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Exact Controllability of the Wave Equation with Neumann Boundary Condition and Time-Dependent Coefficients(*)

Marcelo Moreira Cavalcanti(1)

Résumé. — On considère la contrôlabilité exacte frontière de l'équation

$$\frac{\partial}{\partial t}\left(\alpha(t)\frac{\partial y}{\partial t}\right)-\sum_{j=1}^{n}\frac{\partial}{\partial xj}\left(\beta(t)a(x)\frac{\partial y}{\partial x_j}\right)=0 \quad \text{dans } \Omega\times\,]0\,,T[\,,$$

lorsque le contrôle est de type Neumann et Ω est un ouvert borné connexe de $\mathbb{R}^n$. On utilise la méthode HUM (*Hilbert Uniqueness Method*) de J.-L. Lions.

Mots-clés : contrôlabilité exacte, Neumann, coefficients dépendant du temps.

Abstract. — In this paper we study the exact boundary controllability for the equation

$$\frac{\partial}{\partial t}\left(\alpha(t)\frac{\partial y}{\partial t}\right)-\sum_{j=1}^{n}\frac{\partial}{\partial xj}\left(\beta(t)a(x)\frac{\partial y}{\partial x_j}\right)=0 \quad \text{in } \Omega\times\,]0\,,T[\,,$$

when the control action is of Neumann type and Ω is a bounded domain in $\mathbb{R}^n$. The result is obtained by applying HUM (Hilbert Uniqueness Method) due to J.-L. Lions

Key-words : Exact controllability, Neumann, time-dependent coefficients.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 bondary Γ, and let Q be the finite cylinder $\Omega \times \,]0\,,T[$ with lateral boundary $\Sigma = \Gamma \times \,]0\,,T[$. We consider the following system with inhomogenous boundary conditions

$$\begin{cases} (\alpha(t)y')' + A(t)y = 0 & \text{in } Q \\ \dfrac{\partial y}{\partial \nu_A} = v & \text{on } \Sigma \\ y(0) = y^0 \text{ and } y'(0) = y^1 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where

$$A(t) = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\beta(t)a(x)\frac{\partial}{\partial x_j}\right). \tag{1.2}$$

The problem of the exact controllability for the system (1.1) is formulated as follows.

PROBLEM 1.1.— *Given $T > 0$ large enough, it is possible, for each pair of initial data $\{y^0, y^1\}$ defined in a suitable space, to find a control v such that solution $y = y(x,t)$ of (1.1) satisfies the condition*

$$y(T) = y'(T) = 0\,.$$

Let us note that when $\alpha(t) = \beta(t) = a(x) = 1$, Problem 1.1 was studied by J.-L. Lions [12] by using HUM and also by I. Lasiecka and R. Triggiani [10] by using the ontoness approach. Many other authors studied the exact controllability of distributed systems with time-dependent or x-dependent coefficients. Among them, we can cite J. Lagnese [8] who firstly worked in boundary controllability of distributed systems with time dependent coefficients and C. Bardos, G. Lebeau and J. Rauch [2] whose work treats to the geometric optics approach in the case of space dependent coefficients. In this direction, we can also cite, V. Komornik [11] who presents an elementary and constructive method to obtain the optimal estimates needed in HUM (Hilbert Uniqueness Method) for the exact controllability of some linear evolution systems, R. Fuentes [7], L. A. Medeiros [14], M. Milla Miranda [15], M. Milla Miranda and L. A. Medeiros [16], J. A. Soriano [17], among others.

In this work we prove that system (1.1) is exactly controllable by making use of HUM, *c.f.* J.-L. Lions [12]. For this end, we employ the multiplier technique to obtain the inverse inequality. When the coefficients depend on time, after making appropriated hypothesis on them, the inverse inequality still remains true; but since standard arguments are not applicable, the regularity of backward problem requires a new proof which is the main task of this work (Theorem 5.1).

In fact, the goal of this work is to show that HUM can be applied to the case of *time-dependent coefficients* with Neumann boundary condition. In order to simplify the computations, we consider the simple operators defined by (1.2). Howover, with appropriated changes, we can extend our results to those ones given by

$$A(t) = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(a(x,t)\frac{\partial}{\partial x_j}\right)$$

with $a(x,t) \geq \xi_0 > 0$ in $\overline{\Omega} \times (0,\infty)$. Now, when we consider the matricial operators

$$A(t) = -\sum_{i,\,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j}(x,t)\frac{\partial}{\partial x_j}\right)$$

the usual arguments cannot be applied even if $i = j$ and $a_{i,j}(x,t) = a_i(x)$ (Remark 1, Sect. 4).

Our paper is divided on sixth chapters. In Section 2, we give notations and state the principal result. In Section 3, we consider the homogeneous problem and in Section 4 we establish the inverse inequality. In Section 5, we study the backward problem and in the last section (Sect. 6) we apply HUM.

2. Notations and Main Result

Let $x^0 \in \mathbb{R}^n$, $\nu(x)$ the unit exterior normal vector at $x \in \Gamma$, $m(x) = x - x^0$, $x \in \mathbb{R}^n$ and

$$R(x^0) = \max\left\{\|m(x)\|; x \in \overline{\Omega}\right\} .$$

In what follows the symbol "$\cdot$" denotes the inner product in $\mathbb{R}^n$. Let us define

$$\Gamma(x^0) = \{x \in \Gamma \mid m(x)\cdot\nu(x) > 0\}$$
$$\Gamma_*(x^0) = \{x \in \Gamma \mid m(x)\cdot\nu(x) \leq 0\} = \Gamma \setminus \Gamma(x^0)$$

$$\Sigma(x^0) = \Gamma(x^0) \times \,]0,T[\quad \text{and} \quad \Sigma_*(x^0) = \Gamma_*(x^0) \times \,]0,T[\, = \Sigma \setminus \Sigma(x^0).$$

Let us introduce some notations that will be used throughout this work. We are going to denote $(\cdot\,,\,\cdot)$ and $|\cdot|$ the inner-product and the norm of $L^2(\Omega)$ respectively. The norm in $H^1(\Omega)$ will be denoted by $\|\cdot\|$.

Let A be the operator defined by the triple $\{H^1(\Omega),\, L^2(\Omega), a(u,v)\}$ where

$$a(u,v) = \sum_{j=1}^{n} \int_\Omega a(x) \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_j}\, dx\,, \quad \forall\, u\,,\; v \in H^1(\Omega)$$

and

$$D(A) = \left\{ u \in H^2(\Omega) \;\Big|\; \frac{\partial u}{\partial \nu_A} = 0 \text{ on } \Gamma \right\}.$$

We recall that the Spectral Theorem for self-adjoint operators guatantees the existence of a complete orthonormal system (ω_ν) of $L^2(\Omega)$ given by the eigenfunctions of A. If (λ_ν) are the corresponding eigenvalues of A, then $\lambda_\nu \to +\infty$ as $\nu \to +\infty$. Besides,

$$D(A) = \left\{ u \in L^2(\Omega) \;\Big|\; \sum_{\nu=1}^{+\infty} \lambda_\nu^2 |(u,\omega_\nu)|^2 < +\infty \right\}$$

and

$$Au = \sum_{\nu=1}^{+\infty} \lambda_\nu (u,\omega_\nu)\,\omega_\nu\,, \quad \forall\, u \in D(A).$$

Considering in $D(A)$ the norm given by the graph, that is,

$$\|u\|_{D(A)} = \left(|u|^2 + |Au|^2\right)^{1/2},$$

it turns that (ω_ν) is a complete system in $D(A)$. In fact, it is known that (ω_ν) is also a complete system in $H^1(\Omega)$. Now, since A is positive, given $\delta > 0$ one has

$$D(A^\delta) = \left\{ u \in L^2(\Omega) \Big| \sum_{\nu=1}^{+\infty} \lambda_\nu^{2\delta} |(u,\omega_\nu)|^2 < +\infty \right\}$$

and

$$A^\delta u = \sum_{\nu=1}^{+\infty} \lambda_\nu^\delta (u, \omega_\nu)\, \omega_\nu\,, \quad \forall\, u \in D(A^\delta)\,.$$

In $D(A^\delta)$ we consider the natural topology given by the norm

$$\|u\|_{D(A^\delta)} = \left(|u|^2 + |A^\delta u|^2\right)^{1/2}.$$

We observe that such operators are also self-adjoint, that is,

$$(A^\delta u, v) = (u, A^\delta v)\,, \quad \forall\, u\,,\; v \in D(A^\delta)\,,$$

$D(A^{1/2}) = H^1(\Omega)$ and $D(A^0) = L^2(\Omega)$. We note that $A(t) = \beta(t)A$. Here, we are using the same symbol for both operators to simplify the notation.

We make the following hypotheses :

(H1) α, $\beta \in W^{1,\infty}_{\text{loc}}(0,\infty)$, α', $\beta' \in L^1(0,\infty)$,

$$\alpha(t) \geq \alpha_0 > 0 \quad \text{and} \quad \beta(t) \geq \beta_0 > 0\,, \qquad \forall\, t \geq 0$$

and $a \in C^1(\overline{\Omega})$ with $a(x) \geq a_0 > 0$, $\forall\, x \in \overline{\Omega}$;

(H2) if $n > 1$,

$$\|\nabla a\|_{C^0(\overline{\Omega})} < a_0 \big(R(x^0)\big)^{-1}\,;$$

(H3) if $n = 1$,

$$\exists\; 0 < \gamma < 1 \quad \text{such that} \quad \|\nabla a\|_{C^0(\overline{\Omega})} < \gamma a_0 \big(R(x^0)\big)^{-1}.$$

Now we are in position to state our main result. Consider the system,

$$\begin{cases} (\alpha(t)y')' + A(t)y = 0 & \text{in } Q \\ \dfrac{\partial y}{\partial \nu_A} = \begin{cases} v_0 & \text{on } \Sigma(x^0) \\ v_1 & \text{on } \Sigma_*(x^0) \end{cases} & \\ y(0) = y^0 \text{ and } y'(0) = y^1 & \text{in } \Omega. \end{cases} \tag{2.1}$$

We have the following result.

THEOREM 2.1.— *Suppose that assumptions (H1)-(H3) are satisfied. Then there exists a time* $T_0 > 0$ *such that for* $T > T_0$ *and initial data* $\{y^0, y^1\} \in L^2(\Omega) \times \left(H^1(\Omega)\right)'$, *there exists a control*

$$v_0 \in \left(H^1\left(0, T; L^2(T(x^0))\right)\right)' \quad \text{and} \quad v_1 \in L^2\left(0, T; \left(H^1(\Gamma_*(x^0))\right)'\right)$$

such that the ultra-weak solution (the solution of (2.1) is defined by the transposition method, see [13]) $y = y(x,t)$ *of (2.1) satisfies*

$$y(T) = y'(T) = 0\,.$$

3. The Homogeneous Problem

In this section we present a standard result and a new one about the solutions of the following homogeneous system

$$\begin{cases} \left(\alpha(t)\theta'\right)' + A(t)\theta = f & \text{in } Q \\ \dfrac{\partial\theta}{\partial\nu_A} = 0 & \text{on } \Sigma \\ \theta(0) = \theta^0 \text{ and } \theta'(0) = \theta^1 & \text{in } \Omega. \end{cases} \tag{3.1}$$

We have the following results.

THEOREM 3.1.— *Suppose that assumption (H1) holds. Then, given* $k \in \{0, 1, 2\}$ *and*

$$\{\theta^0, \theta^1, f\} \in D(A^{(k+1)/2}) \times D(A^{k/2}) \times L^1(0, T\,;\, D(A^{k/2}))$$

problem (3.1) possesses a unique solution $\theta : Q \to \mathbb{R}$ *such that,*

$$\theta \in C^0([\,0\,,T\,]\,;\, D(A^{(k+1)/2})) \cap C^1([\,0\,,T\,]\,;\, D(A^{k/2}))\,.$$

Moreover, the linear map

$$\begin{aligned} &D(A^{(k+1)/2}) \times D(A^{k/2}) \times L^1(0, T\,;\, D(A^{k/2})) \longrightarrow \\ &\qquad \longrightarrow C^0([\,0\,,T\,]\,;\, D(A^{(k+1)/2})) \times C^1([\,0\,,T\,]\,;\, D(A^{k/2})) \\ &\qquad\qquad \{\theta^0, \theta^1, f\} \longmapsto \{\theta, \theta'\} \end{aligned}$$

is continuous.

