_{j=0}^{j_0} \psi_j(y) + \psi(y) \quad \text{where } \psi_j = P_t^{(3)} \cdot \varphi_j \text{ and } \psi = P_t^{(3)} \cdot \varphi.$$

By choosing the cut-off function $\eta$ in (1.17) with sufficiently small support, we may assume that the kernel $K_t$ is supported in $B(e,\tau)$. As in [9] we have for $x = \exp X$.

$$|X^J P_t^{(3)}(x)| \leq C \sum_{j=0}^{|J|} t\|X\|^{|J|-j} \left(t^2 + \|X\|^2\right)^{j-(n+1)/2-|J|}, \quad 0 < t \leq t_0.$$

For $\alpha \geq 0$ and $2^{j-1}t < \|X\| < 2^{j+1}t$ one easily sees that

$$t\|X\|^\alpha \left(t^2 + \|X\|^2\right)^{-(n+1)/2-\alpha} \leq \frac{t^{\alpha+1}2^{j\alpha}}{2^{j(n+1)+2j\alpha}t^{n+1+2\alpha}} \leq C2^{-j}(2^j t)^{-n-\alpha}$$

whence

$$X^J \psi_j(x) \leq \sum_{|\alpha|+|\beta|=|J|} C_{\alpha,\beta} |X^\alpha P_t^{(3)}(x) X^\beta \varphi_j(x)|$$
$$\leq C2^{-j}(2^j t)^{-n-|\alpha|}(2^j t)^{-|\beta|} = C2^{-j}(2^j t)^{-n-|J|}.$$

Hence,

$$\left|\frac{\mathrm{d}^N}{\mathrm{d}s^N} P_s * \psi_j(y)\right| \le C\|P_s\|_{L^1(G)} 2^{-j}\left(2^j t\right)^{-n-N} \le C2^{-j}\left(2^j t\right)^{-n-N}$$

for all $s > 0$ and $y \in G$. Thus $\|P_s * \psi_j\|_\infty \le C2^{-j}\left(2^j t\right)^{-n}$. We deduce that $\psi_j = C2^{-j}\widetilde{\psi}_j$ where $\widetilde{\psi}_j \in K_m(x)$. We similarly obtain

$$\left\|\frac{\mathrm{d}^N}{\mathrm{d}s^N} P_s * \psi\right\|_{L^\infty(G)} \le C$$

for all $s > 0$ and $\|\psi\|_\infty \le C$. Therefore $C\psi = \widetilde{\psi}$ for $\widetilde{\psi} \in K_m(x)$ where $C$ depends on $N$ and $\tau$ but not on $j$. Thus

$$\sup_{0\le t\le t_0} \left|P_t^{(3)} * f\right| \le C\sum_{j=0}^{\infty} 2^{-j}\langle f, \widetilde{\psi}_j\rangle + Cf^*(x) + C\langle f, \widetilde{\psi}\rangle \le Cf^*(x)$$

which completes the proof. □

Although there is no pointwise converse of Proposition 2.3, it is still true that the grand maximal function $f^*$ is dominated pointwise by the tangential maximal function $P_M^{**}f$ as we shall show by a standard technique involving an auxiliary kernel closely related to the Poisson kernel but better adapted for the approximation of smooth functions.

Let $L$ be an integer to be specified later and let $\{\ell_0, \ell_1, \ldots, \ell_L\}$ be distinct positive numbers with $1/4 < \ell_j < 1/2$. There exist $c_0, c_1, \ldots, c_L$ such that $\sum_{j=0}^L c_j = 1$ and $\sum_{j=0}^L c_j \ell_j^s = 0$, $s \in \{1, 2, \ldots, L\}$. Define

$$\sigma_t(x, w) = \sum_{j=0}^{L} c_j P_{\ell_j t}(w^{-1}x)\,, \quad \sigma_t^- = \sigma_{1/2} - \sigma_t \quad \text{and} \quad \sigma_t^+ = \sigma_{1/2} + \sigma_t\,.$$

For these kernels we will need to show that there exists a constant $C$ such that

$$\|P_{\ell_j t}\|_{L^\infty(G)} \le Ct^{-n}\,, \quad 0 < t \le 1\,,\ 0 \le j \le L\,. \tag{2.19}$$

Indeed,

$$\left|P_t(x)\right| \le C\sum_{\lambda\in\Lambda} e^{-t\|\lambda+\rho\|} d_\lambda^2 \le Ct^{-(2m+\ell)}\,,$$

whence (2.19).

We will also need the estimate

$$\left|\frac{\mathrm{d}^k}{\mathrm{d}u^k}\,P_u(w^{-1}x)\right| \le Cd(x,w)^{-(n+k)}\,, \quad k\in\mathbb{N}\,,\ w\in G\,;\ d(x,w)\ge 9t\,. \tag{2.20}$$

We may easily reduce this to the case where $w = e$, $x \in B(e,\tau) - B(e,9t)$, and $0 < u \le \tau$. Using (1.9), elementary calculations give

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}s^k}\,P_s(x) &= \sum_{j=0}^{[(k+1)/2]} C_j s^{1+(k-2j)} \int_0^\infty e^{-s^2/4u} W_u(x)\, u^{-3/2-(k-j)}\,\mathrm{d}u \\ &= \sum I_j\end{aligned}$$

($k$ odd) where

$$I_j = C_j s^{1+(k-2j)} \left(\int_0^1 + \int_1^\infty\right) e^{-s^2/4u} u^{-3/2-k+j} W_u(x)\,\mathrm{d}u = I_j^{(1)} + I_j^{(2)}\,.$$

In Proposition 2.3 we obtained $|I_j^{(2)}| \le C$. For $I_j^{(1)}$; we estimate

$$s^{1+k-2j} \int_0^1 e^{-s^2/4u} u^{-3/2-k+j} \Delta_{q,u} K_u(x)\,\mathrm{d}u\,.$$

Since $\left|\Delta_{q,u} K_u(x)\right| \le C u^{-n/2} e^{-\|x\|^2/4u}$ from [9], we have

$$\begin{aligned}\left|I_j^{(1)}\right| &\le C s^{1+k-2j} \int_0^\infty e^{-s^2/4u} u^{-3/2-k+j-n/2} e^{-\|x\|^2/4u}\,\mathrm{d}u \\ &\le C \frac{s^{1+k-2j}}{\left(s^2+\|x\|^2\right)^{-1/2-k+j-n/2}}\,.\end{aligned}$$

Now if $\|x\| < s$, then $\left|I_j^{(1)}\right| \le s^{-1-n-2k+2j+1+k-2j} \le \left\|x\right\|^{-n-k}$; the same estimate is immediate if $\|x\| \ge s$ and (2.20) follows.

If $d(x,w) \le gt$ it is an immediate consequence of (2.20) that

$$\left|\sigma_t(x,w)\right| \le C \frac{t^{L+1}}{d(x,w)^{L+n+1}}\,. \tag{2.21}$$

If $d(x,w) > 9t$,

$$\sigma_t(x,w) =$$

$$= \sum_{k=0}^{L} \frac{\mathrm{d}^k}{\mathrm{d}u^k} P_u(w^{-1}x)\Big|_{u=0} \sum_{j=0}^{L} C_j \ell_j^k + \frac{\mathrm{d}^{L+1}}{\mathrm{d}u^{L+1}} \sigma_u(x,w)\Big|_{u=\theta t} \frac{t^{L+1}}{(L+1)!}$$

and (2.21) again follows from (2.20).

LEMMA .— *Suppose* $\varphi \in C^L(G)$ *and*

$$\left| \frac{\mathrm{d}^{L+1}}{\mathrm{d}s^{L+1}} (P_s * \varphi)(x) \right| \leq h^{-n-L-1} . \tag{2.22}$$

*Then*

$$\left| \int_G \sigma_t^-(x,w)\,\varphi(w)\,\mathrm{d}w \right| \leq C t^{L+1} h^{-n-L-1} . \tag{2.23}$$

*If (2.22) is valid only for* $s \in (0,\tau)$, *then (2.23) remains valid for* $0 < t \leq \tau$. *If, in addition to (2.22),* $\varphi \in B(x,h)$, $\|\varphi\|_{L^1(G)} \leq 1$ *and* $M < L+1$, *then*

$$\int_G \left| \int_G \sigma_{ah}(y,w)\,\varphi(w)\,\mathrm{d}w \right| \left(1 + \frac{d(x,w)}{ah}\right)^M \mathrm{d}y \leq C(a,M) , \quad a > 0 \tag{2.24}$$

*and*

$$\int_G \left| \int_G \sigma_{th}^-(y,w)\,\varphi(w)\,\mathrm{d}w \right| \left(1 + \frac{d(x,w)}{h}\right)^M \mathrm{d}y \leq C t^{L+1} . \tag{2.25}$$

*Proof.*— Since $P_s * \varphi$ is harmonic on $G^+$ and all its derivatives up to order $L$ are continuous in $G^+$ we have

$$\int_G \sigma_t(x,w)\,\varphi(w)\,\mathrm{d}w = \varphi(x) + \sum_{j=0}^{L} C_j \frac{\mathrm{d}^{L+1}}{\mathrm{d}u^{L+1}} P_u * \varphi(x)\Big|_{u=\theta_j \ell_j t} \frac{(\ell_j t)^{L+1}}{(L+1)!}$$

whence

$$\left| \int_G \sigma_t^-(x,w)\,\varphi(w)\,\mathrm{d}w \right| \leq$$

$$\leq \sum_{j=0}^{L} |C_j| \left| \frac{\mathrm{d}^{L+1}}{\mathrm{d}u^{L+1}} P_u * \varphi(x) \right|_{u=\theta_j \ell_j t} \frac{(\ell_j t)^{L+1}}{(L+1)!} +$$

$$+ \sum_{j=0}^{L} |C_j| \left| \frac{\mathrm{d}^{L+1}}{\mathrm{d}u^{L+1}} P_u * \varphi(x) \right|_{u=\theta_j \ell_j t/2} \frac{(\ell_j t/2)^{L+1}}{(L+1)!}$$

$$\leq C t^{L+1} h^{-n-L-1} .$$

Note that the left side of (2.24) is bounded by

$$\left(1+\frac{9}{a}\right)^{M}\int_{B(x,9h)}\int_{G}\left|\sigma_{ah}(y,w)\,\varphi(w)\right|\,\mathrm{d}w\,\mathrm{d}y+$$
$$+\left(\frac{1}{9}+\frac{1}{a}\right)^{M_h-M}\int_{G-B(x,9h)}\int_{G}\frac{(ah)^{L+1}}{d(y,w)^{n+L+1}}\left|\varphi(w)\right|d(x,y)^{M}\,\mathrm{d}w\,\mathrm{d}y$$

the first term of which is obviously bounded by a constant depending only on $M$ and $a$. For the second term, notice that since $y \notin B(x,9h)$ and $\operatorname{supp}\varphi \subset B(x,h)$ it follows that $d(x,y) \geq 2d(y,w)$ and $d(y,w) \leq 2d(x,y)$ and the second term is bounded by

$$C_a h^{-M-L-1}\int_{G}\left|\varphi(w)\right|\int_{G-B(x,9h)} d(x,y)^{M-L-1-n}\,\mathrm{d}w\,\mathrm{d}y \leq C_a\,.$$

