Annales de la faculté des sciences de Toulouse

George Dinca
Panagiotis D. Panagiotopoulos
Gabriela Pop

An existence result on noncoercive hemivariational inequalities

Annales de la faculté des sciences de Toulouse 6e série, tome 6, no 4 (1997), p. 609-632

<http://www.numdam.org/item?id=AFST_1997_6_6_4_609_0>

Annales de la Faculté des Sciences de Toulouse Vol. VI, n° 4, 1997

An Existence Result on Noncoercive Hemivariational Inequalities(*)

George Dinca(1), Panagiotis D. Panagiotopoulos(2) and Gabriela Pop(1)

Résumé. — On donne des résultats concernant l'existence de la solution des inégalités hémivariationnelles non coercives sur des ensembles convexes dans un espace hilbertien de dimension infinie. Les démonstrations utilisent essentiellement l'existence de la solution en dimension finie et certaines techniques de régularisation.

Abstract. — In the present paper noncœrcive hemivariational inequalities on convex sets are studied. Applying the method of recession cones using a regularisation procedure and beginning from a finite dimensional problem we derive sufficient conditions for the existence of the solution.

1. Introduction

The theory of hemivariational inequalities has begun some years ago with the works of the second author (see [20]) concerning the derivation of variational expressions for nonconvex nonsmooth energy functions. Such variational expressions are called hemivariational inequalities and their derivation is based on the notion of generalized gradient of F. H. Clarke [7]. For a complete list of references on the subject we refer to [21].

The aim of the present paper is to give an existence result for noncœrcive hemivariational inequalities on convex sets in a real Hilbert space. For variational inequalities which are noncœrcive such results can be found in

(*) Reçu le 20 septembre 1995

(1) Faculty of Mathematics, Bucharest University, Str. Academiei 14, 70109 Bucharest (Romania)

(2) Department of Civil Engineering, Aristotle University, GR-54006 Thessaloniki (Greece)

the works of Fichera ([10], [11]), and Lions and Stampacchia [17]. In [2], Baiocchi, Gastaldi and Tomarelli extend and unify these well-known results. There is partial overlapping of their results and those contained in Brezis and Haraux [3] and Brezis and Nirenberg [4]. Some ideas from a paper of Goeleven [12] will be used in the present paper. Our framework is the following:

Ω is an open bounded subset in $\mathbb{R}^n$, V is an infinite dimensional real Hilbert space with the scalar product $(\cdot\,,\cdot)$ and norm $\|\cdot\|$, such that

$$V \subset L^2(\Omega) \subset V^*\,, \quad \text{the injection of } V \text{ in } L^2(\Omega) \text{ being compact}\,; \tag{1}$$

the duality pairing between V and V^* will be denoted by $\langle\cdot\,,\cdot\rangle$;

$$a : \{u, v\} \longrightarrow a(u, v) = \langle Tu, v\rangle \quad \text{with } T \in \mathcal{L}(V, V^*)\,, \tag{2}$$

is a bilinear continuous form on V with values in $\mathbb{R}$;

$$K \subset V \quad \text{is a closed, non-empty convex set}\,; \tag{3}$$

$$\ell : v \to \langle \ell, v\rangle \quad \text{is a linear continuous functional on } V\,; \tag{4}$$

$\beta \in L^\infty(\mathbb{R})$, $\beta(\xi \pm 0)$ exist for any $\xi \in \mathbb{R}$ and there is $\xi_0 \in \mathbb{R}$ such that

$$\operatorname*{ess\,sup}_{(-\infty,-\xi_0)} \beta \le 0 \le \operatorname*{ess\,sup}_{(\xi_0,+\infty)} \beta\,; \tag{5}$$

j is defined by

$$j(\xi) = \int_0^\xi \beta(t)\,\mathrm{d}t\,. \tag{6}$$

We consider the functions:

$$\overline{\beta}_\mu(\xi) = \operatorname*{ess\,sup}_{|\xi_1-\xi|\le\mu} \beta(\xi_1) \quad \text{and} \quad \underline{\beta}_\mu(\xi) = \operatorname*{ess\,sup}_{|\xi_1-\xi|\le\mu} \beta(\xi_1)\,.$$

They are increasing and decreasing functions of μ, respectively; therefore, the limits for $\mu \to 0_+$ exist. We denote them by $\overline{\beta}(\xi)$ and $\underline{\beta}(\xi)$ respectively. Chang [6] has shown that if $\beta \in L^\infty_{\text{loc}}$ and $\beta(\xi \pm 0)$ exist for every $\xi \in \mathbb{R}$, then a locally Lipschitz function $j : \mathbb{R} \to \mathbb{R}$ can be determined by simple integration, i.e., $j(\xi) = \int_0^\xi \beta(t)\,\mathrm{d}t$, such that

$$(\partial_C j)(\xi) = \left[\,\underline{\beta}(\xi)\,,\, \overline{\beta}(\xi)\right], \tag{7}$$

where $(\partial_C j)(\xi)$ stands for the generalized gradient of Clarke for j at point ξ (cf. Clarke [7], Aubin [1]).

According to Rockafellar (see [23, p. 62]), we call recession cone of K (asymptotic cone, according to Bourbaki's book, [5, p. 125]) the set

$$K^{\infty} = \bigcap_{\alpha>0} \alpha(K - x_0)\,, \quad x_0 \in K \tag{8}$$

(see also Baiocchi, Gastaldi and Tomarelli [2, p. 623]).

This definition is independent of x_0. The following results hold:

$$K^{\infty} \text{ is a closed convex cone with vertex at the origin;} \tag{9}$$

$$\text{if } K \text{ is a cone, then } K^{\infty} = K; \tag{10}$$

$$\text{if } 0 \in K, \text{ then } K^{\infty} \subset K. \tag{11}$$

An element $w \in V$ belongs to K^{∞} if and only if either of the following conditions is satisfied:

$$x + w \in K\,, \quad \forall\, x \in K\,; \tag{12}$$

$$x + \alpha w \in K\,, \quad \forall\, x \in K\,, \forall\, \alpha \geq 0\,; \tag{13}$$

$$\exists\, x_0 \in K \;:\; x_0 + \alpha w \in K\,, \quad \forall\, \alpha \geq 0\,. \tag{14}$$

Let $T^* \in \mathcal{L}(V, V^*)$ be the adjoint of T:

$$a(u,v) = \langle Tu, v\rangle$$
$$a(v,u) = \langle T^* u, v\rangle\,, \quad \forall\, u\,,\; v \in V\,.$$

We denote by

$$N(T,K) = K^{\infty} \cap \operatorname{Ker}(T + T^*)$$

and by

$$K^{+}(T,K) = \left\{x^* \in V^* \mid \langle x^*, y\rangle \geq 0\,, \;\forall\; y \in N(T,K)\right\}.$$

We say that $K^{+}(T,K)$ is solid if its topological interior is not empty:

$$\overbrace{K^{+}(T,K)}^{\circ} \neq \emptyset\,.$$

Remark 1.1.— Let us assume that $N(T,K) \neq \{0\}$ and denote by

$$\operatorname{int}\{K^+(T,K)\} = \{x^* \in V^* \mid \langle x^*, y\rangle > 0\,,\ \forall\, y \in N(T,K)\,,\ y \neq 0\}\,.$$

Then we get

1) $\overbrace{K^+(T,K)}^{\circ} \subset \operatorname{int}\{K^+(T,K)\}$;

2) if $\operatorname{Ker}(T+T^*)$ is finite dimensional, then $\overbrace{K^+(T,K)}^{\circ} = \operatorname{int}\{K^+(T,K)\}$.

Proof

1) Let $x_0^* \in \overbrace{K^+(T,K)}^{\circ}$. There exists $\varepsilon > 0$ such that $x_0^* + x^* \in K^+(T,K)$ for any $x^* \in V^*$ for which $\|x^*\| \leq \varepsilon$. Let $y \in N(T,K)$, $y \neq 0$ and $x_1^* \in V^*$ be defined as follows:

$$\langle x_1^*, \zeta\rangle = -\frac{(y,\zeta)}{\|y\|}\,\varepsilon\,, \quad \forall\, \zeta \in V\,.$$

Since $\|x_1^*\| \leq \varepsilon$ we get $x_0^* + x_1^* \in K^+(T,K)$. Consequently $\langle x_0^* + x_1^*\,,\, y\rangle \geq 0$, which implies

$$\langle x_0^*, y\rangle \geq \langle -x_1^*, y\rangle = \|y\|\,\varepsilon > 0\,,$$

the last inequality entailing $x_0^* \in \operatorname{int}\{K^+(T,K)\}$.