Theorem 3.1 can be proved in a standard way by applying the Faedo-Galerkin method and using the spectral considerations given in Section 2.

Next we consider the homogeneous problem

$$\begin{cases} (\alpha(t)\theta')' + A(t)\theta = f & \text{in } Q \\ \dfrac{\partial\theta}{\partial\nu_A} = 0 & \text{on } \Sigma \\ \theta(0) = \theta'(0) = 0 & \text{in } \Omega \end{cases} \tag{3.2}$$

which will be used in the study of the regularity of the solution of (2.1).

THEOREM 3.2.— *Given $f \in \mathcal{D}(0,T;\, D(A))$, the unique solution of problem (3.2) satisfies for every $t \in [0\,,T]$*

$$|\alpha^{1/2}A^{1/2}\theta'(t) - \alpha^{-1/2}A^{1/2}f(t)|_{L^2(\Omega)} + |A\theta(t)|_{L^2(\Omega)} \leq C\|f\|_{L^1(0,T\,;D(A))}$$

and

$$|\alpha^{1/2}\theta'(t) - \alpha^{-1/2}f(t)|_{L^2(\Omega)} + |A^{1/2}\theta(t)|_{L^2(\Omega)} \leq C\|f\|_{L^1(0,T\,;H^1(\Omega))}$$

where $C = C(T)$.

Proof.— Since $\theta^0 = \theta^1 = 0$ and $f' \in \mathcal{D}(0,T;\, D(A))$, from Theorem 3.1 the above problem has a unique solution θ such that

$$\theta \in C^0([0\,,T]\,;\, D(A^{3/2})) \cap C^1([0\,,T]\,;\, D(A))\,. \tag{3.3}$$

Besides, this solution satisfies the identity

$$\begin{aligned} &\frac{1}{2}\left\{\alpha(t)|A^{1/2}\theta'(t)|^2 + \beta(t)|A\theta(t)|^2\right\} = \\ &= \frac{1}{2}\int_0^t \beta'(s)|A\theta(s)|^2\,\mathrm{d}s - \frac{1}{2}\int_0^t \alpha'(s)|A\theta'(s)|^2\,\mathrm{d}s + \\ &+ \int_0^t (A^{1/2}f'(s)\,,\, A^{1/2}\theta'(s))\,\mathrm{d}s\,. \end{aligned} \tag{3.4}$$

From (3.3) we get $A\theta \in C^0([0\,,T]\,;\, D(A^{1/2}))$ and therefore

$$(\alpha\theta')' \in L^1(0,T\,;\, D(A^{1/2}))\,.$$

This togheter with assumption (H1) implies that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\alpha^{-1}(s)A^{1/2}f(s)\,,\,\alpha(s)A^{1/2}\theta'(s)\right) =$$

$$= -\left(\frac{\alpha'(s)}{\alpha^2(s)}\,A^{1/2}f(s)\,,\,\alpha(s)A^{1/2}\theta'(s)\right) +$$

$$+\left(\alpha^{-1}(s)A^{1/2}f'(s)\,,\,\alpha(s)A^{1/2}\theta'(s)\right) +$$

$$+\left(\alpha^{-1}(s)A^{1/2}f(s)\,,\,A^{1/2}\Big(\big(\alpha(s)\theta'(s)\big)'\Big)\right).$$

Integrating this equality and noting that $f(0)=0$ we have

$$\int_0^t \left(A^{1/2}f'(s)\,,\,A^{1/2}\theta'(s)\right)\mathrm{d}s =$$

$$= \left(\alpha^{-1}(t)A^{1/2}f(t)\,,\,\alpha(t)A^{1/2}\theta'(t)\right) +$$

$$+\int_0^t \left(\alpha'(s)\alpha^{-1}(s)A^{1/2}f(s)\,,\,A^{1/2}\theta'(s)\right)\mathrm{d}s +$$

$$-\int_0^t \left(\alpha^{-1}(s)A^{1/2}f(s)\,,\,A^{1/2}\Big(\big(\alpha(s)\theta'(s)\big)'\Big)\right)\mathrm{d}s\,.$$

Replacing $(\alpha\theta')'$ by $f' - \beta A\theta$ in the last integral we obtain

$$\int_0^t \left(A^{1/2}f'(s)\,,\,A^{1/2}\theta'(s)\right)\mathrm{d}s =$$

$$= \left(A^{1/2}(t)\,,\,A^{1/2}\theta'(t)\right) + \int_0^t \left(\alpha'(s)\alpha^{-1}(s)A^{1/2}f(s)\,,\,A^{1/2}\theta'(s)\right)\mathrm{d}s +$$

$$-\int_0^t \left(\alpha^{-1}(s)A^{1/2}f(s)\,,\,A^{1/2}f'(s)\right)\mathrm{d}s +$$

$$+\int_0^t \left(\alpha^{-1}(s)A^{1/2}f(s)\,,\,\beta(s)A^{1/2}\big(A\theta(s)\big)\right)\mathrm{d}s\,. \tag{3.5}$$

Now integrating by part and noting that $f(0)=0$,

$$\int_0^t \left(\alpha^{-1}(s)A^{1/2}f(s)\,,\,A^{1/2}f'(s)\right)\mathrm{d}s =$$

$$= \frac{1}{2}\left(\alpha^{-1}(t)A^{1/2}f(t)\,,\,A^{1/2}f(t)\right) + \tag{3.6}$$

$$+\frac{1}{2}\int_0^t \left(\alpha'(s)\alpha^{-2}(s)A^{1/2}f(s)\,,\,A^{1/2}f(s)\right)\mathrm{d}s\,.$$

Replacing (3.6) in (3.5) we have

$$
\begin{aligned}
&\int_0^t \big(A^{1/2}f'(s)\,,\,A^{1/2}\theta'(s)\big)\,\mathrm{d}s = \\
&\quad= \big(A^{1/2}f(t)\,,\,A^{1/2}\theta'(t)\big) + \int_0^t \big(\alpha'(s)\alpha^{-1}(s)A^{1/2}f(s)\,,\,A^{1/2}\theta'(s)\big)\,\mathrm{d}s\,+ \\
&\qquad - \frac{1}{2}\big(\alpha^{-1}(t)A^{1/2}f(t)\,,\,A^{1/2}f(t)\big) + \\
&\qquad - \frac{1}{2}\int_0^t \big(\alpha'(s)\alpha^{-2}(s)A^{1/2}f(s)\,,\,A^{1/2}f(s)\big)\,\mathrm{d}s\,+ \\
&\qquad + \int_0^t \Big(\alpha^{-1}(s)A^{1/2}f(s)\,,\,\beta(s)A^{1/2}\big(A\theta(s)\big)\Big)\,\mathrm{d}s\,.
\end{aligned}
\tag{3.7}
$$

From (3.4) and (3.7) it follows that

$$
\begin{aligned}
&\frac{1}{2}\,\big|\alpha^{1/2}(t)A^{1/2}\theta'(t)\,,\,\alpha^{-1/2}(t)A^{1/2}f(t)\big|^2 + \frac{1}{2}\,\beta(t)\big|A\theta(t)\big|^2 = \\
&\quad= \frac{1}{2}\int_0^t \beta'(s)\big|A\theta(s)\big|^2\,\mathrm{d}s - \frac{1}{2}\int_0^t \alpha'(s)\big|A^{1/2}\theta'(s)\big|^2\,\mathrm{d}s\,+ \\
&\qquad + \int_0^t \big(\alpha'(s)\alpha^{-1}(s)A^{1/2}f(s)\,,\,A^{1/2}\theta'(s)\big)\,\mathrm{d}s\,+ \\
&\qquad - \frac{1}{2}\int_0^t \big(\alpha'(s)\alpha^{-2}(s)A^{1/2}f(s)\,,\,A^{1/2}f(s)\big)\,\mathrm{d}s\,+ \\
&\qquad + \int_0^t \Big(\alpha^{-1}(s)A^{1/2}f(s)\,,\,\beta(s)A^{1/2}\big(A\theta(s)\big)\Big)\,\mathrm{d}s\,.
\end{aligned}
$$

Defining $\alpha^{1/2}\theta' - \alpha^{-1/2}f = \varphi$ and replacing θ' by $\alpha^{-1/2}\varphi + \alpha^{-1}f$ in the above expression we obtain

$$
\begin{aligned}
&\frac{1}{2}\,\big|A^{1/2}\varphi(t)\big|^2 + \frac{1}{2}\,\beta(t)\big|A\theta(t)\big|^2 = \\
&\quad= \frac{1}{2}\int_0^t \beta'(s)\big|A\theta(s)\big|^2\,\mathrm{d}s - \frac{1}{2}\int_0^t \alpha'(s)\alpha^{-1}\big|A^{1/2}\varphi(s)\big|^2\,\mathrm{d}s\,+ \\
&\qquad + \int_0^t \alpha^{-1}(s)\beta(s)\big(Af(s)\,,\,A\theta(s)\big)\,\mathrm{d}s\,.
\end{aligned}
\tag{3.8}
$$

From hypotheses (H1), (H2) and (3.8) there exists a constant $C > 0$ independent of f and θ such that

$$\begin{aligned}\frac{1}{2}\left|A^{1/2}\varphi(t)\right|^2 + \frac{1}{2}\left|A\theta(t)\right|^2 \leq \\ \leq C\left\{\int_0^t \left|A\theta(s)\right|^2 \mathrm{d}s + \frac{1}{2}\int_0^t \left|A^{1/2}\varphi(s)\right|^2 \mathrm{d}s + \right. \\ \left. + \int_0^t \left|Af(s)\right| \left(\left|A\theta(s)\right| + \left|A^{1/2}\varphi(s)\right|\right) \mathrm{d}s\right\}\end{aligned}$$

Applying Gronwall's inequality twice (first we consider the Gronwall inequality $(1/2)g^2(t) \leq \int_0^t m(s)g(s)\,\mathrm{d}s$ where $g(t) = \left|A^{1/2}\varphi(t)\right| + \left|A(\theta)\right|$ and $m(t) = 2C\left(g(t) + \left|Af(t)\right|\right)$ and after the usual one), we get

$$\left|A^{1/2}\varphi(t)\right| + \left|A\theta(t)\right| \leq C\|f\|_{L^1(0,T\,;D(A))}\,, \quad \forall\, t \in [\,0\,,T\,]\,.$$

In a similar way we also infer that

$$\left|\varphi(t)\right| + \left|A\theta(t)\right| \leq C\|f\|_{L^1(0,T\,;H^1(\Omega))}\,, \quad \forall\, t \in [\,0\,,T\,]\,.$$

Using the definition on φ we obtain the desired inequalities. □

4. The Inverse Inequality

In this section we construct a special T_0 time depending on n, $R(x^0)$, on the functions $\alpha(t)$, $\beta(t)$, $a(t)$ and also on the geometry of Ω.

