ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

LEONARDO MORA
NEPTALÍ ROMERO

Persistence of homoclinic tangencies for area-preserving maps

Annales de la faculté des sciences de Toulouse 6e série, tome 6, nº 4 (1997), p. 711-725
<http://www.numdam.org/item?id=AFST_1997_6_6_4_711_0>

Annales de la Faculté des Sciences de Toulouse Vol. VI, n° 4, 1997

Persistence of Homoclinic Tangencies for Area-Preserving Maps(*)

LEONARDO MORA(1) and NEPTALÍ ROMERO(2)

RÉSUMÉ. — Nous prouvons que dans une variété symplectique bidimensionnelle M, l'existence de courbes lisses invariantes dans le monde des applications symplectiques de M est un mécanisme pour créer des ouverts contenant un ensemble dense d'applications exhibant des tangences homocliniques.

ABSTRACT. — In a 2-dimensional symplectic manifold M we show that the presence of smooth invariant curves in the world of symplectic maps of M is a mechanism to create open sets containing a dense set of maps exhibiting homoclinic tangencies.

1. Introduction

In 1970, S. Newhouse [N2] proved the existence of an open set $\mathcal{U} \subset \mathrm{Diff}^s(M)$, $s \geq 2$, where M is a 2-dimensional compact manifold, with the following property: there exists a dense subset of $\mathcal{U}$ such that each $g : M \hookleftarrow$ in this subset exhibits homoclinic tangencies (tangential intersections between the stable set and unstable set, $W^s(p)$ and $W^u(p)$ respectively, of a hyperbolic periodic point p). We call such a set $\mathcal{U} \subset \mathrm{Diff}^s(M)$, with the last property, an open set of "persistence of homoclinic tangencies", from now on OSPHT.

Later, in 1979 [N3], he proved that a mechanism to create this kind of sets is the unfolding of a dissipative homoclinic tangency. More precisely, for every $f \in \mathrm{Diff}^s(M)$, with a homoclinic tangency associated to a dissipative hyperbolic periodic point p ($\left|\det \mathrm{D}f^n(p)\right| < 1$, where n is the minimal period of p), there exists $\mathcal{U}$ an OSPHT such that $f \in \overline{\mathcal{U}}$.

(*) Reçu le 19 septembre 1995

(1) Instituto Venezolano de Investigaciones Científicas, Departamento de Matemáticas, Apartado 21827, Caracas VE-1020-A (Venezuela)

(2) Departamento de Matemáticas, Decanato de Ciencias UCOLA, Apartado 400, Barquisimeto VE (Venezuela)

Here we present a mechanism to generate OSPHT's in the world of symplectic diffeomorphisms; we show that the presence of a smooth invariant curve generates, for nearby maps, this kind of open sets. To be more precise, let M be a 2-dimensional compact manifold with w a symplectic 2-form on M and denote by Diff^s_w the space of C^s diffeomorphisms that preserve w, then we have the following result.

THEOREM 1.— *Let $f \in \mathrm{Diff}^\infty_w(M)$ admit a C^∞ closed invariant curve γ such that the rotation number $\omega = p(f|_\gamma)$ is irrational. Then for every $s \geq 4$ there exists $\mathcal{U} \subset \mathrm{Diff}^s_w(M)$ an OSPHT such tant $f \in \overline{\mathcal{U}}$. Moreover, there is a residual subset $\mathcal{V}$ of $\mathcal{U}$ such that every $f \in \mathcal{V}$ has an invariant smooth curve which is accumulated by elliptic points.*

The method to prove Theorem 1 is different from *the dissipative case*. The wild hyperbolic sets mechanism used to produce persistence of homoclinic tangencies is replaced by the rich structure around a smooth invariant curve, obtained from KAM theory [Bo], combined with the following two propositions.

PROPOSITION 2.— *For $f \in \mathrm{Diff}^\infty_w(M)$ and γ a C^∞ invariant curve assume that:*

(i) ω satisfies a diophantine condition : there exist $\beta \geq 0$ and $C > 0$ such that for every $p/q \in \mathbb{Q}$ then

$$\left|\omega - \frac{p}{q}\right| > \frac{C}{q^{2+\beta}},$$

(ii) f satisfies a twist condition along γ (see Sect. 2),

(iii) there exist $\widetilde{\mathcal{U}} \subset \mathrm{Diff}^s_w(M)$, such that for each $g \in \widetilde{\mathcal{U}}$ there is a continuation curve γ_g of γ which is invariant by g and with the same rotation number ω.

Then there exists $\mathcal{U} \subset \widetilde{\mathcal{U}}$ an OSPHT and for a residual set in $\mathcal{U}$, the continuation curve γ_g is the limit of elliptic periodic orbits.

Remark.— The same conclusion can be obtained in Proposition 2 if we replace the invariant curve γ by a collection of disjoint curves $\{\gamma_i\}_{i=0}^{n-1}$ such that $f(\gamma_i) = \gamma_{i+1}$ and $f(\gamma_{n-1}) = \gamma_0$. Just take f^n, apply Proposition 2 and pull back $\mathcal{U}$ by the map $f \to f^n$.

