Annales de la faculté des sciences de Toulouse

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Annales de la faculté des sciences de Toulouse 6^e série, tome 7, n° 1 (1998), p. 51-85

<http://www.numdam.org/item?id=AFST_1998_6_7_1_51_0>

Annales de la Faculté des Sciences de Toulouse Vol. VII, n° 1, 1998

Asymptotic expansion of the Bergman kernel for weakly pseudoconvex tube domains in $\mathbb{C}^2$ (*)

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RÉSUMÉ. — Dans cet article nous donnons un développement asymptotique du noyau de Bergman pour certains domaines de tube qui sont faiblement pseudo-convexes et de type fini dans $\mathbb{C}^2$.

Notre formule asymptotique montre que la singularité du noyau de Bergman à points faiblement pseudo-convexe est essentiellement représentée par deux variables; de plus, un certain éclatement réel est nécessaire pour comprendre cette singularité.

L'expression du développement asymptotique par rapport à chaque variable est analogue à celle de C. Fefferman dans le cas strictement pseudo-convexe.

Nous démontrons aussi un même résultat pour le noyau de Szegö.

ABSTRACT. — In this paper we give an asymptotic expansion of the Bergman kernel for certain weakly pseudoconvex tube domains of finite type in $\mathbb{C}^2$. Our asymptotic formula asserts that the singularity of the Bergman kernel at weakly pseudoconvex points is essentially expressed by using two variables; moreover certain real blowing-up is necessary to understand its singularity. The form of the asymptotic expansion with respect to each variable is similar to that in the strictly pseudoconvex case due to C. Fefferman. We also give an analogous result in the case of the Szegö kernel.

AMS Classification : 32A40, 32F15, 32H10.

KEY-WORDS : Bergman kernel, Szergö kernel, weakly pseudoconvex, of finite type, tube, asymptotic expansion, real blowing-up, admissible approach region.

(*) Reçu le 3 décembre 1997, accepté le mars 1998

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1. Introduction

The purpose of this paper is to give an asymptotic expansion of the Bergman kernel for certain class of weakly pseudoconvex tube domains of finite type in $\mathbb{C}^2$. We also give an analogous result for the Szegö kernel for the same class of tube domains.

Let Ω be a domain with smooth boundary in $\mathbb{C}^n$. The Bergman space $B(\Omega)$ is the subspace of $L^2(\Omega)$ consisting of holomorphic L^2-functions on Ω. The Bergman projection is the orthogonal projection $\mathbb{B} : L^2(\Omega) \to B(\Omega)$. We can write $\mathbb{B}$ as an integral operator

$$\mathbb{B}f(z) = \int_\Omega K(z,w)f(w)\,\mathrm{d}V(w) \quad \text{for } f \in L^2(\Omega),$$

where $K : \Omega \times \Omega \to \mathbb{C}$ is the *Bergman kernel* of the domain Ω and $\mathrm{d}V$ is the Lebesgue measure on Ω. In this paper we restrict the Bergman kernel on the diagonal of the domain and study the boundary behavior of $K(z) = K(z,z)$.

Although there are many explicit computations for the Bergman kernels of specific domains ([2], [9], [33], [11], [26], [6], [13], [22], [23], [34], [3]), it seems difficult to express the Bergman kernel in closed form in general. Therefore appropriate approximation formulas are necessary to know the boundary behavior of the Bergman kernel. From this viewpoint the following studies have great success in the case of strictly pseudoconvex domains. Assume Ω is a bounded strictly pseudoconvex domain. Hörmander [32] showed that the limit of $K(z)\,d(z-z^0)^{n+1}$ at $z^0 \in \partial\Omega$ equals the determinant of the Levi form at z^0 times $n!/4\pi^n$, where d is the Euclidean distance. Moreover Diederich ([14], [15]) obtained analogous results for the first and mixed second derivatives of $K(z)$. Fefferman [21] and Boutet de Monvel and Sjöstrand [7] gave the following asymptotic expansion of the Bergman kernel of Ω:

$$K(z) = \frac{\varphi(z)}{r(z)^{n+1}} + \psi(z)\log r(z), \tag{1.1}$$

where $r \in C^\infty(\overline{\Omega})$ is a defining function of Ω (i.e., $\Omega = \{r > 0\}$ and $|dr| > 0$ on $\partial\Omega$) and φ, $\psi \in C^\infty(\overline{\Omega})$ can be expanded asymptotically with respect to r.

On the other hand, there are not so strong results in the weakly pseudoconvex case. Let us recall important studies in this case. Many estimates of the size of the Bergman kernel have been obtained ([29], [52], [19], [8], [45], [30], [16], [31], [46], [10], [53], [25], [47]). In particular Catlin [8] gave a complete estimate from above and below for domains of finite type in $\mathbb{C}^2$. Recently Boas, Straube and Yu [4] computed nontangential boundary limit in the sense of Hörmander for bounded domains which are known variously as h-extendible [56] and semiregular [17]. This class contains many kinds of domains of finite type. Diederich and Herbort [18] gave an alternative proof of the result of [4]. Though the above studies about *estimate* and *boundary limit* are detailed, a clear asymptotic formula like (1.1) is yet to be obtained. In this paper we give an asymptotic expansion of the Bergman kernel for certain class of weakly pseudoconvex tube domains of finite type in $\mathbb{C}^2$. Gebelt [24] and Haslinger [28] recently computed certain asymptotic formulas for the special cases, but the method of our expansion is different from theirs.

Our main idea to analyse the Bergman kernel is to introduce certain real blowing-up. Let us briefly indicate how this blowing-up works for the Bergman kernel at a weakly pseudoconvex point z^0. Since the set of strictly pseudoconvex points are dense on the boundary of the domain of finite type, it is a serious problem to resolve the difficulty caused by the influence of strictly pseudoconvex points near z^0. This problem can be avoided by restricting the argument on a non-tangential cone in the domain ([29], [19], [30], [16], [4]). We surmount the difficulty in the case of certain class of tube domains in the following. By blowing up at the weakly pseudoconvex point z^0, we introduce two new variables. The Bergman kernel can be developped asymptotically in terms of these variables in the sense of Sibuya [55] (see also Majima [44]). The expansion, regarded as a function of the first variable, has the form of Fefferman's expansion (1.1), and hence it reflects the strict pseudoconvexity. The characteristic influence of the weak pseudoconvexity appears in the expansion with respect to second variables. Though the form of this expansion is similar to (1.1), we must use m-th root of the defining function, i.e. $r^{1/m}$, as the expansion variable when z^0 is the type $2m$. We remark that a similar situation occurs in the case of another class of domains in [24].

Our method of computation is based on the studies [21], [7], [5] and [51]. In [21], Fefferman selected the ball $\left\{z \in \mathbb{C}^n \mid \sum_{j=1}^n |z_j|^2 < 1\right\}$ as a model domain to approximate strictly pseudoconvex domain. We consider $\left\{(z_1, z_2) \in \mathbb{C}^2 \mid \Im z_2 > g[\Im z_1]^{2m}\right\}$ with $g > 0$, $m = 2, 3, \ldots$ as a *model* domain. The starting point of our analysis is certain integral representation in [40], [48], [54] and [27]. After introducing the blowing-up to this representation, we compute the asymptotic expansion by using the stationary phase method. For the above computation, it is necessary to localize the Bergman kernel near a weakly pseudoconvex point. This localization can be obtained in a fashion similar to the case of some class of Reinhardt domains ([5], [51]).

This paper is organized as follows. Our main theorem is established in Section 2. The next three sections prepare the proof of the theorem. First an integral representation is introduced, which is a clue to our analysis in Section 3. Second the usefulness of our blowing-up is shown by using a simple tube domain $\{(z_1, z_2) \in \mathbb{C}^2 \mid \Im z_2 > [\Im z_1]^{2m}\}$, $m = 2, 3, \ldots$, in Section 4. This domain is considered to be a model domain for more general case. Third a localization lemma is established in Section 5, which is necessary to the computation in the proof of our theorem. Our main theorem is proved in Section 6. After an appropriate localization (§ 6.1) and the blowing-up at a weakly pseudoconvex point, an easy computation shows that certain two propositions are sufficient to prove our theorem (§ 6.2). In order to prove these propositions, we compute the asymptotic expansion of two functions by using the stationary phase method (§ 6.3, 6.4). The rest of Section 6 (§ 6.5, 6.6) is devoted to proving two propositions. In Section 7 an analogous theorem about the Szegö kernel is established.

2. Statement of main result

Given a function $f \in C^\infty(\mathbb{R})$ satisfying that:

$$\begin{aligned}&f'' \geq 0 \text{ on } \mathbb{R} \text{ and } f \text{ has the form in some neighborhood of } 0:\\ &f(x) = x^{2m} g(x) \text{ where } m = 2, 3, \ldots, g(0) > 0 \text{ and } xg'(x) \leq 0\,.\end{aligned} \tag{2.1}$$

Let $\omega_f \subset \mathbb{R}^2$ be a domain defined by $\omega_f = \{(x, y) \mid y > f(x)\}$. Let $\Omega_f \subset \mathbb{C}^2$ be the tube domain over ω_f, i.e.,

$$\Omega_f = \mathbb{R}^2 + i\omega_f\,.$$

Let $\pi : \mathbb{C}^2 \to \mathbb{R}^2$ be the projection defined by $\pi(z_1, z_2) = (\Im z_1, \Im z_2)$. Set $\mathrm{O} = (0, 0)$. It is easy to check that Ω_f is a pseudoconvex domain; moreover $z^0 \in \partial\Omega_f$, with $\pi(z^0) = \mathrm{O}$, is a weakly pseudoconvex point of type $2m$ (or $2m - 1$) in the sense of Kohn or D'Angelo and $\partial\Omega_f \setminus \pi^{-1}(\mathrm{O})$ is strictly pseudoconvex near z^0.

Now we introduce the transformation σ, which plays a key role on our analysis. Set $\Delta = \{(\tau, \varrho) \mid 0 < \tau \leq 1\,,\ \varrho > 0\}$. The transformation σ: $\overline{\omega_f} \to \overline{\Delta}$ is defined by

$$\sigma : \begin{cases} \tau = \chi\left(1 - \dfrac{f(x)}{y}\right) \\ \varrho = y\,, \end{cases} \tag{2.2}$$

where the function $\chi \in C^\infty([0\,,1))$ satisfies the conditions:

$$\chi'(u) \geq \frac{1}{2} \text{ on } [0\,,1] \text{ and } \chi(u) = \begin{cases} u & \text{for } 0 \leq u \leq \dfrac{1}{3} \\ 1 - (1-u)^{1/2m} & \text{for } 1 - \dfrac{1}{3^{2m}} \leq u \leq 1. \end{cases}$$

Then $\sigma \circ \pi$ is the transformation from $\overline{\Omega}$ to $\overline{\Delta}$.

The transformation σ induces an isomorphism of $\omega_f \cap \{x \geq 0\}$ (or $\omega_f \cap \{x \leq 0\}$) on to Δ. The boundary of ω_f is transfered by σ in the following:

$$\sigma\big((\partial\omega_f)\backslash\{\mathrm{O}\}\big) = \{(0, \varrho) \mid \varrho > 0\} \quad \text{and} \quad \sigma^{-1}\big(\{(\tau, 0) \mid 0 \leq \tau \leq 1\}\big) = \{\mathrm{O}\}\,.$$

This indicates that σ is the real blowing-up of $\partial\omega_f$ at O, so we may say that $\sigma \circ \pi$ is the blowing-up at the weakly pseudoconvex point z^0. Moreover σ patches the coordinates (τ, ϱ) on ω_f, which can be considered as the polar coordinates around O. We call τ the angular variable and ϱ the radial variable, respectively. Note that if z approaches some strictly (resp. weakly) pseudoconvex points, $\tau(\pi(z))$ (resp. $\varrho(\pi(z))$) tends to 0 on the coordinates (τ, ϱ).

The following theorem asserts that the singularity of the Bergman kernel of Ω_f at z^0, with $\pi(z^0) = \mathrm{O}$, can be essentially expressed in terms of the polar coordinates (τ, ϱ).

THEOREM 2.1.— *The Bergman kernel of Ω_f has the form in some neighborhood of z^0:*

$$K(z) = \frac{\Phi(\tau, \varrho^{1/m})}{\varrho^{2+1/m}} + \widetilde{\Phi}(\tau, \varrho^{1/m}) \log \varrho\,, \tag{2.3}$$

where $\Phi \in C^\infty((0\,,1] \times [0\,,\varepsilon))$ and $\widetilde{\Phi} \in C^\infty([0\,,1] \times [0\,,\varepsilon))$ with some $\varepsilon > 0$.