Taking into account the regularity of Γ, we can define on Γ a unit exterior normal vector field $\nu(x)$ on class C^1. In the same way we can define a family of $(n-1)$ tangent vector field $\{\tau^1(x), \ldots, \tau^{n-1}(x)\}$ of class C^1 such that the family $\{\nu(x), \tau^1(x), \ldots, \tau^{n-1}(x)\}$ defines an orthonormal basis for $\mathbb{R}^n$, for each $x \in \Gamma$. If $\varphi : \overline{\Omega} \to \mathbb{R}$ is a regular function, we have

$$\frac{\partial \varphi}{\partial x_j} = \nu_j \frac{\partial \varphi}{\partial \nu} + \sum_{k=1}^{n-1} \tau_j^k \frac{\partial \varphi}{\partial \tau^k} \quad \text{on } \Gamma\,,\ j = 1, \ldots, n\,, \tag{4.1}$$

where

$$\frac{\partial \varphi}{\partial \nu} = \nabla\varphi \cdot \nu \quad \text{and} \quad \frac{\partial \varphi}{\partial \tau^k} = \nabla\varphi \cdot \tau^k\,.$$

Defining

$$\sigma_j(\varphi) = \sum_{k=1}^{n-1} \tau_j^k \frac{\partial \varphi}{\partial \tau^k} \tag{4.2}$$

we obtain from (4.1) and (4.2)

$$\frac{\partial \varphi}{\partial x_j} = \nu_j \frac{\partial \varphi}{\partial \nu} + \sigma_j \varphi \quad \text{on } \Gamma\,,\ j = 1, \ldots, n\,. \tag{4.3}$$

We observe that when $\partial\varphi/\partial\nu_A = 0$ on Γ then $\partial\varphi/\partial\nu = 0$ since

$$\frac{\partial \varphi}{\partial \nu_A} = a(x)\frac{\partial \varphi}{\partial \nu} \quad \text{on } \Gamma \quad \text{and} \quad a(x) \geq a_0 > 0\,.$$

Then, defining $\nabla_\sigma \varphi = (\sigma_1\varphi, \ldots, \sigma_n\varphi)$, we obtain from (4.3)

$$\nabla_\sigma \varphi = \nabla \varphi \quad \text{on } \Gamma \tag{4.4}$$

and consequently

$$|\nabla\varphi|^2 = |\nabla_\sigma\varphi|^2 = \sum_{j=1}^{n} |\sigma_j\varphi|^2 \quad \text{on } \Gamma\,. \tag{4.5}$$

Remark 1.— At this point we observe that when A is a matricial operator that is, when it is given by

$$A(t) = -\sum_{i,\,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_j}\right)$$

then we have

$$\frac{\partial y}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial y}{\partial x_j}\nu_i$$

and therefore if $\partial y/\partial\nu_A = 0$ we do not have necessarely that $\partial y/\partial\nu = 0$ and consequently we cannot use the identity

$$|\nabla y|^2 = |\nabla_\sigma y|^2 \quad \text{on } \Sigma_0$$

even if $i = j$ and $a_{ij}(x,t) = a_j(x)$. As this identity plays an essential role to prove the inverse inequality, this case requires another treatment which will not be considered in this work.

If $\varphi \in H^2(\Omega)$ we can define in a natural way a continuous linear operator

$$\sigma_j^1 : H^2(\Omega) \longrightarrow H^{1/2}(\Gamma) \tag{4.6}$$

such that

$$\sigma_j^1 \varphi = \sigma_j \varphi \quad \text{on } \Gamma\,,\ \forall\ \varphi \in C^2(\overline{\Omega})\,. \tag{4.7}$$

In addition, we can also define a continuous linear operator

$$\sigma_j^2 : H^1(\Gamma_0) \longrightarrow L^2(\Gamma_0) \tag{4.8}$$

where Γ_0 is a nonempty open subset of Γ (sometimes the whole Γ) such that

$$\sigma_j^2 \varphi\big|_{\Gamma_0} = (\sigma_j \varphi)\big|_{\Gamma_0} \quad \text{on } \Gamma_0\,,\ \forall\ \varphi \in C^2(\overline{\Omega})\,. \tag{4.9}$$

Thus, from (4.7), (4.9) and by density arguments it results that

$$(\sigma_j^1 u)\big|_{\Gamma_0} = \sigma_j^2 u\big|_{\Gamma_0} \quad \text{on } \Gamma_0\,,\ \forall\ u \in H^2(\Omega)\,. \tag{4.10}$$

Considering the above equality we are able to define the tangential gradient

$$\begin{aligned} \nabla_\sigma u &= \left((\sigma_1^1 u)\big|_{\Gamma_0}, \dots, (\sigma_n^1 u)\big|_{\Gamma_0} \right) \\ &= \left(\sigma_1^2 u\big|_{\Gamma_0}, \dots, \sigma_n^2 u\big|_{\Gamma_0} \right), \quad \forall\ u \in H^2(\Omega)\,. \end{aligned}$$

Dropping the index "2" in (4.8) to simplify the notation, we define the adjoint operator

$$\sigma_j^* : L^2(\Gamma_0) \longrightarrow \left(H^1(\Gamma_0)\right)'$$

$$\left\langle \sigma_j^* \psi\,,\, \varphi \right\rangle = \left(\psi, \sigma_j \varphi\right)_{L^2(\Gamma_0)}, \quad \forall\ \varphi \in H^1(\Gamma_0) \tag{4.11}$$

and from (4.8) and (4.11) we obtain the continuous linear operator

$$\begin{aligned} -\Delta_{\Gamma_0} : H^1(\Gamma_0) &\longrightarrow \left(H^1(\Gamma_0)\right)' \\ \varphi &\longmapsto -\Delta_{\Gamma_0} \varphi = \sum_{j=1}^{n} (\sigma_j^* \circ \sigma_j) \varphi \end{aligned}$$

where "o" denotes composition.

Hence, for all φ, $\psi \in H^1(\Gamma_0)$,

$$\langle -\Delta_{\Gamma_0}\varphi\,,\,\psi\rangle = \int_{\Gamma_0} \nabla_\sigma\varphi\cdot\nabla_\sigma\psi\,\mathrm{d}\Gamma\,. \tag{4.12}$$

In particular

$$\langle -\Delta_{\Gamma_0}\varphi\,,\,\varphi\rangle = \int_{\Gamma_0} |\nabla_\sigma\varphi|^2\,\mathrm{d}\Gamma\,. \tag{4.13}$$

THEOREM 4.1.— *Let θ be the weak solution (it means that the initial data $\{\theta^0,\theta^1\}\in H^1(\Omega)\times L^2(\Omega)$) of the Problem (3.1). Then, if $f=0$,*

$$e^{-C_0}E(0)\le E(t)\le e^{C_0}E(0)\,,\quad \forall\, t\ge 0\,,$$

where

$$C_0=\max\{\alpha_0^{-1},\beta_0^{-1}\}\int_0^{+\infty}\left(|\alpha'(t)|+|\beta'(t)|\right)\,\mathrm{d}t\,.$$

and

$$E(t)=\frac{1}{2}\left(\int_\Omega \alpha(t)|\theta'(x,t)|^2\,\mathrm{d}x+\int_\Omega \beta(t)a(x)|\nabla\theta(x,t)|^2\,\mathrm{d}x\right)\,. \tag{4.14}$$

Proof.— We consider first that $\{\theta^0,\theta^1\}\in D(A)\times H^1(\Omega))$. Then, from Theorem 3.1, there exists a unique solution θ in the class

$$\theta\in C^0([0\,,T]\,;\,D(A))\cap C^1([0\,,T]\,;\,H^1(\Omega))\,.$$

Multiplying (3.1), by $\theta'(t)$ we obtain

$$\alpha'(t)|\theta'(t)|^2+\alpha(t)\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\theta'(t)|^2+\beta(t)\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|a^{1/2}(x)\nabla\theta(t)|^2=0\,.$$

Integrating this expression from 0 to t and then integrating by parts we get

$$\begin{aligned}&\frac{1}{2}\left(\alpha'(t)|\theta'(t)|^2+\beta(t)|a^{1/2}(x)\nabla\theta(t)|^2\right)=\\&\quad=\frac{1}{2}\left(\alpha(0)|\theta^1|^2+\beta(0)|a^{1/2}(x)\nabla\theta^0|^2\right)+\\&\qquad-\frac{1}{2}\int_0^t\alpha'(s)|\theta'(s)|^2\,\mathrm{d}s+\frac{1}{2}\int_0^t\beta'(s)|a^{1/2}(x)\nabla\theta(s)|^2\,\mathrm{d}s\,.\end{aligned}$$

Taking into account (4.14) we can rewrite the above expression as follows

$$0 \leq E(t) = E(0) - \frac{1}{2}\int_0^t \alpha'(s)\left|\theta'(s)\right|^2 \mathrm{d}s + \\ + \frac{1}{2}\int_0^t \beta'(s)\left|a^{1/2}(x)\nabla\theta(s)\right|^2 \mathrm{d}s\,.$$

On the other hand, differentiating $E(t)$ we have

$$E'(t) = -\frac{1}{2}\,\alpha'(t)\left|\theta'(t)\right|^2 + \frac{1}{2}\,\beta'(t)\left|a^{1/2}(x)\nabla\theta(t)\right|^2.$$

and

$$\left|E'(t)\right| \leq \max\{\alpha_0^{-1},\beta_0^{-1}\}\left(\left|\alpha'(t)\right| + \left|\beta'(t)\right|\right) \times \\ \times\left(\alpha(t)\left|\theta'(t)\right|^2 + \beta(t)\left|a^{1/2}(x)\nabla\theta(t)\right|^2\right).$$

So

$$\left|E'(t)\right| \leq G(t)E(t)$$

where

$$G(t) = \max\{\alpha_0^{-1},\beta_0^{-1}\}\left(\left|\alpha'(t)\right| + \left|\beta'(t)\right|\right).$$

The above inequality gives,

$$-G(t)E(t) \leq E'(t) \leq G(t)E(t)\,. \tag{4.15}$$

Now, considering

$$C_0 = \int_0^{+\infty} G(s)\,\mathrm{d}s$$

it follows from (4.15) that

$$e^{-C_0}E(0) \leq E(t) \leq e^{C_0}E(0)\,, \quad \forall\, t \geq 0\,.$$

Finally, considering

$$\{\theta^0,\theta^1\} \in H^1(\Omega)\times L^2(\Omega)$$

we obtain the desired result by using density arguments. □

THEOREM 4.2.— *Let $q = (q_k)_{1\le k\le n}$ be a vector field such that $q \in (C^1(\overline{\Omega}))^n$. Then each weak solution ϕ of Problem (3.1) satifies*

$$\frac{1}{2}\int_\Sigma q_k\nu_k\left(\alpha(t)|\phi'(t)|^2 - \beta(t)a(x)|\nabla_\sigma\phi(t)|^2\right) d\Sigma =$$
$$= \left(\alpha(t)\phi'(t)\,,\, q_k\frac{\partial\phi(t)}{\partial x_k}\right)\Big|_0^T + \frac{1}{2}\int_Q \alpha(t)\frac{\partial q_k}{\partial x_k}|\phi'|^2\, dx\, dt +$$
$$-\frac{1}{2}\int \beta(t)a(x)\frac{\partial q_k}{\partial x_k}|\nabla\phi|^2\, dx\, dt + \int_Q \beta(t)a(x)\frac{\partial\phi}{\partial x_i}\frac{\partial q_k}{\partial x_i}\frac{\partial\phi}{\partial x_k}\, dx\, dt +$$
$$-\frac{1}{2}\int \beta(t)\frac{\partial a(x)}{\partial x_k}q_k|\nabla\phi|^2\, dx\, dt - \int_Q f q_k\frac{\partial\phi}{\partial x_k}\, dx\, dt\,.$$

Proof.— First we prove the identity for the strong (it means that the initial data $\{y^0, y^1\} \in D(A)\times H^1(\Omega)$) solutions of (3.1) and then the result follows by a density arguments. So, let us suppose that

$$\phi \in C^0([0\,,T]\,;\,D(A)) \cap C^1([0\,,T]\,;\,H^1(\Omega))\,.$$

By multiplying the first equation of (3.1) by $q_k\partial\phi/\partial x_k$ and integrating over Q,

$$\int_Q (\alpha(t)\phi')' q_k\frac{\partial\phi}{\partial x_k}\, dx\, dt - \int_Q \beta(t)\frac{\partial}{\partial x_i}\left(a(x)\frac{\partial\phi}{\partial x_i}\right) q_k\frac{\partial\phi}{\partial x_k}\, dx\, dt = \\ = \int_Q f q_k\frac{\partial\phi}{\partial x_k}\, dx\, dt\,. \tag{4.16}$$