PROPOSITION 3.— *Let $f \in \mathrm{Diff}^{\infty}_{w}(M)$. Then for each $s \geq 1$, we have:*

(i) if f exhibits a C^{∞} invariant curve with an irrational rotation number, then for each $\varepsilon > 0$ there exists $\overline{f}$ $C^{s}\varepsilon$-near to f such that $\overline{f}$ exhibits homoclinic tangencies;

(ii) if f exhibits a homoclinic tangency associated to a hyperbolic periodic orbits, then for each $\varepsilon > 0$ there exists $\overline{f}$ $C^{s}\varepsilon$-near to f such that $\overline{f}$ has a generic (in the KAM sense) elliptic periodic point; in particular $\overline{f}$ exhibits C^{∞} invariant curves.

The same conclusion of Theorem 1 holds if we replace the assumption of the presence of an invariant curve by the presence of some homoclinic tangency associated to a hyperbolic periodic point p.

COROLLARY 4.— *Assume that $f \in \mathrm{Diff}^{s}_{w}(M)$, $s \geq 4$, has a hyperbolic periodic point p and that f exhibits a homoclinic tangency associated to p, then there exists $\mathcal{U} \subset \mathrm{Diff}^{s}_{w}(M)$ an OSPHT such that $f \in \overline{\mathcal{U}}$. Moreover, there is a residual subset $\mathcal{V}$ of $\mathcal{U}$ such that every $f \in \mathcal{V}$ has an invariant smooth curve which is accumulated by elliptic points.*

A consequence of Corollary 4 is the creation of infinitely many elliptic islands accumulating KAM curves. However, these elliptic points do not accumulate at the hyperbolic point which unfolds the homoclinic tangency. A related question in the unfolding of a homoclinic tangency is whether the OSPHT's can be constructed generating elliptic islands which accumulate at the hyperbolic periodic point. Some partial results concerning the previous question were obtained in [D]. Moreover, it seems possible to answer the question above by using [MR] and the methods of proof in the dissipative case.

This paper is organized as follows: In Section 2, Birkhoff's normal form and KAM theorem are recalled. The proof of Proposition 3, using some tools of [Z], is presented in Section 3. Finally, in Section 4 we prove Proposition 2 and Theorem 1.

2. Birkhoff's normal form theorem and KAM theorem

Let f be an area-preserving C^{r} diffeomorphism of the annulus $\mathbb{A} = \mathbb{S}^{1} \times \mathbb{R}$, with $r \geq 4k + 4$ and $k \geq 0$; here and in what follows we identify $\mathbb{S}^{1}$ with

$\mathbb{S}^1 \times \{0\}$. Assume that $f(\mathbb{S}^1) = \mathbb{S}^1$ and that $f|_{\mathbb{S}^1} = R_\omega$ the rotation with angle ω. So we can write

$$f(\theta, r) = \big(\theta + \omega + ra(\theta, r)\,,\, rb(\theta, r)\big)\,. \tag{1}$$

We say that $\omega \in \mathbb{R}$ satisfies a diophantine condition if there exist $\beta \geq 0$ and $C > 0$ such that for every $p/q \in \mathbb{Q}$ then $|\omega - p/q| > C/q^{2+\beta}$. Let $D(C, \beta)$ be the set of these numbers with C and β fixed. We recall that the set $D(\beta) = \bigcup_{C \geq 0} D(C, \beta)$ has total Lebesgue measure, i.e., $m\big(D(\beta) \cap [0\,,1]\big) = 1$ when $\beta > 0$.

The following version of Birkhoff's normal form theorem says that if ω satisfies a diophantine condition then after an area-preserving change of coordinates the term $ra(\theta, r)$ in (1) can be written as a polynomial function in r plus higher order terms in r. More precisely, letting

$$\mathbb{A}_\delta = \big\{(\theta, r) \mid \theta \in \mathbb{S}^1\,,\, |r| < \delta\big\}\,,$$

we have the following result.

THEOREM 5.— *For each $n \leq k$ there exists $h_n : \mathbb{A}_\delta \to \mathbb{A}$ a C^{r-4n} area-preserving map letting $\mathbb{S}^1$ invariant and such that $\widehat{f}_n = h_n^{-1} \circ f \circ h_n$ has the following form*

$$\widehat{f}_n(\theta, r) = \big(\theta + \omega + a_1 r + a_2 r^2 + \cdots + a_n r^n + \mathrm{O}(r^{n+1})\,,\, r + \mathrm{O}(r^{n+1})\big)\,.$$

Proof.— For a proof in the C^∞ case see appendices 1 and 2 of [Do]. The finite-differentiability case follows the same lines as the C^∞ case but it is necessary to use lemma 8.1 of [H]. □

Remark.— In the case that f is C^∞ all the changes of coordinates are also C^∞, and we can choose n as large as we want.

Now consider a C^∞ symplectic diffeomorphism $\widetilde{f} \in \mathrm{Diff}^\infty_w$ with an invariant C^∞ curve γ. We define the twist condition along γ as follows: we say that $\widetilde{f}$ satisfies a *twist condition along* γ if there exists a transversal unit vector field X on γ such that $w\big(\mathrm{D}\widetilde{f}X(p)\,,\, X\big(\widetilde{f}(p)\big)\big) > 0$ for all $p \in \gamma$. When $\rho\big(\widetilde{f}|_\gamma\big)$ satisfies a diophantine condition it is well known that after a symplectic change of coordinates, $\widetilde{f}$ restricted to a neighborhood V of γ has the form (1) with $X(\theta, 0) = (0, 1)$. In this case a symplectic diffeomorphism of the annulus $\widetilde{f}$ satisfies a twist condition along γ if and only if

$$a_1 = \int a(\theta, 0)\, \mathrm{d}\theta \neq 0\,.$$

This number does not depend on the symplectic change of coordinates used to put $\widetilde{f}$ in the form (1) and it is called the first Birkhoff coefficient.