We also note that the left side of (2.25) is bounded by

$$10^{M}\int_{B(x,9h)}\left|\int_{G}\sigma_{\overline{th}}(y,w)\,\varphi(w)\,\mathrm{d}w\right|\,\mathrm{d}y+$$
$$+\,Ch^{-M}\int_{G-B(x,9h)}\int_{G}\frac{(th)^{L+1}}{d(y,w)^{n+L+1}}\left|\varphi(w)\right|d(x,y)^{M}\,\mathrm{d}w\,\mathrm{d}y\,,$$

the first term of which is bounded by

$$C\int_{B(x,9h)}(ht)^{L+1}h^{-n-L-1}\,\mathrm{d}y \leq Ct^{L+1}$$

and the second term is similarly bounded as per the treatment given for (2.24). □

**Proposition 2.4.**— *If $N = L+1 > M > n/p$, then $f^*(x) \leq CP^{**}_{M,\varepsilon_0}f(x)$, $f \in \mathcal{S}'(G)$, with $C$ independent of $f$.*

*Proof.*— Let $\varphi \in K_N(x)$ with $\operatorname{supp}\varphi \subset B(x,h)$. We may without loss of generality suppose that $h \leq 1$. Note that  satisfies the assumptions in the last part of the lemma. Choose $2^{-k_0}h \leq \varepsilon_0$. Let

$$\sigma_t^*\sigma_s(v) = \int_G \sigma_t(y,v)\,\sigma_s(y,e)\,\mathrm{d}y\,.$$

Then

$$\sigma_t * \sigma_t = \left(\sum_{j=0}^{L} C_j P_{\ell_j t}\right) * \left(\sum_{i=0}^{L} C_i P_{\ell_i t}\right) = \sum_{i,j=0}^{L} C_i C_j P_{(\ell_j+\ell_i)t}$$

whence $\lim_{t\to 0} \sigma_t * \sigma_t * \varphi(x) = \varphi(x)$. In particular, if we set $h_0 = 2^{-k_0} h$,

$$\sigma_h * \sigma_h * \varphi(y) = \int_G \int_G \sigma_h(y,v)\, \sigma_h(v,w)\, \varphi(w)\, \mathrm{d}w\, \mathrm{d}v$$

and define $\sigma_h^+ * \sigma_h^- * \varphi$ similarly, we have

$$\begin{aligned}\varphi(x) &= \lim_{k\to\infty} \sigma_{2^{-k}h} * \sigma_{2^{-k}h} * \varphi(x)\\ &= \sigma_{h_0} * \sigma_{h_0} * \varphi(x) + \sum_{k=k_0}^{\infty} \sigma^+_{2^{-k}h} * \sigma^-_{2^{-k}h} * \varphi(x)\end{aligned}$$

and therefore for any $f \in \mathcal{S}'(G)$

$$\begin{aligned}\langle f, \varphi\rangle &= \langle f\,,\, \sigma_{h_0} * \sigma_{h_0} * \varphi\rangle + \sum_{k=k_0}^{\infty} \langle f\,,\, \sigma^+_{2^{-k}h} * \sigma^-_{2^{-k}h} * \varphi\rangle\\ &= \langle \sigma_{h_0} * f\,,\, \sigma_{h_0} * \varphi\rangle + \sum_{k=k_0}^{\infty} \langle \sigma^+_{2^{-k}h} * f\,,\, \sigma^-_{2^{-k}h} * \varphi\rangle\,.\end{aligned}$$

Define

$$\sigma^{**}_{M,\varepsilon_0} f(x) = \sup_{\substack{0<t\leq\varepsilon_0\\ v\in G}} \left|\left\langle f\,,\, \sigma_t(\,\cdot\,,v)\right\rangle\right| \left(\frac{t}{t-d(x,v)}\right)^M$$

and define $(\sigma^+)^{**}_{M,\varepsilon_0} f(x)$ by replacing $\sigma_t$ above with $\sigma_t^+$. Since

$$\left\langle f\,,\, \sigma_t(\,\cdot\,,v)\right\rangle = \sum_{j=0}^{L} c_j P_{\ell_j t} * f(v)$$

we get

$$\sigma^{**}_{M,\varepsilon_0} f(x) \leq C P^{**}_{M,\varepsilon_0} f(x) \quad \text{and} \quad (\sigma^+)^{**}_{M,\varepsilon_0} f(x) \leq C P^{**}_{M,\varepsilon_0} f(x)\,.$$

Therefore, in view of the preceding lemma,

$$\begin{aligned}
&\left|\langle \sigma_{h_0} * f\,,\, \sigma_{h_0} * f\rangle\right| \le \\
&\le C\sigma^{**}_{M,\varepsilon_0} f(x) \int_G \left| \int_G \sigma_{h_0}(v,w)\,\varphi(w)\,\mathrm{d}w \right| \left(1 + \frac{d(x,v)}{h_0}\right)^M \mathrm{d}v \\
&\le C\sigma^{**}_{M,\varepsilon_0} f(x)\,.
\end{aligned}$$

Also,

$$\begin{aligned}
&\left|\langle \sigma^+_{2^{-k}h} * f\,,\, \sigma^-_{2^{-k}h} * \varphi\rangle\right| \le \\
&\le (\sigma^+)^{**}_{M,\varepsilon_0} f(x) \int_G \left| \int_G \sigma^-_{2^{-k}h}(v,w)\,\varphi(w)\,\mathrm{d}w \right| \left(1 + \frac{d(x,v)}{2^{-k}h}\right)^M \mathrm{d}v \\
&\le C 2^{(M-L-1)k} P^{**}_{M,\varepsilon_0} f(x)\,.
\end{aligned}$$

It follows that $\left|\langle f, \varphi\rangle\right| \le C P^{**}_{M,\varepsilon_0} f(x)$ ($\varphi \in K_N(x)$) with $C$ independent of $\varphi$. We may finally conclude that $f^*(x) \le C P^{**}_{M,\varepsilon_0} f(x)$. □

For $0 < p < \infty$ the Hardy space $H^p(G)$ is the collection of all distributions $f \in \mathcal{S}'(G)$ for which $P^+ f \in L^p(G)$. The $H^p$ "norm" of $f$ is defined by $\|f\|_{H^p(G)} = \|P^+ f\|_{L^p(G)}$. Although not a norm in general, $\|\cdot\|_{H^p(G)}$ provides a complete metrizable topology in $H^p(G)$. The spaces $H^p(G)$ $(1 < p < \infty)$ are known, to be Lebesgue spaces ([1], [15]) and so we assume that $0 < p \le 1$ in the remainder. The next theorem can be culled from the preceding results.

THEOREM 2.5.— *Let $f$ be a distribution in $\mathcal{S}'(G)$. Suppose that $M > n/p$ and $N > n/p$ with $N$ even. The following are equivalent:*

*i)* $f \in H^p(G)$;

*ii)* $P^+ f \in L^p(G)$;

*iii)* $P^+_\varepsilon f \in L^p(G)$ *for some* $\varepsilon > 0$;

*iv)* $P^+_\varepsilon f \in L^p(G)$, $\varepsilon > 0$;

*v)* $f^* \in L^p(G)$;

*vi)* $P^{**}_{M,\varepsilon} f \in L^p(G)$, $0 < \varepsilon \le \tau$;

*vii)* $P^*_\varepsilon f \in L^p(G)$, $0 < \varepsilon \le \tau$.

*Moreover, we have, for* $0 < \varepsilon < \tau$ *and suitable constants* $C_\varepsilon$ *and* $C$*:*

$$\begin{aligned}\|f\|_{H^p(G)} &= \\ &= \|P^+f\|_{L^p(G)} \le C_\varepsilon \|P_\varepsilon^* f\|_{L^p(G)} \le C_\varepsilon \|P_{M,\varepsilon}^{**} f\|_{L^p(G)} \le C_\varepsilon \|P_\varepsilon^* f\|_{L^p(G)} \\ &\le C_\varepsilon \|P_{2\varepsilon}^+ f\|_{L^p(G)} \le C_\varepsilon \|P_{2\delta}^+ f\|_{L^p(G)} \le C \|f^*\|_{L^p(G)} \le C \|f\|_{H^p(G)} .\end{aligned}$$

THEOREM 2.6.— $\mathcal{S}(G)$ *is dense in* $H^p(G)$.

*Proof.*— Let $f \in H^p(G)$. For fixed $t > 0$ the function $P_t * f(x)$ is in $\mathcal{S}(G)$. By the semigroup property we have $P_s*(P_t*f-f) = P_{s+t}*f-P_s*f$ whence $P^+(f - P_t * f)(x) \le 2P^+f(x)$; the right side is finite almost everywhere since $P^+f$ is in $L^p(G)$. By proving that for almost every $x$, $\lim_{t\to 0} P^+(f - P_t * f)(x) = 0$, the theorem will follow from Lebesgue's theorem. Since for harmonic functions existence of non-tangential limits and nontangential boundedness are almost everywhere equivalent [17], $t \to P_t f(x)$, $t \in (0\,,1]$, admits a uniformly continuous extension to $[0\,,1]$ for almost every $x$ in $G$. This implies that for almost every $x \in G$, given $\varepsilon > 0$ there exists a $t_0 = t_0(x, \varepsilon)$ such that

$$\left|(P_{s+t} * f - P_s * f)(x)\right| < \varepsilon\,, \quad 0 < t \le t_0 .$$

Therefore $\lim_{t\to 0}\left|P^+(f - P_t * f)(x)\right| = 0$. □

## 3. Atomic characterization of $H^p(G)$

Elements in the closed unit ball of $L^\infty(G)$ will be called exceptional atoms. In order to define regular atoms, we consider a faithful finite dimensional unitary representation $\pi$ of $G$; one such exists by consequence of the Peter–Weyl theorem and the choice of $\pi$ will not matter as will be seen in Theorems 3.3 and 3.4. We may therefore identify $G$ with a Lie subgroup of $U_L = \{U \in \mathrm{GL}(L, \mathbb{C}) \mid {}^t\overline{U} \cdot U = e\}$ for some $L$ and by extension regard $G$ as a real submanifold of the real vector space $E$ underlying $\mathrm{End}(\mathbb{C}^L)$. For $0 < p \le 1$ let $n_0 = \left[n(p^{-1} - 1)\right]$. We let $\mathcal{P}_s(E)$ denote the vector space of polynomials on $E$ of degree no larger than $s \in \mathbb{N}$; for a given $p$ we will take $s = n_0$ as above. As in [5], we define a regular $(p, q)$-atom for $0 < p \le 1 \le q \le \infty$ to be an element $a$ of $L^q(G)$ satisfying the following three support, size, and cancellation properties:

(i) $\operatorname{supp} a \subset B(x, \rho)$ for some $\rho > 0$;

(ii) $\|a\|_{L^q(G)} \leq \rho^{n(1/q-1/p)}$;

(iii) $\int_G a(x) P(\pi(x))\,\mathrm{d}x = 0,\ P \in \mathcal{P}_{n_0}(E)$.