Integrating by parts the left side of equality (4.16) we get

$$\int_Q (\alpha(t)\phi')' q_k\frac{\partial\phi}{\partial x_k}\, dx\, dt = \\ = \left(\alpha(t)\phi'(t)\,,\, q_k\frac{\partial\phi(t)}{\partial x_k}\right)\Big|_0^T - \int_Q \alpha(t)q_k\phi'\frac{\partial\phi'}{\partial x_k}\, dx\, dt\,. \tag{4.17}$$

On the other hand, since

$$\int_Q \alpha(t)q_k\phi'\frac{\partial\phi'}{\partial x_k}\, dx\, dt = \frac{1}{2}\int_Q \alpha(t)q_k\frac{\partial}{\partial x_k}(\phi')^2\, dx\, dt$$

we have from (4.17) that

$$\begin{aligned} &\int_Q (\alpha(t)\phi')' q_k \frac{\partial \phi}{\partial x_k}\, \mathrm{d}x\, \mathrm{d}t = \\ &= \left(\alpha(t)\phi'(t)\,,\, q_k \frac{\partial \phi(t)}{\partial x_k}\right)\Big|_0^T - \frac{1}{2}\int_Q \alpha(t) q_k \frac{\partial \phi'}{\partial x_k}\,(\phi')^2\, \mathrm{d}x\, \mathrm{d}t\,. \end{aligned} \tag{4.18}$$

We also have

$$\begin{aligned} &\frac{1}{2}\int_Q \alpha(t) q_k \frac{\partial}{\partial x_k}\,(\phi')^2\, \mathrm{d}x\, \mathrm{d}t = \\ &= -\frac{1}{2}\int_Q \alpha(t)\frac{\partial q_k}{\partial x_k}\,|\phi'|^2\, \mathrm{d}x\, \mathrm{d}t + \frac{1}{2}\int_\Sigma \alpha(t) q_k |\phi'|^2 \nu_k\, \mathrm{d}\Sigma\,. \end{aligned} \tag{4.19}$$

Thus, combining (4.19) and (4.18) we obtain

$$\begin{aligned} &\int_Q (\alpha(t)\phi')' q_k \frac{\partial \phi}{\partial x_k}\, \mathrm{d}x\, \mathrm{d}t = \\ &= \left(\alpha(t)\phi'(t)\,,\, q_k \frac{\partial \phi(t)}{\partial x_k}\right)\Big|_0^T + \frac{1}{2}\int_Q \alpha(t)\frac{\partial q_k}{\partial x_k}\,|\phi'|^2\, \mathrm{d}x\, \mathrm{d}t + \\ &\quad - \frac{1}{2}\int_\Sigma \alpha(t) q_k |\phi'|^2 \nu_k\, \mathrm{d}\Sigma\,. \end{aligned} \tag{4.20}$$

Now, evaluating the right side of (4.16) we have from Green identity

$$\begin{aligned} &-\int_Q \beta(t)\frac{\partial}{\partial x_i}\left(a(x)\frac{\partial \phi}{\partial x_i}\right) q_k \frac{\partial \phi}{\partial x_k}\, \mathrm{d}x\, \mathrm{d}t = \\ &= \int_Q \beta(t) a(x)\frac{\partial \phi}{\partial x_i}\,\frac{\partial q_k}{\partial x_i}\,\frac{\partial \phi}{\partial x_k}\, \mathrm{d}x\, \mathrm{d}t - \frac{1}{2}\int_Q \beta(t)\frac{\partial a}{\partial x_k}\, q_k |\nabla\phi|^2\, \mathrm{d}x\, \mathrm{d}t + \\ &\quad -\frac{1}{2}\int_Q \beta(t) a(x)\frac{\partial q_k}{\partial x_k}\,|\nabla\phi|^2\, \mathrm{d}x\, \mathrm{d}t + \frac{1}{2}\int_\Sigma \beta(t) a(x) q_k \nu_k |\nabla\phi|^2\, \mathrm{d}x\, \mathrm{d}t\,. \end{aligned} \tag{4.21}$$

Combining (4.16), (4.20), (4.21) and (4.5) we obtain the desired identity. □

The T_0 time which Theorem 2.1 is defined by

$$\begin{aligned}
T_0 &= T(x^0, \alpha, \beta, a) \\
&= 2\max\{\alpha_0^{-1}, \beta_0^{-1} a_0^{-1}\}\, e^{C_0} R(x^0) \|\alpha\|_{L^\infty(0,T)} \times \\
&\quad \times \left(1 - \|\nabla a\|_{C^0(\overline{\Omega})} a_0^{-1} R(x^0)\right)^{-1} \quad \text{if } n > 1 \\
&= 2\max\{\alpha_0^{-1}, \beta_0^{-1} a_0^{-1}\}\, e^{C_0} R(x^0) \|\alpha\|_{L^\infty(0,T)} \times \\
&\quad \times \left(\gamma - \left\|\frac{\partial a}{\partial x}\right\|_{C^0(\overline{\Omega})} a_0^{-1} R(x^0)\right)^{-1} \quad \text{if } n = 1
\end{aligned}$$

and uniquely depends on n, $R(x^0)$, $\alpha(t)$, $\beta(t)$, $a(t)$ and on the geometry of Ω.

THEOREM 4.3.— *Suppose that hypotheses (H1), (H2) and (H3) hold and that $T > T_0$. Then for each weak solution ϕ of (3.1) with $f = 0$ there exists $C > 0$ such that:*

(i) if $n > 1$ then

$$\begin{aligned}
&\|\phi^0\|^2_{H^1(\Omega)} + |\phi^1|^2_{L^2(\Omega)} \leq \\
&\leq C\left\{\int_\Sigma m\cdot\nu\left(\alpha(t)|\phi'|^2 - \beta(t)a(x)|\nabla_\sigma\phi|^2\right)\,\mathrm{d}\Sigma + \right.\\
&\quad \left. + \int_\Gamma m\cdot\nu\left(|\phi(0)|^2 + |\phi(T)|^2\right)\,\mathrm{d}\Gamma\right\} ;
\end{aligned}$$

(ii) if $n = 1$ then

$$\begin{aligned}
&\|\phi^0\|^2_{H^1(\Omega)} + |\phi^1|^2_{L^2(\Omega)} \leq \\
&\leq C\left\{\int_\Sigma m\alpha(t)|\phi'|^2\,\mathrm{d}\Sigma + \int_\Gamma m\left(|\phi(0)|^2 + |\phi(T)|^2\right)\,\mathrm{d}\Gamma\right\}.
\end{aligned}$$

Proof.— By using the identity given in Theorem 4.2 with $q(x) = m(x) = x - x^0$, we get after some calculations

$$
\begin{aligned}
&\frac{1}{2}\int_\Sigma m\cdot\nu\Big(\alpha(t)\big|\phi'\big|^2-\beta(t)a(x)\big|\nabla_\sigma\phi\big|^2\Big)\,\mathrm{d}\Sigma=\\
&\quad=\big(\alpha(t)\phi'(t)\,,\,m\cdot\nabla\phi(t)\big)\Big|_0^T+\frac{n}{2}\int_Q\alpha(t)\big|\phi'\big|^2\,\mathrm{d}x\,\mathrm{d}t\,+\\
&\quad\;-\frac{n}{2}\int_Q\beta(t)a(x)\big|\nabla\phi\big|^2\,\mathrm{d}x\,\mathrm{d}t\,+\\
&\quad\;+\int_Q\beta(t)a(x)\big|\nabla\phi\big|^2\,\mathrm{d}x\,\mathrm{d}t-\frac{1}{2}\int_Q\beta(t)(\nabla a\cdot m)\big|\nabla\phi\big|^2\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}
\tag{4.22}
$$

On the other hand,

$$
\begin{aligned}
&\frac{n}{2}\int_Q\Big(\alpha(t)\big|\phi'\big|^2-\beta(t)a(x)\big|\nabla\phi\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}t=\\
&\quad=\frac{n-1}{2}\int_Q\Big(\alpha(t)\big|\phi'\big|^2-\beta(t)a(x)\big|\nabla\phi\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,+\\
&\quad\;+\int_0^T E(t)\,\mathrm{d}t-\int_Q\beta(t)a(x)\big|\nabla\phi\big|^2\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}
\tag{4.23}
$$

Multiplying the first equation of (3.1) by ϕ and integrating on Q we have

$$
\big(\alpha(t)\phi'(t)\,,\,\phi(t)\big)\Big|_0^T=\int_0^T\Big(\alpha(t)\big|\phi'\big|^2-\beta(t)\big|a^{1/2}(x)\,\nabla\phi\big|^2\Big)\,\mathrm{d}t\,.
\tag{4.24}
$$

Replacing (4.24) in (4.23) it follows that

$$
\begin{aligned}
&\frac{n}{2}\int_Q\Big(\alpha(t)\big|\phi'\big|^2-\beta(t)a(x)\big|\nabla\phi\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}t=\\
&\quad=\left(\alpha(t)\phi'(t)\,,\,\frac{n-1}{2}\,\phi(t)\right)\Big|_0^T+\\
&\quad\;+\int_0^T E(t)\,\mathrm{d}t-\int_Q\beta(t)a(x)\big|\nabla\phi\big|^2\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}
\tag{4.25}
$$

Now, substituting (4.25) in (4.22) we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_\Sigma m\cdot\nu\,\Big(\alpha(t)\big|\phi'\big|^2-\beta(t)a(x)\big|\nabla_\sigma\phi\big|^2\Big)\,\mathrm{d}\Sigma=\\
&\quad=\left(\alpha(t)\phi'(t)\,,\,m\cdot\nabla\phi(t)+\frac{n-1}{2}\,\phi(t)\right)\Big|_0^T+\\
&\quad\;+\int_0^T E(t)\,\mathrm{d}t-\int_Q\beta(t)(\nabla a\cdot m)\big|\nabla\phi\big|^2\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}
\tag{4.26}
$$

Since $R(x^0) = \max\{\|m(x)\| \,;\, x \in \overline{\Omega}\}$, from hypothesis (H1) we have

$$\frac{1}{2}\int_Q \beta(t)(\nabla a \cdot m)|\nabla\phi|^2 \,\mathrm{d}x\,\mathrm{d}t \le \|\nabla a\|_{C^0(\overline{\Omega})} R(x^0) a_0^{-1} \int_0^T E(t)\,\mathrm{d}t\,. \quad (4.27)$$

From (4.26) and (4.27) we get

$$\begin{aligned}&\left(\alpha(t)\phi'(t)\,,\, m\cdot\nabla\phi(t) + \frac{n-1}{2}\,\phi(t)\right)\Big|_0^T + \\ &\qquad + \left(1 - \|\nabla a\|_{C^0(\overline{\Omega})} a_0^{-1} R(x^0)\right)\int_0^T E(t)\,\mathrm{d}t \le \\ &\le \frac{1}{2}\int_\Sigma m\cdot\nu\left(\alpha(t)|\phi'|^2 - \beta(t)a(x)|\nabla_\sigma\phi|^2\right)\,\mathrm{d}\Sigma\,,\end{aligned}$$

and from hypothesis (H2) and Theorem 4.1 we obtain

$$\begin{aligned}&\left(\alpha(t)\phi'(t)\,,\, m\cdot\nabla\phi(t) + \frac{n-1}{2}\,\phi(t)\right)\Big|_0^T + \\ &\qquad + \left(1 - \|\nabla a\|_{C^0(\overline{\Omega})} a_0^{-1} R(x^0)\right) e^{-C_0} E(0) \le \qquad (4.28) \\ &\le \frac{1}{2}\int_\Sigma m\cdot\nu\left(\alpha(t)|\phi'|^2 - \beta(t)a(x)|\nabla_\sigma\phi|^2\right)\,\mathrm{d}\Sigma\,,\end{aligned}$$

Next, we estimate the expression

$$z(t) = \left(\alpha(t)\phi'(t)\,,\, m\cdot\nabla\phi(t) + \frac{n-1}{2}\,\phi(t)\right)\,, \quad \forall\, t\in[0\,,T]\,.$$