Now we recall the KAM theorem and remark some facts that we will use in the sequel. Let $f : \mathbb{A}_\delta \to \mathbb{A}$ be a C^∞ map of the annulus. We say that f has the intersection property if for each curve γ in $\mathbb{A}_\delta$ non homotopically trivial we have that $f(\gamma) \cap \gamma \neq \emptyset$. If f admits an invariant curve which is non homotopically trivial and preserves a symplectic form w then it is easy to see that f has the intersection property. Let $s \geq 4$ and $t \in C^\infty((-\delta, \delta), \mathbb{R})$. For each $(\nu, \mu) \in C^s(\mathbb{A}_\delta, \mathbb{R})^2$ let $T_{\nu,\mu} : \mathbb{A}_\delta \to \mathbb{A}$ be the map

$$(\theta, r) \longmapsto \big(\theta + t(r) + \nu(\theta, r)\,,\, r + \mu(\theta, r)\big)\,.$$

THEOREM 6.— *Let $r_0 \in (-\delta, \delta)$ and assume:*

(a) $\dot{t} > 0$, $T_{\nu,\mu}$ satisfies a twist condition;

(b) $\alpha = t(r_0) \in D(c, \beta)$, $\alpha = t(r_0)$ satisfies a diophantine condition;

(c) $T_{\nu,\mu}$ satisfies the intersection property for every (ν, μ) in a neighborhood of $(0,0)$.

Let $s > 2\beta + 3$, then there exists a neighborhood W in $C^s(\mathbb{A}_\delta, \mathbb{R})^2$ of $(0,0)$ such that, for all $(\nu, \mu) \in W$, one can find $\gamma \in C^{s-2(1+\beta)}(\mathbb{S}^1, \mathbb{R})$ and $h \in \mathrm{Diff}^{s-2(1+\beta)}(\mathbb{S}^1)$ with

(i) $\Gamma = \{(\theta, \gamma(\theta)) \mid \theta \in \mathbb{S}^1\}$ is invariant under $T_{\nu,\mu}$;

(ii) $T_{\nu,\mu}|_\Gamma$ is $C^{s-2(1+\beta)}$ conjugated to the rotation $R_\alpha(\theta) = \theta + \alpha (\mathrm{mod}) 1$ by the following conjugation $\theta \to \big(h(\theta)\,,\, \gamma \circ h(\theta)\big)$.

See [Bo] and [SZ] for a proof.

Remarks

- The neighborhood W depends *a priori* on $\alpha = t(r_0)$ (in fact on $\big(\mathrm{d}t(r_0)/\mathrm{d}r\big)^{-1}$) but it can be proved that if r_0 varies in a compact set K, such that $t(K) \subset D(\beta)$ then we can choose W depending just on K. Because of $D(\beta)$ has total Lebesgue measure, this is what gives the rich structure (lots ot other invariant curves) around an invariant curve.
- We have the following regularity statement: if ν, μ are C^∞ then γ is C^∞, see [SZ].

3. Invariant curves and homoclinic tangencies

In this section our goal is to give the proof of Proposition 3, which in turn is a consequence of the following proposition.

PROPOSITION 7.— *Let $f : \mathbb{A}_\delta \to \mathbb{A}$ be a C^∞ area-preserving map of the annulus which leaves invariant some C^∞ curve*

$$\Lambda = \left\{ (\theta, \Psi(\theta)) \mid \theta \in \mathbb{S}^1 \right\}$$

where $\Psi : \mathbb{S}^1 \to \mathbb{R}$, and such that $f\big|_\Lambda$ has an irrational rotation number. Then for $s \geq 1$ and each $\varepsilon > 0$, f can be ε-approximated in the C^s-topology by one F which exhibits homoclinic tangencies and such that for some $\delta' < \delta$ we have $F\big|_{(\mathbb{A}_\delta \backslash \mathbb{A}_{\delta'})} = f\big|_{(\mathbb{A}_\delta \backslash \mathbb{A}_{\delta'})}$.

Proof of Proposition 3

Item (ii) follows from [N3], see also [MR], so we will only prove item (i). Because f and γ are C^∞, we can find a tubular neighborhood U of γ such that there is $h : U \to \mathbb{A}_\delta$ for which $h(\gamma) : \mathbb{S}^1 \times \{0\} \subset \mathbb{A}_\delta$ and $h^*(\mathrm{d}\theta \wedge \mathrm{d}r) = \omega$. So making use of Proposition 7 the result follows. □

To prove Proposition 7 we need first some preliminary results presented in the following subsection.