2) Next we prove that in case $\operatorname{Ker}(T+T^*)$ is finite dimensional, the inclusion

$$\operatorname{int}\{K^+(T,K)\} \subset \overbrace{K^+(T,K)}^{\circ}$$

holds. Since $\operatorname{int}\{K^+(T,K)\} \subset K^+(T,K)$, it is enough to show that $\operatorname{int}\{K^+(T,K)\}$ is open (or, equivalently $V^* \setminus \operatorname{int}\{K^+(T,K)\}$ is closed).

In order to prove it let $(x_n^*)_n \subset V^* \setminus \operatorname{int}\{K^+(T,K)\}$ be a sequence such that $x_n^* \to x^*$ in V^*. Then, for any $n \in \mathbb{N}$, there exists $y_n \in N(T,K)$, $y_n \neq 0$, such that $\langle x_n^*, y_n\rangle \leq 0$. Let the sequence $\zeta_n = y_n/\|y_n\| \in N(T,K)$. Since $\operatorname{Ker}(T+T^*)$ is finite dimensional, we can assume (passing to a subsequence if necessary) that $\zeta_n \to \zeta \in N(T,K)$, $\|\zeta\| = 1$. Using $\langle x_n^*, \zeta_n\rangle \leq 0$, we obviously get $\langle x^*, \zeta\rangle \leq 0$, where $\zeta \in N(T,K)$, $\zeta \neq 0$, that is $x^* \in V^* \setminus \operatorname{int}\{K^+(T,K)\}$.

DEFINITION 1.1.— *We say that $T \in \mathcal{L}(V, V^*)$ is "positive plus" on K if the following conditions are fulfilled:*

(i) $\langle Tx, x\rangle \geq 0$, $\forall\, x \in K$;

(ii) $x \in K^\infty$ *and*

$$\langle Tx, x\rangle = 0 \Longrightarrow x \in \mathrm{Ker}(T + T^*)\,.$$

Remark 1.2 (Mironescu, [18]).— If $T \in \mathcal{L}(V, V^*)$, then

$$\langle Tx, x\rangle \geq 0\,, \quad \forall\, x \in K \quad \Longleftrightarrow \quad \langle Tx, x\rangle \geq 0\,, \quad \forall\, x \in K \cup K^\infty\,.$$

Obviously, only the "only if" part should be proved.

Let $x \in K^\infty$. If $x = 0$, then $\langle Tx, x\rangle = 0$.

Suppose $x \neq 0$ and let x_0 be any point in K. Then, for any $t \geq 0$, $x_0 + tx \in K$. Consequently,

$$\begin{aligned}&\big\langle T(x_0 + tx)\,,\, x_0 + tx\big\rangle = \\ &\qquad = \langle Tx_0, x_0\rangle + t\big(\langle Tx_0, x\rangle + \langle Tx, x_0\rangle\big) + t^2\langle Tx, x\rangle \geq 0\,, \quad \forall\, t \geq 0\,.\end{aligned}$$

Taking the limit as $t \to \infty$, we further get $\langle Tx, x\rangle \geq 0$.

According to this remark, the Definition 1.1 is equivalent to Goeleven's definition [12] where the condition (i) is formulated as $\langle Tx, x\rangle \geq 0$, $\forall\, x \in K \cup K^\infty$.

The Definition 1.1 of "positive plus" operators seems to be a technical one. Note that the classe of "positive plus" operators, basically introduced by Goeleven in [12], includes the classes of copositive plus matrices (cf. Lemke [16]), of copositive plus operators (cf. Gowda and Seidman [14]) and also the class of semicoercive operators (cf. Fichera [11]; see also the concluding remarks of the present paper).

2. The main result

THEOREM 2.1.— *Suppose that conditions (1)-(5) are satisfied. Moreover, let the following conditions hold true:*

(i) T *is positive plus on* K*;*

(ii) $K^+(T, K)$ *is solid;*

(iii) the map $x \longrightarrow \langle Tx, x\rangle$ *is weakly lower semicontinuous;*

(iv) every sequence $(x_n)_n \subset V$ *such that* $\|x_n\| = 1$ *and* $\liminf \langle Tx_n, x_n\rangle = 0$ *has a subsequence* $(x_{n_k})_k$*, such that* $x_{n_k} \xrightarrow{w} x$ *in* V *and* $x \neq 0$.

Then, for every $\ell \in T(K) - \overbrace{K^+(T,K)}^{\circ}$*, the solution set of hemivariational inequality:*

$$u \in K\,, \quad a(u, v-u) + \int_\Omega (\mathrm{D}_C j)(u)(v-u)\,\mathrm{d}\Omega \geq \langle \ell\,,\, v-u\rangle\,, \quad \forall\, v \in K\,, \tag{15}$$

is nonempty.

(In (15),

$$(\mathrm{D}_C j)(x)h = \limsup_{\substack{y \to x \\ \lambda \to 0_+}} \frac{j(y+\lambda h) - j(y)}{\lambda}$$

stands for the directional Clarke derivative).

Remark 2.1.— Let us remark first, that hypothesis (iv) in Theorem 2.1 implies that $\mathrm{Ker}(T+T^*)$ is of finite dimension.

Indeed, in case it were not true, there would exist $(x_n)_n \subset \mathrm{Ker}(T+T^*)$ such that

$$\|x_n\| = 1 \quad \text{and} \quad x_n \xrightarrow{w} 0 \text{ in } V\,.$$

But $\langle Tx_n, x_n\rangle = 0$, hence, using the hypothesis (iv), it follows that there exists a subsequence $(x_{n_k})_k \subset (x_n)_n$ such that

$$x_{n_k} \xrightarrow{w} x \neq 0\,,$$

which obviously contradicts the previous equation.

Remark 2.2.— From the Remarks 1.1, 2.1 and using the hypothesis (iv) in the Theorem 2.1 too, it follows that if $N(T,K) \neq \{0\}$, then

$$\begin{aligned} \overbrace{K^+(T,K)}^{\circ} &= \mathrm{int}\{K^+(T,K)\} \\ &= \{x^* \in V^* \mid \langle x^*, y\rangle > 0\,,\ \forall\, y \in N(T,K)\,,\ y \neq 0\}\,. \end{aligned}$$

The previously derived result will be systematically used as an argument in proving the Theorem 2.1.

We will prove first the following intermediate result.

PROPOSITION 2.1.— *Let us assume the hypotheses of Theorem 2.1 to be satisfied and let $\varepsilon \in (0,1)$ be arbitrarily chosen. Then, there is $u^\varepsilon \in K$ (not necessarily unique) such that:*

$$\langle Tu^\varepsilon\,,\, v - u^\varepsilon\rangle + \int_\Omega \beta_\varepsilon(u^\varepsilon)(v - u^\varepsilon)\,\mathrm{d}\Omega \geq \langle \ell\,,\, v - u^\varepsilon\rangle\,, \quad \forall\, v \in K\,, \tag{16}$$

where β_ε is the mollification of β, i.e., $\beta_\varepsilon = p_\varepsilon * \beta$, with

$$p_\varepsilon(\,\cdot\,) = \frac{1}{\varepsilon}\, p\left(\frac{\cdot}{\varepsilon}\right)\,, \quad p \in \mathcal{D}(-1\,,1)\,, \quad p \geq 0$$

and

$$\int_{-\infty}^{\infty} p(\xi)\,\mathrm{d}\xi = 1\,.$$

Proof

Step 1. Since $K^+(T,K)$ is solid (hypothesis (ii)), we get $\overbrace{K^+(T,K)}^{\circ} \neq \emptyset$. Using the hypothesis $\ell \in T(K) - \overbrace{K^+(T,K)}^{\circ}$, we further infer that there exists $x_0 \in K$ such that

$$Tx_0 - \ell \in \overbrace{K^+(T,K)}^{\circ}\,. \tag{17}$$

Step 2. Denote by $\mathcal{F}_{x_0}$ the family of all finite dimensional subspaces of V which contain x_0. If $F \in \mathcal{F}_{x_0}$, then there is $u_F^\varepsilon \in K \cap F$ such that

$$\langle Tu_F^\varepsilon\,,\, v - u_F^\varepsilon\rangle + \int_\Omega \beta_\varepsilon(u_F^\varepsilon)(v - u_F^\varepsilon)\,\mathrm{d}\Omega \geq \langle \ell\,, v - u_F^\varepsilon\rangle\,, \quad \forall\, v \in K \cap F\,. \tag{18}$$