From hypothesis (H1) and Theorem 4.1, we get,

$$\begin{aligned}|z(t)| \le \|\alpha\|_{L^\infty(0,T)}\Big\{&\max\{\alpha_0^{-1}, \beta_0^{-1}a_0^{-1}\}\, e^{C_0} R(x^0) - \frac{n^2-1}{8R(x^0)}|\phi(t)|^2 + \\ &+ \frac{n-1}{4R(x^0)}\int_\Gamma m\cdot\nu|\phi(t)|^2\,\mathrm{d}\Gamma\Big\}\,,\end{aligned} \quad (4.29)$$

and from (4.29) we obtain

$$\begin{aligned}&\left|\left(\alpha(t)\phi'(t)\,,\, m\cdot\nabla\phi(t) + \frac{n-1}{2}\,\phi(t)\right)\Big|_0^T\right| \le \\ &\le \|\alpha\|_{L^\infty(0,T)}\Big\{2\max\{\alpha_0^{-1}, \beta_0^{-1}a_0^{-1}\}\, e^{C_0} R(x^0) + \\ &\qquad - \frac{n^2-1}{8R(x^0)}\left(|\phi(0)|^2 + |\phi(T)|^2\right) + \qquad (4.30) \\ &\qquad + \frac{n-1}{4R(x^0)}\int_\Gamma m\cdot\nu\left(|\phi(0)|^2 + |\phi(T)|^2\right)\,\mathrm{d}\Gamma\Big\}\,.\end{aligned}$$

From the above inequality we get

$$\begin{aligned}
&\Big(\Big(1-\|\nabla a\|_{C^0(\overline{\Omega})}R(x^0)a_0^{-1}\Big)\,e^{-C_0}T+\\
&\qquad -2\max\{\alpha_0^{-1},\beta_0^{-1}a_0^{-1}\}\,e^{C_0}R(x^0)\|\alpha\|_{L^\infty(0,T)}\Big)\,E(0)+\\
&\qquad +\frac{n^2-1}{8R(x^0)}\|\alpha\|_{L^\infty(0,T)}\Big(|\phi(0)|^2+|\phi(T)|^2\Big)\le\\
&\le\Big(\alpha(t)\phi'(t)\,,\,m\cdot\nabla\phi(t)+\frac{n-1}{2}\phi(t)\Big)\Big|_0^T+\\
&\quad+\Big(1-\|\nabla a\|_{C_0(\overline{\Omega})}R(x^0)a_0^{-1}\Big)\,e^{-C_0}E(0)T+\\
&\quad+\frac{n-1}{4R(x^0)}\|\alpha\|_{L^\infty(0,T)}\int_\Gamma m\cdot\nu\Big(|\phi(0)|^2+|\phi(T)|^2\Big)\,\mathrm{d}\Gamma\,,
\end{aligned}$$

which together (4.28) implies that

$$\begin{aligned}
&\Big(\Big(1-\|\nabla a\|_{C^0(\overline{\Omega})}R(x^0)a_0^{-1}\Big)\,e^{-C_0}T+\\
&\qquad -2\max\{\alpha_0^{-1},\beta_0^{-1}a_0^{-1}\}\,e^{C_0}R(x^0)\|\alpha\|_{L^\infty(0,T)}\Big)\,E(0)+\\
&\qquad +\frac{n^2-1}{8R(x^0)}\|\alpha\|_{L^\infty(0,T)}\Big(|\phi(0)|^2+|\phi(T)|^2\Big)\le\\
&\le\frac{1}{2}\int_\Gamma m\cdot\nu\Big(\alpha(t)|\phi'|^2-\beta(t)a(x)|\nabla_\sigma\phi|^2\Big)\,\mathrm{d}\Gamma+\\
&\quad+\frac{n-1}{4R(x^0)}\|\alpha\|_{L^\infty(0,T)}\int_\Gamma m\cdot\nu\Big(|\phi(0)|^2+|\phi(T)|^2\Big)\,\mathrm{d}\Gamma\,.
\end{aligned}$$

This gives (i).

To prove (ii), we consider the identity

$$\begin{aligned}
&\frac{1}{2}\int_Q\Big(\alpha(t)|\phi'|^2-\beta(t)a(x)|\nabla\phi|^2\Big)\,\mathrm{d}x\,\mathrm{d}t=\\
&\quad=\frac{\gamma}{2}\int_Q\Big(\alpha(t)|\phi'|^2+\beta(t)a(x)|\nabla\phi|^2\Big)\,\mathrm{d}x\,\mathrm{d}t+\\
&\quad+\frac{1-\gamma}{2}\int_Q\Big(\alpha(t)|\phi'|-\beta(t)a(x)|\nabla\phi|^2\Big)\,\mathrm{d}x\,\mathrm{d}t+\\
&\quad+(1-\gamma)\int_Q\beta(t)a(x)|\nabla\phi|^2\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}\tag{4.31}$$

Then, it follows from (4.22) and (4.31) that

$$
\begin{aligned}
&\big(\alpha(t)\phi'(t)\,,\, m\cdot\nabla\phi(t)\big)\big|_0^T+\frac{\gamma}{2}\int_Q\Big(\alpha(t)\big|\phi'\big|^2+\beta(t)a(x)\big|\nabla\phi\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}t+\\
&\quad+\frac{1-\gamma}{2}\int_Q\Big(\alpha(t)\big|\phi'\big|^2\beta(t)a(x)\big|\nabla\phi\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}t+\\
&\quad+(1-\gamma)\int_Q\beta(t)a(x)\big|\nabla\phi\big|^2\,\mathrm{d}x\,\mathrm{d}t+\\
&\quad-\frac{1}{2}\int_Q\beta(t)\,\nabla a\cdot m\big|\nabla\phi\big|^2\,\mathrm{d}x\,\mathrm{d}t=\\
&=\frac{1}{2}\int_\Sigma\alpha(t)\,m\big|\phi'\big|^2\,\mathrm{d}\Sigma\,.
\end{aligned}
$$

From (H3) we have that $0<\gamma<1$ and therefore,

$$
\begin{aligned}
&\big(\alpha(t)\phi'(t)\,,\, m\cdot\nabla\phi(t)\big)\big|_0^T+\frac{\gamma}{2}\int_Q\Big(\alpha(t)\big|\phi'\big|^2+\beta(t)a(x)\big|\nabla\phi\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}t+\\
&\quad+\frac{1-\gamma}{2}\int_Q\Big(\alpha(t)\big|\phi'\big|^2-\beta(t)a(x)\big|\nabla\phi\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}t+\\
&\quad-\frac{1}{2}\int_Q\beta(t)\,\nabla a\cdot m\big|\nabla\phi\big|^2\,\mathrm{d}x\,\mathrm{d}t\le\\
&\le\frac{1}{2}\int_\Sigma\alpha(t)\,m\big|\phi'\big|^2\,\mathrm{d}\Sigma\,.
\end{aligned}
\tag{4.32}
$$

Then, by making use of the same arguments of (4.27) and (4.28), from (4.32) we obtain

$$
\begin{aligned}
&\left(\alpha(t)\phi'(t)\,,\, m\cdot\nabla\phi(t)+\frac{1-\gamma}{2}\,\phi(t)\right)\Big|_0^T+\\
&\qquad+\left(\gamma-\|\nabla a\|_{C_0(\overline{\Omega})}R(x^0)a_0^{-1}\right)e^{-C_0T}E(0)\le\\
&\le\frac{1}{2}\int_\Sigma\alpha(t)\,m\big|\phi'\big|^2\,\mathrm{d}\Sigma\,.
\end{aligned}
$$

Defining

$$
z(t)=\left(\alpha(t)\phi'(t)\,,\, m\cdot\nabla\phi(t)+\frac{1-\gamma}{2}\,\phi(t)\right)\Big|_0^T,
$$

from hypothesis (H2) and using similar arguments to the case $n>1$, we obtain (ii). □

THEOREM 4.4 (Inverse Inequality). — *Suppose that (H1)-(H3) hold and let $T > T_0$. Then for each strong solution ϕ of (3.1) with $f = 0$ there exists $C > 0$ such that:*

(i) if $n > 1$

$$\|\phi^0\|^2_{H^1(\Omega)} + \|\phi^1\|^2_{L^2(\Omega)} \leq$$
$$\leq C\left\{\int_{\Sigma(x^0)} \left(|\phi|^2 + |\phi'|^2\right) \mathrm{d}\Sigma + \int_{\Sigma_*(x^0)} |\nabla_\sigma \phi|^2 \,\mathrm{d}\Sigma\right\};$$

(ii) if $n = 1$

$$\|\phi^0\|^2_{H^1(\Omega)} + \|\phi^1\|^2_{L^2(\Omega)} \leq C \int_{\Sigma(x^0)} \left(|\phi|^2 + |\phi'|^2\right) \mathrm{d}\Sigma .$$

Proof. — We are going to prove the case (i) since (ii) is analougous. Dropping the terms which give negative contributions in Theorem 4.3 one has

$$\|\phi^0\|^2_{H^1(\Omega)} + \|\phi^1\|^2_{L^2(\Omega)} \leq$$
$$\leq C_1\left\{\int_{\Sigma(x^0)} \left(|\phi|^2 + |\phi'|^2\right) \mathrm{d}\Sigma + \int_{\Sigma_*(x^0)} |\nabla_\sigma \phi|^2 \,\mathrm{d}\Sigma + \right. \tag{4.33}$$
$$\left. + \int_{\Gamma(x^0)} \left(|\phi(0)|^2 + |\phi'(T)|^2\right) \mathrm{d}\Gamma\right\}.$$

On the other hand, there exists a constant $C_2 > 0$ such that

$$\int_{\Gamma(x^0)} \left(|\phi(0)|^2 + |\phi'(T)|^2\right) \mathrm{d}\Sigma \leq C_2 \int_{\Sigma(x^0)} \left(|\phi|^2 + |\phi'|^2\right) \mathrm{d}\Sigma . \tag{4.34}$$

Indeed, since ϕ is a regular solution of (3.1), then

$$\phi \in C^0([0, T]; D(A)) \cap C^1([0, T]; H^1(\Omega))$$

and therefore

$$\phi|_\Sigma \in C^0([0, T]; H^{3/2}(\Gamma)) \quad \text{and} \quad \phi'|_\Sigma \in C^0([0, T]; H^{1/2}(\Gamma)) . \tag{4.35}$$

Defining

$$h(t) = \left|\phi(t)\right|^2_{L^2(\Gamma(x^0))}, \quad \forall\, t \in [\,0\,,T\,]$$

we have

$$h'(t) = 2\big(\phi(t)\,,\,\phi'(t)\big)^2_{L^2(\Gamma(x^0))}, \quad \forall\, t \in [\,0\,,T\,]$$

and from (4.35) it follows that h, $h' \in L^2(0,T)$ and hence $h \in C^0([\,0\,,T\,])$. Let $t_0 \in [\,0\,,T\,]$ be a minimizer of h. Thus

$$h(t) - h(t_0) = \int_{t_0}^{t} h'(s)\,\mathrm{d}s$$

and consequently

$$h(t) \le h(t_0) + \int_0^T \left|\phi(s)\right|^2_{L^2(\Gamma(x^0))} \mathrm{d}s + \int_0^T \left|\phi'(s)\right|^2_{L^2(\Gamma(x^0))} \mathrm{d}s\,. \tag{4.36}$$

But, since t_0 is a minimizer, we have

$$\int_0^T h(t)\,\mathrm{d}t \ge h(t_0)T$$

and then

$$h(t_0) \le \frac{1}{T_0}\int_0^T h(t)\,\mathrm{d}t\,. \tag{4.37}$$

Thus, from (4.36) and (4.37) we obtain

$$h(t) \le C'\left\{\int_0^T \left|\phi(s)\right|^2_{L^2(\Gamma(x^0))} \mathrm{d}s + \int_0^T \left|\phi'(s)\right|^2_{L^2(\Gamma(x^0))} \mathrm{d}s\right\},$$

$\forall\, t \in [\,0\,,T\,]$, which proves (4.34). Combining (4.33) and (4.34) we get the desired result. □

5. The Backward Problem

Let $T > T_0$ where T_0 is defined in the previous section, and consider the following homogeneous problem:

$$\begin{cases} (\alpha(t)\phi')' + A(t)\phi = 0 & \text{in } Q \\ \dfrac{\partial\phi}{\partial\nu_A} = 0 & \text{on } \Sigma \\ \phi(0) = \phi^0 \text{ and } \phi'(0) = \phi^1 & \text{on } \Omega, \end{cases} \tag{5.1}$$

According to the inverse inequality given in Theorem 4.4, the expression

$$\left\|\{\phi^0,\phi^1\}\right\|_* = \left\{\int_{\Sigma(x^0)}\left(|\phi|^2+|\phi'|^2\right)\mathrm{d}\Sigma + \int_{\Sigma_*(x^0)}|\nabla_\sigma\phi|^2\,\mathrm{d}\Sigma\right\}^{1/2} \tag{5.2}$$

defines a norm in $D(A)\times H^1(\Omega)$.