3.1 Preliminaries

Let $f : \mathbb{A}_\delta \to \mathbb{A}$ be a C^∞ area-preserving map of the annulus which leaves $\mathbb{S}^1$ invariant, i.e., $f(\mathbb{S}^1) = \mathbb{S}^1$. We assume that $f\big|_{\mathbb{S}^1} = R_\omega$ with $\omega = p/n$ where p, n are relatively prime and

$$\begin{aligned} f(\theta, r) &= \left(\theta + \omega + a_1 r + a_2 r^2 + \cdots + a_n r^n + \mathrm{O}(r^{n+1})\,,\, r + \mathrm{O}(r^{n+1})\right) \\ &= f_n(\theta, r) + \mathrm{O}(r^{n+1})\,, \end{aligned}$$

with $a_1 > 0$. Since f leaves $\mathbb{S}^1$ invariant (see [Do]) we have that locally around $\mathbb{S}^1$, $f(\theta, r) = (\Theta, R)$ is described by a generating function $h(\theta, R)$ in the following way

$$f(\theta, r) = (\Theta, R) \quad \text{iff} \quad \begin{cases} r = \dfrac{\partial h}{\partial \theta}\,, \\ \Theta = \dfrac{\partial h}{\partial R}\,. \end{cases}$$

It is easy to check that

$$h_n(\theta, R) = (\theta, \omega)R + \frac{a_1}{2} R^2 + \frac{a_2}{3} R^2 + \cdots + \frac{a_n}{n+1} R^{n+1}$$

is the generating function of f_n. From this we get that the generating function of f has the form $h(\theta, R) = h_n(\theta, R) + O(R^{n+2})$.

We follow Moser and Zehnder to make a perturbation of f. Consider the following two parameter family of generating functions

$$\begin{aligned} h_{\varepsilon,\gamma}(\theta, R) &= h(\theta, R) - \varepsilon R + \gamma \cos(2\pi n\theta) R^{n+1} \\ &= (\theta + \omega - \varepsilon)R + \frac{a_1}{2} R_2 + \frac{a_2}{3} R^2 + \cdots + \frac{a_n}{n+1} R^{n+1} + \\ &\quad + \gamma \cos(2\pi n\theta) R^{n+1} + O(R^{n+2}) \,. \end{aligned} \tag{2}$$

This family generates, for ε and γ small enough, the following two parameter family of diffeomorphisms $f_{\varepsilon,\gamma} : \mathbb{A}_{\delta/2} \to \mathbb{A}$ with

$$\begin{aligned} f_{\varepsilon,\gamma}(\theta, r) = \big(&\theta + \omega - \varepsilon + a_1 r + a_2 r^2 + \cdots + a_n r^n + \\ &+ (n+1)\gamma \cos(2\pi\theta n) r^n + O(r^{n+1}) \,, \\ &r + 2\pi\gamma n \sin(2\pi n\theta)(r^{n+1}) + O(r^{n+2})\big) \,. \end{aligned} \tag{3}$$

Observe that by the way we made the perturbation, $\mathbb{S}^1$ continues to be invariant for the family$f_{\varepsilon,\gamma}$.

PROPOSITION 8.— *Assume that $a_1 \neq 0$, then for ε and γ small enough, $f_{\varepsilon,\gamma}$ has two n-periodic orbits $\{h_i(\varepsilon,\gamma)\}_{i=0}^{n-1}$, $\{e_i(\varepsilon,\gamma)\}_{i=0}^{n-1}$, which satisfy:*

(a) $\{h_i(\varepsilon,\gamma)\}_{i=0}^{n-1}$ is a hyperbolic n-periodic orbit with

$$h_i(\varepsilon, \gamma) \longrightarrow \left(\frac{i}{n}\,, 0\right)$$

for γ fixed and $\varepsilon \to 0$;

(b) $\{e_i(\varepsilon,\gamma)\}_{i=0}^{n-1}$ is an elliptic n-periodic orbit with

$$e_i(\varepsilon, \gamma) \to \left(\frac{2i+1}{2n}\,, 0\right)$$

for γ fixed and $\varepsilon \to 0$;

(c) there exist $\overline{\delta} > 0$ *and*

$$\psi_{h_i}(\varepsilon, \gamma) \in W^s_{\mathrm{loc}}(h_i(\varepsilon, \gamma)) \cap W^u_{\mathrm{loc}}(h_{i+1}(\varepsilon, \gamma))$$

for which we have

$$\psi_{h_i}(\varepsilon, \gamma) \longrightarrow \psi_{h_i}(0, \gamma) \in \left(\frac{2i+1}{2n} - \overline{\delta}\,,\ \frac{2i+1}{2n} + \overline{\delta}\right) \times \{0\}\,,$$

where $\overline{\delta}$ *does not depend on* ε *when this is small enough;*

(d) the angle

$$\angle\left(T_{\psi_{h_i}} W^s_{\mathrm{loc}}(h_i(\varepsilon, \gamma))\,,\ T_{\psi_{h_i}} W^u_{\mathrm{loc}}(h_{i+1}(\varepsilon, \gamma))\right) \longrightarrow 0$$

for γ *fixed and* $\varepsilon \to 0$.

The proof of this proposition is contained in [Z], so we only present the construction of the periodic points and shows how the homoclinic points are found and refer to [Z] for the rest of the details.