Moreover Φ is written in the form on the set $\{\tau > \alpha\varrho^{1/2m}\}$ with some $\alpha > 0$: for every nonnegative integer μ_0

$$\Phi(\tau, \varrho^{1/m}) = \sum_{\pi=0}^{\mu_0} c_\mu(\tau)\varrho^{\mu/m} + R_{\mu_0}(\tau, \varrho^{1/m})\varrho^{\mu_0/m+1/2m} \tag{2.4}$$

where

$$c_\mu(\tau) = \frac{\varphi_\mu(\tau)}{\tau^{3+2\mu}} + \psi_\mu(\tau)\log\tau\,, \tag{2.5}$$

for φ_μ, $\psi_\mu \in C^\infty([0\,,1])$, φ_0 is positive on $[0\,,1]$ and R_{μ_0} satisfies

$$\left|R_{\mu_0}(\tau, \varrho^{1/m})\right| \le C_{\mu_0}\left[\tau - \alpha\varrho^{1/2m}\right]^{-4-2\mu_0}$$

for some positive constant C_{μ_0}.

Let us describe the asymptotic expansion of the Bergman kernel K in more detail. Considering the meaning of the variables τ, ϱ, we may say that each expansion with respect to τ or $\varrho^{1/m}$ is induced by the strict or weak pseudoconvexity, respectively. Actually the expansion (2.5) has the same form as the one of Fefferman (1.1). By (2.4)-(2.5), in order to see the characteristic influence of the weak pseudoconvexity on the singularity of the Bergman kernel K, it is sufficient to argue about K on the region

$$\mathcal{U}_\alpha = \left\{z \in \mathbb{C}^2 \mid \tau \circ \pi(z) > \alpha^{-1}\right\}, \quad \alpha > 1\,.$$

This is because $\mathcal{U}_\alpha$ is the *widest* region where the coefficients $c_\mu(\tau)$'s are bounded. We call $\mathcal{U}_\alpha$ an admissible approach region of the Bergman kernel of Ω_f at z^0. The region $\mathcal{U}_\alpha$ seems deeply connected with the admissible approach regions studied in [41], [42], [1], [43], etc. We remark that on the region $\mathcal{U}_\alpha$, the exchange of the expansion variable $\varrho^{1/m}$ for $r^{1/m}$, where r is a defining function of Ω_f (e.g., $r(x,y) = y - f(x)$), gives no influence on the form of the expansion on the region $\mathcal{U}_\alpha$.

Now let us compare the asymptotic expansion (2.3) on $\mathcal{U}_\alpha$ with Fefferman's expansion (1.1). The essential difference between them only appears in the expansion variable (i.e., $r^{1/m}$ in (2.3) and r in (1.1)). A similar phenomenon occurs in subelliptic estimates for the $\overline{\partial}$-Neumann problem. As is well-known, the finite-type condition is equivalent to the condition that a subelliptic estimate holds, i.e.,

$$|||\phi|||_\varepsilon^2 \le C\left(\left\|\overline{\partial}\phi\right\|^2 + \left\|\overline{\partial}^*\phi\right\|^2 + \|\phi\|^2\right), \quad \varepsilon > 0$$

(refer to [39] for the details). Let ε_0 be the best possible order of subellipticity. In two dimensional case, $\varepsilon_0 = 1/2$ in the strictly pseudoconvex case and $\varepsilon_0 = 1/2m$ in the weakly pseudoconvex case of type $2m$. The difference between these two cases only appears in the value of ε. From this viewpoint, our expansion (2.3) seems to be a natural generalization of Fefferman's expansion (1.1) in the strictly pseudoconvex case.

Remarks

1) From the viewpoint of the studies ([4], [17]), let us consider the limit of $\varrho^{2+1/m}K(z)$ at z^0. This limit on a nontangential cone is

$$c_0(1) = \varphi_0(1) = \frac{g(0)^{1/m}}{(4\pi)^2}\,\Gamma\left(2+\frac{1}{m}\right)\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-w^{2m}+vw}\,\mathrm{d}w\right)^{-1}\mathrm{d}v\,.$$

The above integral seems impossible to be changed into simpler form (see [39]). We remark that if we do not restrict the region for the approach, the above limit is $c_0(\tau)$, which depends on the angular variable τ.

2) The idea of the blowing-up σ is originally introduced in the study of the Bergman kernel of the domain $\left\{z \in \mathbb{C}^n \mid \sum_{j=1}^n |z_j|^{2m_j} < 1\right\}$ ($m_j \in \mathbb{N}, m_n \neq 1$) in [34]. Since the above domain has high homogeneity, the asymptotic expansion with respect to the radial variable does not appear (see also Sect. 4). Recently the author [37] also gave an analogous result in the case of the decoupled tube domain $\left\{z \in \mathbb{C}^{n+1} \mid \Im z_{n+1} > \sum_{j=1}^n [\Im z_j]^{2m_j}\right\}$ ($m_j \in \mathbb{N}$, $m_n \neq 1$).

3) If we consider the Bergman kernel on the region $\mathcal{U}_\alpha$, then we can remove the condition $xg'(x) \leq 0$ in (2.1). Namely even if the condition $xg'(x) \leq 0$ is not satisfied, we can still obtain (2.3)-(2.4) in the theorem where c_μ's are bounded on $\mathcal{U}_\alpha$. But, for our method in this paper, the condition $xg'(x) \leq 0$ is necessary to obtain the asymptotic expansion with respect to τ.

4) From the definition of asymptotic expansion of functions of several variables in [55] and [44], the expansion in the theorem is not complete. In order to get a complete asymptotic expansion, we must take a further blowing-up at the point $(\tau, \varrho) = (0,0)$. The real blowing-up $(\tau, \varrho) \mapsto (\tau, \varrho\tau^{-2m})$ is sufficient for this purpose.

Notation. — In this paper, we use c, c_j or C for various constants without further comment.

3. Integral representation

In this section we give an integral representation of the Bergman kernel, which is a clue to our analysis. Korányi [40], Nagel [48], Saitoh [54] and Haslinger [27] obtain similar representations of Bergman kernels or Szegö kernels for certain tube domains.

In this section, we assume that $f \in C^\infty(\mathbb{R})$ is a function such that $f(0) = 0$ and $f''(x) \geq 0$. The tube domain $\Omega_f \subset \mathbb{C}^2$ is defined as in Section 2. Let Λ, $\Lambda^* \subset \mathbb{R}^2$ be the cones defined by

$$\Lambda = \{(x,y) \mid (tx,ty) \in \omega_f \text{ for any } t > 0\}$$
$$\Lambda^* = \{(\zeta_1,\zeta_2) \mid x\zeta_1 + y\zeta_2 > 0 \text{ for any } (x,y) \in \Lambda\}$$

respectively. We call Λ^* the dual cone of ω_f. Actually Λ^* can be computed explicitly:

$$\Lambda^* = \{(\zeta_1,\zeta_2) \mid -R^-\zeta_2 < \zeta_1 < R^+\zeta_2\},$$

where $(R^\pm)^{-1} = \lim_{x\to\mp\infty} f(x)|x|^{-1} > 0$, respectively. We allow that $R^\pm = \infty$. If $\lim_{|x|\to\infty} f(x)|x|^{-1-\varepsilon} > 0$ with some $\varepsilon > 0$, then $R^\pm = \infty$, i.e., $\Lambda^* = \{(\zeta_1,\zeta_2) \mid \zeta_2 > 0\}$.

The Bergman kernel of Ω_f is expressed in the following. Set $(x,y) = (\Im z_1, \Im z_2)$;

$$K(z) = \frac{1}{(4\pi)^2} \iint_{\Lambda^*} e^{-x\zeta_1 - y\zeta_2} \frac{\zeta_2}{\mathrm{D}(\zeta_1,\zeta_2)}\, d\zeta_1\, d\zeta_2\,, \tag{3.1}$$

where

$$\mathrm{D}(\zeta_1,\zeta_2) = \int_{-\infty}^{\infty} e^{-\xi\zeta_1 - f(\xi)\zeta_2}\, d\xi\,. \tag{3.2}$$

The above representation can be obtained by applying the argument of Korányi [40] and Saitoh [54] to our case, so we omit the proof.

4. Analysis on a model domain

Let $\omega_0 \cup \mathbb{R}^2$ be a domain defined by $\omega_0 = \{(x,y) \mid y > gx^{2m}\}$, where $m = 2, 3, \ldots$ and $g > 0$. Set $\Omega_0 = \mathbb{R}^2 + i\omega_0$. F. Haslinger [28] computes the asymptotic expansion of the Bergman kernel of Ω_0 (not only on the diagonal but also off the diagonal). In his result Fefferman's expansion only appears.

In this paper, we consider Ω_0 as a model domain for the study of singularity of the Bergman kernel for more general domains. The following proposition shows the reason why we take Ω_0 as a model domain. Set $(x, y) = (\Im z_1, \Im z_2)$.

PROPOSITION 4.1.— *The Bergman kernel K of Ω_0 has the form:*

$$K(z) = \frac{\Phi(\tau)}{\varrho^{2+1/m}}, \tag{4.1}$$

where $\tau = \chi(1 - gx^{2m}y^{-1})$, $\varrho = y$ (see (2.2)) and

$$\Phi(\tau) = \frac{\varphi(\tau)}{\tau^3} + \psi(\tau)\log\tau,$$

with φ, $\psi \in C^\infty([0, 1])$ and φ is positive on $[0, 1]$.

Proof.— Normalizing the integral representation (3.1) and introducing the variables $t = g^{1/2m}xy^{-1/2m}$, $\varrho = y^{1/2m}$, we have (4.1) where

$$\Phi(\tau) = \frac{2m}{(4\pi)^2}\, g^{1/m} \int_0^\infty e^{-s^{2m}} L(ts)s^{4m+1}\, ds, \tag{4.2}$$

$$L(u) = \int_{-\infty}^{\infty} e^{uv} \frac{1}{\phi(v)}\, dv,$$
$$\phi(v) = \int_{-\infty}^{\infty} e^{-w^{2m}+vw}\, dw.$$

It turns out from (4.2) and the definition of τ that $\Phi \in C^\infty((0, 1])$.

Now let $\widehat{\Phi}$ be defined by

$$\widehat{\Phi}(\tau) = \frac{2m}{(4\pi)^2} \int_1^\infty e^{-s^{2m}} L(ts)s^{4m+1}\, ds. \tag{4.3}$$

If we admit Lemma 6.2 in Paragraph 6.4 below, we have

$$L(u) = u^{2m-2}\, e^{u^{2m}}\, \widetilde{L}(u)\,, \tag{4.4}$$

where $\widetilde{L}(u) \sim \sum_{j=0}^{\infty} c_j u^{-2mj}$ as $u \to \infty$. Substituting (4.4) into (4.3), we have

$$\begin{aligned}\widehat{\Phi}(\tau) &= \frac{2m}{(4\pi)^2}\, t^{2m-2} \int_1^{\infty} e^{-[1-t^{2m}]s^{2m}}\, \widetilde{L}(ts) s^{6m-1}\, \mathrm{d}s \\ &= \frac{1}{(4\pi)^2} \int_1^{\infty} e^{-\chi^{-1}(\tau)\sigma}\, \widehat{L}(\tau,\sigma)\sigma^2\, \mathrm{d}\sigma\,.\end{aligned}$$

Since $\widehat{L}(\tau,\sigma) \sim \sum_{j=0}^{\infty} c_j(\tau)\sigma^{-j}$ as $\sigma \to \infty$ for $c_j \in C^{\infty}([0\,,\,1])$, we have

$$\widehat{\Phi}(\tau) = \frac{\widehat{\varphi}(\tau)}{\tau^3} + \widehat{\psi}(\tau)\log\tau\,,$$

with $\widehat{\varphi}$, $\widehat{\psi} \in C^{\infty}([0\,,\,1])$ and $\widehat{\varphi}$ is positive on $[0\,,\,1]$.

Finally since the difference between Φ and $\widehat{\Phi}$ is smooth on $[0\,,\,1]$, we can obtain Proposition 4.1. □

5. Localization lemma

In this section, we prepare a lemma, which is necessary for the proof of Theorem 2.1. This lemma shows that the singularity of the Bergman kernel for certain class of domains is determined by the local information about the boundary. The method of the proof is similar to the case of some class of Reinhardt domains ([5], [51]). Throughout this section, j stands for 1 or 2.

Let f_1, $f_2 \in C^{\infty}(\mathbb{R})$ be functions such that $f_j(0) = f_j'(0) = 0$, $f_j'' \geq 0$ on $\mathbb{R}$ and $f_1(x) = f_2(x)$ on $|x| < \delta$. Let $\omega_j \subset \mathbb{R}^2$ be a domain defined by $\omega_j = \{(x,y) \mid y > f_j(x)\}$. Set $\Omega_j = \mathbb{R}^2 + i\omega_j \subset \mathbb{C}^2$.

LEMMA 5.1.— *Let K_j be the Bergman kernels of Ω_j for $j = 1, 2$, respectively. Then we have*

$$K_1(z) - K_2(z) \in C^{\omega}(U)\,,$$

where U is some neighborhood of z^0.