So, we are able to define the Hilbert space

$$F = \overline{D(A)\times H^1(\Omega)}^{\|\cdot\|_*} \tag{5.3}$$

equipped with the topology

$$\left\|\{\phi^0,\phi^1\}\right\|_F = \lim_{\nu\to\infty}\left\|\{\phi^0_\nu,\phi^1_\nu\}\right\|_* \tag{5.4}$$

where $\left(\{\phi^0_\nu,\phi^1_\nu\}\right)_{\nu\in\mathbb{N}}$ is any Cauchy sequence in $\left(D(A)\times H^1(\Omega)\,,\,\|\cdot\|_*\right)$ defined by the equivalence relation

$$\{\phi^0_\nu,\phi^1_\nu\}\sim\{\psi^0_\nu,\psi^1_\nu\}\iff\lim_{\nu\to\infty}\left\|\{\phi^0_\nu-\psi^0_\nu\,,\,\phi^1_\nu-\psi^1_\nu\}\right\|_*=0\,.$$

For every $\{\phi^0_\nu,\phi^1_\nu\}\in D(A)\times H^1(\Omega)$ we have

$$\left\|\{\phi^0,\phi^1\}\right\|_*\le C_1\left\|\{\phi^0,\phi^1\}\right\|_{D(A)\times H^1(\Omega)}$$

and

$$\left\|\{\phi^0,\phi^1\}\right\|_{H^1(\Omega)\times L^2(\Omega)}\le C_2\left\|\{\phi^0,\phi^1\}\right\|_*\,.$$

Now, since $D(A)\times H^1(\Omega)$ in dense in F, we have

$$D(A)\times H^1(\Omega)\hookrightarrow F\hookrightarrow H^1(\Omega)\times L^2(\Omega)\,, \tag{5.5}$$

where the inclusion are continuous and dense.

We can observe that by the construction of F,

$$\{\phi^0,\phi^1\}\in F\iff\int_{\Sigma(x^0)}\left(|\phi|^2+|\phi'|^2\right)\,\mathrm{d}\Sigma+\int_{\Sigma_*(x^0)}|\nabla_\sigma\phi|^2\,\mathrm{d}\Sigma<\infty\,,$$

which means that if $\{\phi^0,\phi^1\}\in F$ then

$$\phi|_{\Sigma(x^0)}\,,\ \phi'|_{\Sigma(x^0)}\in L^2(\Sigma(x^0))\quad\text{and}\quad\nabla_\sigma\phi|_{\Sigma_*(x^0)}\in\left(L^2(\Sigma_*(x^0))\right)^n \tag{5.6}$$

and

$$\phi\big|_{\Sigma_*(x^0)} \in L^2\Big(0,T;\, H^1(\Gamma_*(x^0))\Big)\,. \tag{5.7}$$

The proof of the above regularities are given in the appendix.

We then consider the backward problem

$$\begin{cases} (\alpha(t)\psi')' + A(t)\psi = 0 & \text{in } Q \\ \dfrac{\partial\psi}{\partial\nu_A} = \begin{cases} \beta^{-1}\left(-\phi + \dfrac{\partial}{\partial t}(\phi')\right) & \text{on } \Sigma(x^0) \\ \beta^{-1}\Delta_{\Gamma_*(x^0)}\phi & \text{on } \Sigma_*(x^0) \end{cases} \\ \psi(T) = \psi'(T) = 0 & \text{on } \Omega, \end{cases} \tag{5.8}$$

where ϕ is the unique solution of Problem (5.1) with initial data $\{\phi^0, \phi^1\} \in F$.

We observe that the operator $\partial/\partial t$ is well defined on $\Sigma(x^0)$ taking into account (5.6) and considering its meaning: $\forall\, w \in H^1\Big(0,T;\, L^2\big(\Gamma(x^0)\big)\Big)$

$$\begin{aligned} &\left\langle \frac{\partial}{\partial t}(\phi')\,,\, w \right\rangle_{(H^1(0,T;L^2(\Gamma(x^0))))'\,,\,H^1(0,T;L^2(\Gamma(x^0)))} = \\ &\qquad = -\int_0^T \int_{\Gamma(x^0)} \phi' w' \,\mathrm{d}\Gamma\,\mathrm{d}t\,. \end{aligned} \tag{5.9}$$

It is important to note this operator is not taken in the distributional sense.

On the other hand from (5.7) we obtain

$$\Delta_{\Gamma_*(x^0)}\phi \in L^2\left(0,T;\, \Big(H^1\big(\Sigma_*(x^0)\big)\Big)'\right)\,. \tag{5.10}$$

Let ψ be the solution of (5.8) defined by the transposition method, which will be precised later. Let $\{\phi^0, \phi^1\} \in F$, $f \in L^1\big(0,T;\, H^1(\Omega)\big)$ and let $\theta : Q \to \mathbb{R}$ be the unique solution of

$$\begin{cases} (\alpha(t)\theta')' + A(t)\theta = f & \text{in } Q \\ \dfrac{\partial\theta}{\partial\nu_A} = 0 & \text{on } \Sigma \\ \theta(0) = \theta^0 \text{ and } \theta'(0) = \theta^1 & \text{on } \Omega. \end{cases} \tag{5.11}$$

Multiplying (5.11) by ψ and integrating by parts, we obtain formally

$$\int_Q f\psi\, \mathrm{d}x\, \mathrm{d}t =$$
$$= -\int_\Omega \alpha(0)\theta'(0)\psi(0)\, \mathrm{d}x + \int_\Omega \alpha(0)\theta(0)\psi'(0)\, \mathrm{d}x + \int_\Sigma \beta(t)\frac{\partial\psi}{\partial\nu_A}\, \theta\, \mathrm{d}\Sigma_0\,. \tag{5.12}$$

Replacing $\partial\psi/\partial\nu_A$ by its value given in (5.8) we get from (4.12) and (5.9)

$$\int_\Sigma \beta(t)\frac{\partial\psi}{\partial\nu_A}\, \theta\, \mathrm{d}\Sigma = -\int_{\Sigma(x^0)} (\phi\theta + \phi'\theta')\, \mathrm{d}\Sigma - \int_{\Sigma_*(x^0)} \nabla_\sigma\phi\cdot\nabla_\sigma\theta\, \mathrm{d}\Sigma\,.$$

Observing this expression we define the functionnal

$$L(\theta^0,\theta^1,f) = -\int_{\Sigma(x^0)} (\phi\theta + \phi'\theta')\, \mathrm{d}\Sigma - \int_{\Sigma_*(x^0)} \nabla_\sigma\phi\cdot\nabla_\sigma\theta\, \mathrm{d}\Sigma \tag{5.13}$$

Thus, from (5.12) and (5.13) we obtain formally that

$$\begin{aligned}\int_Q f\psi\, \mathrm{d}x\, \mathrm{d}t + \int_\Omega \alpha(0)\theta'(0)\psi(0)\, \mathrm{d}x - \int_\Omega \alpha(0)\theta(0)\psi'(0)\, \mathrm{d}x = \\ = L(\theta^0,\theta^1,f)\,.\end{aligned} \tag{5.14}$$

Considering Theorem 3.1 and the construction of F, we have that the functional given by (5.13) is continuous, that is,

$$L \in F' \times \Big(L^1(0,T;\, H^1(\Omega))\Big)'\,. \tag{5.15}$$

Indeed, firt of all we note that the solution θ of (5.11) verifies $\theta = \theta_1 + \theta_2$, where θ_1 and θ_2 are, respectively, the solutions of the following problems:

$$\begin{cases} \big(\alpha(t)\theta_1'\big)' + A(t)\theta_1 = 0 & \text{in } Q \\ \dfrac{\partial\theta_1}{\partial\nu_A} = 0 & \text{on } \Sigma \\ \theta_1(0) = \theta^0 \text{ and } \theta_1'(0) = \theta^1 & \text{in } \Omega. \end{cases}$$

and

$$\begin{cases} \big(\alpha(t)\theta_2'\big)' + A(t)\theta_2 = f & \text{in } Q \\ \dfrac{\partial\theta_2}{\partial\nu_A} = 0 & \text{on } \Sigma \\ \theta_2(0) = \theta_2'(0) = 0 & \text{in } \Omega. \end{cases}$$

Besides, from (5.13) we can write for all $\{\phi^0, \phi^1\} \in D(A) \times H^1(\Omega)$ and $i = 1, 2$

$$L(\theta^0, \theta^1, f) = \\ = \sum_{i=1}^{2} \left(\int_{\Sigma(x^0)} (\phi\theta_i + \phi'\theta_i)\, \mathrm{d}\Sigma + \int_{\Sigma_*(x^0)} \nabla_\sigma \phi\, \nabla_\sigma \theta_i \, \mathrm{d}\Sigma \right) \tag{5.16}$$

and therefore from (5.2) and (5.16) we obtain

$$\left|L(\theta^0, \theta^1, f)\right| \leq \\ \leq C_1 \left\|\{\phi^0, \phi^1\}\right\|_F \sum_{i=1}^{2} \left(\int_{\Sigma(x^0)} \left(|\phi\theta_i|^2 + |\phi'\theta_i|^2 \right) \mathrm{d}\Sigma + \right. \\ \left. + \int_{\Sigma_*(x^0)} \left|\nabla_\sigma \phi\, \nabla_\sigma \theta_i\right|^2 \mathrm{d}\Sigma \right)^{1/2} . \tag{5.17}$$

From (5.17) and Theorem 3.1 we have

$$\left|L(\theta^0, \theta^1, f)\right| \leq C_2 \left(\left\|\{\phi^0, \phi^1\}\right\|_F^2 + \|f\|^2_{L^1(0,T\,;\,H^1(\Omega))} \right)^{1/2} . \tag{5.18}$$

By density arguments we conclude that inequality (5.18) is valid for all $\{\phi^0, \phi^1\}$, $\{\theta^0, \theta^1\} \in F$ which proves (5.15).

It follows that there exists a unique triple $\{\rho^0, \rho^1, \psi\}$ such that

$$\left\{\alpha(0)\rho^1, -\alpha(0)\rho^0\right\} \in F, \qquad \psi \in L^\infty\Big(0, T\,;\, (H^1(\Omega))'\Big)$$

and verifies

$$\int_0^T \langle \psi(t)\,,\, f(t) \rangle_{(H^1(\Omega))', H^1(\Omega)} + \Big\langle \left\{-\alpha(0)\rho^1, \alpha(0)\rho^0\right\}, \{\theta^0, \theta^1\} \Big\rangle_{F', F} = \\ = - \left(\int_{\Sigma(x^0)} (\phi\theta + \phi'\theta')\, \mathrm{d}\Sigma + \int_{\Sigma_*(x^0)} \nabla_\sigma \phi\, \nabla_\sigma \theta \, \mathrm{d}\Sigma \right) . \tag{5.19}$$

DEFINITION.— *The unique function ψ that satisfies (5.19) in called solution by transposition of Problem (5.8).*

Now we state the main result of this section, which is a consequence of Theorem 3.2.