Since $G$ has finite diameter, (i) is only meaningful in conjunction with (ii). For such pairs $(p, q)$; $H_a^{p,q}(G)$ is the space of all $f \in \mathcal{S}'(G)$ admitting a decomposition $f = \sum_{k=1}^{\infty} c_k a_k$ $(\sum_{k=1}^{\infty} |c_k|^p < \infty)$ where each $a_k$ is either a regular $(p, q)$-atom or an exceptional atom. The "norm" $\|f\|_{H_a^{p,q}}$ is the infimum of $\left(\sum |c_k|^p\right)^{1/p}$ over all such decompositions of $f$. These definitions are evidently valid in the larger class of compact Lie groups; we will in particular use the definition for the unitary groups $U_L$. On the face of it, these atomic Hardy spaces seem to depend on the particular embedding, but, by obtaining their equivalence with maximal function based Hardy space, we will show this to be illusory. In the more general context of spaces of homogeneous type it is known that the index $q$ is ultimately superfluous: $H_a^{p,q}(G) = H_a^{p,\infty}(G)$ [8]; therefore we need only consider $H_a^{p,\infty}(G)$. By a $p$-atom we mean either an exceptional atom or a regular $(p, \infty)$-atom. The constant $(n-1)/2$, arising frequently, will be denoted by $\nu$. Our first result is that maximal Bochner–Riesz operators of $p$-atoms are uniformly bounded in $L^p(G)$ for suitable exponents. We let

$$S_+^\delta a(x) = \sup_{t>0} \left|S_t^\delta * a(x)\right|.$$

Proposition 3.1.— *There exists a $\delta > \nu$ and a constant $C$ depending only on $G$, $\delta$ and $p$ such that $\|S_+^\delta a\|_{L^p(G)} \leq G$ for all $p$-atoms $a$.*

*Proof.*— If $a$ is exceptional, then

$$\|S_t^\delta * a\|_{L^\infty(G)} \leq \|a\|_{L^\infty(G)} \|S_t^\delta\|_{L^1(G)} \leq C \quad \text{for all } t > 0$$

and the conclusion follows for these atoms. For a regular $p$-atom $a$, we have

$$\|S_+^\delta a\|_{L^\infty(G)} \leq C\rho^{-n/p} \quad \text{for all } \delta > \nu\,.$$

Therefore,

$$\int_{d(e,x)\leq 2\rho} |S_+^\delta a(x)|^p\,\mathrm{d}x \leq C$$

and we need only estimate

$$\int_{d(e,x)>2\rho} |S_+^\delta a(x)|^p\,\mathrm{d}x\,.$$

Fix $N$ as in [5, p. 108]. For $t \leq 2N\rho$ we argue as in [5] to obtain

$$\left|S_t^\delta * a(x)\right| \leq C\rho^{-n/p} \int_{B(x,\rho)} \left|S_t^\delta(y)\right| \mathrm{d}y$$

and

$$\left|S_t^\delta(y)\right| \leq Ct^{\delta-\nu} \left(d(e,y)^{-\delta-\nu-1} + \left|D^{-1}(y)\right|\right) .$$

Since $d(e,y) \geq d(e,x) - d(y,x)$ and $d(y,x) \leq \rho \leq d(e,x)/2$ we have $d(y,x) \geq \rho \geq t/2N$. Therefore,

$$\left|S_t^\delta * a(x)\right| \leq$$
$$\leq C\rho^{-n(1-1/p)} t^{\delta-\nu} \left( d(e,x)^{-(\nu+1+\delta)} + \left|B(x,\rho)\right|^{-1} \int_{B(x,\rho)} \left|D^{-1}(y)\right| \mathrm{d}y \right) .$$

By setting $\delta = n/p + \delta_0 - \nu - 1$, the task of finding $\delta > \nu$ becomes equivalent to that of finding $\delta_0 > 0$. In terms of this parameter our estimate above gives

$$\left|S_t^\delta * a(x)\right| \leq C\rho^{\delta_0} \left( d(e,x)^{-\delta_0-n/p} + \left|B(x,\rho)\right|^{-1} \int_{B(x,\rho)} \left|D^{-1}(y)\right| \mathrm{d}y \right) .$$

for $0 < t \leq 2N\rho$.

Now suppose that $t \geq 2N\rho$. The argument of Proposition 6.2 of [5] provides a Taylor polynomial $T^x_{n_0}(S^\delta_y)(y)$ such that

$$\begin{aligned}
&\left|S_t^\delta * a(x)\right| = \\
&= \left| \int_{d(e,\rho)} a(y)\left(S_t^\delta(xy^{-1}) - T^x_{n_0}(S_t^\delta)(y)\right) \mathrm{d}y \right| \\
&\leq C\rho^n \|a\|_{L^\infty(G)} \rho^{n_0+1} \sup\left\{ \left|X^J S_t^\delta(y)\right| \,\middle|\, y \in B(x,\rho)\,,\ |J| \leq n_0 + 1 \right\} \\
&\leq C\rho^{n+n_0+1-n/p} t^{-n_0-1} \cdot t^{\delta_0+n(1/p-1)} \left| d(e,x)^{-\delta_0-n/p} + \left|\Delta^R(x)^{-1}\right| \right|
\end{aligned}$$

where $\Delta^R$ is as in [5] and $R = 1/t$. Since $-n_0 - 1 + n(1/p-1) < 0$ we can choose $\delta_0$ sufficiently small that $-n_0 - 1 + n(1/p-1) + \delta_0 < 0$. Then, since $t \geq 2N\rho$, we have

$$t^{-n_0-1+\delta_0+n(1/p-1)} \leq C\rho^{-n_0-1+\delta_0+n(1/p-1)}$$

and

$$\begin{aligned}\left|S_t^\delta * a(x)\right| &\le C\rho^{\delta_0}\left(d(e,x)^{-\delta_0-n/p} + \left|\Delta^R(g)^{-1}\right|\right)\\ &\le C\rho^{\delta_0}\left(d(e,x)^{-\delta_0-n/p} + \left|D^{-1}(x)\right|\right).\end{aligned}$$

Thus

$$\begin{aligned}&\left|S_+^\delta a(x)\right| \le\\ &\le \rho^{\delta_0}\left(d(e,x)^{-\delta_0-n/p} + \left|D^{-1}(x)\right| + \left|B(x,\rho)\right|^{-1}\int_{B(x,\rho)}\left|D^{-1}(y)\right| \mathrm{d}y\right)\end{aligned}$$

for $d(e,x) \ge 2\rho$ and $C$ independent of the particular $p$-atom $a$. Therefore,

$$\begin{aligned}\left\|S_+^\delta a\right\|_{L^p(G)}^p &\le C + \int_{d(e,x)\ge 2\rho}\rho^{\delta_0 p} d(e,x)^{-n-\delta_0 p}\,\mathrm{d}x\, +\\ &\quad + \int_{d(e,x)\ge 2\rho}\left|D^{-p}(x)\right| \mathrm{d}x + \int_G \rho^{-np}\left(\int_{B(x,\rho)}\left|D^{-1}(y)\right| \mathrm{d}y\right)^p \mathrm{d}x\\ &\le C + C\left(\int_{B(x,\rho)}\int_G\left|D^{-1}(yx)\right| \mathrm{d}y\,\mathrm{d}x\right)^p \rho^{-np} \le C\,. \square\end{aligned}$$

In analogy with the maximal operators $P^+$ and $S_+^\delta$ we define the radial maximal heat operator

$$W^+ f(x) = \sup_{t>0}\left|W_t * f(x)\right|.$$

We prove in Proposition 3.2 that $W^+$ and $P^+$, like $S_+^\delta$, are uniformly bounded on $p$-atoms; this is the standard procedure for establishing the containment $H^{p,\infty}(G) \subset H^p(G)$ which is recorded in Theorem 3.3 which follows.

PROPOSITION 3.2.— *There is a constant $C$ depending only on $G$ and $p$ such that $\left\|P^+ a\right\|_{L^p(G)} \le C$ and $\left\|W^+ a\right\|_{L^p(G)} \le C$ for all $p$-atoms $a$.*

*Proof.*— Let $a$ be a $p$-atom and define the maximal operators $W_1$ and $W_2$ by

$$W_1 f(x) = \sup_{0<t\le 5}\left|W_t * f(x)\right| \quad\text{and}\quad W_2 f(x) = \sup_{5\le t<\infty}\left|W_t * f(x)\right|.$$

For $t \geq 5$ there is a Taylor polynomial $T^x_{n_0}\big(W_t(y)\big)$ (as in Proposition 3.1) such that

$$\begin{aligned}
&\big|W_t * a(x)\big| = \\
&\quad = \left|\int_{B(e,\rho)} a(y)\big(K_t(xy^{-1}) - T^x_{n_0}(W_t)(y)\big)\,\mathrm{d}y\right| \\
&\quad \leq C\rho^{1+n(1-1/p)+n_0} \sup\Big\{\big|X^J W_t(y)\big| \;\Big|\; y \in B(x,\rho)\,,\ |J| \leq 1 + n_0\Big\} \\
&\quad \leq C\big\|W_t\big\|_{H^{2,s}(G)}
\end{aligned}$$

for some $s > n/2 + 1 + n_0$. Therefore

$$\big\|W_t * a(x)\big\|_{L^\infty(G)} \leq C\left(\sum_{\lambda\in\Lambda} e^{-t\|\lambda\|^2}\big\|\lambda\big\|^{2s+n}\right)^{1/2} \leq C\big(1 + t^{-(s+n)}\big)\,.$$

This shows that

$$\big\|W_2 a\big\|_{L^\infty(G)} \leq C\,. \tag{3.1}$$

By (1.15),

$$W_1 a(x) \leq C \sup_{0<t\leq 5}\left|\int_0^\infty s^\delta\, e^{-s} S^\delta_{t/s} * a(x)\,\mathrm{d}s\right| \leq C S^\delta_+ a(x)\,.$$

Thus

$$\big\|W^+ a\big\|_{L^\infty(G)} \leq \big\|W_1 a\big\|_{L^\infty(G)} + \big\|W_2 a\big\|_{L^\infty(G)} \leq C$$

by Proposition 3.1 and (3.1). By (1.9) we have

$$\begin{aligned}
P^+ a(x) &= \sup_{t>0} C\left|\int_0^\infty u^{-1/2}\, e^{-u} W_{t^2/4u} * a(x)\,\mathrm{d}u\right| \\
&\leq C\int_0^\infty u^{-1/2}\, e^{-u}\,\mathrm{d}u \cdot W^+ a(x)
\end{aligned}$$

and the required bound for $P^+$ follows from that for $W^+$. □

THEOREM 3.3.— *There exists a constant $C$ depending only on $p$ and $G$ such that $\|f\|_{H^p_H(G)} \leq C\|f\|_{H^{p,\infty}_a(G)}$ for all $f$ in $H^{p,\infty}_a(G)$. In particular, $H^{p,\infty}_a(G) \subset H^p(G)$.*

*Proof.*— Each $f \in H^{p,\infty}_a(G)$ has a decomposition $f = \sum c_j a_j$ where $\{a_j\}$ is a sequence of $p$-atoms and $\sum |c_j|^p < \infty$. Thus

$$\begin{aligned} \|P^+ f\|_p^p &= \int_G \sup_{t>0} \left|P_t * \sum c_j a_j\right|^p \mathrm{d}x \\ &\leq \sum |c_j|^p \int_G P^+ a(x)^p \,\mathrm{d}x \leq C \sum |c_j|^p . \ \square \end{aligned}$$

It remains to establish the reverse inequality between the norms of $H^{p,\infty}_a(G)$ and $H^p(G)$; that will complete the proof of the equivalence of the two Hardy spaces. The idea of the proof below is not new, but minor errors have appeared in the two instances that we know of where it has been used. We first state two lemmas, the proofs of which use only standard partition of unity techniques and are omitted. We then consider the classical unitary group $U_n$ (where, in this discussion, $n$ is arbitrary and not $\dim_{\mathbb{R}} G$) and prove Lemma 3 which establishes atomic decompositions for Hardy functions on the unitary groups. Although the technicalities of Lemma 3 are specific to unitary groups, the broad generalities of the decomposition have been previously carried out for balls and spheres. The case of general compact Lie groups may quickly be reduced to that of Lemma 3.