Proof. — Let Λ_j^* be the dual cone of ω_j, i.e.,

$$\Lambda_j^* = \left\{(\zeta_1, \zeta_2) \mid -R_j^- \zeta_2 < \zeta_1 < R_j^+ \zeta_2\right\},$$

where $\left(R_j^{\pm}\right)^{-1} = \lim_{x \to \mp\infty} f(x)\left|x\right|^{-1}$, respectively (Sect. 3).
Let $K_j[\Delta](x,y)$ be defined by

$$K_j[\Delta](x,y) = \frac{1}{(4\pi)^2} \iint_\Delta e^{-y\zeta_2 - x\zeta_1} \frac{\zeta_2}{\mathrm{D}_j(\zeta_1,\zeta_2)} \, \mathrm{d}\zeta_1 \, \mathrm{d}\zeta_2 ,$$

where $\Delta \subset \mathbb{R}^2$ and $\mathrm{D}_j(\zeta_1,\zeta_2) = \int_{-\infty}^{\infty} e^{-\zeta_2 f_j(\xi) - \zeta_1 \xi} \, \mathrm{d}\xi$.
Set $\Lambda_\varepsilon = \left\{(\zeta_1,\zeta_2) \mid |\zeta_1| < \varepsilon\zeta_2\right\}$, where $\varepsilon > 0$ is small. Now the following claims (i), (ii) imply Lemma 5.1. Set $\mathrm{O} = (0,0)$;

(i) $K_j[\Lambda_j^*] \equiv K_j[\Lambda_\varepsilon]$ modulo $C^\omega\left(\{\mathrm{O}\}\right)$ for any $\varepsilon > 0$,

(ii) $K_1[\Lambda_{\varepsilon_0}] \equiv K_2[\Lambda_{\varepsilon_0}]$ modulo $C^\omega\left(\{\mathrm{O}\}\right)$ for some $\varepsilon_0 > 0$.

In fact if we substitute $(x,y) = (\Im z_1, \Im z_2)$, then $K_1 = K_1[\Lambda_1^*] \equiv K_1[\Lambda_{\varepsilon_0}] \equiv K_2[\Lambda_{\varepsilon_0}] \equiv K_2[\Lambda_2^*] = K_2$ modulo $C^\omega\left(\{z^0\}\right)$.

Let us show the above claims.

(i) Set $\Lambda_\varepsilon^{\pm} = \left\{(\zeta_1,\zeta_2) \mid 0 < \varepsilon\zeta_2 < \pm\zeta_1 < R_j^{\pm}\zeta_2\right\}$, respectively. Since $K_j[\Lambda_j^*] - K_j[\Lambda_\varepsilon] = K_j[\Lambda_\varepsilon^+] + K_j[\Lambda_\varepsilon^-]$, it is sufficient to show $K_j[\Lambda_\varepsilon^{\pm}] \in C^\omega\left(\{\mathrm{O}\}\right)$. We only consider the case of $K_j[\Lambda_\varepsilon^+]$. Changing the integral variables, we have

$$K_j[\Lambda_\varepsilon^+](x,y) = \frac{1}{(4\pi)^2} \int_0^\infty \int_\varepsilon^{R_j^+} H_j(\zeta,\eta;x,y) \, \mathrm{d}\zeta \, \mathrm{d}\eta ,$$

where

$$H_j(\zeta,\eta;x,y)\,(= H_j) = e^{-y\eta + x\eta\zeta} \frac{\eta^2}{E_j(\zeta,\eta)} ,$$

$$E_j(\zeta,\eta) = \int_{-\infty}^{\infty} e^{-\eta(f_j(\xi) - \zeta\xi)} \to \xi .$$

It is an important remark that $K_j[\Lambda_\varepsilon^+]$ is real analytic on the region where H_j is integrable on $\left\{(\zeta,\eta) \mid \zeta > \varepsilon ,\ \eta > 0\right\}$.

If we take $\delta_1 > 0$ such that $\left|f_j(\xi)\xi^{-1}\right| < (1/2)\varepsilon$ if $|\xi| < \delta_1$, then we have

$$\begin{aligned} E_j(\zeta,\eta) &\geq 2\int_0^{\delta_1} e^{\eta\xi(\zeta - f(\xi)\xi^{-1})}\,\mathrm{d}\xi \\ &\geq 2\int_0^{\delta_1} e^{(1/2)\varepsilon\eta\xi}\,\mathrm{d}\xi \geq \frac{4}{\varepsilon\eta}\left(e^{(1/2)\delta_1\varepsilon\eta} - 1\right). \end{aligned} \tag{5.1}$$

By (5.1), we have $H_j(\zeta,\eta;x,y) \leq C\eta\, e^{-(y - x\zeta + (1/2)\delta_1\varepsilon)\eta}$ for $\eta \geq 1$. This inequality implies that if $x < 0$ and $y > -(1/2)\delta_1\varepsilon$, then H_j is integrable on $\{(\zeta,\eta) \mid \zeta > \varepsilon\,,\ \eta > 0\}$. Thus $K_j[\Lambda_\varepsilon^+]$ is real analytic on

$$\omega_j \cup \left\{(x,y) \mid x < 0\,,\ y > -\frac{1}{2}\delta_1\varepsilon\right\} =: \omega_j^+\,.$$

By regarding x, y as two complex variables, $K_j[\Lambda_\varepsilon^+](x,y)$ is holomorphic on $\omega_j^+ + i\mathbb{R}^2$, so the shape of ω_j^+ implies that $K_j[\Lambda_\varepsilon^+]$ can be extended holomorphically to a region containing some neighborhood of $\{\mathrm{O}\} + i\mathbb{R}^2$. Consequently we have $K_j[\Lambda_\varepsilon^+] \in C^\omega(\{\mathrm{O}\})$.

(ii) Changing the integral variables, we have

$$K_1[\Lambda_\varepsilon](x,y) - K_2[\Lambda_\varepsilon](x,y) = \frac{1}{(4\pi)^2}\int_0^\infty\int_{-\varepsilon}^{\varepsilon}(H_1 - H_2)\,\mathrm{d}\zeta\,\mathrm{d}\eta\,.$$

We remark that $K_1[\Lambda_\varepsilon] - K_2[\Lambda_\varepsilon]$ is real analytic on the region where $H_1 - H_2$ is integrable on $\{(\zeta,\eta) \mid |\zeta| < \varepsilon\,,\ \eta > 0\}$. To find a positive number ε_0 satisfying (ii), let us consider the integrability of

$$|H_1 - H_2| = \eta^2\, e^{-y\eta + x\eta\zeta}\,\frac{\left|E_2(\zeta,\eta) - E_1(\zeta,\eta)\right|}{\left|E_1(\zeta,\eta)\cdot E_2(\zeta,\eta)\right|}\,. \tag{5.2}$$

First we give an estimate of $\left|E_2(\zeta,\eta) - E_1(\zeta,\eta)\right|$. Let $\varepsilon_1 > 0$ be defined by $\left|f_j(\xi)\xi^{-1}\right| \geq \varepsilon_1$ if $|\xi| \geq \delta$. If $|\zeta| \leq \varepsilon_1/2$, then

$$\int_{|\xi|\geq\delta} e^{-\eta\xi(f_j(\xi)\xi^{-1} - \zeta)}\,\mathrm{d}\xi \leq 2\int_\delta^\infty e^{-(1/2)\varepsilon_1\eta\xi}\,\mathrm{d}\xi \leq \frac{4}{\varepsilon_1\eta}\,e^{-(1/2)\delta\varepsilon_1\eta}\,. \tag{5.3}$$

Therefore (5.3) implies

$$\left|E_2(\zeta,\eta) - E_1(\zeta,\eta)\right| \leq \sum_{j=1}^{2}\int_{|\xi|\geq\delta} e^{-\eta(f_j(\xi) - \zeta\xi)}\,\mathrm{d}\xi \leq \frac{8}{\varepsilon_1\eta}\,e^{-(1/2)\delta\varepsilon_1\eta}\,. \tag{5.4}$$

Second, we give an estimate of $E_j(\zeta,\eta)$. By Taylor's formula, we can choose $\varepsilon_2 > 0$ satisfying the following. If $|\zeta| < \varepsilon_2$, then there is a function $\alpha_j(\zeta) = \alpha_j$ ($\alpha_j(0) = 0$) such that $f_j'(\alpha_j) = \zeta$ and moreover there is a bounded function $R_j(\zeta,\xi)$ on $[-\varepsilon_2\,,\,\varepsilon_2] \times [-\xi_0\,,\,\xi_0]$, with some $\xi_0 > 0$, such that

$$F_j(\xi)\,(:= f_j(\xi) - \zeta\xi) = F_j(\alpha_j) + R_j(\zeta, \xi - \alpha_j)(\xi - \alpha_j)^2$$

with $F_j(\alpha_j) \le 0$. Then if $|\zeta| < \varepsilon_2$, we have

$$\begin{aligned} E_j(\zeta,\eta) = \int_{-\infty}^{\infty} e^{-\eta F_j(\xi)}\,\mathrm{d}\xi &\ge e^{-\eta F_j(\alpha_j)} \int_{-\xi_0}^{\xi_0} e^{-\eta R_j(\zeta,\xi)\xi^2}\,\mathrm{d}\xi \\ &\ge \frac{C}{\sqrt{\eta}}\, e^{-\eta F_j(\alpha_j)} \ge \frac{C}{\sqrt{\eta}}\,. \end{aligned} \tag{5.5}$$

Now we set $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$. Then by putting (5.2), (5.4) and (5.5) together, we have

$$|H_1 - H_2| \le C\eta^2\, e^{-\eta(y - x\zeta + (1/2)\delta\varepsilon_1)} \quad \text{for } |\zeta| < \varepsilon_0\,,\ \eta > 0\,.$$

This inequality implies that if $\varepsilon_0|x| - y - (1/2)\delta\varepsilon_1 > 0$, then $|H_1 - H_2|$ is integrable on $\{(\zeta,\eta) \mid |\zeta| < \varepsilon_0\,,\ \eta > 0\}$. Hence $K_1[\Lambda_{\varepsilon_0}] - K_2[\Lambda_{\varepsilon_0}]$ is real analytic on the region $\{(x,y) \mid \varepsilon_0|x| - y - (1/2)\delta\varepsilon_1 > 0\}$, which contain $\{O\}$.

This completes the proof of Lemma 5.1. □

6. Proof of Theorem 2.1

In this section we give a proof of Theorem 2.1. The definitions of f, ω_f and Ω_f are given as in Section 2.

6.1 Localization

From the previous section, it turns out that the singularity of the Bergman kernel of Ω_f at z^0 is determined by the local information about $\partial\Omega_f$ near z^0. Thus we construct an appropriate domain whose boundary coincides $\partial\Omega_f$ near z^0 for the computation below.

We can easily construct a function $\widetilde{g} \in C^\infty(\mathbb{R})$ such that

$$\widetilde{g}(x) = \begin{cases} g(x) & \text{for } |x| \le \delta \\ \frac{9}{10} g(0) & \text{for } |x| \ge 1 \end{cases} \tag{6.1}$$

and

$$0 \le -x\widetilde{g}'(x), \quad \left|x^2 \widetilde{g}''(x)\right| < \frac{1}{5} g(0) \quad \text{for } x \in \mathbb{R}, \tag{6.2}$$

for some small positive constant $\delta < 1$. Note that

$$\frac{9}{10} g(0) \le \widetilde{g}(x) \le g(0).$$

Set $\widetilde{f}(x) = x^{2m}\widetilde{g}(x)$ and $\omega_{\widetilde{f}} = \left\{(x,y) \in \mathbb{R}^2 \mid y > \widetilde{f}(x)\right\}$. Let $\Omega_{\widetilde{f}} \subset \mathbb{C}^2$ be the tube domain over $\omega_{\widetilde{f}}$, i.e., $\Omega_{\widetilde{f}} = \mathbb{R}^2 + i\omega_{\widetilde{f}}$. Here we remark that the boundary of $\Omega_{\widetilde{f}}$ is strictly pseudoconvex off the set $\left\{(z_1, z_2) \mid \Im z_1 = \Im z_2 = 0\right\}$. In fact we can easily check that $\widetilde{f}''(x) > 0$ if $x \ne 0$ by (6.1) and (6.2).

Let $\widetilde{K}$ be the Bergman kernel of $\Omega_{\widetilde{f}}$. In order to obtain Theorem 2.1, it is sufficient to consider the singularity of the Bergman kernel $\widetilde{K}$ near z^0 by Lemma 5.1.

6.2 Two propositions and the proof of Theorem 2.1

A clue to our analysis of the Bergman kernel is the integral representation in Section 3. Normalizing this representation, the Bergman kernel $\widetilde{K}$ of $\Omega_{\widetilde{f}}$ can be expressed in the following:

$$\widetilde{K}(z) = \frac{2m}{(4\pi)^2} g(0)^{1/m} \int_0^\infty e^{-yu^{2m}} P(x,u) u^{4m+1}\, du$$

with

$$P(x,u) = \int_{-\infty}^\infty e^{g(0)^{1/2m} xuv} \frac{1}{\phi(v, u^{-1})}\, dv,$$

$$\phi(v, X) = \int_{-\infty}^\infty e^{-\widehat{g}(Xw)w^{2m} + vw}\, dw,$$

where $\widehat{g}(g(0)^{1/2m}x) = \widetilde{g}(x)/g(0)$. In order to prove Theorem 2.1, it is sufficient to consider the following function $\overline{K}$ instead of $\widetilde{K}$:

$$\overline{K}(z) = \frac{2m}{(4\pi)^2} g(0)^{1/m} \int_1^\infty e^{-yu^{2m}} P(x,u) u^{4m+1}\, du. \tag{6.3}$$

In fact the difference between $\widetilde{K}$ and $\overline{K}$ is smooth.