For the proof, let $\mathcal{J} \cdot F \to F^*$, defined as follows:

$$\langle \mathcal{J}u, v\rangle_{F,F^*} = \langle Tu, v\rangle + \int_\Omega \beta_\varepsilon(u) v\,\mathrm{d}\Omega - \langle \ell, v\rangle\,, \tag{19}$$

F being endowed with the norm induced from V. We shall show that $\mathcal{J}$ is continuous. Indeed, if $u_n \xrightarrow{V} u$, then is easily seen that

$$\left\|\mathcal{J}u_n - \mathcal{J}u\right\|_{F^*} \leq \left\|Tu_n - Tu\right\|_{V^*} + c\left\|\beta_\varepsilon(u_n) - \beta_\varepsilon(u)\right\|_{L^2(\Omega)}\,,$$

c being the constant used to express the continuous imbedding of V into $L^2(\Omega)$, and the problem reduces to showing that

$$\beta_\varepsilon(u_n) \xrightarrow{L^2(\Omega)} \beta_\varepsilon(u)\,.$$

But, the injection of V into $L^2(\Omega)$ being continuous, we have

$$u_n \xrightarrow{L^2(\Omega)} u$$

and, consequently, there exists a subsequence $(u_{n_k})_k \subset (u_n)_n$ such that $u_{n_k} \to u$ a.e. on Ω. It is easily seen that for any t_1, $t_2 \in \mathbb{R}$, we have:

$$\left|\beta_\varepsilon(t_1) - \beta_\varepsilon(t_2)\right| \leq \left\|\beta\right\|_{L^\infty(\mathbb{R})} \cdot \left\|p'_\varepsilon\right\|_{C([-\varepsilon\,,\varepsilon])} |t_1 - t_2| \left(|t_1 - t_2| + 2\varepsilon\right). \quad (20)$$

Consequently,

$$\beta_\varepsilon(u_{n_k}) \longrightarrow \beta_\varepsilon(u) \quad \text{a.e. on } \Omega\,.$$

On the other hand, $\left|\beta_\varepsilon(t)\right| \leq \left\|\beta\right\|_{L^\infty(\mathbb{R})}$, for any $t \in \mathbb{R}$ and, from the classical dominated convergence theorem, it follows that

$$\beta_\varepsilon(u_{n_k}) \xrightarrow{L^2(\Omega)} \beta_\varepsilon(u)\,.$$

So, we have the following result: if $u_n \xrightarrow{V} u$, then there exists a subsequence $(u_{n_k})_k \subset (u_n)_n$ such that

$$\beta_\varepsilon(u_{n_k}) \xrightarrow{L^2(\Omega)} \beta_\varepsilon(u)\,.$$

It follows from this that

$$\beta_\varepsilon(u_n) \xrightarrow{L^2(\Omega)} \beta_\varepsilon(u)\,.$$

Indeed, if one assumes on the contrary that the above limit is false, then one easily obtains a contradiction to the result stated above.

For each $n \in \mathbb{N}$, let $\overline{B}_n = \left\{u \in V \mid \|u\| \leq n\right\}$. As $x_0 \in K \cap F$, for n sufficiently large, the compact convex set $K_n = K \cap F \cap \overline{B}_n$ contains x_0. By using a classical result (see Theorem 3.1 in Kinderlehrer and Stampacchia [15]) for any such n there is $u_n \in K_n$ such that

$$\left\langle \mathcal{J} u_n\,,\, v - u_n \right\rangle_{F,F^*} \geq 0\,, \quad \forall\, v \in K_n\,,$$

that is

$$\left\langle T u_n\,,\, v - u_n \right\rangle + \int_\Omega \beta_\varepsilon(u_n)(v - u_n)\,\mathrm{d}\Omega \geq \left\langle \ell\,, v - u_n \right\rangle, \quad \forall\, v \in K_n\,. \quad (21)$$

We shall show that the sequence $(u_n)_n$ is bounded. Suppose to the contrary, that $(u_n)_n$ is unbounded, i.e. (passing to a subsequence if necessary) $\|u_n\| \to \infty$ and let

$$x_n = \frac{u_n}{\|u_n\|}, \quad \ell_n = \frac{\ell}{\|u_n\|}.$$

Passing (if necessary) to a subsequence, we can suppose that

$$x_n \xrightarrow{V} \widetilde{x}, \quad \|\widetilde{x}\| = 1 \quad \text{and} \quad \ell_n \to 0.$$

Let us show that $\widetilde{x} \in N(T, K) = K^\infty \cap \operatorname{Ker}(T + T^*)$. We show first that $\widetilde{x} \in K^\infty$. According to (13) it is sufficient to show that for any $v \in K$ and $\alpha \geq 0$, $v + \alpha\widetilde{x} \in K$.

Let $v \in K$ and $\alpha \geq 0$. For n sufficiently large, we have $\alpha/\|u_n\| \in [0\,,1]$ and, by the convexity of K,

$$\left(1 - \frac{\alpha}{\|u_n\|}\right) v + \frac{\alpha}{\|u_n\|}\, u_n \in K$$

that is

$$\left(1 - \frac{\alpha}{\|u_n\|}\right) v + \alpha x_n \in K.$$

Passing to the limit one obtains $v + \alpha\widetilde{x} \in K$.

As $\widetilde{x} \in K^\infty$ and T is positive plus on K, in order to prove that $\widetilde{x} \in \operatorname{Ker}(T + T^*)$ it is sufficient to prove that $\langle T\widetilde{x}\,, \widetilde{x}\rangle = 0$.

According to (21) we have

$$\langle Tu_n\,, x_0 - u_n\rangle + \int_\Omega \beta_\varepsilon(u_n)(x_0 - u_n)\,\mathrm{d}\Omega \geq \langle \ell\,, x_0 - u_n\rangle\,, \tag{22}$$

for n sufficiently large.

At this moment we need the following auxiliary result (see the proof of Lemma 2.1 in Panagiotopoulos [22] or Rauch and McKenna [19]): there are the constants $\rho_1 > 0$, $\rho_2 = \|\beta\|_{L^\infty(\mathbb{R})}$, which do not depend of ε, such that:

$$\begin{aligned} \xi\beta_\varepsilon(\xi) &\geq 0 \quad \text{for } |\xi| > \rho_1 \text{ and } \varepsilon \in (0,1), \\ &\geq -\rho_1\rho_2 \quad \text{for any } \xi \in \mathbb{R} \text{ and } \varepsilon \in (0,1). \end{aligned} \tag{23}$$

Consequently, one has

$$\int_\Omega \beta_\varepsilon(u_n)(x_0 - u_n)\,\mathrm{d}\Omega \le \left\|\beta\right\|_{L^\infty(\mathbb{R})} \cdot \left\|x_0\right\|_{L^1(\Omega)} + \rho_1\rho_2 \operatorname{mes}\Omega\,. \tag{24}$$

By combining (22) and (24) one obtains

$$\left\langle Tu_n\,,\, x_0 - u_n\right\rangle + \left\|\beta\right\|_{L^\infty(\mathbb{R})} \cdot \left\|x_0\right\|_{L^1(\Omega)} + \rho_1\rho_2 \operatorname{mes}\Omega \ge \left\langle \ell\,,\, x_0 - u_n\right\rangle. \tag{25}$$

Now, divide by $\left\|u_n\right\|^2$ in (25) and pass to the limit. One obtains $\langle T\widetilde{x}\,,\,\widetilde{x}\rangle \le 0$ that is, in fact, $\langle T\widetilde{x}\,,\,\widetilde{x}\rangle = 0$ and, consequently, $\widetilde{x} \in \operatorname{Ker}(T + T^*)$.