LEMMA 1.— *There exist constants $A$, $\alpha$, $\beta$, $\gamma$ and $\delta$ with $0 < \delta < \gamma < 1 < \beta < \alpha$ such that for any nonempty open subset $\Omega$ of $G$ there is a collection of balls $\{B(x_j, r_j)\}_j$, $0 < r_j \leq \varepsilon_0$, for which*

*i)* $x_j^{-1} B(x_j, r_j) \subset B(e, \varepsilon_0)$,

*ii)* $B(x_j, \alpha r_j)$ *is not contained in* $\Omega$,

*iii)* $U_j B(x_j, \gamma r_j) = \Omega$,

*iv)* $B(x_j, \delta r_j) \cap B(x_i, \delta r_j) = \phi$, $i \neq j$,

*v)* $B(x_j, \beta r_j) \subset \Omega$,

*vi)* $B(x_i, r_i)$ *intersects at most* $A$ *of the* $B(x_j, r_j)$ *for all* $i$.

LEMMA 2.— *Given* $\Omega$ *and* $\{B(x_j, r_j)\}_j$ *as in Lemma 1, there is a collection of positive functions* $\{\varphi_j\}_j \subset \mathcal{S}(G)$ *with*

*1)* $\operatorname{supp} \varphi_j \subset B(x_j, r_j)$,

*2)* $\sum \varphi_j = \chi_\Omega$,

*3) For each* $j$ *there exists a* $y_j \in G \setminus \Omega$ *such that* $C\varphi_j / \|\varphi_j\|_{L^1(G)} \in K_N(y_j)$.

Next we consider the classical unitary reductive group

$$U_n = \left\{U \in \mathrm{GL}(n, \mathbb{C}) \mid {}^t\overline{U}U = e\right\}.$$

The identity map on $U_n$ is a faithful finite dimensional unitary representation of $U_n$ which allows us to embed $U_n$ in $E = \mathbb{R}^{2n^2}$. For each $f \in \mathcal{S}(G) \cap H^p(U_n)$ let $\Omega_k = \Omega_k(f) = \left\{x \in G \mid f^*(x) > 2^k\right\}$. For each fixed $k$. Let $x_{jk}$ , $r_{jk}$ and $\varphi_{jk}$ be the items of the previous two lemmas associated to $\Omega_k$. Let $P(x : j, k)$ be the unique element of $\mathcal{P}_s(E)$ such that

$$\int_G \big(f(x) - P(x : j, k)\big) P\big(\pi(x)\big) \varphi_{jk}(x) \,\mathrm{d}x = 0$$

for each $P$ in $\mathcal{P}_s(E)$ and let $P(x : i, j, k)$ be the unique element in $\mathcal{P}_s(E)$ such that

$$\int_G \big(f(x) - P(x : j, k)\big) P\big(\pi(x)\big) \varphi_{ik}(x) \varphi_{jk}(x) \,\mathrm{d}x =$$
$$= \int_G P(x : i, j, k)\, P\big(\pi(x)\big) \varphi_{jk}(x) \,\mathrm{d}x$$

for all $P$ in $\mathcal{P}_s(E)$. Of course, in the definition of these polynomials we have suppressed dependence on $f$, $s$, $G$ and $\pi$.

LEMMA 3.— *For* $G = U_n$ *if* $x \in B(x_{jk}, Cr_{jk})$, *then* $\left|P(x : i, k)\right| \leq C2^k$ *and* $\left|P(x : i, j, k)\right| \leq C2^k$.

*Proof.*— We will denote the general element of $U_n$ by $x = (u_{ij} + i \cdot v_{ij})$ where $u_{ij}$, $v_{ij} \in \mathbb{R}$. It is an elementary fact that $(u_{ij})$ or $(v_{ij})$ may be taken as a coordinate system for $U_n$; we will give explicit arguments in the cases where the latter provide the more convenient coordinates, writing $u_{ij}(v)$ and $x(v) = \big(u_{ij}(v) + i \cdot v\big)$, $v = (v_{ij})$, accordingly. A general $2n^2$-tuple of nonnegative integers will be denoted $\bar{\imath} = (\kappa_{ij}, \xi_{ij})$ with $\sum_{i,j}(\kappa_{ij}, \xi_{ij}) = |\bar{\imath}|$. We order these multi-indices so that $\bar{\jmath} < \bar{k}$ if $|\bar{\jmath}| < |\bar{k}|$; we need not specify the ordering of $\bar{\jmath}$ and $\bar{k}$ when they have equal length. Let $\{p_{\bar{\imath}} \mid 0 \leq |\bar{\imath}| \leq s\}$ be the basis of $\mathcal{P}_s(E)$ such that

$$p_{\bar{\imath}}(x) = \left(\prod_i u_{ii}^{\kappa_{ii}}\right)\left(\prod_{i\neq j}(1-u_{ij})^{\kappa_{ij}}\right)\left(\prod_{i,j} v_{ij}^{\xi_{ij}}\right) \quad \text{for } \sum \xi_{ij} > 0$$

and

$$p_{\bar{\imath}}(x) = \left(\prod_i (1-u_{ii})^{\kappa_{ii}}\right)\prod_{i\neq j}(u_{ij})^{\kappa_{ij}} \quad \text{for } \sum \xi_{ij} = 0\,.$$

For the first set of polynomials that is, those for which $\sum \xi_{ij} > 0$, we will use $v = (v_{ij}) \in \mathbb{R}^{n^2}$ as a coordinate system of $U_n$. For the second set of polynomials, $(u_{ij})$ is the more convenient coordinate system; the discussion for these, differing in no essential way from that of the first set of polynomials, is omitted. Let $\{Q_{\bar{\imath}}\}$ be the orthonormal system obtained by orthonormalizing $\{p_{\bar{\imath}}\}$ with respect to the measure $d\mu_{jk}(x) = \varphi_{ij}(x) \cdot \|\varphi_{jk}\|_{L^1}^{-1}\, \mathrm{d}x$ :

$$Q_{\bar{\imath}}(x) = \frac{P_{\bar{\imath}}(x) - \sum_{\bar{\jmath}<\bar{\imath}} \widehat{p_{\bar{\imath}}}(\bar{\jmath})Q_{\bar{\jmath}}(x)}{\left\|p_{\bar{\imath}} - \sum_{\bar{\jmath}<\bar{\imath}} \widehat{p_{\bar{\imath}}}(\bar{\jmath})Q_{\bar{\jmath}}\right\|_{L^2(d\mu_{jk})}} \tag{3.2}$$

where $\widehat{p_{\bar{\imath}}}(\bar{\jmath}) = \langle p_{\bar{\imath}}, Q_{\bar{\jmath}}\rangle_{L^2(d\mu_{jk})}$. We denote the numerator of (3.2) by $q_{\bar{\imath}}$.

We assert that $\{Q_{\bar{\imath}}\}$ is uniformly bounded on $B(x_{jk}, Cr_{jk})$ independently of $\varphi_{jk}$ and $r_{jk}$. Using a standard argument we need only show that $|Q_{\bar{\imath}}(u)| \leq C$ for $u \in B(e, Cr_{jk})$ and $\operatorname{supp}\varphi_{jk} \subset B(e, Cr_{jk})$. The proof of this is by induction on $\bar{\imath}$, the case $|\bar{\imath}| = 0$ being obvious. We will drop the subscript $j$ during this argument. For any multi-index $\bar{\imath}$ we first estimate the numerator of (3.2). If $\sum \xi_{ij} > 0$, one easily sees that $|p_{\bar{\imath}}(x)| \leq Cr^{\sum \xi_{ij}}$ for $x \in B(e, Cr)$ and that

$$\left|\widehat{p_{\bar{\imath}}}(\bar{k})\right| \leq \sup\left\{ |p_{\bar{\imath}}(x)|\, |Q_{\bar{k}}(x)| \,\middle|\, x \in B(e, Cr)\right\}.$$

It follows that $\left|\widehat{p_{\bar{\imath}}}(\bar{k})\right| \leq Cr^{\sum \xi_{ij}}$. Therefore,

$$\sup\left\{ |q_{\bar{\imath}}(x)| \,\middle|\, x \in B(e, Cr)\right\} \leq Cr^{\sum \xi_{ij}}$$

when $\bar{\imath} = (\kappa_{ij}, \xi_{ij})$ with $\sum \xi_{ij} > 0$. Also,

$$\|q_{\bar{\imath}}\|^2 \geq Cr^{-n^2} \int_{B(e,\delta r)} \left| p_{\bar{\imath}}(x) - \sum_{\bar{k}<\bar{\imath}} \widehat{p_{\bar{\imath}}}(\bar{k}) Q_{\bar{k}} \right|^2 dx$$

$$\geq Cr^{-n^2} \int_{B(e,\delta r)} \left| \prod_{i,j} v_{ij}^{\xi_{ij}} - \prod_{i=1}^{n} u_{ii}^{-\kappa_{ii}} \prod_{i\neq j} (1-u_{ij})^{-\kappa_{ij}} \sum_{\bar{k}<\bar{\imath}} \widehat{p_{\bar{\imath}}}(\bar{k}) Q_{\bar{k}} \right|^2 dx.$$

Since $c_1\, dv = c_1 \prod dv_{ij} \leq dx \leq c_2\, dv$, there exist $C$ and $\delta_0 > 0$ such that $\|q_{\bar{\imath}}\|^2 \geq Cr^{\sum \xi_{ij}} A_r$ where

$$A_r = \int_{B(0,1/2)} \left| \prod_{ij} v_{ij}^{\xi_{ij}} - \prod_{i} u_{ii}^{-\kappa_{ii}}(\delta_0 r v) \prod_{i\neq j} \left(1-u_{ij}(\delta_0 r v)\right)^{-\kappa_{ij}} \times \right.$$

$$\left. \times\ (\delta_0 r)^{-\sum \xi_{ij}} \sum_{\bar{k}<\bar{\imath}} \widehat{p_{\bar{\imath}}}(\bar{k}) Q_{\bar{k}}\left(x(\delta_0 r v)\right) \right|^2 dv\,.$$

Our assertion concerning the uniform boundeness of the $Q_{\bar{\imath}}$ will therefore follow from the existence of a constant $M$ such that

$$A_r \geq M\,, \quad 0 < r\,. \tag{3.3}$$

If (3.3) were invalid, we could find a sequence $\{r_n\}$ such that $A_{r_n} < 1/n$ and $\lim r_n = r_0 \in [0\,, \varepsilon_0]$. If $r_0 > 0$ we can find an $\varepsilon_1 > 0$ such that $\varepsilon_1 \leq r_n \leq \varepsilon_0$ for all $n$. But since $\{\varphi_n\}$ is uniformly bounded and equicontinuous, we may, by the Ascoli–Arzela theorem, assume without loss of generality that $\varphi_n(x) \to \varphi(x)$. This gives us a pair $(r_0, \varphi)$ with $\|\varphi\|_\infty \leq 1$, $\varphi \geq 0$, and $\operatorname{supp}\varphi \subset B(e, r_0)$. Now we have by [10]

$$0 = \lim_{k\to\infty} A_{r_k} = \underline{\lim}\, A_{r_k} \geq E(v_{ij}) > 0$$

where $E(v_{ij})$ is the best approximation in $L^2(B(0,1/2))$ of $\prod_{i,j} v_{ij}^{\xi_{ij}}$ by means of the family of functions

$$\left\{ \prod_{i,j} v_{ij}^{\ell_{ij}} \prod_{i=1}^{n} u_{ii}^{m_{ii}-\kappa_{ii}}(\delta_0 r_0 v) \prod_{i\neq j} \left(1-u_{ij}(\delta_0 r_0 v)\right)^{m_{ij}-\kappa_{ij}} \right\}$$

$$0 \le \sum (\ell_{ij} + m_{ij}) \le |\bar{\imath}| - 1 \quad \text{and} \quad \sum (\kappa_{ij} + \xi_{ij}) = |\bar{\imath}| .$$