By introducing the variables $t_0 = g(0)^{1/2m} xy^{-1/2m}$, $\xi = y^{1/2m}$ to the integral representation (6.3), we have

$$\overline{K}(z) = \frac{2m}{(4\pi)^2} \xi^{-4m-2} g(0)^{1/m} \int_\xi^\infty e^{-s^{2m}} L(t_0, \xi; s) s^{4m+1}\, \mathrm{d}s\,, \tag{6.4}$$

where

$$L(t_0, \xi; s) = \int_{-\infty}^\infty e^{t_0 sv} \frac{1}{\phi(v, \xi s^{-1})}\, \mathrm{d}v\,. \tag{6.5}$$

We divide the integral in (6.4) into two parts:

$$\overline{K}(z) = \frac{2m}{(4\pi)^2} g(0)^{1/m} \xi^{-4m-2} \{K^{\langle 1\rangle}(z) + K^{\langle 2\rangle}(z)\}\,, \tag{6.6}$$

where

$$K^{\langle 1\rangle}(z) = \int_1^\infty e^{-s^{2m}} L(t_0, \xi; s) s^{4m+1}\, \mathrm{d}s\,, \tag{6.7}$$

$$K^{\langle 2\rangle}(z) = \int_\xi^1 e^{-s^{2m}} L(t_0, \xi; s) s^{4m+1}\, \mathrm{d}s\,. \tag{6.8}$$

Since the function $(\phi(v, X))^{-1}$ is smooth function of X on $[0\,, 1]$, for any positive integer μ_0

$$\frac{1}{\phi(v, X)} = \sum_{\mu=0}^{\mu_0} a_\mu(v) X^\mu + r_{\mu_0}(v, X) X^{\mu_0+1}\,, \tag{6.9}$$

where

$$\begin{aligned} a_\mu(v) &= \frac{1}{\mu!} \frac{\partial^\mu}{\partial X^\mu} \frac{1}{\phi(v, X)}\bigg|_{X=0}\,, \\ r_{\mu_0}(v, X) &= \frac{1}{\mu_0!} \int_0^1 (1-p)^{\mu_0} \frac{\partial^{\mu_0+1}}{\partial Y^{\mu_0+1}} \frac{1}{\phi(v, Y)}\bigg|_{Y=Xp} \mathrm{d}p\,. \end{aligned} \tag{6.10}$$

Substituting (6.9) into (6.5), we have

$$L(t_0, \xi; s) = \sum_{\mu=0}^{\mu_0} L_\mu(t_0 s) \xi^\mu s^{-\mu} + \widetilde{L}_{\mu_0}(t_0, \xi; s) \xi^{\mu_0+1} s^{-\mu_0-1}\,, \tag{6.11}$$

where

$$L_\mu(u) = \int_{-\infty}^{\infty} e^{uv} a_\mu(v)\,\mathrm{d}v\,,$$

$$\widetilde{L}_{\mu_0}(t_0,\xi;s) = \int_{-\infty}^{\infty} e^{t_0 sv} r_{\mu_0}(v,\xi s^{-1})\,\mathrm{d}v\,.$$

Moreover substituting (6.11) into (6.7)-(6.8), we have

$$K^{\langle j\rangle}(z) = \sum_{\mu=0}^{\mu_0} K_\mu^{\langle j\rangle}(\tau,\xi)\xi^\mu + \widetilde{K}_{\mu_0}^{\langle j\rangle}(\tau,\xi)\xi^{\mu_0+1}\,,$$

for $j=1,\,2$ where

$$K_\mu^{\langle 1\rangle}(\tau,\xi) = \int_1^{\infty} e^{-s^{2m}} L_\mu(t_0 s) s^{4m+1-\mu}\,\mathrm{d}s\,,$$

$$\widetilde{K}_{\mu_0}^{\langle 1\rangle}(\tau,\xi) = \int_1^{\infty} e^{-s^{2m}} \widetilde{L}_{\mu_0}(t_0,\xi;s) s^{4m-\mu_0}\,\mathrm{d}s\,,$$

$$K_\mu^{\langle 2\rangle}(\tau,\xi) = \int_\xi^{1} e^{-s^{2m}} L_\mu(t_0 s) s^{4m+1-\mu}\,\mathrm{d}s\,,$$

$$\widetilde{K}_{\mu_0}^{\langle 2\rangle}(\tau,\xi) = \int_\xi^{1} e^{-s^{2m}} \widetilde{L}_{\mu_0}(t_0,\xi;s) s^{4m-\mu_0}\,\mathrm{d}s\,.$$

The following two propositions are concerned with the singularities of the above functions. Their proofs are given in Paragraphs 6.5 and 6.6.

PROPOSITION 6.1

(i) For any nonnegative integer k_0, $K_\mu^{\langle 1\rangle}$ is expressed in the form:

$$K_\mu^{\langle 1\rangle}(\tau,\xi) = \sum_{k=0}^{k_0} c_{\mu,k}(\tau)\xi^k + \widetilde{K}_{\mu,k_0}^{\langle 1\rangle}(\tau,\xi)\xi^{k_0+1}\,,$$

where

$$c_{\mu,k}(\tau) = \frac{\varphi_{\mu,k}(\tau)}{\tau^{3+\mu+k}} + \psi_{\mu,k}(\tau)\log\tau\,,$$

for $\varphi_{\mu,k}$, $\psi_{\mu,k} \in C^\infty([0\,,1])$ and $\widetilde{K}_{\mu,k_0}^{\langle 1\rangle}$ satisfies $|\widetilde{K}_{\mu,k_0}^{\langle 1\rangle}(\tau,\xi)| < C_{\mu,k_0}(\tau-\alpha\xi)^{-4-\mu-k_0}$ for some positive constants C_{μ,k_0} and α.

(ii) $\widetilde{K}_{\mu_0}^{\langle 1\rangle}$ satisfies $|\widetilde{K}_{\mu_0}^{\langle 1\rangle}(\tau,\xi)| < C_{\mu_0}(\tau-\alpha\xi)^{-4-\mu_0}$ for some positive constants C_{μ_0} and α.

PROPOSITION 6.2

(i) (a) For $0 \le \mu \le 4m+1$, $K_\mu^{\langle 2\rangle} \in C^\infty([0,1]\times[0,\varepsilon))$.

(b) For $\mu \ge 4m+2$, $K_\mu^{\langle 2\rangle}$ *can be expressed in the form:*

$$K_\mu^{\langle 2\rangle}(\tau,\xi)\xi^{-4m-2+\mu} = H_\mu(\tau,\xi)\xi^{-4m-2+\mu}\log\xi + \widetilde{H}_\mu(\tau,\xi),$$

where H_μ, $\widetilde{H}_\mu \in C^\infty([0,1]\times[0,\varepsilon))$.

(ii) For any positive integer r, *there is a positive integer* μ_0 *such that*

$$\widetilde{K}_{\mu_0}^{\langle 2\rangle}(\tau,\xi)\xi^{-4m-1+\mu_0} \in C^r([0,1]\times[0,\varepsilon)).$$

First by Proposition 6.1, $K^{\langle 1\rangle}$ can be expressed in the form:

$$K^{\langle 1\rangle}(z) = \sum_{\mu=0}^{\mu_0} c_\mu(\tau)\xi^\mu + R_{\mu_0}(\tau,\xi)\xi^{\mu_0+1}, \tag{6.12}$$

where c_μ's are expressed as in (2.5) in Theorem 2.1 and R_{μ_0} satisfies $|R_{\mu_0}| < C_{\mu_0}(\tau-\alpha\xi)^{-4-\mu_0}$.

Next by Proposition 6.2, $K^{\langle 2\rangle}(z)$ can be expressed in the form: for any positive integer r,

$$\xi^{-4m-2}K^{\langle 2\rangle}(z) = H(\tau,\xi)\log\xi + \widetilde{H}(\tau,\xi), \tag{6.13}$$

where $H \in C^\infty([0,1]\times[0,\varepsilon))$ and $\widetilde{H} \in C^r([0,1]\times[0,\varepsilon))$.

Hence putting (6.6), (6.12) and (6.13) together, we can obtain Theorem 2.1. Note that $K(z)$ is an even function of ξ. □

6.3 Asymptotic expansion of a_μ

By a direct computation in (6.10), $a_\mu(v)$ can be expressed in the following form:

$$a_\mu(v) = \sum_{|\alpha|=\mu} C_\alpha \frac{\phi^{[\alpha_1]}(v)\cdots\phi^{[\alpha_\mu]}(v)}{\phi(v)^{\mu+1}}, \tag{6.14}$$

where

$$\phi^{[k]}(v) = \left.\frac{\partial^k}{\partial X^k}\phi(v,X)\right|_{X=0},$$

C_α's are constants depending on $\alpha = (\alpha_1, \ldots, \alpha_\mu) \in \mathbb{Z}^\mu_{\geq 0}$ and $|\alpha| = \sum_{j=1}^\mu \alpha_j$. Since

$$\frac{\partial^k}{\partial X^k}\phi(v,X) = \int_{-\infty}^{\infty}\left\{w^k\sum_{j=1}^{k}c_{kj}(Xw)w^{2mj}\right\}e^{-\widehat{g}(Xw)w^{2m}+vw}\,dw\,,$$

for $k \geq 1$ where $c_{kj} \in C^\infty(\mathbb{R})$, we have

$$\phi^{[k]}(v) = \frac{\partial^k}{\partial X^k}\phi(v,X)\bigg|_{X=0} = \sum_{j=1}^{k}c_{kj}(0)\,\phi_{2mj+k}(v)\,, \tag{6.15}$$

for $k \geq 1$ where

$$\phi_\ell(v) = \int_{-\infty}^{\infty} w^\ell\, e^{-w^{2m}+vw}\,dw\,.$$

Here the following lemma is concerned with the asymptotic expansion of ϕ_ℓ at infinity.

LEMMA 6.1.— *Set* $a = \left[(2m)^{-1/(2m-1)} - (2m)^{-2m/(2m-1)}\right] > 0$. *Then we have*

$$\phi_\ell(v) \sim v^{(1-m+\ell)/(2m-1)}\cdot\exp\{av^{2m/(2m-1)}\}\cdot\sum_{j=0}^{\infty}c_j v^{-2mj/(2m-1)}$$

as $v \to +\infty$ *for* $\ell \geq 0$.

The proof of the above lemma will be given soon later.

Lemma 6.1 and (6.15) imply

$$\frac{\phi^{[k]}(v)}{\phi(v)} \sim v^{(2m+1)k/(2m-1)}\cdot\sum_{j=0}^{\infty}c_j v^{-2mj/(2m-1)} \quad \text{as } v \to \infty \text{ for } k \geq 1\,.$$

Moreover, we have

$$\frac{\phi^{[\alpha_1]}(v)\cdots\phi^{[\alpha_\mu]}(v)}{\phi(v)^\mu} \sim v^{(2m+1)\mu/(2m-1)}\cdot\sum_{j=0}^{\infty}c_j v^{-2mj/(2m-1)} \quad \text{as } v \to \infty\,. \tag{6.16}$$

Therefore (6.14), (6.16) and Lemma 6.1 imply

$$a_\mu(v) \sim v^{(m-1+(2m+1)\mu)/(2m-1)} \cdot \exp\{-av^{2m/(2m-1)}\} \sum_{j=0}^{\infty} c_j v^{-2mj/(2m-1)}$$

as $v \to \infty$.

Proof of Lemma 6.1.— Changing the integral variable, we have

$$\phi_\ell(v) = v^{(\ell+1)/(2m-1)} \int_{-\infty}^{\infty} t^\ell \, e^{-\widetilde{v}p(t)} \, \mathrm{d}t \,, \tag{6.17}$$

where $\widetilde{v} = v^{2m/(2m-1)}$ and $p(t) = t^{2m} - t$. We divide (6.17) into two parts:

$$\phi_\ell(v) = v^{(\ell+1)/(2m-1)} \{ I_1(\widetilde{v}) + I_2(\widetilde{v}) \} \,, \tag{6.18}$$

with

$$I_1(\widetilde{v}) = \int_{|t-\alpha| \le \delta} t^\ell \, e^{-\widetilde{v}p(t)} \, \mathrm{d}t \quad \text{and} \quad I_2(\widetilde{v}) = \int_{|t-\alpha| > \delta} t^\ell \, e^{-\widetilde{v}p(t)} \, \mathrm{d}t \,,$$

where $\delta > 0$ is small and $\alpha = (2m)^{-1/(2m-1)}$. Note that $p'(\alpha) = 0$.