THEOREM 5.1.— *The unique solution by transposition ψ of the Problem (5.8) has the following regularity:*

$$\psi \in L^\infty\Big(0,T\,;\,(H^1(\Omega))'\Big) \cap W^{1,\infty}\Big(0,T\,;\,(D(A))'\Big),$$

and

$$\{\psi'(0), \psi(0)\} \in F'.$$

In a addition, the linear map

$$\{\phi^0, \phi^1\} \in F \longmapsto \{\alpha(0)\psi'(0)\,,\,-\alpha(0)\psi(0)\} \in F'$$

is continuous.

Proof.— For every $f \in \mathcal{D}\big(0,T\,;\,D(A)\big)$ we have

$$L(0,0,f') = -\int_{\Sigma(x^0)} (\phi\theta + \phi'\theta')\,\mathrm{d}\Sigma - \int_{\Sigma_*(x^0)} \nabla_\sigma\phi \cdot \nabla_\sigma\theta\,\mathrm{d}\Sigma\,,$$

where θ and ϕ are, respectively, solutions of

$$\begin{cases} \big(\alpha(t)\theta'\big)' + A(t)\theta = f' & \text{in } Q \\ \dfrac{\partial\theta}{\partial\nu_A} = 0 & \text{on } \Sigma \\ \theta(0) = \theta'(0) = 0 & \text{in } \Omega \end{cases} \tag{5.20}$$

and

$$\begin{cases} \big(\alpha(t)\phi'\big)' + A(t)\phi = 0 & \text{in } Q \\ \dfrac{\partial\phi}{\partial\nu_A} = 0 & \text{on } \Sigma \\ \phi(0) = \phi^0 \text{ and } \phi'(0) = \phi^1 & \text{in } \Omega. \end{cases} \tag{5.21}$$

By the definition of F and from Theorem 3.1 it follows that

$$\big|L(0,0,f')\big| \le C\left(\|\theta\|_{L^1(0,T\,;\,D(A))} + \|\theta'\|_{L^1(0,T\,;\,H^1(\Omega))}\right). \tag{5.22}$$

Indeed, it is sufficient to prove (5.22) when the initial data $\{\phi^0, \phi^1\} \in D(A) \times H^1(\Omega)$ because by density arguments we conclude the same when $\{\phi^0, \phi^1\} \in F$.

We have by Schwarz inequality and Theorem 3.1

$$
\begin{aligned}
&\left|L(0,0,f')\right| \leq \\
&\leq C_1 \int_0^T \left(\int_{\Gamma(x^0)} |\phi|^2 \, \mathrm{d}\Gamma + \int_{\Gamma(x^0)} |\phi'|^2 \, \mathrm{d}\Gamma + \int_{\Gamma_*(x^0)} |\nabla_\sigma \phi| \, \mathrm{d}\Gamma \right)^{1/2} , \\
&\left(\int_{\Gamma(x^0)} |\theta|^2 \, \mathrm{d}\Gamma + \int_{\Gamma(x^0)} |\theta'|^2 \, \mathrm{d}\Gamma + \int_{\Gamma_*(x^0)} |\nabla_\sigma \theta| \, \mathrm{d}\Gamma \right)^{1/2} \leq \\
&\leq \|\phi^0, \phi^1\|_{D(A)\times H^1(\Omega)} \left(\|\theta\|_{L^1(0,T;D(A))} + \|\theta'\|_{L^1(0,T;H^1(\Omega))} \right)
\end{aligned}
$$

which concludes (5.22).

On the other hand, from Theorem 3.2 we get

$$
\|\theta\|_{L^1(0,T;D(A))} + \|\theta'\|_{L^1(0,T;H^1(\Omega))} \leq C\|f\|_{L^1(0,T;D(A))} \tag{5.23}
$$

which is the crucial point for control problems involving time-dependent coefficients.

In fact, before we prove (5.23) we observe that in the right side of equation (5.20) we have f' while in the right side of (5.23) we have f. Besides, we note that when the coefficients do not depend on time (see for example the most simple case for the wave equation), it is not difficult to obtain the above inequality by using Theorem 3.1 and the following standard argument.

If ω is a solution to problem

$$
\begin{cases}
\omega'' - \Delta\omega = f & \text{in } Q \\
\dfrac{\partial\omega}{\partial\nu} = 0 & \text{on } \Sigma \\
\omega(0) = \omega'(0) = 0 & \text{in } \Omega
\end{cases}
$$

with $f \in \mathcal{D}\big(0,T;\, D(A)\big)$, then $\theta = \omega'$ is the solution of

$$
\begin{cases}
\theta'' - \Delta\theta = f & \text{in } Q \\
\dfrac{\partial\theta}{\partial\nu_A} = 0 & \text{on } \Sigma \\
\theta(0) = 0 \text{ and } \theta'(0) = 0 & \text{in } \Omega.
\end{cases}
$$

But in our case, where we have time-dependent coefficients, this arguments fails completely and we need to solve it in other way. From Theorem 3.2 and observing that $\theta(t) = \int_0^t \theta'(s)\,\mathrm{d}s$ after some calculations we obtain

$$
\|\theta\|_{L^1(0,T;L^2(\Omega))} \leq k_1\|f\|_{L^1(0,T;D(A))}
$$

$$
\|A\theta\|_{L^1(0,T;L^2(\Omega))} \leq k_2\|f\|_{L^1(0,T;D(A))}
$$

which implies

$$\|\theta\|_{L^1(0,T;D(A))} \le k_3\|f\|_{L^1(0,T;D(A))}. \tag{5.24}$$

In addition

$$\|\theta'\|_{L^1(0,T;L^2(\Omega))} \le k_4\|f\|_{L^1(0,T;D(A))}$$
$$\|A^{1/2}\theta'\|_{L^1(0,T;L^2(\Omega))} \le k_5\|f\|_{L^1(0,T;D(A))}$$

and therefore

$$\|\theta'\|_{L^1(0,T;H^1(\Omega))} \le k_6\|f\|_{L^1(0,T;D(A))}. \tag{5.25}$$

From (5.24) and (5.25) we get (5.23). Combining (5.22) and (5.23) we obtain

$$|L(0,0,f')| \le C\|f\|_{L^1(0,T;D(A))}, \quad \forall\, f \in \mathcal{D}(0,T;\, D(A))$$

which is sufficient to prove the desired regularity, that is,

$$\psi' \in L^\infty\Big(0,T;\, \big(D(A)\big)'\Big). \tag{5.26}$$

In fact, let us define

$$S(f) = -L(0,0,f'), \quad \forall\, f \in \mathcal{D}(0,T;\, D(A))$$

Since $\mathcal{D}(0,T;\, D(A))$ is dense in $L^1(0,T;\, D(A))$, we can consider the unique linear continuous extension $\overline{S}$ of S, given by

$$\overline{S}(f) = S(f) = -L(0,0,f), \quad \forall\, f \in \mathcal{D}(0,T;\, D(A)) \tag{5.27}$$

and, consequently, if follows that

$$\overline{S} \in \Big(L^1(0,T;\, D(A))\Big)' = L^\infty\Big(0,T;\, \big(D(A)\big)'\Big). \tag{5.28}$$

Now, given $f = \varphi\theta$ with $\varphi \in D(A)$ and $\theta \in \mathcal{D}(0,T)$, according to (5.13), (5.19), (5.27) and considering the fact that $\theta^0 = \theta^1 = 0$ we obtain,

$$\begin{aligned}\langle \overline{S}, \varphi\theta\rangle &= \langle S, \varphi\theta\rangle = -L(0,0,f')\\ &= -\int_0^T \langle \psi(t), f'(t)\rangle\, dt = -\int_0^T \langle \psi(t), \varphi\rangle \theta'(t)\, dt.\end{aligned}$$

So, by (5.28) it follows that

$$\int_0^T \langle \overline{S}(t)\,,\,\varphi\rangle\theta(t)\,\mathrm{d}t = -\int_0^T \langle \psi(t)\,,\,\varphi\rangle\theta'(t)\,\mathrm{d}t$$

which implies that

$$\left\langle \int_0^T \overline{S}(t)\theta(t)\,\mathrm{d}t\,,\,\varphi \right\rangle = \left\langle -\int_0^T \psi(t)\theta'(t)\,\mathrm{d}t\,,\,\varphi \right\rangle, \quad \forall\, \varphi \in D(A)\,.$$

Therefore $\overline{S} = \psi'$ in $\mathcal{D}'\Big(0,T\,;\,\big(D(A)\big)'\Big)$ and (5.26) is then proved.

One observes that if in (5.19) we consider $f = \varphi(\alpha\eta') + \beta A(\varphi\eta)$; $\theta = \varphi\eta$ with $\varphi \in D(A^{3/2})$, $\eta \in D(0,T)$ and $\phi^0 = \phi^1 = 0$, we have

$$\big(\alpha\psi'\big)' + A(t)\psi = 0 \quad \text{in } L^\infty\Big(0,T\,;\,\big(D(A^{3/2})\big)'\Big)\,.$$

Since $\alpha(t) \geq \alpha_0 > 0$, if follows that

$$\psi'' \in L^\infty\Big(0,T\,;\,\big(D(A^{3/2})\big)'\Big) \tag{5.29}$$

Then, from (5.19), (5.26) and (5.29) we obtain

$$\psi \in C_s\Big(0,T\,;\,\big(H^1(\Omega)\big)'\Big) \cap C^0\Big([0\,,\,T];\,\big(D(A)\big)'\Big)$$

and

$$\psi' \in C_s\Big(0,T\,;\,\big(D(A)\big)'\Big) \cap C^0\Big([0\,,\,T];\,\big(D(A^{3/2})\big)'\Big)$$

(see for example, J.-L. Lions and E. Magenes [13, Vol. 1, Lemma 8.1]) which makes $\psi(0)$ and $\psi'(0)$ meaningful.

Using the regularity of ψ, considering $f = \varphi\big(\alpha\eta'\big)' + \beta A(\varphi\eta)$ and $\theta = \varphi\eta$ where $\varphi \in D(A^{3/2})$, $\eta \in C^2\big(0,T\big)$ (first we get, for instance, $\eta(t) = \big(T-t\big)^2 t$ and secondly we can consider $\eta(t) = \big(T-t\big)^2$), we obtain from (5.19) with $\phi^0 = \phi^1 = 0$,

$$\psi(0) = \rho^0 \quad \text{and} \quad \psi'(0) = \rho^1\,.$$

Finally, considering $f = 0$ in (5.19) we conclude that

$$\big\|\big\{\alpha(0)\psi'(0)\,,\,-\alpha(0)\psi(0)\big\}\big\|_{F'} \leq C\big\|\{\phi^0,\phi^1\}\big\|_F\,, \quad \forall\,\{\phi^0,\phi^1\} \in F\,.$$

which ends the proof. □

6. HUM and Exact Controllability

Let us define the linear operator $\Lambda : F \to F'$ given by

$$\Lambda\{\phi^0, \phi^1\} = \{\alpha(0)\psi'(0)\,,\, -\alpha(0)\psi(0)\}\,, \tag{6.1}$$

which is continuous in view of Theorem 5.1.