This contradiction forces $r_0 = 0$ and we have

$$0 = \underline{\lim}\, A_{r_k} \ge$$

$$\ge \int_{B(0,1/2)} \underline{\lim} \left| \prod_{i,j} v_{ij}^{\xi_{ij}} - \prod_i u_{ii}^{-\kappa_{ii}}(\delta_0 r_k v) \prod_{i \ne j} \left(1 - u_{ij}(\delta_0 r_k v)\right)^{-\kappa_{ij}} \right.$$

$$\left. \times \ (\delta_0 r_k)^{\sum \xi_{ij}} \sum_{\bar{\jmath} < \bar{\imath}} \widehat{p_{\bar{\imath}}}(\bar{\jmath}) Q_{\bar{\jmath}}\left(x(\delta_0 r_k v)\right) \right|^2 \, dv .$$

But $(u_{i,j})$ tends to the identity matrix as $(v_{ij}) \to 0$ and therefore

$$\underline{\lim} \sum_{\bar{\jmath} < \bar{\imath}} \frac{\widehat{p_{\bar{\imath}}}(\bar{\jmath})}{(\delta_0 r_k)^{\sum \xi_{ij}}} Q_{\bar{\jmath}}\left(x(\delta_0 r_k v)\right) = \prod v_{ij}^{\xi_{ij}}$$

for almost all $(v_{ij}) \in B(0,1/2)$. Furthermore $\widehat{p_{\bar{\imath}}}(\bar{\jmath}) = \mathrm{O}\big(r_k^{\sum \xi_{ij}}\big)$ and by our induction hypothesis we have

$$\sum_{\bar{\jmath} < \bar{\imath}} Q_{\bar{\jmath}}\left(x(\delta_0 r_k v)\right) \widehat{p_{\bar{\imath}}}(\bar{\jmath}) (\delta_0 r_k)^{-\sum \xi_{ij}} =$$

$$= \sum_{(\lambda,\eta) < \bar{\imath}} C_{\lambda\eta} \prod_i u_{ii}^{\lambda_{ii}}(\delta_0 r_k v) \prod_{i \ne j} \left(1 - u_{ij}(\delta_0 r_k v)\right)^{\lambda_{ij}} \prod_{i,j} (r \delta_0 v_{ij})^{\eta_{ij}}$$

where the constants $C_{\lambda\eta}$ depend on $r_k$ but in such a way that $C_{\lambda\eta} = \mathrm{O}(1)$. Therefore, we can choose $r_{k_m}$ such that

$$\prod_{i,j} v_{ij}^{\xi_{ij}} = \underline{\lim} \sum_{\bar{\jmath} < \bar{\imath}} Q_{\bar{\jmath}}\left(x(\delta_0 r_{k_m} v)\right) \frac{\widehat{p_{\bar{\imath}}}(\bar{\jmath})}{(\delta_0 r_{k_m})} = C .$$

This contradiction establishes (3.3).

Now we may complete the proof of Lemma 3. Since the $Q_{\bar{\imath}}$'s are uniformly bounded and since $C \cdot \varphi / \|\varphi\|_{L^1(G)} \in K_N(y)$ for some $y \notin \Omega_k$ we also have $C \cdot Q_{\bar{\imath}}\left(\varphi / \|\varphi\|_{L^1(G)}\right) \in K_N(y)$. Now

$$P(x : i, k) = \sum_{\bar{\imath}} \int f(z) \overline{Q_{\bar{\imath}}(z)} \frac{\varphi_{ik}(z)}{\|\varphi_{ik}\|_{L^1(G)}} \, dz \, Q_{\bar{\imath}}(x)$$

and therefore

$$\big|P(x:i,k)\big| \le \sum_{\bar{\imath}} Cf^*(y)\big|Q_{\bar{\imath}}(x)\big| \le C2^k \quad \text{if } x \in B(v_{ik}, Cr_{ik})\,.$$

Since

$$P(x:i,j,k) = \sum_{\bar{\imath}} \int \big(f(y) - P(y:j,k+1)\big)\varphi_{ik}(z)\overline{Q_{\bar{\imath}}(z)}\frac{\varphi_{jk}(z)}{\|\varphi_{jk}\|_{L^1}}\,\mathrm{d}z\,Q_{\bar{\imath}}(x)$$

and $\{Q_{\bar{\imath}}\}_{\bar{\imath}}$ is orthonormal with respect to the measure $d\mu_{jk}$, the remaining estimate follows analogously. □

THEOREM 3.4.— *If $f \in H^p(G)$ then $f$ admits a decomposition $f = \sum_j c_j a_j$ where the $a_j$'s are $p$-atoms and $\big(\sum_j |c_j|^p\big)^{1/p} \le C\|f\|_{H^p(G)}$ for some constant $C$ depending only on $p$ and $G$.*

*Proof.*— Fix $f \in H^p(G) \cap \mathcal{S}(G)$. In the notation already established, we may use the embedding $\pi : G \to U(L, \mathbb{C})$ to transfer the estimates of Lemma 3 to $G$:

$$\big|P(x:i,k)\big| \le C2^k \quad \text{and} \quad \big|P(x:i,j,k)\big| \le C2^k$$

for all $x \in B(x_{jk}, r_{jk})$. The atomic decomposition is now obtained in a standard way. We write

$$f = f\cdot\chi_{G\setminus\Omega_k} + \sum_i P(\,\cdot\,:i,k)\varphi_{ik} + \sum_i \big(f - P(\,\cdot\,:i,k)\big)\varphi_{ik} = g_k + \sum_i b_{ik}\,.$$

We have $\|g_k\|_{L^\infty(G)} \le C2^k$ and $b_{ik} = 0$ for every $(i,k)$ for which $\|f\|_{L^\infty(G)} \le 2^k$. Choose an $h$ such that

$$2^h \le \|f^*\|_{L^p(G)}\,, \quad f = g_h + \sum_{k=h}^{\infty}(g_{k+1} - g_k)$$

and $g_h = C2^h a_h$ with $a_h$ an exceptional atom. We observe that

$$g_{k+1} - g_k = \sum_i \big(f - P(\,\cdot\,:i,k)\big)\varphi_{ik} - \sum_j \big(f - P(\,\cdot\,:j,k+1)\big)\varphi_{j,k+1}\,.$$

Also, $\Omega_k \supset \Omega_{k+1}$ and $\Omega_k = \phi$ as soon as $2^k \geq \|f^*\|_{L^\infty(G)} \geq \|f\|_{L^\infty(G)}$. Moreover, by the geometry of the covering, if $B(x_{ik}, r_{ik}) \cap B(x_{j,k+1}, r_{j,k+1} \neq \emptyset$, then $r_{j,k+1} \leq C r_{ik}$ (for $C = (1+\alpha)/(\beta - 1)$). Therefore, for fixed $j$, only a finite number of balls of the collection $\{B(x_{ik}, r_{ik})\}_i$ intersect $B(x_{j,k+1}, r_{j,k+1})$ and only a finite number of the $P(\cdot\ : i, j, k+1)$ are different from 0. Since $\sum \varphi_{ik}\varphi_{j,k+1} = \varphi_{j,k+1}$ we have

$$\int_G \sum_i P_{ij}(x : i, j, k+1) P(\pi(x)) \varphi_{jk}(x)\, dx =$$

$$= \int_G \left(f - P_j^{k+1}(x) P(\pi(x))\right) \sum_i \varphi_{ik}(x) \varphi_{j+1,k+1}(x)\, dx = 0$$

for any $P \in \mathcal{P}_s(E)$. This implies that $\sum_i P(\cdot\ : i, j, k+1) = 0$. Hence

$$\begin{aligned} g_{k+1} - g_k &= \sum_i \left((f - P(\cdot\ : i, k))\varphi_{ik}\right) + \\ &\quad - \sum_j \left((f - P(\cdot\ : j, k+1))\varphi_{ik} - P(\cdot\ : i, j; k)\right)\varphi_{j,k+1}\big) \\ &= C 2^k |B(x_{ik}, r_{ik})|^{1/p} a_{ik} \end{aligned}$$

where each $a_{ik}$ is a $(p, \infty)$-atom with support contained in $B(x_{ik}, r_{ik})$. Consequently,

$$f = C 2^h a_h + \sum_{k=h}^{\infty} \sum_i C 2^k |B(x_{ik}, r_{ik})|^{1/p} a_{ik} \tag{3.4}$$

and since $C$ is independent of $f$ and since this right side of (3.4), being locally a finite sum, converges to $f$ pointwise and in each Lebesgue space $L^q(G)$ ($q < \infty$), this series furnishes an atomic decomposition of $f$. Movever,

$$\begin{aligned} |C2^h|^p + C \sum_{k=h}^{\infty} \sum_i \left(|B(x_i^k, r_i^k)|^{1/p} 2^k\right)^p &\leq C\|f^*\|_p^p + C \sum_{k=h}^{\infty} 2^{kp} |\Omega_k| \\ &\leq C\|f^*\|_p^p . \end{aligned}$$

This proves the theorem for $f \in \mathcal{S}(G)$. The transition to an arbitrary distribution $f$ in $H^P(G)$ is now, in view of Theorem 2.11, a routine denseness argument. Since there is a sequence $\{f_n\} \subset \mathcal{S}(G)$ such that $f_n = \sum_j c_{jn} a_{jn}$ (an atomic decomposition),

$$\|f_n\|_{H^p(G)} \leq 2^{-n} \|f\|_{H^p(G)}, \quad f = \sum f_n,$$

$$\left( \sum_j |c_{jn}|^p \right)^{1/p} \leq C \|f_n\|_{H^p(G)},$$

we have $f = \sum_{j,n} c_{jn} a_{jn}$ and

$$\left( \sum_{j,n} |c_{jn}|^p \right)^{1/p} \leq C \left( \sum_n \|f_n\|^p_{H^p(G)} \right)^{1/p} \leq C \|f\|_{H^p(G)}$$

which concludes the proof of Theorem 3.4. □

## 4. Heat kernels and Hardy space

As in Euclidean space, it is useful to have a heat kernel characterization of Hardy space. Define the kernel

$$\widetilde{W}_t(x) = \sum_{\lambda \in \Lambda} e^{-t\|\lambda+\rho\|^2} d_\lambda \chi_\lambda(x) = e^{-t\|\rho\|} W_t(x), \tag{4.1}$$

the second equality exhibiting the positivity of $\widetilde{W}_t(x)$. The associated maximal operator is $\widetilde{W}^+ f = \sup_{t>0} |\widetilde{W}_t * f|$. We define a nontangential maximal operator by

$$\widetilde{W}^* f(x) = \sup_{d(x,y)^2 < t} |\widetilde{W}_t * f(y)|.$$

It is convenient to have some auxiliary operators:

$$U(x,t) = t^{1/2} |\nabla_x (f * \widetilde{W}_t)(x)| \tag{4.2}$$

$$U^*(x) = \sup_{d(x,y)^2 < t} U(y,t). \tag{4.3}$$

Then we have the following result:

PROPOSITION 4.1.— *For any $p > 0$ there exists a constant $C$ such that*

$$\|\widetilde{W}^* f\|_{L^p(G)} \le C \|\widetilde{W}^+ f\|_{L^p(G)} \, .$$

*Proof.*— By an argument that originates in [11], the proposition follows from the following inequality

$$\|U^*(x)\|_{L^p(G)} \le C \|\widetilde{W}^* f\|_{L^p(G)} \tag{4.4}$$

which in turn, by Propositions 2.1 and 2.2, follows from

$$U^*(x) \le C \widetilde{W}_N^{**} f(x) \, , \quad N > \frac{n}{p} \tag{4.5}$$

where

$$\widetilde{W}_N^{**} f(x) = \sup_{(y,t) \in G^+} \left|\widetilde{W}_t * f(y)\right| \left(\frac{t^{1/2}}{d(x,y) + t^{1/2}}\right)^N \, .$$

By the semigroup property, $f * \widetilde{W}_t = (f * \widetilde{W}_{t/2}) * \widetilde{W}_{t/2}$.