First we consider the function I_1. By Taylor's formula, we have

$$\begin{aligned} I_1(\widetilde{v}) &= \int_{|t-\alpha|<\delta} t^\ell \exp\left\{ -\widetilde{v}\Big(p(\alpha) + \widetilde{p}(t-\alpha)(t-\alpha)^2 \Big) \right\} \mathrm{d}t \\ &= e^{a\widetilde{v}} \int_{|t| \le \delta} (t-\alpha)^\ell \, e^{-\widetilde{v}\widetilde{p}(t)t^2} \, \mathrm{d}t \,, \end{aligned}$$

where $a = -p(\alpha) = \left[(2m)^{-1/(2m-1)} - (2m)^{-2m/(2m-1)} \right] > 0$ and $\widetilde{p}(t) = \int_0^1 (1-u) p''(ut+\alpha) \, \mathrm{d}u$. Set $s = \widetilde{p}(t)^{1/2} t$ for $|t| \le \delta$ and $\delta_\pm = \widetilde{p}(\pm\delta)^{1/2}\delta$, respectively. Then there is a function $\widehat{p} \in C^\infty([-\delta_-\,, \delta_+])$ such that $t = \widehat{p}(s)$ and $\widehat{p}' > 0$. Changing the integral variable, we have

$$I_1(\widetilde{v}) = \int_{-\delta_-}^{\delta_+} e^{-\widetilde{v}s^2} \, \Psi(s) \, \mathrm{d}s \,,$$

where $\Psi(s) = (\widehat{p}(s) + \alpha)^\ell \cdot \widehat{p}'(s)$. Since $\Psi \in C^\infty([-\delta_-\,, \delta_+])$, we have

$$\begin{aligned} I_1(v) &= \widetilde{v}^{-1/2} \int_{-\delta_-\widetilde{v}^{-1/2}}^{\delta_+\widetilde{v}^{1/2}} e^{-u^2} \, \Psi(u\widetilde{v}^{-1/2}) \, \mathrm{d}u \\ &\sim \widetilde{v}^{-1/2} \sum_{j=0}^{\infty} c_j \widetilde{v}^{-j} \quad \text{as } \widetilde{v} \to \infty \,. \end{aligned} \tag{6.19}$$

We remark that $\int_{-\infty}^{\infty} e^{-u^2} u^j \, \mathrm{d}u = 0$ if $j \in \mathbb{Z}$ is odd.

Next we consider the function I_2. Let p_d be the function defined by $p_d(t) = d|t-\alpha| - a$ where $d > 0$. We can choose $d > 0$ such that $p(t) \geq p_d(t)$ for $|t-\alpha| > \delta$. Then we have

$$\begin{aligned} \left|I_2(\widetilde{v})\right| &\leq \int_{|t-\alpha|>\delta} e^{-\widetilde{v}p_d(t)}\,\mathrm{d}t \\ &\leq 2C\,e^{a\widetilde{v}} \int_\delta^\infty e^{-d\widetilde{v}t}\,\mathrm{d}t \leq 2C\widetilde{v}^{-1}\,e^{(a-d\delta)\widetilde{v}}\,. \end{aligned} \tag{6.20}$$

Finally putting (6.18), (6.19) and (6.20) together, we have the asymptotic expansion in Lemma 6.1 □

6.4 Asymptotic expansion of L_μ

Let $A \in C^\infty(\mathbb{R})$ be an even or odd function (i.e., $A(-v) = A(v)$ or $-A(v)$) satisfying

$$A(v) \sim v^{n/(2m-1)}\cdot\exp\{-av^{2m/(2m-1)}\}\cdot\sum_{j=0}^{\infty} c_j v^{-2mj/(2m-1)} \quad \text{as } v \to +\infty\,,$$

where $n \in \mathbb{N}$ and the constant a is as in Lemma 6.1. $L \in C^\omega(\mathbb{R})$ be the function defined by

$$L(u) = \int_{-\infty}^{\infty} e^{uv} A(v)\,\mathrm{d}v\,. \tag{6.21}$$

In this section we give the asymptotic expansion of L at infinity.

LEMMA 6.2.— *$L(u) \sim u^{m-1+n}\cdot e^{u^{2m}}\cdot\sum_{j=0}^\infty c_j u^{-2mj}$ as $u \to +\infty$.*

Remark.— Lemmas 6.1 and 6.2 imply that for μ, $\ell \geq 0$,

$$\begin{aligned} L_\mu^{(\ell)}(u) &= \frac{\mathrm{d}^\ell}{\mathrm{d}u^\ell} L_\mu(u) \\ &\sim u^{2m-2+(2m+1)\mu+(2m-1)\ell}\cdot e^{u^{2m}}\cdot\sum_{j=0}^{\infty} c_j u^{-2mj} \quad \text{as } u \to +\infty\,. \end{aligned} \tag{6.22}$$

Proof.— We only show Lemma 6.2 in the case where A is an even function. Let $\widehat{A} \in C^\infty(\mathbb{R} \setminus \{0\})$ be defined by

$$A(v) = v^{n/(2m-1)} \cdot \exp\{-av^{2m/(2m-1)}\} \cdot \widehat{A}(v^{2m/(2m-1)}), \qquad (6.23)$$

then $\widehat{A}(x) \sim \sum_{j=0}^{\infty} c_j x^{-j}$ as $x \to \infty$. Substituting (6.23) into (6.21), we have

$$L(u) = \int_{-\infty}^{\infty} \exp\Big\{-a|v|^{2m/(2m-1)} + uv\Big\}\, |v|^{n/(2m-1)} \widehat{A}\Big(|v|^{2m/(2m-1)}\Big)\, dv .$$

Changing the integral variable and setting $q(t) = at^{2m} - t^{2m-1}$, we have

$$L(u) = (2m-1)u^{2m+n-1} \int_{-\infty}^{\infty} e^{-u^{2m}q(t)} \widehat{A}(t^{2m}u^{2m}) t^{2m+n-2}\, dt . \quad (6.24)$$

Now we divide the integral in (6.24) into two parts:

$$L(u) = (2m-1)u^{2m+n-1}\{J_1(\widetilde{u}) + J_2(\widetilde{u})\}, \qquad (6.25)$$

with

$$J_1(\widetilde{u}) = \int_{|t-\beta|\le\delta} e^{-\widetilde{u}q(t)} \widehat{A}(\widetilde{u}t^{2m}) t^{2m+n-2}\, dt ,$$

$$J_2(\widetilde{u}) = \int_{|t-\beta|>\delta} e^{-\widetilde{u}q(t)} \widehat{A}(\widetilde{u}t^{2m}) t^{2m+n-2}\, dt ,$$

where $\widetilde{u} = u^{2m}$, $\delta > 0$ is small and $\beta = (2m-1)(2ma)^{-1}$. Note that $q'(\beta) = 0$.

First we consider the function J_1. By Taylor's formula, we have

$$J_1(\widetilde{u}) = e^{\widetilde{u}} \int_{|t|\le\delta} e^{-\widetilde{u}\widetilde{q}(t)t^2} \widehat{A}\Big(\widetilde{u}(t+\beta)^{2m}\Big)\, (t+\beta)^{2m+n-2}\, dt ,$$

where $\widetilde{q}(t) = \int_0^1 (1-v)q''(vt+\alpha)\, dv$. Note that $q(0) = -1$. Set $s = \widetilde{q}(t)^{1/2}t$ for $|t| \le \delta$ and $\widetilde{\delta}_\pm = \widetilde{q}(\pm\delta)^{1/2}\delta$, respectively. Then there is a function $\widehat{q} \in C^\infty([-\widetilde{\delta}_-, \widetilde{\delta}_+])$ such that $t = \widehat{q}(s)$ and $\widehat{q}' > 0$. Changing the integral variable, we have

$$J_1(\widetilde{u}) = \int_{-\widetilde{\delta}_-}^{\widetilde{\delta}_+} e^{-\widetilde{u}s^2} \widetilde{\Psi}(s, \widetilde{u})\, ds , \qquad (6.26)$$

where
$$\widetilde{\Psi}(s,\widetilde{u}) = \widehat{A}\Big(\widetilde{u}\big(\widehat{q}(s)+\beta\big)^{2m}\Big)\,\big(\widehat{q}(s)+\beta\big)^{2m+n-2}\widehat{q}'(s)\,.$$

Since $\widehat{A}(x) \sim \sum_{j=0}^{\infty} c_j x^{-j}$ as $x \to \infty$, we have

$$\widetilde{\Psi}(s,\widetilde{u}) \sim \sum_{j=0}^{\infty} c_j(s)\widetilde{u}^{-j} \quad \text{as } \widetilde{u} \to \infty\,, \tag{6.27}$$

where $c_j \in C^{\infty}([-\widetilde{\delta}_-\,,\ \widetilde{\delta}_+])$. Substituting (6.27) into (6.26), we have

$$J_1(\widetilde{u}) \sim \widetilde{u}^{-1/2}\, e^{\widetilde{u}} \sum_{j=0}^{\infty} c_j \widetilde{u}^{-j} \quad \text{as } \widetilde{u} \to \infty\,. \tag{6.28}$$

Next we consider the function J_2. By a similar argument about the estimate of $I_2(\widetilde{v})$ in the proof of Lemma 6.1, we can obtain

$$\big|J_2(\widetilde{u})\big| \le C\widetilde{u}^{-1}\, e^{(1-\varepsilon)\widetilde{u}}\,, \tag{2.29}$$

where ε is a positive constant.

Finally putting (6.25), (6.28) and (6.29) together, we obtain the asymptotic expansion in Lemma 6.2. □

6.5 Proof of Proposition 6.1

We can construct the function $h \in C^{\infty}([0\,,\infty))$ such that if $Y = \widetilde{f}(X)^{1/2m}$, then $X = Yh(Y)$. In fact

$$\frac{\mathrm{d}}{\mathrm{d}X}\left[\widetilde{f}(X)^{1/2m}\right] > 0 \quad \text{for } X \ge 0\,.$$

Set $t = \widetilde{f}(x)^{1/2m}\xi^{-1}$. Then we can write $t_0 = th(t\xi)$. Note that $\widehat{g}(X)^{1/2m}h(X) = 1$ for X, $Y \ge 0$ and hence $h'(Y) \ge 0$ for $Y \ge 0$. Let us prepare two lemmas for the proof of Propositions 6.1 and 6.2.

LEMMA 6.3.— *Assume that the functions a, b and c on* $[0\,,\varepsilon) \times [0\,,1]$ *satisfy* $a(\xi,t_0) = b(\xi,t) = c(\xi,\tau)$. *If one of these functions belongs to* $C^{\infty}([0\,,\varepsilon) \times [0\,,1])$, *then so do the others.*

Proof.— This lemma is directly shown by the relation between three variables t_0, t and τ. □

LEMMA 6.4.— *There is a positive number* α *such that* $1-t_0^{2m} \geq \tau - \alpha\xi$.

We remark that the above constant α is same as that in Proposition 6.2.

Proof.— By definition, we have

$$1-t_0^{2m} = 1-t^{2m}h(t\xi)^{2m} = (1-t^{2m}) - \big(h(t\xi)-1\big)t^{2m}\,.$$

Since $h(0)=1$ and $h'(X)\geq 0$, we have $h(t\xi)^{2m}-1 \leq \alpha t\xi$ for some positive number α. Therefore we have

$$1-t_0^{2m} \geq \tau - \alpha t^{2m+1}\xi \geq \tau - \alpha\xi\,. \ \square$$

Proof of Proposition 6.1

(i) Recall the definition of the function $K_\mu^{\langle 1\rangle}$:

$$K_\mu^{\langle 1\rangle}(\tau,\xi) = \int_1^\infty e^{-s^{2m}} L_\mu(t_0 s)s^{4m+1-\mu}\,\mathrm{d}s\,, \tag{6.30}$$

where

$$L_\mu(u) = \int_{-\infty}^{\infty} e^{uv} a_\mu(v)\,\mathrm{d}v \quad \text{and} \quad a_\mu(v) = \frac{1}{\mu!}\frac{\partial^\mu}{\partial X^\mu}\frac{1}{\phi(v,X)}\bigg|_{X=0}\,.$$

We obtain the Taylor expansion of $L_\mu(t_0;s) = L_\mu\big(tsh(t\xi)\big)$ with respect to ξ:

$$L_\mu(t_0 s) = \sum_{k=0}^{k_0} L_{\mu,k}(t;s)\xi^k + \widetilde{L}_{\mu,k_0}(t,\xi;s)\xi^{k_0+1}\,, \tag{6.31}$$

where

$$L_{\mu,k}(t;s) = \frac{1}{k!}\frac{\partial^k}{\partial\xi^k} L_\mu\big(tsh(t\xi)\big)\bigg|_{\xi=0}\,, \tag{6.32}$$

$$\widetilde{L}_{\mu,k_0}(t,\xi;s) = \frac{1}{k_0!}\int_0^1 (1-p)^{k_0}\,\frac{\partial^{k_0+1}}{\partial X^{k_0+1}} L_\mu\big(tsh(tX)\big)\bigg|_{X=\xi p}\,\mathrm{d}p\,. \tag{6.33}$$

Substituting (6.31) into (6.30), we have

$$K_\mu^{\langle 1\rangle}(\tau,\xi) = \sum_{k=0}^{k_0} K_{\mu,k}(t)\xi^k + \widetilde{K}_{\mu,k_0}(t,\xi)\xi^{k_0+1}\,,$$

where

$$K_{\mu,k}(t) = \int_1^\infty e^{-s^{2m}} L_{\mu,k}(t;s) s^{4m+1-\mu}\,\mathrm{d}s\,, \tag{6.34}$$

$$\widetilde{K}_{\mu,k_0}(t,\xi) = \int_1^\infty e^{-s^{2m}} \widetilde{L}_{\mu,k_0}(t,\xi;s) s^{4m+1-\mu}\,\mathrm{d}s\,. \tag{6.35}$$

First we consider the singularity of $K_{\nu,k}$ at $t=1$.