Let us resume: assuming that the sequence $\left(u_n\right)_n$ is not bounded we infer the existence of $\widetilde{x}$ such that

$$\widetilde{x} \in N(T,K)\,, \quad \left\|\widetilde{x}\right\| = 1\,, \quad x_n = \frac{u_n}{\|u_n\|} \to \widetilde{x}$$

hold. But this entails a contradiction.

Indeed, the case $N(T,K) = \{0\}$ is inconsistent with $\widetilde{x} \in N(T,K)$ and $\|\widetilde{x}\| = 1$.

If $N(T,K) \ne \{0\}$ then, from (17) and Remark 2.2, we get

$$\left\langle Tx_0 - \ell\,,\, \widetilde{x}\right\rangle > 0\,. \tag{26}$$

On the other hand by the positivity of T, $\langle Tu_n, u_n\rangle \ge 0$, so that, taking into account (25) it follows that

$$\langle Tu_n, x_0\rangle + \left\|\beta\right\|_{L^\infty(\mathbb{R})} \cdot \left\|x_0\right\|_{L^1(\Omega)} + \rho_1\rho_2 \operatorname{mes}\Omega \ge \left\langle \ell\,,\, x_0 - u_n\right\rangle.$$

Dividing by $\|u_n\|$ and passing to the limit one obtains

$$\langle T\widetilde{x}, x_0\rangle \ge -\langle \ell, \widetilde{x}\rangle\,.$$

But, taking into account that $\widetilde{x} \in \operatorname{Ker}(T + T^*)$, we also have

$$\langle T\widetilde{x}, x_0\rangle = -\langle T^*\widetilde{x}, x_0\rangle = -\langle Tx_0, \widetilde{x}\rangle$$

therefore,

$$\langle Tx_0, \widetilde{x}\rangle \le \langle \ell, \widetilde{x}\rangle$$

which contradicts (26).

The sequence $(u_n)_n$ with $u_n \in K \cap F \cap \overline{B}_n$ and satisfying (21) is, consequently, bounded. We can suppose (passing to a subsequence if necessary) that

$$u_n \xrightarrow{V} u \in K \cap F\,.$$

To conclude the proof for Step 2 we shall show that such an u satisfies

$$\langle Tu\,,\, v-u\rangle + \int_\Omega \beta_\varepsilon(u)(v-u)\,\mathrm{d}\Omega \geq \langle \ell\,,\, v-u\rangle\,, \quad \forall\, v \in K \cap F\,. \tag{27}$$

Indeed, let v be an arbitrary element in $K \cap F$.

For n sufficiently large, $v \in K_n = K \cap F \cap \overline{B}_n$ and, consequently,

$$\langle Tu_n\,,\, v-u_n\rangle + \int_\Omega \beta_\varepsilon(u_n)(v-u_n)\,\mathrm{d}\Omega \geq \langle \ell\,,\, v-u_n\rangle\,. \tag{28}$$

Taking into account the previous results, we have:

$$u_n \xrightarrow{V} u \Longrightarrow u_n \xrightarrow{L^2(\Omega)} u \Longrightarrow \beta_\varepsilon(u_n) \xrightarrow{L^2(\Omega)} \beta_\varepsilon(u)\,.$$

Consequently, passing to the limit in (28), inequality (27) follows.

Let us underline the result obtained above: if F is a finite dimensional subspace of V such that $x_0 \in F$, then there exists $u_F^\varepsilon \in K \cap F$ such that (18) is satisfied.

Step 3. Let $F \in \mathcal{F}_{x_0}$ and

$$V_F^\varepsilon = \bigcup_{\substack{F' \in \mathcal{F}_{x_0} \\ F' \supset F}} \{u \in K \mid u \text{ satisfies (27) for every } v \in K \cap F'\}\,. \tag{29}$$

For every $F \in \mathcal{F}_{x_0}$, the set V_ε^F is (nonempty and) bounded.

Indeed, let us suppose the contrary: there exist $u_n \in K$, $\|u_n\| \to \infty$ and $F_n' \supset F$ such that:

$$\langle Tu_n\,,\, v-u_n\rangle + \int_\Omega \beta_\varepsilon(u_n)(v-u_n)\,\mathrm{d}\Omega \geq \langle \ell\,,\, v-u_n\rangle\,, \quad \forall\, v \in K \cap F_n'\,. \tag{30}$$

By using the same arguments as above, we have:

$$\begin{aligned} \langle Tu_n\,,\, v-u_n\rangle + \|\beta\|_{L^\infty(\mathbb{R})} \cdot \|v\|_{L^1(\Omega)} + \rho_1\rho_2 \,\mathrm{mes}\,\Omega \geq \\ \geq \langle \ell\,,\, v-u_n\rangle\,, \quad \forall\, v \in K \cap F_n'\,. \end{aligned}$$

As $x_0 \in K \cap F \subset K \cap F'_n$ for any n, we have, in particular,

$$\begin{aligned} \langle Tu_n \,,\, x_0 - u_n \rangle + \|\beta\|_{L^\infty(\mathbb{R})} \cdot \|x_0\|_{L^1(\Omega)} + \rho_1 \rho_2 \operatorname{mes} \Omega \geq \\ \geq \langle \ell \,,\, x_0 - u_n \rangle \,, \quad \forall\, n \in \mathbb{N} \,. \end{aligned} \tag{31}$$

Let $x_n = u_n / \|u_n\|$. Passing to a subsequence, if necessary, we can suppose

$$x_n \xrightarrow{w} \widetilde{x} \,.$$

We shall prove that $\widetilde{x} \in N(T, K) = K^\infty \cap \operatorname{Ker}(T + T^*)$.

We show first that $\widetilde{x} \in K^\infty$.

To do it, let v be an arbitrary element in K and $\alpha \geq 0$. For n sufficiently large, we have $\alpha / \|u_n\| \in [\,0\,,\,1\,]$ and, consequently,

$$\left(1 - \frac{\alpha}{\|u_n\|}\right) v + \frac{\alpha}{\|u_n\|}\, u_n \in K \,,$$

which can also be written as

$$\left(1 - \frac{\alpha}{\|u_n\|}\right) v + \alpha x_n \in K \,.$$

Taking the weak limit, it follows that $v + \alpha \widetilde{x} \in K$ (being convex and closed, K is weakly closed), which implies $\widetilde{x} \in K^\infty$.

As T is positive plus on K, in order to prove that $\widetilde{x} \in \operatorname{Ker}(T + T^*)$, it is sufficient to show that $\langle T\widetilde{x}, \widetilde{x} \rangle = 0$. To do it, let us divide (31) by $\|u_n\|^2$. One obtains:

$$\begin{aligned} \left\langle Tx_n \,,\, \frac{x_0}{\|u_n\|} - x_n \right\rangle + \frac{1}{\|u_n\|^2} \left(\|\beta\|_{L^\infty(\mathbb{R})} \cdot \|x_0\|_{L^1(\Omega)} + \rho_1 \rho_2 \operatorname{mes} \Omega \right) \geq \\ \geq \frac{1}{\|u_n\|} \left\langle \ell \,,\, \frac{x_0}{\|u_n\|} - x_n \right\rangle \,. \end{aligned}$$

Taking the sup-limit in this inequality one has:

$$\liminf_{n \to \infty} \langle Tx_n, x_n \rangle \leq 0 \,.$$

On the other hand, by using the positivity of T and condition (iii), it follows that:

$$0 \le \langle T\widetilde{x}\,,\,\widetilde{x}\rangle \le \liminf_{n\to\infty}\langle Tx_n, x_n\rangle \le 0\,,$$

therefore, $\langle T\widetilde{x}, \widetilde{x}\rangle = 0$.

Let us observe that $\widetilde{x} \neq 0$. This is a simple consequence of

$$\|x_n\| = 1\,, \quad \liminf_{n\to\infty}\langle Tx_n, x_n\rangle = 0$$

and hypotesis (iv).