Proof.— We begin by making the following change of coordinates $\ell(\theta, \rho) = (\theta, \varepsilon\rho) = (\theta, r)$ which allows us to see what happens in a microscopic neighborhood of $\mathbb{S}^1$. In terms of θ and ρ, $\widetilde{f} = \ell^{-1} \circ f \circ \ell$ is written as

$$\begin{aligned}
\widetilde{f}_{\varepsilon,\gamma}(\theta, \rho) = \Big(&\theta + \frac{p}{n} - \varepsilon + a_1\varepsilon\rho + \cdots + a_n(\varepsilon\rho)^n + \\
&+ (n+1)\gamma\cos(2\pi\theta n)(\varepsilon\rho)^n + \mathrm{O}\big((\varepsilon\rho)^{n+1}\big)\,, \\
&\rho + 2\pi\gamma n \sin(2\pi\theta n)\varepsilon^n\rho^{n+1} + \mathrm{O}\big(\varepsilon^{n+1}\rho^{n+2}\big)\Big) \\
= \Big(&\theta + \frac{p}{n} - \varepsilon + a_1\varepsilon\rho + \cdots + a_n(\varepsilon\rho)^n + \\
&+ (n+1)\gamma\cos(2\pi\theta n)(\varepsilon\rho)^n + \mathrm{O}\big(\varepsilon^{n+1}\big)\,, \\
&\rho + 2\pi\gamma n \sin(2\pi\theta n)\varepsilon^n\rho^{n+1} + \mathrm{O}\big(\varepsilon^{n+1}\big)\Big)\,.
\end{aligned} \tag{4}$$

We get for the n-th iterate of $\widetilde{f}_{\varepsilon,\gamma}$ the following expression

$$\begin{aligned}
\widetilde{f}^{\,n}_{\varepsilon,\gamma}(\theta, \rho) = (&\theta + p - n\varepsilon + na_1\epsilon\rho + \mathrm{O}(\varepsilon^2)\,, \\
&\rho + 2\pi\gamma n^2 \sin(2\pi\theta n)\varepsilon^n\rho^{n+1} + \mathrm{O}(\varepsilon^{n+1}))\,,
\end{aligned} \tag{5}$$

where

$$O(\varepsilon^2) = na_2\varepsilon^2\rho^2 + \cdots + na_n(\varepsilon\rho)^n + \\ + n(n+1)\gamma\cos(2\pi\theta n)(\varepsilon\rho)^n + O(\varepsilon^{n+1}).$$

The fixed points of $\widetilde{f}^n_{\varepsilon,\gamma}$ are the solutions of the equations

$$\theta = \theta + p - n\varepsilon + na_1\varepsilon\rho + O(\varepsilon^2) \tag{6}$$

$$\rho = \rho + 2\pi\gamma n^2 \sin(2\pi\theta n)\varepsilon^n\rho^{n+1} + O(\varepsilon^{n+1}). \tag{7}$$

The fact that $a_1 \neq 0$ and the implicit function theorem imply that there exists $\rho(\varepsilon)$ a solution of (6) which equals $1/a_1$ when $\varepsilon = 0$. Using this solution in (7) we get $2n$ solutions $\{h_i(\varepsilon), e_i(\varepsilon)\}$ with $i = 1, \ldots, n$ which equal

$$\left(\frac{i}{n}, \frac{1}{a_1}\right), \left(\frac{2i+1}{2n}, \frac{1}{a_1}\right)$$

respectively when $\varepsilon = 0$. Since p, n are relative primes, the uniqueness part of the implicit function theorem gives that

$$\{e_i(\varepsilon) = (e_i(\varepsilon), \rho(\varepsilon))\} \quad \text{and} \quad \{h_i(\varepsilon) = (h_i(\varepsilon), \rho(\varepsilon))\}$$

are actually part of a n-periodic orbit. To determine the nature of these orbits we make another change of coordinates. Let ϕ be any of the points $\{e_i, h_i\}_{i=0}^{n-1}$ and let $\widetilde{\ell}(\psi, x) = (\psi + \phi, \rho(\varepsilon) + \varepsilon^{(n-1)/2}x)$ then $\widehat{f}_{\varepsilon,\gamma} = \widetilde{\ell}^{-1} \circ \widetilde{f}^n_{\varepsilon,\gamma} \circ \widetilde{\ell}$ takes the following form

$$\widehat{f}(\psi, x) = (\psi + \widehat{f}_1(\varepsilon, \psi, \varepsilon^{(n-1)/2}x), x + \widehat{f}_2(\varepsilon, \psi, \varepsilon^{(n-1)/2}x)), \tag{8}$$

where

$$\widehat{f}_1(\varepsilon, \psi, y) = -n\varepsilon + na_1\varepsilon(\rho(\varepsilon) + y) + \cdots + na_n\varepsilon^n(\rho(\varepsilon) + y)^n + \\ + n(n+1)\gamma\cos(2\pi\psi n)\varepsilon^n(\rho(\varepsilon) + y)^n + O(\varepsilon^{n+1}) \tag{9}$$

and

$$\widehat{f}_2(\varepsilon, \psi, y) = (-1)^\sigma 2\pi\gamma n^2 \sin(2\pi\psi n)\varepsilon^{n-(n-1)/2}(\rho(\varepsilon) + y)^{n+1} + \\ + O(\varepsilon^{n-(n-1)/2+1}), \tag{10}$$

with $\sigma = \pm 1$ depending on the value of ϕ. From (9) and the fact $\widehat{f}_1(\varepsilon, 0, 0) = (0, 0)$ we get

$$\widehat{f}_1(\varepsilon, \psi, y) = \varphi_0(\varepsilon, \psi) + \varphi_1(\varepsilon, \psi)y + \varphi_2(\varepsilon, \psi)y^2 \tag{11}$$

with

$$\varphi_0(\varepsilon,\psi) = \widehat{f}_1(\varepsilon,\psi,0) = \mathrm{O}(\varepsilon^{n+1}),$$