By a direct computation in (6.32), we have

$$L_{\mu,k}(t;s) = \sum_{\ell=1}^{k} h_\ell(t) s^\ell L_\mu^{(\ell)}(ts)\,,$$

where $h_\ell \in C^\infty([0\,,1])$ (which depends on μ, k). We define the function $S_{\mu,k}(t;s^{2m})$ by

$$L_{\mu,k}(t;s) = s^{2m-2+(2m-1)\mu+2mk}\, e^{t^{2m}s^{2m}} S_{\mu,k}(t;s^{2m})\,. \tag{6.36}$$

Lemma 6.2 implies

$$S_{\mu,k}(t;s^{2m}) \sim \sum_{j=0}^\infty c_j(t) s^{-2mj} \quad \text{as } s\to\infty\,, \tag{6.37}$$

where $c_j \in C^\infty([0\,,1])$.

Substituting (6.36) into (6.34), we have

$$\begin{aligned} K_{\mu,k}(t) &= \int_1^\infty e^{-(1-t^{2m})s^{2m}} S_{\mu,k}(t;s^{2m}) s^{2m-2+2m\mu+2mk}\,\mathrm{d}s \\ &= \frac{1}{2m}\int_1^\infty e^{-\chi^{-1}(\tau)\sigma} S_{\mu,k}(t;\sigma)\sigma^{2+\mu+k}\,\mathrm{d}\sigma\,. \end{aligned} \tag{6.38}$$

Moreover substituting (6.37) into (6.38), we have

$$K_{\mu,k}(t) = \frac{\varphi_{\mu,k}(\tau)}{\tau^{3+\mu+k}} + \psi_{\mu,k}(\tau)\log\tau\,,$$

where $\varphi_{\mu,k}$, $\psi_{\mu,k} \in C^\infty([0\,,1])$.

Next we obtain the inequality $\left|\widetilde{K}^{\langle 1\rangle}_{\mu,k}(\tau,\xi)\right| \leq C_{\mu,k_0}\left[\tau-\alpha\xi\right]^{-4-\mu-k_0}$ for some positive constant C_{μ,k_0}. By a direct computation in (6.33), we have

$$\frac{\partial^{k_0+1}}{\partial X^{k_0+1}} L_\mu\big(tsh(tX)\big) = \sum_{\ell=1}^{k_0+1} \widetilde{h}_\ell(t,X)s^\ell L_\mu^{(\ell)}\big(tsh(tX)\big)\,,$$

where $\widetilde{h}_\ell$ are bounded functions (depending on μ, k_0). Since $h'(X)\geq 0$, we can obtain

$$\begin{aligned}\left|\widetilde{L}_{\mu,k_0}(t,\xi;s)\right| &\leq C\sum_{\ell=1}^{k_0+1} s^\ell L_\mu^{(\ell)}\big(tsh(ts)\big)\\ &\leq Cs^{k_0+1}L_\mu^{(k_0+1)}(t_0 s)\\ &\leq Cs^{2m-2+(2m+1)\mu+2m(k_0+1)}\,e^{t_0^{2m}s^{2m}}\end{aligned}\tag{6.39}$$

for $s\geq 1$. Substituting (6.39) to (6.35), we obtain

$$\begin{aligned}\widetilde{K}_{\mu,k_0}(t,\xi) &\leq C\int_1^\infty e^{-(1-t_0^{2m})s^{2m}}s^{6m-2+2m\mu+2m(k_0+1)}\,\mathrm{d}s\\ &\leq C\big(1-t_0^{2m}\big)^{-4-\mu-k_0}\leq C\big(\tau-\alpha\xi\big)^{-4-\mu-k_0}\end{aligned}$$

by Lemma 6.4. This completes the proof of Proposition 6.2(i).

(ii) Recall the definition of the function $\widetilde{K}^{\langle 1\rangle}_\mu$:

$$\widetilde{K}^{\langle 1\rangle}_{\mu_0}(\tau,\xi) = \int_1^\infty e^{-s^{2m}}\widetilde{L}_{\mu_0}(t_0,\xi;s)s^{4m-\mu_0}\,\mathrm{d}s\,.\tag{6.40}$$

where

$$\widetilde{L}_{\mu_0}(t_0,\xi;s) = \int_{-\infty}^\infty e^{t_0 sv}r_{\mu_0}(v,\xi s^{-1})\,\mathrm{d}v\,,\tag{6.41}$$

$$r_{\mu_0}(v,X) = \frac{1}{\mu_0!}\int_0^1 (1-p)^{\mu_0}\,\frac{\partial^{\mu_0+1}}{\partial Y^{\mu_0+1}}\,\frac{1}{\phi(v,Y)}\bigg|_{Y=Xp}\,\mathrm{d}p\,.\tag{6.42}$$

The following lemma is necessary to obtain the estimate of $\widetilde{K}^{\langle 1\rangle}_\mu$ in (ii).

LEMMA 6.5.— *For $v\in\mathbb{R}$, $X\geq 0$,*

$$\left|r_{\mu_0}(v,X)\right| < C\left|v\right|^{((2m+1)(\mu_0+1)+m-1)/(2m-1)}\,e^{-a|v|^{2m/(2m-1)}}\,.$$

We remark that the constant a is as in Lemma 6.1. The proof of the above lemma is given soon later.

Applying Lemma 6.5 to (6.41), we have

$$\begin{aligned}
&\left|\widetilde{L}_{\mu_0}(t_0,\xi,s)\right| \leq \\
&\quad \leq C\int_{-\infty}^{\infty} |v|^{(2m+1)(\mu_0+1)+m-1/(2m-1)}\, e^{t_0 sv - a|v|^{2m/(2m-1)}}\, \mathrm{d}v \qquad (6.43)\\
&\quad \leq C s^{2m-2(2m+1)(\mu_0+1)}\, e^{t_0^{2m}s^{2m}} .
\end{aligned}$$

The second inequality is given by Lemma 6.2. Moreover substituting (6.43) into (6.40), we have

$$\begin{aligned}
\left|\widetilde{K}_{\mu_0}^{\langle 1\rangle}(t,\xi)\right| &\leq C\int_1^{\infty} e^{-(1-t_0^{2m})s^{2m}}\, s^{8m-1+2m\mu_0}\,\mathrm{d}s \\
&\leq C\left(1-t_0^{2m}\right)^{-4-\mu_0} \leq C\left(\tau-\alpha\xi\right)^{-4-\mu_0},
\end{aligned}$$

by Lemma 6.4. Therefore we obtain the estimate of $\widetilde{K}_{\mu_0}^{\langle 1\rangle}$ in Proposition 6.1(ii). □

Proof of Lemma 6.5. — We only consider the case where v is positive. The proof for the case where v is negative is given in the same way.

By a direct computation, we have

$$\frac{\partial^{\mu_0+1}}{\partial X^{\mu_0+1}}\frac{1}{\phi(v,X)} = \sum_{|\alpha|=\mu_0+1} C_\alpha \frac{\phi^{[\alpha_1]}(v,X)\cdots\phi^{[\alpha_{\mu_0+1}]}(v,X)}{\phi(v,X)^{\mu_0+2}}, \qquad (6.44)$$

where

$$\phi^{[k]}(v,X) = \frac{\partial^k}{\partial X^k}\phi(v,X)$$

and C_α's are constants depending on $\alpha = (\alpha_1,\ldots,\alpha_{\mu_0+1}) \in \mathbb{Z}_{\geq 0}^{\mu_0+1}$. By a direct computation, the function $\phi^{[k]}$'s are expressed in the form:

$$\phi^{[k]}(v,X) = \sum_{|\beta|=k} c_\beta F_\beta(\widetilde{v},\widetilde{X}), \qquad (6.45)$$

where $\widetilde{v} = v^{2m/(2m-1)}$, $\widetilde{X} = Xv^{-1/2m}$, c_β's are constants and

$$\begin{aligned}
F_\beta(\widetilde{v},\widetilde{X}) &= \int_{-\infty}^{\infty} e^{-\widetilde{g}(Xw)w^{2m}+vw}\, w^\gamma \prod_\beta \widehat{g}^{(\beta_k)}(Xw)\,\mathrm{d}w \\
&= \widetilde{v}^{\gamma+1/2m}\int_{-\infty}^{\infty} e^{-\widetilde{v}p(s,\widetilde{X})}\, s^\gamma \prod_\beta \widehat{g}^{(\beta_k)}(\widetilde{X}s)\,\mathrm{d}s,
\end{aligned} \qquad (6.46)$$

with $p(s, \widetilde{X}) = \widehat{g}(\widetilde{X}s)s^{2m} - s$, $\beta_k \in \mathbb{N}$ and $\gamma \in \mathbb{N}$ depending on $\beta = \left(\beta_k\right)_k$. In order to apply the stationary phase method to the above integral, we must know the location of the critical points of the function $p(\,\cdot\,, \widetilde{X})$. The lemma below gives the information about it.

LEMMA 6.6.— *There exists a function $\alpha \in C^\infty\big([\,0\,,\,\infty)\big)$ such that*

$$\frac{\partial}{\partial s} p\big(\alpha(\widetilde{X}), \widetilde{X}\big) = 0 \quad \textit{and} \quad \alpha_0 \leq \alpha(\widetilde{X}) \leq \alpha_1 \quad \textit{for } \widetilde{X} \geq 0\,, \tag{6.47}$$

where $\alpha_0 = \big(2m\big)^{-1/(2m-1)}$ *and* $\alpha_1 = \big(2m \cdot 4/5\big)^{-1/(2m-1)}$.

Proof.— By a direct computation, we have

$$\begin{aligned}
\frac{\partial}{\partial s} p(s, \widetilde{X}) &= s^{2m-1}\big(\widehat{g}'(\eta)\eta + 2m\widehat{g}(\eta)\big) - 1\\
\frac{\partial^2}{\partial s^2} p(s, \widetilde{X}) &= s^{2m-2}\big(\widehat{g}''(\eta)\eta^2 + 4m\widehat{g}'(\eta)\eta + 2m(2m-1)\widehat{g}(\eta)\big)
\end{aligned}$$

where $\eta = \widetilde{X}s$. It is easy to obtain the following inequalities by using the conditions (6.1)-(6.2):

$$\frac{\partial}{\partial s} p(\alpha_0, \widetilde{X}) \leq 0 < \frac{\partial}{\partial s} p(\alpha_1, \widetilde{X}) \quad \text{for } \widetilde{X} \geq 0 \tag{6.48}$$

$$\frac{\partial^2}{\partial s^2} p(s, \widetilde{X}) \geq c > 0 \quad \text{on } [\,\alpha_0\,,\,\alpha_1\,] \times [\,0\,,\,\infty) \tag{6.49}$$

for some positive constant c. Then the inequalities (6.48)-(6.49) imply the claim in Lemma 6.6 by the implicit function theorem. □

Now we divide the integral in (6.46) into two parts.

$$F_\beta(\widetilde{v}, \widetilde{X}) = \widetilde{v}^{\,\gamma+1/2m}\big\{I_1(\widetilde{v}, \widetilde{X}) + I_2(\widetilde{v}, \widetilde{X})\big\}\,,$$

where

$$\begin{aligned}
I_1(\widetilde{v}, \widetilde{X}) &= \int_{|t-\alpha(\widetilde{X})|\leq\delta} e^{-\widetilde{v}p(t,\widetilde{X})t^2} t^\gamma \prod_\beta \widehat{g}^{\,(\beta_k)}(\widetilde{X}t)\,\mathrm{d}t\,,\\
I_2(\widetilde{v}, \widetilde{X}) &= \int_{|t-\alpha(\widetilde{X})|>\delta} e^{-\widetilde{v}p(t,\widetilde{X})t^2} t^\gamma \prod_\beta \widehat{g}^{\,(\beta_k)}(\widetilde{X}t)\,\mathrm{d}t\,,
\end{aligned}$$

where $\delta > 0$ is small.