Let us resume: assuming that V_F^{ε} were not bounded, the element $\widetilde{x}$ obtained as above would be such that, simultaneously, $\widetilde{x} \neq 0$ and $\widetilde{x} \in N(T,K)$. But this entails a contradiction. Indeed, the case $N(T,K) = \{0\}$ is inconsistent with $\widetilde{x} \in N(T,K)$ and $\widetilde{x} \neq 0$. If $N(T,K) \neq \{0\}$ then, from (17) and Remark 2.2, we obtain

$$\big\langle Tx_0 - \ell\,,\,\widetilde{x}\big\rangle > 0\,. \tag{32}$$

On the other hand, by the positivity of T, one has

$$\langle Tu_n, u_n\rangle \ge 0\,.$$

By combining this last inequality with (31) and dividing by $\|u_n\|$ one derives:

$$\begin{aligned}\langle Tx_n, x_0\rangle + \frac{1}{\|u_n\|}\Big(\|\beta\|_{L^\infty(\mathbb{R})}\cdot\|x_0\|_{L^1(\Omega)} + \rho_1\rho_2 \operatorname{mes}\Omega\Big) \ge \\ \ge \left\langle \ell\,,\,\frac{x_0}{\|u_n\|} - x_n\right\rangle.\end{aligned} \tag{33}$$

Passing to the limit in (33) (and taking into the account that operator T is "weak to weak continuous") one obtains:

$$\langle T\widetilde{x}, x_0\rangle \ge -\langle \ell, \widetilde{x}\rangle\,.$$

With the technique used for Step 2 we derive from this the inequality below

$$\big\langle Tx_0 - \ell\,,\,\widetilde{x}\big\rangle \le 0\,,$$

which contradicts (32).

Step 4. Now we are able to indicate an element $u^\varepsilon \in K$ which satisfies (16). To do it, we proceed as follows: according to the previous result, for every $F \in \mathcal{F}_{x_0}$, V_F^ε is a (nonempty) bounded set contained in K. Consequently, the weak closure of V_F^ε, $\overline{V_F^\varepsilon}^w$ is weakly compact and also contained in K.

Fix $F_0 \in \mathcal{F}_{x_0}$ and consider the family $\{\overline{V_F^\varepsilon}^w \cap \overline{V_{F_0}^\varepsilon}^w \mid F \in \mathcal{F}_{x_0}\}$ (it is a family of weakly closed subsets contained in the weakly compact set $\overline{V_{F_0}^\varepsilon}^w$). Obviously, this family has the finite intersection property. Consequently,

$$\bigcap_{F\in\mathcal{F}_{x_0}} \left(\overline{V_F^\varepsilon}^w \cap \overline{V_{F_0}^\varepsilon}^w\right) = \bigcap_{F\in\mathcal{F}_{x_0}} \overline{V_F^\varepsilon}^w \neq \emptyset .$$

Consider an element $u^\varepsilon \in \bigcap_{F\in\mathcal{F}_{x_0}} \overline{V_F^\varepsilon}^w$. We shall show that such an element satisfies (16).

Indeed, let $v \in K$ and $F \in \mathcal{F}_{x_0}$ such that $v \in F$. Because $u^\varepsilon \in \overline{V_F^\varepsilon}^w$, there is a sequence $\left(u_n\right)_n \subset V_F^\varepsilon$ such that $u_n \xrightarrow{w} u^\varepsilon$. Taking (29) into account, this implies: there exist $F_n \in \mathcal{F}_{x_0}$, $F_n \supset F$ such that

$$\big\langle Tu_n\,,\, w - u_n\big\rangle + \int_\Omega \beta_\varepsilon(u_n)(w-u_n)\,\mathrm{d}\Omega \geq \big\langle \ell\,,\, w-u_n\big\rangle\,, \qquad \forall\, w \in K\cap F_n\,,\ \forall\, n\in\mathbb{N}\,.$$

In particular,

$$\big\langle Tu_n\,,\, v - u_n\big\rangle + \int_\Omega \beta_\varepsilon(u_n)(v-u_n)\,\mathrm{d}\Omega \geq \big\langle \ell\,,\, v-u_n\big\rangle\,, \quad \forall\, n\in\mathbb{N}\,. \tag{34}$$

Because $u_n \xrightarrow{w} u^\varepsilon$ and T is weak to weak continuous,

$$\langle Tu_n, v\rangle \longrightarrow \langle Tu^\varepsilon, v\rangle\,. \tag{35}$$

Moreover, by using hypothesis (iii), it follows that

$$\langle Tu^\varepsilon, u^\varepsilon\rangle \leq \liminf_{n\to\infty} \langle Tu_n, u_n\rangle\,. \tag{36}$$

Finally, we shall show that

$$\limsup_{n\to\infty} \int_\Omega \beta_\varepsilon(u_n)(v-u_n)\,\mathrm{d}\Omega \leq \int_\Omega \beta_\varepsilon(u^\varepsilon)(v-u^\varepsilon)\,\mathrm{d}\Omega\,. \tag{37}$$

Indeed, because the imbedding of V into $L^2(\Omega)$ is compact, we have:

$$u_n \xrightarrow{w} u^\varepsilon \Longrightarrow u_n \xrightarrow{L^2(\Omega)} u^\varepsilon$$
$$\Longrightarrow \text{(passing possibly to a subsequence) } u_n \longrightarrow u^\varepsilon \text{ a.e. on } \Omega\,.$$

By using estimation (20), it follows that $\beta_\varepsilon(u_n) \to \beta_\varepsilon(u^\varepsilon)$ a.e. on Ω.

On the other hand

$$\beta_\varepsilon(u_n)(v-u_n) \leq \rho_1\rho_2 + \|\beta\|_{L^\infty(\mathbb{R})} \cdot |v| \quad \text{a.e. on } \Omega\,,$$

therefore, by using Fatou's lemma one has:

$$\begin{aligned}\limsup_{n\to\infty} \int_\Omega \beta_\varepsilon(u_n)(v-u_n)\,\mathrm{d}\Omega &\leq \int_\Omega \limsup_{n\to\infty} \beta_\varepsilon(u_n)(v-u_n)\,\mathrm{d}\Omega \\ &= \int_\Omega \beta_\varepsilon(u^\varepsilon)(v-u^\varepsilon)\,\mathrm{d}\Omega\end{aligned}$$

which is precisely (37).

Passing to the sup-limit in (34), and taking into account (35), (36) and (37), we obtain (16) and the proof is complete.

Now we are able to give the following proof.

Proof of Theorem 2.1

Let $\ell \in T(K) - \overbrace{K^+(T,K)}^{\circ}$ and let $x_0 \in K$ be such that

$$Tx_0 - \ell \in \overbrace{K^+(T,K)}^{\circ}\,.$$

According to Proposition 2.1, for every $\varepsilon \in (0,1)$ the set $\{u^\varepsilon \mid u^\varepsilon \text{ satisfies } (16)\}$ is nonempty. Moreover, we shall show that

$$\mathcal{U} = \{u^\varepsilon \in K \mid \varepsilon \in (0,1)\,,\ u^\varepsilon \text{ satisfies } (16)\}$$

is bounded.

Indeed, if we suppose on the contrary that this family is unbounded, we have: there exists a sequence $(\varepsilon_n)_n \subset (0,1)$ and a sequence $(u_{\varepsilon_n})_n \subset K$ such that

$$\begin{aligned}\big\langle Tu_{\varepsilon_n}\,,\, v - u_{\varepsilon_n}\big\rangle + \int_\Omega \beta_{\varepsilon_n}(u_{\varepsilon_n})(v-u_{\varepsilon_n})\,\mathrm{d}\Omega \geq \\ \geq \big\langle \ell\,,\, v - u_{\varepsilon_n}\big\rangle\,, \quad \forall\, v\in K\,,\, \forall\, n\in\mathbb{N}\,,\end{aligned} \tag{38}$$

with $\|u_{\varepsilon_n}\| \to \infty$.