Hence, for any $(y,t) \in G^+$ such that $d(x,y)^2 < t$, we have

$$\begin{aligned} &\left|X_i (f * \widetilde{W}_t)(y)\right| \le \\ &\quad \le C \widetilde{W}_N^{**} f(x) \int_G t^{-N/2} (t^{1/2} + d(y,z))^N \left|X_i \widetilde{W}_{t/2}(z^{-1} y)\right| \mathrm{d}z \, . \end{aligned}$$

Therefore, (4.5) will follow by showing there is a $C$ independent of $t$ in (4.6):

$$I_t = \int_G t^{-N/2+1/2} (t^{1/2} + d(y,e))^N \left|X_i \widetilde{W}_{t/2}(y)\right| \mathrm{d}y \le C \, . \tag{4.6}$$

From Section 1 we recall that

$$\widetilde{W}_{t/2}(y) = C_\delta \left(\int_0^{Mt} + \int_{Mt}^\infty\right) \left(s^\delta \, e^{-s} S_{t/2s}^\delta(y)\right) \mathrm{d}s$$

for any $\delta > 0$ and $M > 0$. After substituting into (4.5) we arrive at $I_t = I_1 + I_2$ with $I_1$ an integral over $G \times (0, Mt)$ and $I_2$ an integral over

$M \times [\,Mt\,, \infty)$. We vill estimate $I_1$ and $I_2$ separately. First, we break up $I_2$ further by

$$I_2 = C\int_{Mt}^{\infty} s^{\delta}\, e^{-s} t^{-(N-1)/2} \int_{d(y,e)^2 \geq t/2s} + \\ + \int_{d(y,e)^2 < t/2s} +\big(t^{1/2} + d(y,e)\big)^N \big|X_i S^{\delta}_{t/2s}(y)\big|\, \mathrm{d}y\, \mathrm{d}s\,,$$

denoting the first summand by $J_1$ and the second by $J_2$. By Theorem 5.4 of [5], for $\delta > 2N$, we obtain by a simple calculation

$$J_2 \leq C\int_{Mt}^{\infty} s^{\delta+1/2}\, e^{-s}\big(1 + (2s)^{-1/2}\big)^N\, \mathrm{d}s \leq C_M\,.$$

Again using the same theorem in [5] we get

$$J_1 \leq C\int_{Mt}^{\infty} s^{\delta}\, e^{-s} t^{-(N-1)/2} \int_{d(y,e)^2 > t/2s} \big(t^{1/2} + d(y,e)\big)^N \times \\ \times \left(\frac{2s}{t}\right)^{n/4-1/4-\delta/2} \Big(d\big(y,e\big)^{-n/2-1/2-\delta} + \Delta^{(R)}\big(y\big)^{-1}\Big)\, \mathrm{d}y\, \mathrm{d}s$$

where $\Delta^{(R)} = \big(2s/t\big)^{-\mu^R} D^R$, $\mu^R$ and $D^R$ are as in [5].

Since $\big(t^{1/2} + d(y,e)\big)^N \leq d\big(y,e\big)^N \big(1 + (2s)^{1/2}\big)^N$ we obtain

$$J_1 \leq C\int_{Mt}^{\infty} s^{\delta}\, e^{-s} t^{-(N-1)/2}\big(1 + (2s)^{1/2}\big)^N \left(\frac{2s}{t}\right)^{n/4-1/4-\delta/2} \times \\ \times \int_{d(y,e) > (t/2s)^{1/2}} \Big(d\big(y,e\big)^{N-n/2-1/2-\delta} + d\big(y,e\big)^N \Delta^{(R)}\big(y\big)^{-1}\Big)\, \mathrm{d}y\, \mathrm{d}s\,.$$

Let $\delta = (n-1)/2 + \delta_0$ for some $\delta_0 > N$. Then

$$\int_{Mt}^{\infty} s^{-\delta}\, e^{-s} t^{-(N-1)/2}\big(1 + (2s)^{1/2}\big)^N \left(\frac{2s}{t}\right)^{-\delta_0/2} \times \\ \times \int_{d(y,e) > (t/2s)^{1/2}} d\big(y,e\big)^{N-n-1-\delta_0}\, \mathrm{d}y\, \mathrm{d}s \leq \\ \leq C\int_{Mt}^{\infty} s^{\delta}\, e^{-s}\big(1 + (2s)^{1/2}\big)^N \left(\frac{t}{2s}\right)^{\delta_0/2-(N-1)/2} \big(2s\big)^{-(N-1)/2}\, \mathrm{d}s + \\ + C\int_{Mt}^{\infty} s^{\delta}\, e^{-s}\big(1 + (2s)^{1/2}\big)^N \big(2s\big)^{-(N-1)/2}\, \mathrm{d}s \leq C_M\,.$$

Also, from the definition of $\Delta^{(R)}$ one easily sees that

$$\int_{Mt}^{\infty} s^{\delta}\, e^{-s} t^{-(N-1)/2} \big(1 + (2s)^{1/2}\big)^N \Big(\frac{2s}{t}\Big)^{n/4-1/4-\delta/2} \times$$
$$\times \int_{d(y,e)^2 > t/2s} d(y,e)^N \Delta^{(R)}(y)^{-1}\, \mathrm{d}y\, \mathrm{d}s \le C_M\,.$$

This shows that $I_2 \le C_M$. For $I_1$ we use the estimate [5, Theorem 5.2],

$$\big\| X_i S^{\delta}_{t/2s}(y) \big\|_{L^\infty(G)} \le C \Big(\frac{s}{t}\Big)^{(n+1)/2}\,.$$

When $t \ge 2$, $\big(s/t\big)^{(n+1)/2} \le C_M (s/t)^{1/2}$ and $\big(t^{1/2} + d(y,e)\big)^N \le C_M t^{N/2}$, whence

$$I_1 \le C_M \int_0^{\infty} s^{\delta - 1/2}\, e^{-s}\, \mathrm{d}s \le C_M\,.$$

When $t < 2$,

$$I_1 \le C \int_0^{Mt} s^{\delta + (n+1)/2} t^{-(N-1)/2-(n+2)/2}\, e^{-s}\, \mathrm{d}s \le C_M\,.$$

This shows that $|I_t| \le C$ for any $t > 0$ completing the proof of (4.6). □

By consequence we see that for $f \in L^p(G)$, $1 < p < \infty$,

$$\big\| \widetilde{W}^* f \big\|_{L^p(G)} \le C_p \big\| f \big\|_{L^p(G)} \tag{4.7}$$

and

$$\lim_{\substack{t \to 0 \\ d^2(x,y) < t}} \widetilde{W}_t * f(y) = f(x) \quad \text{for almost every } x \text{ in } G\,. \tag{4.8}$$

For $N$ a positive integer and $x \in G$ we define a (new) class $K_N(x)$ in a way similar to that of Section 2:

$$K_N(x) = \big\{ \varphi \in \mathcal{S}(G) \mid \varphi \text{ satisfies (i), (ii), and (iii)} \big\}$$

(i) $\operatorname{supp} \varphi \subset B(x,h)$,

(ii) $\displaystyle \sup_{(x,t) \in G+} \left| \frac{\mathrm{d}^N}{\mathrm{d}t^N} (W_t * \varphi)(x) \right| \le h^{-2N-n}$,

(iii) $\big\| \varphi \big\|_{\infty} \le h^{-n}$.

We draw attention to the $2N$ in (ii). For a distribution $f$ in $\mathcal{S}'(G)$ and $\varphi \in K_N(x)$, we use the notation $\langle f, \varphi \rangle = \int_G f(x)\,\overline{\varphi}(x)$. The corresponding grand maximal function $f^*$ of $f$ in $\mathcal{S}'(G)$ is then

$$f^*(x) = \left\{ \sup_{\varphi} \left| \langle f, \varphi \rangle \right| \;\middle|\; \varphi \in K_N(x) \right\} .$$

By methods too similar to those already given in Section 2 to bear repeating, we have $W^+ f(x) \le C f^*(x)$ for all $x$ in $G$ and therefore, by Proposition 4.1,

$$\|\widetilde{W}^* f\|_{L^p(G)} \le C \|\widetilde{W}^+ f\|_{L^p(G)} \le C \|W^+ f\|_{L^p(G)} \le C \|f^*\|_{L^p(G)} . \quad (4.9)$$

From here there is no difficulty in repeating the arguments of Section 2 and 3 to obtain the analogues of Theorem 2.10 and Theorem 3.5.

THEOREM 4.2.— *The following are equivalent:*

*i)* $f^* \in L^p(G)$,
*ii)* $\|W_1^* f\|_p \le \infty$,
*iii)* $\widetilde{W}_t f \in L^p(G)$,
*iv)* $\widetilde{W}^* f \in L^p(G)$,
*v)* $\widetilde{W}^+ f \in L^p(G)$,
*vi)* $W^+ f \in L^p(G)$.

THEOREM 4.3.— $W^+ f \in L^p(G)$ *if and only if* $f \in H^p_{\mathrm{atom}}(G)$.