$$\varphi_1(\varepsilon,\psi) = \frac{\partial \widehat{f}_1}{\partial y}(\varepsilon,\psi,0) = a_1\varepsilon + \mathrm{O}(\varepsilon^2),$$

$$\varphi_2(\varepsilon,\psi) = \frac{\partial^2 \widehat{f}_1}{\partial^2 y}(\varepsilon,\psi,\widehat{y}) = \mathrm{O}(1)$$

and $0 \le \widehat{y} \le y$. All of these together imply that we can write

$$\begin{aligned} &\widehat{f}(\psi,x) = \\ &= \Big(\psi + na_1\varepsilon^{(n+1)/2}x + \mathrm{O}(\varepsilon^{n/2+1}), \\ &\qquad x + (-1)^\sigma 2\pi \left(\frac{1}{a_1}\right)^{n+1} \gamma n^2 \sin(2\pi\psi n)\varepsilon^{(n+1)/2} + \mathrm{O}(\varepsilon^{n/2+1})\Big). \end{aligned} \tag{12}$$

Now from (12) the jacobien matrix at $(0,0)$ equals to

$$J(\varepsilon) = \begin{pmatrix} 1 + \mathrm{O}(\varepsilon^{n/2+1}) & na_1\varepsilon^{(n+1)/2} + \mathrm{O}(\varepsilon^{n/2+1}) \\ \mathcal{R} + \mathrm{O}(\varepsilon^{n/2+1}) & 1 + \mathrm{O}(\varepsilon^{n/2+1}) \end{pmatrix}$$

where

$$\mathcal{R} = (-1)^\sigma 4\pi^2 \left(\frac{1}{a_1}\right)^{n+1} \gamma n^3 \varepsilon^{(n+1)/2}, \quad \sigma = \begin{cases} 1 & \text{at } e_i \\ 0 & \text{at } h_i. \end{cases}$$

From here it follows that

$$\begin{aligned} \operatorname{tr} J(\varepsilon) &= 2 + \mathrm{O}(\varepsilon^{n/2+1}) \\ \det J(\varepsilon) &= 1 - (-1)^\sigma 4\pi^2 \left(\frac{1}{a_1}\right)^n \gamma n^4 \varepsilon^{n+1} + \mathrm{O}(\varepsilon^{n/2+1}). \end{aligned} \tag{13}$$

So we conclude from (13) that we have an elliptic orbit at $\{e_i\}_{i=0}^{n-1}$ and a hyperbolic orbit at $\{h_i\}_{i=0}^{n-1}$ with eigenvalues given by

$$\begin{aligned} \lambda_s &= 1 - \pi\sqrt{\left(\frac{1}{a_1}\right)^n \gamma n^4 \varepsilon^{(n+1)/2}} + \mathrm{O}(\varepsilon^{n/2+1}) \\ \lambda_u &= 1 + \pi\sqrt{\left(\frac{1}{a_1}\right)^n \gamma n^4 \varepsilon^{(n+1)/2}} + \mathrm{O}(\varepsilon^{n/2+1}). \end{aligned} \tag{14}$$

The local stable (unstable) manifold $W^{s(u)}_{\text{loc}}(0)$ of $(0,0)$ of $\widehat{f}_{\varepsilon,\gamma}$ is described by the following proposition, whose proof follows immediately from Proposition 1 and 2 of [Z].

PROPOSITION 9.— *There exist C_1, C_2 and ε_0 such that for $0 < \varepsilon \leq \varepsilon_0$, the local stable (unstable) manifolds $W^{s(u)}_{\text{loc}}(0)$ are given in $|\psi| \leq 3/4n$ by*

$$
\begin{aligned}
&W^{s(u)}_{\text{loc}}(0) = \operatorname{graph} g^{s(u)} \\
&g^s(\varepsilon,\psi) = -\frac{2}{n}\sqrt{\frac{\gamma}{a_1^{n+2}}}\,\sin(\pi n\psi) + u^s(\varepsilon,\psi)\,, \quad u^s(\varepsilon,0) = 0 \\
&g^u(\varepsilon,\psi) = \frac{2}{n}\sqrt{\frac{\gamma}{a_1^{n+2}}}\,\sin(\pi n\psi) + u^u(\varepsilon,\psi)\,, \quad u^u(\varepsilon,0) = 0
\end{aligned}
\tag{15}
$$

where $\left|u^{s(u)}(\varepsilon,\psi)\right| < C_1\varepsilon$, $\operatorname{Lip}\left(u^{s(u)}\right) < C_2\varepsilon$ and $u^{s(u)}(\varepsilon,0) = 0$.

From this proposition if follows that the $W^{s(u)}_{\text{loc}}(h_i)$ are the graphs of functions $g_1^{s(u)}$ defined on an interval with center at h_i and length equals $3\pi/4n$. To prove part (c) of Proposition 8, Let us show that $W^u_{\text{loc}}(h_i(e)) \cap W^s_{\text{loc}}(h_{i+1}(e)) \neq \emptyset$. We argue by contradiction. Observe that, since $\mathbb{S}^1$ is left invariant by $f_{\varepsilon,\gamma}$, the annulus is decomposed in two regions and the periodic orbit $\{h_i\}_0^{n-1}$ lies in one of these sides (fig. 1).