By using the estimations (23) again, we easily derive from (38)

$$\begin{aligned}\big\langle Tu_{\varepsilon_n}\,,\, v - u_{\varepsilon_n}\big\rangle + \big\|\beta\big\|_{L^\infty(\mathbb{R})}\cdot\big\|v\big\|_{L^1(\Omega)} + \rho_1\rho_2 \operatorname{mes}\Omega \geq \\ \geq \big\langle \ell\,,\, v - u_{\varepsilon_n}\big\rangle\,, \quad \forall\, v\in K\,,\, \forall\, n\in\mathbb{N}\,.\end{aligned} \tag{38'}$$

In particular,

$$\begin{aligned}\big\langle Tu_{\varepsilon_n}\,,\, x_0 - u_{\varepsilon_n}\big\rangle + \big\|\beta\big\|_{L^\infty(\mathbb{R})}\cdot\big\|x_0\big\|_{L^1(\Omega)} + \rho_1\rho_2 \operatorname{mes}\Omega \geq \\ \leq \big\langle \ell\,,\, x_0 - u_{\varepsilon_n}\big\rangle\,,\ \forall\, n\in\mathbb{N}\,.\end{aligned} \tag{39}$$

Let us set $x_n = u_{\varepsilon_n}/\|u_{\varepsilon_n}\|$. We can suppose $x_n \xrightarrow{w} \widetilde{x}$. By using arguments like those used for Step 3 in the proof of Proposition 2.1 (inequality (39) replacing (31) this time) one obtains:

$$\widetilde{x}\in N(T,K)\,,\quad \widetilde{x}\neq 0 \quad\text{and}\quad \big\langle Tx_0 - \ell\,,\, \widetilde{x}\big\rangle \leq 0\,.$$

But this entails a contradiction. Indeed, the case $N(T,K) = \{0\}$ is inconsistent with $\widetilde{x}\in N(T,K)$ and $\widetilde{x}\neq 0$. If $N(T,K)\neq\{0\}$ then, from $Tx_0 - \ell \in \overbrace{K^+(T,K)}^{\circ}$ and Remark 2.2, we get $\big\langle Tx_0 - \ell\,,\, \widetilde{x}\big\rangle > 0$ which is inconsistent with the already established inequality $\big\langle Tx_0 - \ell\,,\, \widetilde{x}\big\rangle \leq 0$.

Now, the family $\mathcal{U} = \big\{u^\varepsilon\in K \mid \varepsilon\in(0,1)\,,\ u^\varepsilon \text{ satisfies (16)}\big\}$ being bounded, there is $\varepsilon_n \to 0_+$ and a sequence $(u_{\varepsilon_n})_n \subset \mathcal{U}$ such that $u_{\varepsilon_n} \xrightarrow{w} u\in K$.

We shall show that u satisfies (15).

Indeed, let v be an arbitrary element in K. We have:

$$\big\langle Tu_{\varepsilon_n}\,,\, v - u_{\varepsilon_n}\big\rangle + \int_\Omega \beta_{\varepsilon_n}(u_{\varepsilon_n})(v-u_{\varepsilon_n})\,\mathrm{d}\Omega \geq \big\langle \ell\,,\, v - u_{\varepsilon_n}\big\rangle\,, \quad \forall\, n\in\mathbb{N}\,. \tag{40}$$

With the (same) arguments used in order to obtain (35), (36), (37) we can write:

$$\limsup_{n\to\infty}\langle Tu_{\varepsilon_n}\,,\,v-u_{\varepsilon_n}\rangle \leq \langle Tu\,,\,v-u\rangle \tag{41}$$

$$\limsup_{n\to\infty}\langle \ell\,,\,v-u_{\varepsilon_n}\rangle = \langle \ell\,,\,v-u\rangle \tag{42}$$

$$\begin{aligned}\limsup_{n\to\infty}\int_\Omega \beta_{\varepsilon_n}(u_{\varepsilon_n})(v-u_{\varepsilon_n})\,\mathrm{d}\Omega &\leq \\ &\leq \int_\Omega \limsup_{n\to\infty}\beta_{\varepsilon_n}(u_{\varepsilon_n})(v-u_{\varepsilon_n})\,\mathrm{d}\Omega\,.\end{aligned} \tag{43}$$

The essential point consists in showing that for any $y\in\mathbb{R}$,

$$\limsup_{n\to\infty}\beta_{\varepsilon_n}(u_{\varepsilon_n})(y-u_{\varepsilon_n}) \leq (\mathrm{D}_Cj)(u)(y-u) \quad \text{a.e. on } \Omega\,. \tag{44}$$

Since we can suppose (passing, possibly, to a subsequence) $u_{\varepsilon_n}\to u$ a.e. on Ω, for the proof of (44) it is sufficient to show that for any $x\in\Omega$ such that $u_{\varepsilon_n}(x)\to u(x)$ and for any convergent subsequence $y_k=\beta_{\varepsilon_n}\big(u_{\varepsilon_n}(x)\big)\big(y-u_{\varepsilon_n}(x)\big)$, inequality

$$\lim_{n\to\infty} y_k \leq (\mathrm{D}_Cj)\big(u(x)\big)\big(y-u(x)\big) \tag{45}$$

holds.

First, we shall show that

$$\liminf_{n\to\infty}\beta_{\varepsilon_n}\big(u_{\varepsilon_n}(x)\big) \quad \text{and} \quad \limsup_{n\to\infty}\beta_{\varepsilon_n}\big(u_{\varepsilon_n}(x)\big)$$

belong to $(\partial_Cj)\big(u(x)\big)$. As a consequence, we shall have:

$$\begin{aligned}\Big(\liminf_{n\to\infty}\beta_{\varepsilon_n}\big(u_{\varepsilon_n}(x)\big)\Big)\,z &\leq (\mathrm{D}_Cj)\big(u(x)\big)z\\ \Big(\limsup_{n\to\infty}\beta_{\varepsilon_n}\big(u_{\varepsilon_n}(x)\big)\Big)\,z &\leq (\mathrm{D}_Cj)\big(u(x)\big)z\,, \quad \forall\, z\in\mathbb{R}\,.\end{aligned} \tag{46}$$

For the proof, let $x\in\Omega$, with $u_{\varepsilon_n}(x)\to u(x)$ and $\mu>0$ be given. There exists $n_{\mu,x}$ such that, for every $n\geq n_{\mu,x}$, one has

$$\big|u_{\varepsilon_n}(x)-u(x)\big| \leq \frac{\mu}{2}\,, \quad 0\leq\varepsilon_n\leq\frac{\mu}{2}\,.$$

Consequently,

$$\beta_{\varepsilon_n}\big(u_{\varepsilon_n}(x)\big) \leq \operatorname*{ess\,sup}_{|t-u_{\varepsilon_n}(x)|\leq\varepsilon_n} \beta(t) \leq \operatorname*{ess\,sup}_{|t-u(x)|\leq\mu} \beta(t) = \overline{\beta}_\mu\big(u(x)\big)\,.$$

Therefore

$$\limsup_{n\to\infty} \beta_{\varepsilon_n}\big(u_{\varepsilon_n}(x)\big) \leq \overline{\beta}_\mu\big(u(x)\big)$$

and, passing to the limit with $\mu \to 0_+$, we derive

$$\limsup_{n\to\infty} \beta_{\varepsilon_n}\big(u_{\varepsilon_n}(x)\big) \leq \overline{\beta}\big(u(x)\big)\,.$$

Analogously, one obtains

$$\underline{\beta}\big(u(x)\big) \leq \liminf_{n\to\infty} \beta_{\varepsilon_n}\big(u_{\varepsilon_n}(x)\big)\,.$$

Thus,

$$\limsup_{n\to\infty} \beta_{\varepsilon_n}\big(u_{\varepsilon_n}(x)\big) \quad \text{and} \quad \liminf_{n\to\infty} \beta_{\varepsilon_n}\big(u_{\varepsilon_n}(x)\big) \in \big[\underline{\beta}\big(u(x)\big)\,,\, \overline{\beta}\big(u(x)\big)\big]$$
$$= \big(\partial_C j\big)\big(u(x)\big)\,.$$

Now, we are able to prove (45).

If $y = u(x)$, then $y - u_{\varepsilon_n}(x) \to 0$. Because $|\beta_\varepsilon(t)| \leq \|\beta\|_{L^\infty(\mathbb{R})}$ for any $t \in \mathbb{R}$ and any $\varepsilon > 0$, it follows that $y_k \to 0$. Thus, (45) is verified.