We now show that any central $\varphi \in \mathcal{S}(G)$ satisfying $\int_G \varphi \,\mathrm{d}x = 1$ also characterizes the $H^p(G)$ spaces. In addition, we will prove that some non-smooth kernels (including Bochner–Riesz kernels) characterize $H^p(G)$ as well. For $\varphi$ satisfying

$$\varphi \in \mathcal{S}(\mathfrak{t}) \text{ is a radial function satisfying } \int_{\mathfrak{t}} \varphi(H)\,\mathrm{d}H = 1 \quad (4.10)$$

we define

$$\varphi_t(x) = \sum_{\lambda \in \Lambda} \widehat{\varphi}\left(t\|\lambda + \rho\|\right) d_\lambda \chi_\lambda(x) , \quad t > 0 . \quad (4.11)$$

THEOREM 4.4.— *If $0 < p < \infty$, then for all $f \in \mathcal{S}'(G)$ the following are equivalent:*

*i)* $\sup\limits_{t>0}\left|\varphi_t * f\right| \in L^p(G)$ *for some $\varphi$ satisfying (4.10),*

*ii)* $\sup\limits_{d(x,y)<t}\left|\varphi_t * f(y)\right| \in L^p(G)$ *for some $\varphi$ satisfying (4.10),*

*iii)* $f^*(x) = \sup\limits_{\phi\in A_0}\ \sup\limits_{d(x,y)<t}\left|\phi_t * f(y)\right| \in L^p(G)$ *where*

$$A_0 = \left\{ \phi \in \mathcal{S}(\mathbb{R}^\ell) \;\middle|\; \int_t (1+|\theta|)^{N^0} \sum_{|\alpha|\le N_0} \left|\frac{\partial^\alpha}{\partial\theta^\alpha}\phi(\theta)\right|^2 \mathrm{d}\theta \le 1 \right\}$$

*for some $N_0 = N_0(p,n)$.*

*Proof.*— For the most part, the proof of the original result for Euclidean spaces [11, Theorem 11] carries over and we need only show that i)→ii). As in [11], it suffices to prove

$$U^*(x) \le C\varphi_M^{**}f(x) \quad \text{for } M > \frac{n}{p} \tag{4.12}$$

where $U^*(x) = \sup_{d(x,y)<t} t\left|\nabla\varphi_t * f(y)\right|$ and

$$\varphi_M^{**}f(x) = \sup_{(y,t)\in G^+} \left|\varphi_t * f(y)\right| \left(\frac{t}{t+d(x,y)}\right)^M .$$

Choose $\zeta \in C^\infty([0,1])$ such that $\zeta(s) = s^N/N!$ for $0 \le s \le 1/2$, $0 \le \zeta(s) \le s^N/N!$ for $1/2 \le s \le 1$ and $\partial_s^j\zeta(1) = 0$ for $0 \le j \le N+1$ where $\partial_s = \partial/\partial s$. Let $(\varphi_s)^{*(N)} = (\varphi_1^{*(N)})_s$ be the $N$-fold convolution product of $\varphi_s$. Let

$$I = (-1)^{N+1}\int_0^1 \zeta(s)\big(\partial_s^{N+1}\varphi_s^{*(N+2)}\big) * \varphi_1 \, \mathrm{d}s .$$

Then by integration by parts we have

$$I = \varphi_1 + \int_0^1 \partial_s^{N+1}\zeta(s)\varphi_s^{*(N+2)} * \varphi_1 \, \mathrm{d}s$$

or

$$\varphi_1 = (-1)^{N+1} \int_0^1 \zeta(s)\big(\partial_s^{N+1} \varphi_s^{*(N+2)}\big) * \varphi_1 \, \mathrm{d}s + \\ - \int_0^1 \partial_s^{N+1} \zeta(s) \varphi_s^{*(N+2)} * \varphi_1 \, \mathrm{d}s\,.$$

But

$$\partial_s^{N+1} \varphi_s^{*(N+2)}(x) = \sum_{\lambda \in \Lambda} \psi(s\|\lambda+\rho\|)\ \widehat{\varphi}(s\|\lambda+\rho\|)\ \|\lambda+\rho\|^{N+1} d_\lambda \chi_\lambda(x)$$

for another radial function $\psi$ in $\mathcal{S}(\mathrm{t})$. Therefore

$$\varphi_t = (-1)^N \int_0^1 \zeta(s) \sum_{\lambda\in\Lambda} \psi(st\|\lambda+\rho\|)\ (t\|\lambda+\rho\|)^{N+1}\, d_\lambda \chi_\lambda * \varphi_{st} * \varphi_t \, \mathrm{d}s + \\ - \int_0^1 \varphi_{st} * \partial_s^{N+1} \zeta(s)\, \varphi_{st}^{*(N+1)} * \varphi_t \, \mathrm{d}s\,.$$

Hence, for any $X \in \mathfrak{g}$

$$\begin{aligned} X\big(f * \varphi_t(y)\big) &= (-1)^{N+1} \int_0^1 f * \varphi_{st} * \zeta(s) \sum_{\lambda\in\Lambda} \psi(st\|\lambda+\rho\|)\, d_\lambda \chi_\lambda * \\ &\quad * t^{N+1} X \sum_{\lambda\in\Lambda} \|\lambda+\rho\|^{N+1} \widehat{\varphi}\,(t\|\lambda+\rho\|)\, d_\lambda * \chi_\lambda(y) \, \mathrm{d}s + \\ &\quad - \int_0^1 f * \varphi_{st} * \partial_s^{N+1} \zeta(s)\, \varphi_{st}^{*(N+1)} * X\varphi_t(y) \, \mathrm{d}s \\ &= J_1(y) + J_2(y)\,. \end{aligned}$$

Now for any fixed $(y,t) \in G^+$ such that $d(x,y) < t$,

$$\begin{aligned} &\big|J_2(y)\big| \le \\ &\le C\varphi_M^{**} f(x) \int_{1/2}^1 \int_G \left(\frac{st}{d(x,z)+st}\right)^{-M} \Big|\big(\varphi_{st}^{*(N+1)} * X\varphi_t\big)(z^{-1}y)\Big| \, \mathrm{d}z \, \mathrm{d}s\,. \end{aligned}$$

By the argument given in the proof of Proposition 4.1, if $d(x,z) < t$,

$$t\big|J_2(y)\big| \le C\varphi_M^{**} f(x) \int_{1/2}^1 \int_G |\varphi_{st}^{*(N+1)}(y)| \, \mathrm{d}y \le C\varphi_M^{**} f(x)\,.$$

If $d(x,z) \geq t$, then

$$\left(\frac{st + d(x,z)}{st}\right)^M \leq C\left(1 + \frac{d(z^{-1}y,e)}{t}\right)^M$$

whence

$$t\int_{1/2}^{1}\int_{d(x,z)\geq t}\left(\frac{st}{d(x,z)+st}\right)^{-M}\left|\varphi_{st}^{*(N+1)} * X\varphi_t(z^{-1}y)\right| \mathrm{d}z\,\mathrm{d}s \leq$$

$$\leq C\sum_{k_1+k_2=0}^{N}\int_{1/2}^{1}\int_G\left|\varphi_{st}^{*(N+1)}(z)\right|\left(\frac{d(z,I)}{t}\right)^{k_1}\times$$

$$\times\int_G t\left(\frac{d(z^{-1}w,e)}{t}\right)^{k_2}\left|X\varphi_t(z^{-1}w)\right|\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}s \leq C\,.$$

Hence, $\left|tJ_2(y)\right| \leq C\varphi_M^{**}f(x)$ whenever $d(x,y) < t$.

We turn to a similar estimate for $J_1(y)$. It's easy to see that

$$\left|tJ_1(y)\right| \leq$$

$$\leq C\varphi_M^{**}f(x)\int_0^1\int_G\left(\frac{st+d(x,z)}{t}\right)^M\Bigg|\sum_{\lambda\in\Lambda}\psi(st\|\lambda+\rho\|)\,d_\lambda\chi_\lambda *$$

$$* \, t^{N+2}X\sum_{\lambda\in\Lambda}\|\lambda+\rho\|^{N+1}\widehat{\varphi}(t\|\lambda+\rho\|)\,d_\lambda\chi_\lambda(z^{-1}y)\Bigg|\,\mathrm{d}z\,\mathrm{d}s\,.$$

For $z^{-1}y$ conjugate to $\exp\theta \in T^\ell$ there exists a differential operator $P(\partial)$ of order $N+1$ such that

$$\sum_{\lambda\in\Lambda}\|\lambda+\rho\|^{N+1}\widehat{\varphi}(t\|\lambda+\rho\|)\,d_\lambda\chi_\lambda(\theta) = P(\partial)\sum_{\lambda\in\Lambda}\widehat{\varphi}(t\|\lambda+\rho\|)\,d_\lambda\chi_\lambda(\theta)\,.$$

Now using Theorems 3.3 and 5.4 of [5], our estimate for $J_2$ and Proposition 4.1, we get $t\left|J_1(y)\right| < C$, $d(x,y) < t$, from which the required result follows. □

We note that from Theorem 4.4, it is easy to see that for $K_t = \sum_{\lambda\in\Lambda} e^{-t^2\|\lambda+\rho\|^2} d_\lambda\chi_\lambda$ and $\varphi$ as in Theorem 4.4, we have $\left\|K^+f\right\|_{L^p(G)} \cong \left\|\varphi^+f\right\|_{L^p(G)}$ and, since $\left\|K^+f\right\|_{L^p(G)} \cong \left\|W^+f\right\|_{L^p(G)}$, we get

$$\left\|\varphi^+f\right\|_{L^p(G)} \cong \left\|K^+f\right\|_{L^p(G)}\,,\quad 0 < p \leq 1\,,\ f \in \mathcal{S}'(G)\,.$$

Heretofore our kernels have been infinitely differentiable. The following theorem shows that such smoothness is not a necessary condition for the characterization of $H^p(G)$.

THEOREM 4.5.— *For the maximal Bochner–Riesz operator* $S_*^\delta f(x) = \sup_{t>0}\left|S_t^\delta * f(x)\right|$,

$$\left\|S_*^\delta f\right\|_{L^p(G)} \simeq \left\|f\right\|_{L^p(G)} \quad \text{if } \delta > \frac{n}{p} - \frac{n+1}{2}.$$

*Proof.*— Suppose that $\delta > n/p - (n+1)/2$. If $f \in H^p(G)$, then $f$ has atomic decomposition $f = \sum c_j a_j$ and

$$\left\|S_*^\delta f\right\|_{L^p(G)} \leq C\left(\sum |c_j|^p\right)^{1/p}\left(\int_G |S_*^\delta a|^p \,\mathrm{d}x\right)^{1/p}.$$

Conversely, if $\left\|S_*^\delta f\right\|_{L^p(G)} < \infty$ then

$$\sup_{t>0}\left|\widetilde{W}_t * f(x)\right| = \sup_{t>0}\left|C\int_0^\infty s^\delta\, e^{-s} S_{t/s}^\delta * f(x)\,\mathrm{d}x\right| \leq C S_*^\delta f(x)\,. \ \square$$

We will now further relax the regularity assumption, admitting even non-smooth kernels, by generalizing a Euclidean result of Y. Han concerning the characterization of Hardy space by kernels satisfying a Dini condition. For a function $\varphi \in C(\mathfrak{t})$ satisfying:

$$\begin{aligned}
&\text{i)} \quad \operatorname{supp}\varphi \subset \{\theta \in \mathfrak{t} \mid \|\theta\| \leq 1\}\,,\\
&\text{ii)} \quad \varphi \text{ has } \frac{n-\ell}{2} \text{ derivatives}\,,\\
&\text{iii)} \quad \varphi \text{ is radial: } \varphi(\theta) = \varphi_0\big(\|\theta\|\big) \text{ for a function } \varphi_0 \text{ of one variable},\\
&\text{iv)} \quad \int_{\mathfrak{t}} \varphi(\theta)\,\mathrm{d}\theta \neq 0\,,
\end{aligned} \tag{4.13}$$

set $\phi(s) = (\mathrm{d}/\mathrm{d}s)^{(n-\ell)/2}\varphi_0(s)$ and define the central kernel $\varphi_t$ on $G$ by

$$\varphi_t(\exp\theta) = t^{-n}\left(\prod_{\alpha\in A}\frac{\langle\alpha,\theta\rangle}{\sin\langle\alpha,\theta\rangle/2}\right)\phi\left(\frac{\|\theta\|}{t}\right), \quad 0 < t \leq 1\,,\ \theta \in \mathfrak{t}. \tag{4.14}$$

We do not assume that the kernel is smooth; instead of a smoothness assumption on $\phi$ we assume that $\phi$ satisfies a Dini condition:

$$\int_0^{\varepsilon} \omega_\phi(\delta)\frac{\mathrm{d}\delta}{\delta} < \infty \quad \text{for some } \varepsilon > 0 \tag{4.15}$$

where $\omega_\phi(\delta) = \sup_{\|\theta-\zeta\|\leq\delta}\left|\phi(\theta) - \phi(\zeta)\right|$. We define a maximal operator $\varphi^*$ on functions on $G$ by

$$\varphi^* f(x) = \sup_{d(x,y)<t\leq 1} \left|\varphi_t * f(y)\right| \tag{4.16}$$

and an $H^1(G)$-type space by

$$H_\varphi(G) = \left\{f \in L^1(G) \mid \varphi^* f \in L^1(G)\right\} ; \tag{4.17}$$

we norm $H_\varphi(G)$ by $\|f\|_{H_\varphi(G)} = \|\varphi^* f\|_{L^1(G)}$.