First we consider the function I_1. By Lemma 6.6 and Taylor's formula, we have

$$
\begin{aligned}
& I_1(\widetilde{v}, \widetilde{X}) = \\
& = e^{a(\widetilde{X})\widetilde{v}} \int_{|t|\le\delta} e^{-\widetilde{v}\widetilde{p}(t,\widetilde{X})t^2} (t+\alpha(\widetilde{X}))^{\gamma} \prod_{\beta} \widehat{g}^{(\beta_k)} \left(\widetilde{X}\left(t+\alpha(\widetilde{X})\right)\right) \mathrm{d}t\,,
\end{aligned}
\tag{6.50}
$$

where $a(\widetilde{X}) = -p\big(\alpha(\widetilde{X}), \widetilde{X}\big)$ and

$$
\widetilde{p}(t, \widetilde{X}) = \int_0^1 (1-u) \frac{\partial^2}{\partial s^2} p\big(ut + \alpha(\widetilde{X}), \widetilde{X}\big)\, \mathrm{d}u\,.
$$

Set $s = \widetilde{p}\big(t, \widetilde{X}\big)^{1/2} t$ on $[-\delta\,,\,\delta] \times [\,0\,,\,\infty)$ and $\delta_{\pm}(\widetilde{X}) = \widetilde{p}\big(\pm\delta, \widetilde{X}\big)^{1/2}\delta$, respectively. Then there is a function $\widehat{p} \in C^{\infty}\big([-\delta_{-}(\widetilde{X})\,,\,\delta_{+}(\widetilde{X})] \times [\,0\,,\,\infty)\big)$ such that $t = \widehat{p}(s, \widetilde{X})$ and $(\partial/\partial s)\widehat{p}(s, \widetilde{X}) > 0$. Changing the integral variable, we have

$$
I_1(\widetilde{v}, \widetilde{X}) = e^{a(\widetilde{X})\widetilde{v}} \int_{-\delta_{-}(\widetilde{X})}^{\delta_{+}(\widetilde{X})} e^{-\widetilde{v}s^2}\, \Psi(s, \widetilde{X})\, \mathrm{d}s\,,
$$

where

$$
\Psi(s, \widetilde{X}) = \big(\widehat{p}(s, \widetilde{X}) + \alpha(\widetilde{X})\big)^{\gamma} \prod_{\beta} \widehat{g}^{(\beta_k)} \left(\widetilde{X}\big(\widehat{p}(s, \widetilde{X}) + \alpha(\widetilde{X})\big)\right) \frac{\partial}{\partial s} \widehat{p}(s, \widetilde{X})\,.
$$

Since $\Psi \in C^{\infty}\big([-\delta_{-}(\widetilde{X})\,,\,\delta_{+}(\widetilde{X})] \times [\,0\,,\,\infty)\big)$, we have

$$
\begin{aligned}
& I_1(\widetilde{v}, \widetilde{X}) \cdot \{\widetilde{v}^{-1/2} e^{a(\widetilde{X})\widetilde{v}}\}^{-1} = \\
& \int_{-\delta_{-}(\widetilde{X})\widetilde{v}^{1/2}}^{\delta_{+}(\widetilde{X})\widetilde{v}^{1/2}} e^{-u^2}\, \Psi(u\widetilde{v}^{-1/2}, \widetilde{X})\, \mathrm{d}u \longrightarrow \sqrt{\pi}\, \Psi(0, \widetilde{X}) \quad \text{as } \widetilde{v} \to \infty\,.
\end{aligned}
\tag{6.51}
$$

Note that

$$
\Psi(0, \widetilde{X}) = \alpha\big(\widetilde{X}\big)^{\gamma} \cdot \left\{\frac{1}{2} \frac{\partial^2}{\partial t^2} p\big(\alpha(\widetilde{X}), \widetilde{X}\big)\right\}^{-1/2} \prod_{\beta} \widehat{g}^{(\beta_k)} \big(\widetilde{X}\alpha(\widetilde{X})\big)\,.
$$

Next we consider the function I_2. In a similar argument about the estimate of $I_2(\widetilde{v})$ in the proof in Lemma 6.1, we can obtain

$$
\big|I_2(\widetilde{v}, \widetilde{X})\big| \le C\widetilde{v}^{-1}\, e^{a(\widetilde{X})\widetilde{v} - \varepsilon\widetilde{v}}\,,
\tag{6.52}
$$

where ε is a positive constant.

Putting (6.51)-(6.52) together, we have

$$\lim_{\widetilde{v}\to\infty} F_\beta(\widetilde{v},\widetilde{X})\{\widetilde{v}^{(\gamma+1-m)/2m}e^{-p(\widetilde{X})\widetilde{v}}\}^{-1} = \sqrt{\pi}\,\widetilde{\Psi}_\beta(0,\widetilde{X}). \tag{6.53}$$

Now under the condition $|\beta| = k$, the number γ in (6.46) attains the maximum value $(2m+1)k$ when $\beta = (1, \ldots, 1)$. Therefore (6.45) and (6.53) imply that

$$\left|\frac{\phi^{[k]}(v,X)}{\phi(v,X)}\right| \le C\widetilde{v}^{(2m+1)k/2m}. \tag{6.54}$$

Moreover (6.44) and (6.53) imply that

$$\left|\frac{\partial^{\mu_0+1}}{\partial X^{\mu_0+1}}\frac{1}{\phi(v,X)}\right| \le C\widetilde{v}^{(2m+1)(\mu_0+1)+m-1/2m}e^{-a(\widetilde{X})\widetilde{v}}.$$

Now we admit the following lemma.

Lemma 6.7.— $a(\widetilde{X}) \ge a(0) = a$.

We remark that the constant a is as in Lemma 6.1. The above lemma implies

$$\left|\frac{\partial^{\mu_0+1}}{\partial X^{\mu_0+1}}\frac{1}{\phi(v,X)}\right| \le C\widetilde{v}^{(2m+1)(\mu_0+1)+m-1/2m}e^{-a\widetilde{v}}. \tag{6.55}$$

Finally substituting (6.55) into (6.42), we can obtain the estimate of r_{μ_0} in Lemma 6.5.

Proof of Lemma 6.7.— The definition of $\alpha(\widetilde{X})$ in (6.47) implies that

$$\begin{aligned} a'(\widetilde{X}) &= -\alpha'(\widetilde{X})\cdot\alpha(\widetilde{X})^{2m-1}\left(\widehat{g}'(\widetilde{X}\alpha(\widetilde{X}))\widetilde{X}\alpha(\widetilde{X}) + 2m\widehat{g}(\widetilde{X}\alpha(\widetilde{X}))\right) \\ &\quad + \alpha'(\widetilde{X}) - g'(\widetilde{X}\alpha(\widetilde{X}))\alpha(\widetilde{X})^{2m+1} \\ &= -\widehat{g}'(\widetilde{X}\alpha(\widetilde{X}))\alpha(\widetilde{X})^{2m+1}. \end{aligned}$$

Since the condition $xg'(x) \le 0$ in (2.1) implies

$$\widetilde{X}a'(\widetilde{X}) = -\widetilde{X}\widehat{g}'(\widetilde{X}\alpha(\widetilde{X}))\alpha(\widetilde{X})^{2m+1} \ge 0,$$

$a(\widetilde{X})$ attains the minimum value when $\widetilde{X} = 0$. It is easy to check that $a(0) = a$. □

6.6 Proof of Proposition 6.2

(i) Recall the definition of the function $K_\mu^{\langle 2\rangle}$:

$$K_\mu^{\langle 2\rangle}(\tau,\xi) = \int_\xi^1 e^{-s^{2m}} L_\mu(t_0 s) s^{4m+1-\mu}\,\mathrm{d}s\,,$$

where

$$L_\mu(u) = \int_{-\infty}^{\infty} e^{uv} a_\mu(v)\,\mathrm{d}v\,.$$

We remark that L_μ extends to an entire function.

By the residue formula, we have

$$K_\mu^{\langle 2\rangle}(\tau,\xi) = h_\mu(t_0)\xi^{-4m-2+\mu}\log\xi + \widetilde{h}_\mu(t_{0,\xi})\,, \tag{6.56}$$

where

$$h_\mu(t_0) = \frac{1}{2\pi i}\oint_{|\zeta|=\delta} e^{-\zeta^{2m}} L_\mu(t_0\zeta)\zeta^{4m+1-\mu}\,\mathrm{d}\zeta \tag{6.57}$$

for δ is a small positive integer and $\widetilde{h}_\mu \in C^\infty([0\,,1]\times[0\,,\varepsilon))$. Here (6.57) implies that $h_\mu \equiv 0$ for $0 \le \mu \le 4m+1$ and $h_\mu \in C^\infty([0\,,1])$ for $\mu \le 4m+2$. By Lemma 6.3, we can obtain (i) in Proposition 6.2.

(ii) Changing the integral variable, we have

$$\xi^{-4m-2+\mu_0}\widetilde{K}_{\mu_0}^{\langle 2\rangle}(\tau,\xi) = \int_1^{\xi^{-1}} e^{-\xi^{2m}u^{2m}}\widetilde{L}_{\mu_0}(t_0,\xi,\xi u)u^{4m-\mu_0}\,\mathrm{d}u\,,$$

$$\widetilde{L}_{\mu_0}(t_0,\xi,\xi u) = \int_{-\infty}^{\infty} e^{t_0\xi uv} r_{\mu_0}(v,u^{-1})\,\mathrm{d}v\,.$$

Note that r_{μ_0} satisfies the inequality in Lemma 6.5.

Keeping the above integrals in mind, we define the function H by

$$H(\alpha,\beta,\gamma,\delta) = t_0^\delta\xi^\alpha\int_1^{\xi^{-1}} e^{-\xi^{2m}u^{2m}}\frac{\partial^\gamma}{\partial X^\gamma}I(t_0\xi u)u^{-\beta-1}\,\mathrm{d}u\,, \tag{6.58}$$

with

$$I(X) = \int_{-\infty}^{\infty} e^{Xv} r(v)\,\mathrm{d}v\,, \tag{6.59}$$

where α, β, γ (≥ 0), δ are integers and the function r satisfies $|r(v)| \le C\,e^{-c|v|^{2m/(2m-1)}}$ for some positive constants c, C. Note that H is a function of (t_0,ξ).

By a direct computation, we have

$$\frac{\partial}{\partial t_0} H(\alpha,\beta,\gamma,\delta) = H(\alpha+1,\, \beta-1,\, \gamma+1,\, \delta)\,, \tag{6.60}$$

$$\begin{aligned}\frac{\partial}{\partial \xi} H(\alpha,\beta,\gamma,\delta) = -e^{-1} t_0^{\delta} \xi^{\alpha+\beta-1} \frac{\partial^{\gamma}}{\partial X^{\gamma}} I(\xi) + \alpha H(\alpha-1,\, \beta,\, \gamma,\, \delta) + \\ - 2mH(\alpha+2m-1,\, \beta-2m,\, \gamma,\, \delta) + \\ + H(\alpha,\, \beta-1,\, \gamma+1,\, \delta+1)\,.\end{aligned} \tag{6.61}$$

Since $\partial^{\gamma} I/\partial X_{\gamma}$ is bounded on $[\,0\,,\,1\,]$, we have

$$\big|H(\alpha,\beta,\gamma,\delta)\big| \le C\big|\xi\big|^{\alpha+\beta}\,, \tag{6.62}$$

by (6.58). By induction, we have

$$\left|\frac{\partial^k}{\partial t_0^k}\frac{\partial^\ell}{\partial \xi^\ell} H(\alpha,\beta,\gamma,\delta)\right| \le C\big|\xi\big|^{\alpha+\beta-\ell}\,,$$

by (6.60), (6.61) and (6.62).

Now if replace $r(v)$ by $r_{\mu_0}(v, Y)$ in (6.59), then

$$\xi^{-4m-2+\mu_0} \widetilde{K}^{\langle 2\rangle}_{\mu_0}(\tau,\xi) = H(0,\, -4m+\mu_0-1,\, 0,\, 0)\,.$$

Therefore if $\mu_0 > 4m+1+\ell$, then

$$\frac{\partial^k}{\partial t_0^k}\frac{\partial^\ell}{\partial \xi^\ell} \widetilde{K}^{\langle 2\rangle}_{\mu_0}(\tau,\xi)$$

is a continuous function of $(t_0, \xi) \in [\,0\,,\,1\,] \times [\,0\,,\,\varepsilon)$. Therefore we can obtain (ii) in Proposition 6.2 by Lemma 6.3.

This completes the proof of Proposition 6.2. □.

7. The Szegö kernel

Let Ω_f be a tube domain satisfying the condition in Section 2. Let $H^2(\Omega_f)$ be the subspace of $L^2(\Omega_f)$ consisting of holomorphic functions F on Ω_f such that

$$\sup_{\varepsilon>0} \int_{\partial\Omega_f} \big|F(z_1,\, z_2+i\varepsilon)\big|^2 \, \mathrm{d}\sigma(z) < \infty\,,$$

where $\mathrm{d}\sigma$ is the measure on $\partial\Omega_f$ given by Lebesgue measure on $\mathbb{C}\times\mathbb{R}$ when we identify $\partial\Omega_f$ with $\mathbb{C}\times\mathbb{R}$ (by the map $(z, t+if(\Im z)) \mapsto (z,t)$). The Szegö projection is the orthogonal projection $\mathbb{S} : L^2(\partial\Omega_f) \to H^2(\Omega_f)$ and we can write

$$\mathbb{S}F(z) = \int_{\partial\Omega_f} S(z,w)F(w)\,\mathrm{d}\sigma(w)\,,$$

where $\mathbb{S} : \Omega_f \times \Omega_f \to \mathbb{C}$ is the *Szegö kernel* of the domain Ω_f. We are interested in the restriction of the Szegö kernel on the diagonal, so we write $S(z) = S(z,z)$.