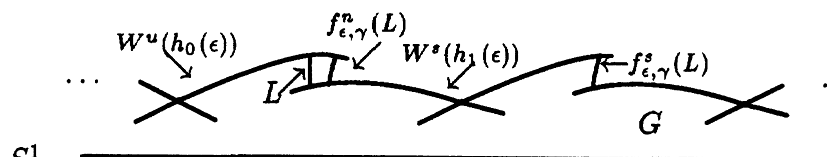

Fig. 1

Now following [Z], we build a curve C_0 in the annulus in the following way: the vertical line $(1/2n, x)$ intersects $W^u_{\text{loc}}(h_0(\varepsilon))$ and $W^s_{\text{loc}}(h_1(\varepsilon))$ in the points P and Q respectively (fig. 1);let C_0 be the path that goes from h_0 until P through $W^u_{\text{loc}}(h_0(\varepsilon))$, then follows by the vertical segment from P until Q and then continues from this point until h_1 through $W^s_{\text{loc}}(h_1(\varepsilon))$. Define now the curve C as being $\bigcup_0^{n-1} f_{\varepsilon,\gamma}(C_0)$. This curve is a non homotopically trivial Jordan curve. Let G be the region bounded by $\mathbb{S}^1$ and this curve. It is easy to see, using the properties of the stable and unstable manifolds described in Proposition 8, that $m(G) > m\big(f_{\varepsilon,\gamma}(G)\big)$ therefore contradicting the area-preserving property.

By (15) the angle between these manifolds at the intersection point goes to zero when ε goes to zero. □

3.2 Proof of Proposition 7

The proof of the proposition will be made through a sequence of steps that consist in making some reductions and perturbations. We dedicate one item to each one.

- We change coordinates with $h(\theta, r) = \big(\theta\, ,\, r - \Psi(\theta)\big) = (\overline{\theta}, \overline{r})$ so that $\overline{f}(\overline{\theta}, \overline{r}) = h \circ f \circ h^{-1}$ has $h(\Lambda) = \mathbb{S}^1$ as an invariant curve. Observe that $\overline{f}$ is C^∞ and

$$\big\|h(\theta, r)\big\|_{C^s} \leq 1 + \big\|\Psi\big\|_{C^s}$$
$$\big\|h^{-1}(\theta, r)\big\|_{C^s} \leq 1 + \big\|\Psi^{-1}\big\|_{C^s}\,,$$

 so if we prove the proposition for $\overline{f}$ then we will also have it proved for f.

- Thus we assume that $f(\mathbb{S}^1) = \mathbb{S}^1$ and $f \mid \mathbb{S}^1$ is conjugated to R_ω with ω an irrational number. Consider $f_\beta(\theta, r) = f(\theta, r) + (\beta, 0)$ then by [H] we can find $\beta_n \to 0$ with $n \to \infty$ such that $f_{\beta_n}(\mathbb{S}^1) = \mathbb{S}^1$ and $f_{\beta_n}\big|_{\mathbb{S}^1}$ has a rotation number $\omega_n = \omega + \beta_n$ satisfying a diophantine condition, and once more by [H] we know that there exists $h_n : \mathbb{S}^1 \hookleftarrow a\ C^\infty$ diffeomorphism, conjugating $f_{\beta_n}|_{\mathbb{S}^1}$ with R_{ω_n}. Consider $H_n(\theta, r) = \big(h_n(\theta)\, ,\, r/h_n'(\theta)\big)$, then $H_n^{-1} \circ f_{\beta_n} \circ H_n = \widehat{f}$ satisfies $\widehat{f}(\mathbb{S}^1) = \mathbb{S}^1$ and $\widehat{f}|_{\mathbb{S}^1} = R_{\omega_n}$. Also these changes of coordinates can be made uniformly in the sense that there is some constant $M_n > 0$ such that

$$\max\Big\{\big\|H_n(\theta, r)\big\|_{C^s}\, ,\, \big\|H_n^{-1}(\theta, r)\big\|_{C^s}\Big\} < M_n\,.$$

 So once more, it is enough to prove the proposition for this map.

- We assume there that $f(\mathbb{S}^1) = \mathbb{S}^1$ and $f\big|_{\mathbb{S}^1} = R_\omega$ with ω satisfying a diophantine condition. By Theorem 5, we can write after a change of coordinates

$$f(\theta, r) = \big(0 + \omega + a_1 r + \cdots + a_n r^n + \mathrm{O}(r^{n+1})\, ,\, r + \mathrm{O}(r^{n+1})\big)\,,$$

we may assume that $a_1 \neq 0$ unless we perturb f in such a way that the new f has $a_1 \neq 0$, even more we choose $a_1 > 0$ (in the case $a_1 < 0$ we take f^{-1}). After this we perturb once again so the rotation number of $f|_{\mathbb{S}^1}$ becomes rational. We apply now the Proposition 8 to get a sequence of maps $f_k \to f$ such that f_k has a hyperbolic periodic orbit $\{h_i(k)\}_{i=1}^n$ with $\psi_{h_i}(k) \in W^s_{\text{loc}}(h_i(k)) \cap W^u_{\text{loc}}(h_{i+1}(k))$, and the angle at point goes to zero as $k \to \infty$. Moreover, $h_i(k) = i/n$ and

$$\psi_{h_i}(k) \to \psi'_{h_i} \in \left(\frac{2i+1}{2n} - \overline{\delta}\,,\ \frac{2i+1}{2n} + \overline{\delta}\right) \times \{0\}\,.$$

So we can use the following lemma (see [N1]).