THEOREM 4.6.— *Suppose that $\varphi \in C(\mathfrak{t})$ satisfies (4.13) and that $\phi = (\mathrm{d}/\mathrm{d}s)^{(n-\ell)/2}\varphi$ is bounded and satisfies (4.15). Then $H_\varphi(G) = H^1(G)$.*

*Proof.*— We may assume that $\int_{\mathfrak{t}} \varphi(\theta)\,\mathrm{d}\theta = 1$. For any exceptional atom $a(x)$ and any $(y,t) \in G^+$, $\left|\varphi_t * a(y)\right| \leq \|\varphi_t\|_{L^1(G)} \leq C$, where $C$ does not depend on $(y,t)$. Let $a(x)$ be a $(1,\infty)$-atom with support in some ball $B$. Without loss of generality, we can assume $B = B(e,\rho)$ for $\rho$ sufficiently small. Now for any $0 < t \leq 1$ and $y \in G$ such that $d(x,y) < t$

$$\begin{aligned}\varphi^* a(x) &= \sup_{d(x,y)<t} \left|\varphi_t * a(y)\right| \\ &\leq \|a\|_\infty \|\varphi_t\|_{L^1(G)} \leq C\left|B(e,\rho)\right|^{-1}.\end{aligned}$$

Therefore, for any $\varepsilon > 0$ as in (4.15), there exists a constant $C = C(\varepsilon)$ independent of $a(x)$ such that

$$\int_{[\|\theta'\|\leq 2\rho/\varepsilon]} \varphi^* a(x)\,\mathrm{d}x \leq C$$

where $[E]$ denotes the subset of $G$ conjugate to $\exp E$ for $E \subset \mathfrak{t}$. For fixed $y \in G$, and for $\xi \in G$, let $\theta = \theta(\xi) \in \mathfrak{t}$ exponentiate to a conjugate of $y\xi^{-1}$. Then

$$\varphi_t * a(y) = C \int_{B(e,\rho)} a(\xi) t^{-n} \left(\prod_{\alpha\in A} \frac{\langle\alpha,\theta\rangle}{\sin\langle\alpha,\theta\rangle/2}\right) \phi\left(\frac{\|\theta\|}{t}\right) \mathrm{d}\xi\,.$$

Notice that $\|\theta\| \leq d(y\xi^{-1}, e) = d(y, \xi)$ and $\phi(\|\theta\|/t)$ is supported in $\theta \leq t$. Thus, for any $\xi \in B(e,\rho)$, we have $\varphi_t * a(y) = 0$ for $0 < t < d(y,\xi)$ and so we need only consider $d(x,y) < t \leq 1$, $d(y,\xi) \leq t$, $x \in G \setminus B(e, 2\rho/\varepsilon)$ and $\xi \in B(e,\rho)$. But for such $x$, $y$, $\xi$ and $t$,

$$\frac{d(x,e)}{2} \leq d(x,e) - d(\xi,e) \leq d(x,\xi) \leq d(x,y) + d(y,\xi) \leq 2t\,.$$

Thus

$$\frac{1}{t} \leq \frac{4}{d(x,e)}\,. \tag{4.18}$$

Now, using Taylor's formula, we obtain

$$\begin{aligned}
\left|\varphi_t * a(y)\right| &\leq C\left|\int_{B(e,\rho)} a(\xi)t^{-n}\phi\left(\frac{\|\theta\|}{t}\right)\,\mathrm{d}\xi\right| + \\
&\quad + C\int_{B(e,\rho)} |a(\xi)|t^{-n}\|\theta\|^2\left|\phi\left(\frac{\|\theta\|}{t}\right)\right|\,\mathrm{d}\xi \\
&\leq P_1 + P_2
\end{aligned}$$

For $\exp\zeta$ conjugate to $y$, we have

$$|P_1| \leq C\|a\|_{L^\infty(G)}\int_{B(e,\rho)} t^{-n}\left|\phi\left(\frac{\|\theta\|}{t}\right) - \phi\left(\frac{\|\zeta\|}{t}\right)\right|\,\mathrm{d}\xi\,.$$

By (6.3) of [5] (as noted in the proof of Proposition 1.1 above) $\|\theta - \zeta\| \leq d(y\xi^{-1}, y) = d(\xi, e)$. This shows that

$$|P_1| \leq C\rho^{-n}t^{-n}\int_{B(e,\rho)} \omega_\phi\left(\frac{d(\xi,e)}{t}\right)\,\mathrm{d}\xi\,.$$

Recall from (4.18) that $t^{-1} \leq 4d(x,e)^{-1}$ and $d(\xi,e) \leq \rho$. Therefore,

$$|P_1| \leq Cd(x,y)^{-n}\omega_\phi\left(\frac{4\rho}{d(x,e)}\right)\,. \tag{4.19}$$

But

$$\begin{aligned}
|P_2| &\leq C\rho^{-n}t^{-n+2}\int_{B(I,\rho)} \phi\left(\frac{\|\theta\|}{t}\right)\|\theta\|^2 t^{-2}\,\mathrm{d}\zeta \\
&\leq Ct^{-n+2} \leq Cd(x,e)^{-n+2}\,.
\end{aligned}$$

Therefore, for $m = -\log_2(\rho/\varepsilon)$ which we may assume is integral,

$$\begin{aligned}
\int_{[\|\theta'\|\geq 2\rho/\varepsilon]} \varphi^* a(x)\,\mathrm{d}x &\leq \\
&\leq \int_{[\|\theta'\|\geq 2\rho/\varepsilon]} d(x,e)^{-n}\, \omega_\phi\left(\frac{4\rho}{d(x,e)}\right)\,\mathrm{d}x + C \\
&\leq C + C\sum_{k=1}^{m} \int_{\rho 2^k/\varepsilon \leq \|\theta'\| \leq \rho 2^{k+1}/\varepsilon} \|\theta'\|^{-\ell}\, \omega_\phi\left(\frac{4\rho}{\|\theta'\|}\right)\,\mathrm{d}\theta' \\
&\leq C + C\sum_{k=1}^{m} \int_{2^k\rho/\varepsilon}^{2^{k+1}\rho/\varepsilon} s^{-1}\, \omega_\phi\left(\frac{4\rho}{s}\right)\,\mathrm{d}s\,,
\end{aligned}$$

and on substituting $s = 4\rho/\delta$

$$\begin{aligned}
\int_{[\|\theta'\|\geq 2\rho/\varepsilon]} \varphi^* a(x)\,\mathrm{d}x &\leq C + C\sum_{k=1}^{m} \int_{2^{-k+1}\varepsilon}^{2^{-k+2}\varepsilon} \omega_\phi(\delta)\,\mathrm{d}\delta \\
&= C + C\int_0^{2\varepsilon} \omega_\phi(\delta)\frac{\mathrm{d}\delta}{\delta}
\end{aligned}$$

as required. Since the constant is independent of the atom, a similar estimate for any $f \in H^1(G)$ follows immediately from the atomic decomposition, yielding

$$\|f\|_{H_\varphi(G)} \leq C\|f\|_{H^1(G)}\,.$$

Conversely, if $f \in H_\varphi(G)$, let $\psi$ be a $C^\infty$ radial function on $\mathfrak{t}$ that is supported in $\|\theta\| \leq 1/2$ and $\int_{\mathfrak{t}} \psi(\theta)\,\mathrm{d}\theta \neq 0$. Let $\Psi(\theta) = \varphi * \psi(\theta)$. Then $\Psi$ is a radial tempered function with $\int_{\mathfrak{t}} \Psi(\theta)\,\mathrm{d}\theta \neq 0$. We define $\psi_t$ and $\Psi_t$ on $G$ by

$$\psi_t(x) = \int_{\lambda\in\Lambda} \widehat{\psi}\left(t\|\lambda+\rho\|\right) d_\lambda \chi_\lambda(x)$$

and

$$\Psi_t(x) = \sum_{\lambda\in\Lambda} \widehat{\Psi}\left(t\|\lambda+\rho\|\right) d_\lambda \chi_\lambda(x)\,.$$

Then $\Psi_t(x) = \varphi_t * \psi_t(x)$. But for any fixed $0 < t \leq 1$ and $y \in G$ such that $d(x,y) < t$ we have

$$\begin{aligned}
\left|f * \varphi_t * \psi_t(x)\right| &\leq C\varphi_{n+1}^{**} f(x) \int_G \left(2 + \frac{d(y\xi^{-1},e)}{t}\right)^{n+1} \left|\psi_t(y\xi^{-1})\right|\,\mathrm{d}\xi \\
&\leq C\varphi_{n+1}^{**} f(x)\,.
\end{aligned}$$

Thus,

$$\left\| \sup_{0<t\leq 1} |f * \Psi_t(x)| \right\|_{L^1(G)} \leq C \|\varphi_{n+1}^{**} f\|_{L^1(G)} \leq C \|\varphi^* f\|_{L^1(G)} .$$

Also, for $t > 1$,

$$\|f * \Psi_t\|_{L^\infty(G)} \leq C \|\Psi_t\|_{L^\infty(G)} \|f\|_{L^1(G)} \leq C \|f\|_{L^1(G)} .$$

Furthermore, since $\lim_{t\to 0} \varphi_t * f(x) = Cf(x)$ for almost every $x \in G$, we get

$$\sup_{t>1} \|f * \Psi_t\|_{L^\infty(G)} \leq C \|\varphi * f\|_{L^1(G)} .$$

By Theorem 4.4, we conclude that $\|f\|_{H^1(G)} \leq C \|\varphi^* f\|_{L^1(G)}$. □

We now pick up two corollaries:

COROLLARY 1.— *Suppose that* $\varphi$ *is radial, compactly supported,* $\int_{\mathrm{t}} \varphi \, \mathrm{d}\theta \neq 0$ *and* $\widehat{\varphi}(\theta) = \mathrm{O}(|\theta|^{-n-\varepsilon_0})$ *as* $\theta \to \infty$ *for some* $\varepsilon_0 > 0$*. Let* $\phi$ *be obtained from* $\varphi$ *as above and suppose that* $\phi$ *satisfies (4.15). Then* $f \in H^1(G)$ *if and only if*

$$\sup_{d(x,y)<t} |\varphi_t * f(y)| \in L^1(G) .$$

COROLLARY 2.— *If* $\varphi$ *and* $\phi$ *are as in Corollary 1, if* $\int_{\mathrm{t}} \varphi \, \mathrm{d}\theta = 1$*, and if* $f \in H^1(G)$*, then*

$$\lim_{t\to 0} \sup_{d(x,y)<t} f * \varphi_t(y) = f(x)$$

*for almost every* $x \in G$.

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