If $y > u(x)$, then

$$\begin{aligned}
\lim_{k\to\infty} y_k &= \left(\lim_{k\to\infty} \beta_{\varepsilon_{n_k}}\big(u_{\varepsilon_{n_k}}(x)\big)\right)\big(y - u(x)\big) \\
&\leq \left(\limsup_{n\to\infty} \beta_{\varepsilon_n}\big(u_{\varepsilon_n}(x)\big)\right)\big(y - u(x)\big) \\
&\leq (\mathrm{D}_C j)\big(u(x)\big)\big(y - u(x)\big)\,,
\end{aligned}$$

the last inequality being justified by (46).

For the case $y < u(x)$, a similar procedure can be used (by using the inf-limit this time). So (44) is proved.

From (43) and (44) it follows that

$$\limsup_{n\to\infty} \int_\Omega \beta_{\varepsilon_n}(u_{\varepsilon_n})(v - u_{\varepsilon_n})\,\mathrm{d}\Omega \leq \int_\Omega (\mathrm{D}_C j)(u)(v-u)\,\mathrm{d}\Omega\,. \qquad (47)$$

Finally, passing to the sup-limit in (40) and taking into account (41), (42), (47), inequality (15) follows.

3. Some comments and remarks

The hypotheses concerning operator T associated to the bilinear form a are those formulated by Goeleven [12] for the study of variational inequality

$$u \in K\,, \quad a(u, v-u) + \phi(v) - \phi(u) \geq \big\langle -\ell\,,\, v-u \big\rangle\,, \quad \forall\, v \in K\,. \tag{48}$$

In Goeleven [12], both V and K are supposed to be separable and the functional $\phi : V \to R$ satisfies the following assumptions

(a) ϕ is convex and lower semicontinuous;

(b) $\phi(\alpha u) = \alpha\phi(u)$, $\forall\, \alpha \geq 0$, $\forall\, u \in K$.

In our paper, instead of inequality (48) we consider the hemivariational inequality (15) and the hypothesis concerning the separability of V and K is removed. Consequently, our technique is quite different from that used by Goeleven in [12].

Note that the assumption on the separability of V and K was deleted also by Goeleven in [13].

Let us also remark that in [12], sufficient conditions are given, which guarantee the satisfaction of hypotheses of Theorem 2.1. To the extent that these conditions concern only operator T, they remain valid in our case.

To complete the picture, we give these conditions (adding some comments too) in the next.

I.— If $T \in \mathcal{L}(V, V^*)$ is positive on V, i.e.,

$$\langle Tx, x \rangle \geq 0\,, \quad \forall\, x \in V\,,$$

or T is compact, assumption (iii) is satisfied.

II.— If T is positive on V, then T is positive plus on K.

III.— Let P be the continuous projector of V onto $\big[\mathrm{Ker}(T+T^*)\big]^{\perp}$. If T is semicoercive on $K \cup K^{\infty}$, i.e.,

$$\langle Tu, u \rangle \geq c\big\|Pu\big\|^2\,, \quad \forall\, u \in K \cup K^{\infty}$$

then T is positive plus on K.

It can be proved without difficulties that T is semicoercive on $K \cup K^{\infty}$ if and only if T is semicoercive on K:

$$\langle Tu, u\rangle \geq c\|Pu\|^2 , \ \forall\, u \in K \cup K^{\infty} \quad \Longleftrightarrow \quad \langle Tu, u\rangle \geq c\|Pu\|^2 , \ \forall\, u \in K .$$

Obviously, we need to prove only the "$\Leftarrow$" part. Let $u_0 \in K^{\infty}$. It follows that $u + tu_0 \in K$, $\forall\, t \geq 0$, $\forall\, u \in K$. Using $\langle T(u + tu_0)\,,\, u + tu_0\rangle \geq c\|P(u + tu_0)\|^2$ we derive

$$\begin{aligned}\langle Tu, u\rangle + t\big(\langle Tu_0, u\rangle + \langle Tu_0, u_0\rangle\big) + t^2\langle Tu_0, u_0\rangle \geq \\ \geq c\left(\|Pu\|^2 - 2t\|Pu\| \cdot \|Pu_0\| + t^2\|Pu_0\|^2\right) , \quad \forall\, t \geq 0 .\end{aligned}$$

therefore

$$\langle Tu_0, u_0\rangle \geq c\|Pu_0\|^2 .$$

IV.— If T is semicoercive on V and $\dim[\operatorname{Ker} T + T^*] < \infty$ then assumption (iv) is satisfied.

Since in Remark 2.1 we showed : hypothesis (iv) $\Rightarrow \dim[\operatorname{Ker} T + T^*] < \infty$, consequently, we have the conclusion: if T is semicoercive on V then the assumption (iv) in Theorem 2.1 holds if and only if $\dim[\operatorname{Ker} T + T^*] < \infty$.

V.— If there exist $\alpha_1 > 0$, $\alpha_2 > 0$ and a real Hilbert space Z with V compactly imbedded in Z such that

$$\langle Tx, x\rangle + \alpha_1\|x\|_Z^2 \geq \alpha_2\|x\|_V^2$$

then the assumption (iv) is satisfied. Notice that, in this case, $\operatorname{Ker}(T + T^*)$ is also finite dimensional (indeed, it is easily seen that if condition (49) holds, then the unit ball in $\operatorname{Ker}(T + T^*)$ is compact).

VI (Proposition 3.1 in [12]).— Let C be a closed subset of V. If T is semicoercive on C and $\dim[\operatorname{Ker}(T + T^*)] < \infty$, then from every sequence $(x_n)_n \subset C$ such that $\|x_n\| = 1$ and $\liminf\langle Tx_n, x_n\rangle = 0$, there exists a subsequence $(x_{n_k})_{n_k}$ such that $w - \lim x_{n_k} = x$ in V and $x \neq 0$.

VII.— If T is symetric ($T = T^*$) then

$$\ell \in T(K) - \overbrace{K^+(T, K)}^{\circ} \quad \text{if and only if} \quad -\ell \in \overbrace{K^+(T, K)}^{\circ} .$$

Indeed,

$$\text{if } \ell \in T(K) - \overbrace{K^+(T,K)}^{\circ} \text{ then } \exists\, x_0 \in K \text{ so that } Tx_0 - \ell \in \overbrace{K^+(T,K)}^{\circ} .$$

Consequently, there exists $\varepsilon > 0$ such that, for any $x^* \in V^*$ with $\|x^*\| < \varepsilon$ one has $\langle Tx_0 - \ell + x^*\,,\, y\rangle \geq 0$, $\forall\, y \in N(T,K)$. As $\langle Tx_0, y\rangle = \langle T^*y, x_0\rangle = 0$ one deduces that $\langle -\ell + x^*\,,\, y\rangle \geq 0$, $\forall\, y \in N(T,K)$, that is $-\ell \in \overbrace{K^+(T,K)}^{\circ}$. For the converse implications, similar arguments can be used.

VIII.— If (a) $\mathrm{Ker}(T+T^*) = \{0\}$ or (b) $K^\infty = \{0\}$, then $\overbrace{K^+(T,K)}^{\circ} = V^*$. Consequently, inequality (15) has a solution for every $\ell \in V^*$. We have (a) if, for example, $\langle Tx, x\rangle > 0$, $\forall\, x \in V$, $x \neq 0$; we have (b) if K is bounded closed convex set.

IX.— Let us examine the particular case $K = V$.

Obviously, in this case

$$K^\infty = V\,, \quad N(T,V)) = \mathrm{Ker}(T+T^*)$$

$$\begin{aligned} K^+(T,V) &= \{x^* \in V^* \mid \langle x^*, y\rangle = 0\,,\ \forall\, y \in \mathrm{Ker}(T+T^*)\} \\ &= [\mathrm{Ker}(T+T^*)]^\perp \end{aligned}$$

and the hypotheses formulated in the Theorem 2.1 become:

(i) T is positive plus on $V \Leftrightarrow T$ is positive on V ($\langle Tu, u\rangle \geq 0$, $\forall\, u \in V$).