LEMMA.— *Let $\varepsilon > 0$ and $s \in \mathbb{N}$. There exists $C(s) > 0$ such that given δ and a linear subspace $H \subset \{v = (v_1, v_2) \mid |v_2| \leq C(s)\delta^{s-1}\varepsilon|v_1|\}$: there exists a C^s area-preserving diffeomorphism $\varphi : \mathbb{A} \hookleftarrow$ such that $\varphi(0) = 0$, $D\varphi\{v_2 = 0\} = H$ and $\varphi(\theta, r) = (\theta, r)$ for $\text{dist}\big((\theta, r), (0,0)\big) \geq \delta$ and $\|\varphi - \text{id}\|_{C^s} \leq \varepsilon$.*

So, we can get perturbations $\widetilde{f}_k$ of f_n with the property that $\widetilde{f}_k$ exhibits homoclinic tangencies and $\widetilde{f}_k \to f$. If the tangency it not quadratic, with a new perturbation, we make it quadratic. □

4. Proof of Theorem 1

Proof of Proposition 2

Let $\widetilde{\mathcal{U}}$ be an open neighborhood of f where the continuation of γ exists, i.e., for each $g \in \widetilde{\mathcal{U}}$ there exists an invariant curve γ_g such that the rotation number of $g \mid \gamma_g$ equals that of $f \mid \gamma$; this neighborhood is provided by KAM theory. Since f and γ are C^∞ we apply Theorem 6 and the remark which follows to conclude the existence of a subset $\mathcal{U}$ of $\widetilde{\mathcal{U}}$ for which the following property holds : for each $g \in \mathcal{U}$ such that g is a C^∞ map, the invariant curve γ_g prolongation of γ is also C^∞. Now Proposition 3 allows us to conclude that this neighborhood is an OSPHT. To see the existence of the residual set we observe first that, by the remark following Theorem 6, for each $g \in \widetilde{\mathcal{U}}$ there are lots of invariant curves, in particular γ_g is the limit of other invariant curves satisfying the *twist condition* and whose rotation numbers satisfy diophantine conditions. We also notice that each C^∞ map

f with an C^∞ invariant curve can be approximated by another one having an elliptic periodic orbit with arbitrary large period. This follows from the proof of Proposition 3. Now in $\mathcal{U}$ consider the subset $\mathcal{U}_m$ of all $g \in \mathcal{U}$ having some elliptic periodic orbit in the $1/m$-neighborhood for γ_g. This set is obvious open and $U_{m+1} \subset U_m$. Also each U_{m+1} is dense in U_m, because of the two previous observations. So the set $R = \bigcap U_m$ is a residual set satisfying the conclusion of Proposition 2, so we are done. □

Proof of Theorem 1

We approximate f by $\widetilde{f}$, a C^∞ map having a generic elliptic periodic orbit; it is a consequence of the proof of Proposition 3. Let $\mathcal{U}_1$ be a set containing $\widetilde{f}$ and for which this elliptic periodic point survives. Choose an invariant C^∞ curve of $\widetilde{f}$ associated to this elliptic periodic point. Observe that this curve is invariant by f^n where n is the period of the elliptic periodic point. By KAM theorem we have a subset $\mathcal{U}$ of $\mathcal{U}_1$, in which the curve survives. Now the remark after Proposition 2 allows us to conclude the proof. □

Acknowledgments

The authors would like to express their gratitude to IMPA/CNPq-Brazil, CONICIT-Venezuela and IVIC for their financial support during the realization of this work and to P. Duarte for helpful conversations on a first draft of this paper. We also would like to thank the referee for comments which helped to improve the paper presentation.

References

[Bo] BOST (J.) .— *Tores invariants des systèmes dynamiques hamiltoniens*, Astérisque **133-134** (1986), pp. 113-156.

[D] DUARTE (P.) .— *Plenty of Elliptic Islands for the Standard Family of Area Preserving Maps*, Ann. Inst. Henri-Poincaré **4.** (1994), pp. 359-409.

[Do] DOUADY (R.) .— *Stabilité ou instabilité des points fixes elliptiques*, Ann. École Normale Supérieure, 4-ième série, **21** (1988), pp. 1-46.

[H] HERMAN (M.) .— *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Publ. Math. I.H.E.S. **49** (1979), pp. 5-234.

[M] MOSER (J.) .— *Stable and Random Motions in Dynamical Systems*, Princeton Univ. Press, Ann. Math. Studies **77** (1973).

[MR] MORA (L.) and ROMERO (N.) .— *KAM-Sructure in Homoclinic Bifurcations*, preprint 1995.

[N1] NEWHOUSE (S.) .— *The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms*, Publ. Math. I.H.E.S. **50** (1979), pp. 101-151.

[N2] NEWHOUSE (S.) .— *Non-density of Axiom $A(a)$ on $\mathbb{S}^2$*, Proc. Symp. Pure Math., Amer. Math. Soc., **14** (1970), pp. 191-202.

[N3] NEWHOUSE (S.) .— *Quasi-elliptic periodic points in conservative dynamical systems*, American J. of Math. **99** (1977), pp. 1061-1087.

[SZ] SALAMON (D.) and ZENHDER (E.) .— *KAM theory in configuration space*, Comment. Math. Helvetici **64** (1989), pp. 84-132.

[Z] ZEHNDER (E.) .— *Homoclinic Points Near Elliptic Fixed Points*, Comm. Pure Appl. Math. **26** (1973), pp. 131-182.