The Szegö kernel of Ω_f has an integral representation:

$$S(z) = \frac{1}{(4\pi)^2}\iint_{\Lambda^*} e^{-x\zeta_1 - y\zeta_2}\frac{1}{D(\zeta_1,\zeta_2)}\,\mathrm{d}\zeta_1\,\mathrm{d}\zeta_2\,,$$

where $(x,y) = (\Im z_1, \Im z_2)$ and $D(\zeta_1,\zeta_2)$ is as in Section 3 (3.2).

We also give an asymptotic expansion of the Szegö kernel of Ω_f. The theorem below can be obtained in a fashion similar to the case of the Bergman kernel, so we omit the proof.

THEOREM 7.1.— *The Szegö kernel of Ω_f has the form in some neighborhood of z^0:*

$$S(z) = \frac{\Phi^S(\tau,\varrho^{1/m})}{\varrho^{1+1/m}} + \widetilde{\Phi}^S(\tau,\varrho^{1/m})\log\varrho\,,$$

where $\Phi^S \in C^\infty([0\,,1]\times[0\,,\varepsilon))$ and $\widetilde{\Phi}^S \in C^\infty([0\,,1]\times[0\,,\varepsilon))$ with some $\varepsilon > 0$.

Moreover Φ^S is written in the form on the set $\{\tau > \alpha\varrho^{1/2m}\}$ with some $\alpha > 0$: for every nonnegative integer μ_0,

$$\Phi^S(\tau,\varrho^{1/m}) = \sum_{\mu=0}^{\mu_0} c_\mu^S(\tau)\varrho^{\mu/m} + R_{\mu_0}^S(\tau,\varrho^{1/m})\varrho^{\mu_0/m+1/2m}\,,$$

where

$$c_\mu^S(\tau) = \frac{\varphi_\mu^S(\tau)}{\tau^{2+2\mu}} + \psi_\mu^S(\tau)\log\tau\,,$$

for φ_μ^S, $\psi_\mu^S \in C^\infty([0\,,1])$, φ_0^S is positive on $[0\,,1]$ and $R_{\mu_0}^S$ satisfies $|R_{\mu_0}^S(\tau,\varrho^{1/m})| \le C_{\mu_0}^S(\tau - \alpha\varrho^{1/2m})^{-3-2\mu_0}$ for some positive constant $C_{\mu_0}^S$.

Acknowlegments

I would like th thank Professors Kazuo Okamoto, Takeo Ohsawa and Katsunori Iwasaki for generosity and several valuable suggestions. I would also like to thank Professor Klas Diederich for giving me some important information. Professor Iwasaki carefully read the manuscript and supplied many corrections and Professor Mitsuhiko Kohno translated my abstract in French.

References

[1] ALADRO (G.) .— *The compatibility of the Kobayashi approach region and the admissible approach region*, Illinois J. Math. **33** (1989), pp. 27-41 .

[2] BERGMAN (S.) .— *Zur Theorie von pseudokonformen Abbildungen*, Math. Sbornik. Akad. Nauk SSSR (1936), pp. 79-96.

[3] BOAS (H. P.), FU (S.) and STRAUBE (E. J.) .— *The Bergman kernel function: Explicit formulas and zeros*, prepint.

[4] BOAS (H. P.), STRAUBE (E. J.) and YU (J.) .— *Boundary limits of the Bergman kernel and metric*, Michigan Math. J. **42** (1995), pp. 449-461.

[5] BOICHU (D.) and CŒURÉ (G.) .— *Sur le noyau de Bergman des domaines de Reinhardt*, Invent. Math. **72** (1983), pp. 131-152.

[6] BONAMI (A.) and LOHOUÉ (N.) .— *Projecteurs de Bergman et Szegö pour une classe de domaines faiblement pseudo-convexes et estimation L^p*, Compositio Math. **46**, n° 2 (1982), pp. 159-226.

[7] BOUTET DE MONVEL (L.) and SJÖSTRAND (J.) .— *Sur la singularité des noyaux de Bergman et de Szegö*, Soc. Math. de France Astérique **34-35** (1976), pp. 123-164.

[8] CATLIN (D.) .— *Estimates of invariant metric on pseudoconvex domains of dimension two*, Math. Z. **200** (1989), pp. 429-266.

[9] CHALMERS (B. L.) .— *On boundary behavior of the Bergman kernel function and related domain functionals*, Pacific J. Math. **29** (1969), pp. 243-250.

[10] CHO (S.) .— *Boundary behavior of the Bergman kernel function on some pseudo-convex domains in $\mathbb{C}^n$*, Trans. of A.M.S. **345** (1994), pp. 803-817.

[11] D'ANGELO (J. P.) .— *A Note on the Bergman Kernel*, Duke Math. J. **45** (1978), pp. 259-265.

[12] D'ANGELO (J. P.) .— *Real hypersurfaces, orders of contact, and applications*, Ann. of Math. **115** (1982), pp. 615-637.

[13] D'ANGELO (J. P.) .— *An explicit computation of the Bergman kernel function*, J. Geom. Analysis **4** (1994), pp. 23-34.

[14] DIEDERICH (K.) .— *Das Randverhalten der Bergmanschen Kernfunktion und Metrik in streng pseudo-konvexen Gebieten*, Math. Ann. **187** (1970), pp. 9-36.

[15] DIEDERICH (K.) .— *Ueber die 1. und 2. Ableitungen der Bergmanschen Kernfunktion und ihr Randverhalten*, Math. Ann. **203** (1973), pp. 129-170.

[16] DIEDERICH (K.) and HERBORT (G.) .— *Geometric and analytic boundary invariants. Comparison results*, J. Geom. Analysis **3** (1993), pp. 237-267.

[17] DIEDERICH (K.) and HERBORT (G.) .— *Pseudoconvex domains of semiregular type*, Contribution to Complex Analysis and Analytic Geometry, Aspects of Mathematics **E26**, Vieweg 1994.

[18] DIEDERICH (K.) and HERBORT (G.) .— *An alternative proof of a theorem of Boas–Straube–Yu*, Complex Analysis and Geometry, Pitman Research notes in Mathematics Series, **366** (1997), pp. 112-118.

[19] DIEDERICH (K.), HERBORT (G.) and OHSAWA (T.) .— *The Bergman kernel on uniformy extendable pseudoconvex domains*, Math. Ann. **273** (1986), pp. 471-478.

[20] ERDÉLYI (A.) .— *Asymptotic expansions*, Dover, New York (1956).

[21] FEFFERMAN (C.) .— *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math. **26** (1974), pp. 1-65.

[22] FRANCSICS (G.) and HANGES (N.) .— *Explicit formulas for the Szegö kernel on certain weakly pseudoconvex domains*, Proc. A.M.S. **123** (1995), pp. 3161-3168.

[23] FRANCSICS (G.) and HANGES (N.) .— *The Bergman kernel of complex ovals and multivariable hypergeometric functions*, to appear in J. Funct. Analysis.

[24] GEBELT (N. W.) .— *The Bergman kernel on certain weakly pseudoconvex domains*, Math. Z. **220** (1995), pp. 1-9.

[25] GONG (S.) and ZHENG (X.) .— *The Bergman kernel function of some Reinhardt domains*, Trans. of A.M.S. **348** (1996), pp. 1771-1803.

[26] GREINER (P. C.) and STEIN (E. M.) .— *On the solvability of some differential operators of* $\Box_b$, Proc. Internat. Conf. (Cortona, Italy, 1976-1977), Scuola Norm. Sup. Pisa (1978), pp. 106-165.

[27] HASLINGER (F.) .— *Szegö kernels of certain unbounded domains in* $\mathbb{C}^2$, Rev. Roumaine Math. Pures Appl. **39** (1994), pp. 939-950.

[28] HASLINGER (F.) .— *Singularities of the Szegö kernels for certain weakly pseudoconvex domains in* $\mathbb{C}^2$, J. Funct. Analysis **129** (1995), pp. 406-427.

[29] HERBORT (G.) .— *Logarithmic growth of the Bergman kernel for weakly pseudoconvex domains in* $\mathbb{C}^3$ *of finite type*, Manuscripta Math. **45** (1983), pp. 69-76.

[30] HERBORT (G.) .— *The growth of the Bergman kernel on pseudoconvex domains of homogeneous finite diagonal type*, Nagoya Math. J. **126** (1992), pp. 1-24.

[31] HERBORT (G.) .— *On the invariant differential metrics near pseudoconvex boundary points where the Levi form has corank one*, Nagoya Math. J. **130** (1993), pp. 25-54.

[32] HÖRMANDER (L.) .— L^2 *estimates and existance theorems for the* $\overline{\partial}$*-operator*, Acta Math. **113** (1965), pp. 89-152.

[33] ISE (M.) .— *On Thullen domains and Hirzebruch manifolds* I, J. Math. Soc. Japan **26** (1974), pp. 508-522.

[34] KAMIMOTO (J.) .— *Singularities of the Bergman kernel for certain weakly pseudoconvex domains*, to appear in the Journal of Math. Sci. the Univ. of Tokyo.

[35] KAMIMOTO (J.) .— *On the singularities of non-analytic Szegö kernels*, preprint.

[36] KAMIMOTO (J.) .— *On an integral of Hardy and Littlewood*, to appear in Kyushu J. of Math.

[37] KAMIMOTO (J.) .— *The Bergman kernel on decoupled pseudoconvex domains*, to appear in Proc. of I.S.A.A.C. Congress, Reproducing kernels and their applications, Kluwer Academic Publishers.

[38] KOHN (J. J.) .— *Boundary behavior of the $\bar{\partial}$ on weakly pseudoconvex manifolds of dimension two*, J. Diff. Geom. **6** (1972), pp. 523-542.

[39] KOHN (J. J.) .— *A survey of the $\bar{\partial}$-Neumann problem*, Proc. Symp. Pure Math. **41** (1984), pp. 137-145.

[40] KORÁNYI (A.) .— *The Bergman kernel function for tubes over convex cones*, Pacific J. Math. **12** (1962), pp. 1355-1359.

[41] KRANTZ (S. G.) .— *Geometric Analysis and Function Spaces*, C.B.M.S., Amer. Math. Soc. **81**.

[42] KRANTZ (S. G.) .— *Fatou theorems on domains in $\mathbb{C}^n$*, Bull. A.M.S. **16** (1987), pp. 93-96.

[43] KRANTZ (S. G.) .— *Invariant metrics and the boundary behavior of holomorphic functions on domains in $\mathbb{C}^n$*, J. Geom. Analysis **1** (1991), pp. 71-97.

[44] MAJIMA (H.) .— *Asymptotic Analysis for Integrable Connections with Irregular Singular Points*, Lec. Notes in Math., Springer **1075** (1984).

[45] MCNEAL (J. D.) .— *Local geometry of decoupled pseudoconvex domains*, Proceedings in honor of Hans Grauert, Aspekte de Mathematik, Vieweg, Berlin (1990), pp. 223-230.

[46] MCNEAL (J. D.) .— *Estimates on the Bergman kernels of convex domains*, Adv. Math. **109** (1994), pp. 108-139.

[47] MCNEAL (J. D.) .— *On large value of L^2 Holomorphic functions*, Math. Res. Letters **3** (1996), pp. 247-259.

[48] NAGEL (A.) .— *Vector fields and nonisotropic metrics*, Beijing Lectures in Harmonic Analysis (E. M. Stein, ed.), Ann. Math. Studies **112**, Princeton University Press, Princeton, NXJ, 1986, pp. 241-306.

[49] NAGEL (A.), STEIN (E. M.) and WAINGER (S.) .— *Boundary behavior of functions holomorphic in domains of finite type*, Natl. Acad. Sci. USA **78** (1981), pp. 6596-6599.

[50] NAGEL (A.), STEIN (E. M.) and WAINGER (S.) .— *Ball and metrics defined by vector fields I: Basic properties*, Acta Math. **155** (1985), pp. 103-147.

[51] NAKAZAWA (N.) .— *Asymptotic expansion of the Bergman kernel for strictly pseudoconvex complete Reinhardt domains in $\mathbb{C}^2$*, Osaka J. Math. **31** (1994), pp. 291-329.

[52] OHSAWA (T.) .— *Boundary behavior of the Bergman kernel function on pseudoconvex domains*, Publ. R.I.M.S., Kyoto Univ. **20** (1984), pp. 897-902.

[53] OHSAWA (T.) .— *On the extension of L^2 holomorphic functions III: negligible weights*, Math. Z. **219** (1995), pp. 215-225.

[54] SAITOH (S.) .— *Fourier–Laplace transforms and the Bergman spaces*, Proc. of A.M.S. **102** (1988), pp. 985-992.

[55] SIBUYA (Y.) .— *Perturbation of linear ordinary differential equations at irregular singular points*, Funkt. Ekv. **11** (1968), pp. 235-246.

[56] YU (J.) .— *Peak functions on weakly pseudoconvex domains*, Indiana Univ. Math. J. **43** (1994), pp. 1271-1295.