For obvious reasons only the part "$\Leftarrow$" has to be proved.

In other words, we have to prove that, if T is positive on V then

$$x \in V \quad \text{and} \quad \langle Tx, x\rangle = 0 \quad \Longrightarrow x \in \mathrm{Ker}(T+T^*)\,.$$

Indeed, let $x \in V$ be such that $\langle Tx, x\rangle = 0$ and let y be an arbitrary element of V. Then, for any $\lambda > 0$, since

$$\langle T(x+\lambda y)\,,\, x+\lambda y\rangle \geq 0$$

we get

$$\langle (T+T^*)x\,,\, y\rangle + \lambda\langle Ty, y\rangle \geq 0\,.$$

Taking the limit as $\lambda \to 0_+$, we obtain

$$\langle (T+T^*)x\,,\,y\rangle \geq 0\,.$$

Since y is an arbitrary element of V, the previously considered inequality entails $\langle (T+T^*)x\,,\,y\rangle = 0$, that is $(T+T^*)x = 0$.

(ii) $K^+(T,V) = \left[\operatorname{Ker}(T+T^*)\right]^{\perp}$ is solid $\Leftrightarrow$ $\operatorname{Ker}(T+T^*) = \{0\}$ $\Leftrightarrow$ $K^+(T,V) = V^*$.

The argument is somehow a standard one.

The assumption (iii) is implied by the assumption (i): if T is positive on V then the mapping $x \to \langle Tx, x\rangle$ is weakly lower semicontinuous.

Note that

$$\big\{(\mathrm{i}) + (\mathrm{ii}) + (\mathrm{iii})\big\} \Leftrightarrow T \text{ is coercive on } V \quad \big(\langle Tu, u\rangle \geq c\big\|u\big\|^2\,,\ \forall\ u \in V\big)\,.$$

The "$\Leftarrow$" is straightforward. Moreover, if T is coercive then there exists no sequences $(x_n)_n$ such that $\|x_n\| = 1$ and $\liminf\langle Tx_n, x_n\rangle = 0$.

The argument to justify the "$\Rightarrow$" part can be as follows. Let $c = \inf\langle Tx, x\rangle$, hence $c \geq 0$. Assuming that $c = 0$, there exists a sequence $(x_n)_n$ such that $\|x_n\| = 1$ and $\langle Tx_n, x_n\rangle \to 0$.

According to (iv) there is a subsequence $(x_{n_k})_k \subset (x_n)_n$ such that

$$x_{n_k} \xrightarrow{w} x \quad \text{and} \quad x \neq 0\,.$$

But, using (i) and its outcome (iii) it folows:

$$0 \leq \langle Tx, x\rangle \leq \underline{\lim}\langle T_{n_k}, x_{n_k}\rangle = 0$$

hence $\langle Tx, x\rangle = 0$, that is $x \in \operatorname{Ker}(T+T^*) = \{0\}$, which is, obviously, a contradiction.

Now, let us notice that since $K^+(T,V) = V^*$, the condition $\ell \in T(K) - \overbrace{K^+(T,K)}^{\circ}$ can be expressed as "for any $\ell \in V^*$".

Finally, using the previously described developments, we arrive at the following conclusion.

THEOREM 3.1.— *Let $a(u,v) = \langle Tu, v\rangle$ be a bilinear continuous coercive form on V and $\beta \in L^{\infty}(\mathbb{R})$ such that the conditions (5) hold. Then, for any $\ell \in V^*$, the solution set of*

$$u \in K\,, \quad a(u, v-u) + \int_{\Omega} (\mathrm{D}_C j)(u)(v-u)\,\mathrm{d}\Omega \geq \big\langle \ell\,,\, v-u \big\rangle\,, \quad \forall\, v \in V\,,$$

where j is defined by $j(\xi) = \int_0^{\xi} \beta(t)\,\mathrm{d}t$, $\forall\, \xi \in \mathbb{R}$, is not empty (to be compared to the Theorem 2.1, [22]).

Acknowledgements

The authors express their thanks to the referee for his valuable suggestions.

References

[1] AUBIN (J.-P.) .— *Gradients généralisés de Clarke*, Ann. Sc. Math. Québec, **II**, n° 2 (1978), pp. 197-252.

[2] BAIOCCHI (C.), GASTALDI (F.) and TOMARELLI (F.) .— *Some Existence Results on Noncoercive Variational Inequalities*, Annali Scuola Norm. Sup. Pisa, Serie IV, **XIII**, n° 4 (1986).

[3] BREZIS (H.) and HARAUX (A.) .— *Image d'une somme d'opérateurs monotones et applications*, Israel J. Math. **23** (1976), pp. 165-186.

[4] BREZIS (H.) and NIRENBERG (L.) .— *Characterizations of the Ranges of Some Nonlinear Operators and Applications to Boundary Value Problems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **4**, n° 5 (1978), pp. 225-326.

[5] BOURBAKI (N.) .— *Éléments de Mathématique. Espaces vectoriels topologiques*, chap. I et II, Act. Sci. Ind. **1189**, Hermann, Paris (1966).

[6] CHANG (K. C.) .— *Variational Methods for Non-differentiable Functionals and their Applications to Partial Differential Equations*, J. Math. Anal. Appl. **80** (1981), pp. 102-129.

[7] CLARKE (F. H.) .— *Optimization and Nonsmooth Analysis*, John Wiley, New York (1983).

[8] DINCA (G.), PANAGIOTOPOULOS (P. D.) and POP (G.) .— *Coercive and Semicoercive Hemivariational Inequalities on Convex Sets*, Vest. R.U.D.N. **2** (1995), pp. 96-110.

[9] DINCA (G.), PANAGIOTOPOULOS (P. D.) and POP (G.) .— *Inégalités hémivariationnelles semi-coercives sur des ensembles convexes*, C. R. Acad. Sci., Paris, Série I, **320** (1995), pp. 1183-1186.

[10] Fichera (G.) .— *Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno*, Atti Accad. Naz. Lincei Mem. Sez., t. I, **8**, n° 7 (1964), pp. 71-140.

[11] Fichera (G.) .— *Boundary value problems in elasticity with unilateral constraints*, Handbuch der Physik **VIa**, n° 2, Springer, Berlin (1972), pp. 347-389.

[12] Goeleven (D.) .— *On the Solvability of Noncoercive Linear Variational Inequalities in Separable Hilbert Spaces*, JOTA **79**, n° 3 (1993), pp. 493-511.

[13] Goeleven (D.) .— *On Noncoercive Variational Inequalities and Some Applications in Unilateral Mechanics*, Ph. D. Degree in Sciences, F.U.N.D.P. (1993).

[14] Gowda (M. S.) and Seidman (I.) .— *Generalized Linear Complementarity Problems*, Mathematical Programming **46** (1990), pp. 329-340.

[15] Kinderlehrer (D.) and Stampacchia (G.) .— *An introduction to Variational Inequalities and Their Applications*, Academic Press, New York (1980).

[16] Lemke (C. E.) .— *Bimatrix Equilibrum Points and Mathematical Programming*, Management Science **11** (1965), pp. 681-689.

[17] Lions (J.-L.) and Stampacchia (G) .— *Variational Inequalities*, Comm. Pure Appl. Math. **20** (1967), pp. 493-519.

[18] Mironescu (P.) Private Communication.

[19] McKenna (P. J.) and Rauch (J.) .— *Strongly Nonlinear Perturbations of Nonnegative Boundary Value Problems with Kernel*, J. Diff. Eq. **28** (1978), pp. 253-265.

[20] PanagiotoPoulos (P. D.) .— *Nonconvex Superpotentials in the Sense of F. H. Clarke and Applications*, Mech. Res. Comm. **8** (1981), pp. 335-340.

[21] Panagiotopoulos (P. D.) .— *Hemivariational Inequalities and Applications in Mechanics and Engineering*, Springer Verlag, Berlin, New York (1993).

[22] Panagiotopoulos (P. D.) .— *Coercive and Semicoercive Hemivariational Inequalities*, Nonlinear Anal. TMA, **16** (1991), pp. 209-231.

[23] Rockafellar (R. T.) .— *Convex Analysis*, Princeton Univ. Press, Princeton (1970).