Considering $f = 0$, $\theta^0 = \phi^0$ and $\theta^1 = \phi^1$ in (5.13) and (5.14), we have

$$\begin{aligned}\langle \Lambda\{\phi^0, \phi^1\}\,,\, \{\phi^0, \phi^1\}\rangle_{F',F} &= \Big\langle \{\alpha(0)\psi'(0)\,,\, -\alpha(0)\psi(0)\}\,,\, \{\phi^0, \phi^1\}\Big\rangle \\ &= \int_{\Sigma(x^0)} \Big(|\phi|^2 + |\phi'|^2\Big)\, \mathrm{d}\Sigma + \int_{\Sigma_{0,*}(x^0)} |\nabla_\sigma \phi|^2\, \mathrm{d}\Sigma\,,\end{aligned}$$

that is,

$$\langle \Lambda\{\phi^0, \phi^1\}\,,\, \{\phi^0, \phi^1\}\rangle_{F',F} = \|\{\phi^0, \phi^1\}\|_F^2\,.$$

This implies immediately that Λ is injective and self-adjoint. Then Λ is an isomorphism from F to F'. Therefore, given $\{y^1, -y^0\} \in F'$ we obtain $\{\alpha(0)y^1\,,\, -\alpha(0)y^0\} \in F'$ and consequently there exists a unique $\{\phi^0, \phi^1\} \in F$ such that

$$\Lambda\{\phi^0, \phi^1\} = \{\alpha(0)y^1\,,\, -\alpha(0)y^0\}\,, \tag{6.2}$$

From (6.1) and (6.2) we have

$$\psi'(0) = y^1 \quad \text{and} \quad \psi(0) = y^0\,. \tag{6.3}$$

Now we are going to finish the proof of Theorem 2.1. Since $\{y^1, -y^0\} \in L^2(\Omega) \times (H^1(\Omega))'$, taking into account the chain

$$\begin{aligned} D(A) \times H^1(\Omega) \hookrightarrow F \hookrightarrow H^1(\Omega) \times L^2(\Omega) \hookrightarrow L^2(\Omega) \times L^2(\Omega) \\ \hookrightarrow (H^1(\Omega))' \times L^2(\Omega) \hookrightarrow F' \hookrightarrow (D(A))' \times (H^1(\Omega))'\,, \end{aligned}$$

we obtain $\{y^1, -y^0\} \in F'$ and therefore in this case we deduce (6.3).

Defining in (2.1) the controls

$$v_0 = \beta^{-1}\left(-\phi + \frac{\partial}{\partial t}(\phi')\right) \quad \text{on } \Sigma(x^0)$$

and

$$v_1 = \beta^{-1}\Delta_{\Gamma_*(x^0)}\phi \quad \text{on } \Sigma_*(x^0)$$

from (6.3), the uniqueness of solutions to Problems (2.1) and

$$\begin{cases} (\alpha(t)\psi')' + A(t)\psi = 0 & \text{in } Q \\ \dfrac{\partial\psi}{\partial\nu_A} = \begin{cases} v_0 & \text{on } \Sigma(x^0) \\ v_1 & \text{on } \Sigma_*(x^0) \end{cases} & \\ \psi(0) = y^0 \text{ and } \psi'(0) = y^1 & \text{in } \Omega \\ \psi(T) = \psi'(T) = 0 & \text{in } \Omega, \end{cases}$$

we finally conclude that

$$y(T) = y'(T) = 0\,.$$

Thus Theorem 2.1 is proved. □

7. Appendix

Since $D(A) \times H^1(\Omega)$ is dense in F, there exist $\{\phi_\nu^0, \phi_\nu^1\} \in D(A) \times H^1(\Omega)$ such that

$$\lim_{\nu\to\infty} \{\phi_\nu^0, \phi_\nu^1\} = \{\phi^0, \phi^1\} \quad \text{in } F \tag{7.1}$$

and therefore, considering the inverse inequality,

$$\lim_{\nu\to\infty} \{\phi_\nu^0, \phi_\nu^1\} = \{\phi^0, \phi^1\} \quad \text{in } H^1(\Omega) \times L^2(\Omega)\,. \tag{7.2}$$

According to Theorem (3.1), for each $\nu \in \mathbb{N}$ there exists $\phi_\nu \in C^0([0\,,T]\,;D(A)) \cap C^1([0\,,T]\,;H^1(\Omega))$ which is the solution of (3.1) with initial data $\{\phi_\nu^0, \phi_\nu^1\} \in D(A) \times H^1(\Omega)$ and $f \in L^1\big(0,T\,;H^1(\Omega)\big)$. Thus, from the linearity of (3.1) we have

$$\begin{aligned} \big\|\phi_\nu - \phi_\mu\big\|_{C^0([0\,,T]\,;H^1(\Omega))} + \big\|\phi_\nu' - \phi_\mu'\big\|_{C^0([0\,,T]\,;L^2(\Omega))} \le \\ \le C\left(\big\|\phi_\nu^0 - \phi_\mu^0\big\| + \big|\phi_\nu^0 - \phi_\mu^0\big|\right) \end{aligned}$$

wich implies that the unique solution $\phi : Q \to \mathbb{R}$ of (3.1) satisfies

$$\begin{aligned} &\lim_{\nu\to\infty} \phi_\nu = \phi \quad \text{in } C^0([0\,,T]\,;H^1(\Omega)) \\ &\lim_{\nu\to\infty} \phi_\nu' = \phi' \quad \text{in } C^0([0\,,T]\,;L^2(\Omega))\,. \end{aligned} \tag{7.3}$$

On the other hand, from (5.2) we obtain,

$$\left\|\{\phi_\nu^0 - \phi_\mu^0\,,\, \phi_\nu^1 - \phi_\mu^1\}\right\|_*^2 = \left(\int_{\Sigma(x^0)} \left(|\phi_\nu - \phi_\mu|^2 + |\phi_\nu' - \phi_\mu'|^2\right) \mathrm{d}\Sigma + \right.$$
$$\left. + \int_{\Sigma_{0,*}(x^0)} |\nabla_\sigma \phi_\nu - \nabla_\sigma \phi_\mu|^2 \,\mathrm{d}\Sigma \right) . \tag{7.4}$$

From the convergence in (7.1) we conclude that the right side of (7.4) converges to zero when μ and ν goes to infinity. So (ϕ_ν), (ϕ_ν') and $(\nabla_\sigma \phi_\nu)$ are, respectively, Cauchy sequences in $L^2(\Sigma(x^0))$, $L^2(\Sigma(x^0))$ and $L^2(\Sigma_*(x^0))$, which proves (5.6).

To prove (5.7) we need the following result.

LEMMA. — $\forall\ R > 0$, $\exists\ C > 0$ *such that*

$$\left\|\{\phi^0, \phi^1\}\right\| \le C \left\|\{\phi^0, \phi^1\}\right\|_*, \quad \forall\ \{\phi^0, \phi^1\} \in D(A) \times H^1(\Omega)$$

satisfying $\left\|\{\phi^0, \phi^1\}\right\|_* \ge R$.

Proof. — Consider $\{\phi^0, \phi^1\} \in D(A) \times H^1(\Omega)$ such that $\left\|\{\phi^0, \phi^1\}\right\|_* \ge R$. So $\{\phi^0, \phi^1\}$ is different from $\{0, 0\}$ and, consequently, it is sufficient to prove that: $\forall\ R > 0$, $\exists\ C > 0$, such that

$$\frac{1}{C} \le \left\|\{\phi^0, \phi^1\}\right\|_*, \quad \forall\ \{\phi^0, \phi^1\} \in D(A) \times H^1(\Omega) \tag{7.5}$$

with

$$\left\|\{\phi^0, \phi^1\}\right\|_{D(A)\times H^1(\Omega)} = 1 \quad \text{and} \quad \left\|\{\phi^0, \phi^1\}\right\|_* \ge R\,.$$

Let us suppose it does not happen, that is, there exists $R_0 > 0$ such that $\forall\ C > 0\ \exists\ \{\phi_C^0, \phi_C^1\} \in D(A) \times H^1(\Omega)$ with $\left\|\{\phi_C^0, \phi_C^1\}\right\|_{D(A)\times H^1(\Omega)} = 1$, $\left\|\{\phi_C^0, \phi_C^1\}\right\|_* \ge R$ and $\left\|\{\phi_C^0, \phi_C^1\}\right\|_* < 1/C$.

In the particular case when $C = 1/R_0$ it follows that

$$R_0 \le \left\|\{\phi_{R_0}^0, \phi_{R_0}^1\}\right\|_* < R_0$$

which is a contradiction. So, (7.5) is proved and consequently the lemma.

Let us consider firstly $\{\phi^0, \phi^1\} \in D(A) \times H^1(\Omega)$ and suppose ϕ is the strong solution of (5.1). Then $\phi \in C^0([0\,,T]\,;\, D(A)) \cap C^1([0\,,T]\,;\, H^1(\Omega))$ and therefore,

$$\phi\big|_\Sigma \in C^0([0\,,T]\,;\, H^{3/2}(\Gamma)) \subset C^0([0\,,T]\,;\, H^1(\Gamma))\,.$$

Thus, from Theorem 3.1 we obtain

$$\Big\|\phi|_{\Sigma_*(x^0)}\Big\|_{L^2(0,T\,;\,H^1(\Gamma_*(x^0)))} \leq k\big\|\{\phi^0,\phi^1\}\big\|_{D(A)\times H^1(\Omega)}\,. \tag{7.6}$$

Consider, now, $\{\phi^0,\phi^1\} \in F$ and ϕ the weak solution of (5.1). If $\{\phi^0,\phi^1\} = \{0,0\}$ then $\phi = 0$ and the regularity in (5.7) follows immediately. Let us consider $\{\phi^0,\phi^1\}$ different from $\{0,0\}$. Since $D(A) \times H^1(\Omega)$ is dense in F there exists $\{\phi^0_\nu,\phi^1_\nu\} \subset D(A) \times H^1(\Omega)$ such that

$$\lim_{\nu\to\infty} \{\phi^0_\nu,\phi^1_\nu\} = \{\phi^0,\phi^1\} \in F\,. \tag{7.7}$$

Defining

$$R_0 = \frac{1}{2}\,\big\|\{\phi^0,\phi^1\}\big\|_F\,,$$

there exists $\{\phi^0_\mu,\phi^1_\mu\}$ subsequence of $\{\phi^0_\nu,\phi^1_\nu\}$ such that

$$\big\|\{\phi^0_\mu,\phi^1_\mu\} - \{\phi^0,\phi^1\}\big\|_F < R_0\,, \quad \forall\,\mu\in\mathbb{N}\,.$$

Therefore

$$\big\|\{\phi^0_\mu,\phi^1_\mu\}\big\|_F = \big\|\{\phi^0_\mu,\phi^1_\mu\}\big\|_* \geq R_0\,. \tag{7.8}$$

Thus, from (7.8) and the above lemma $\exists\, C = C\Big(\big\|\{\phi^0,\phi^1\}\big\|_F\Big) \geq 0$ such that

$$\big\|\{\phi^0_\mu,\phi^1_\mu\}\big\|_{D(A)\times H^1(\Omega)} \leq C\big\|\{\phi^0_\mu,\phi^1_\mu\}\big\|_*\,, \quad \forall\,\mu\in\mathbb{N}\,. \tag{7.9}$$

Let $\{\phi_\mu\}$ be the sequence of strong solutions of (5.1) with initial data $\{\phi^0_\mu,\phi^1_\mu\}$. Then, from (7.6) and (7.9) there exists $C_1 = C_1\Big(\big\|\{\phi^0,\phi^1\}\big\|_F\Big) > 0$ such that

$$\Big\|\phi_\mu|_{\Sigma_*(x^0)}\Big\|_{L^2(0,T\,;\,H^1(\Gamma_*(x^0)))} \leq C_1\big\|\{\phi^0_\mu,\phi^1_\mu\}\big\|_*\,. \tag{7.10}$$

But, from (7.7) we obtain

$$\big\|\{\phi^0_\mu,\phi^1_\mu\}\big\|_F = \big\|\{\phi^0_\mu,\phi^1_\mu\}\big\|_* \leq L\,, \quad \forall\,\mu\in\mathbb{N}\,. \tag{7.11}$$

So, from (7.10) and (7.11) we conclude that

$$\Big\|\phi_\mu|_{\Sigma_*(x^0)}\Big\|_{L^2(0,T\,;\,H^1(\Gamma_*(x^0)))} \leq M\,, \quad \forall\,\mu\in\mathbb{N}\,.$$

Then, there exists a subsequence that we will denote by the same notation $\{\phi_\mu\}$ such that

$$\phi_\mu\big|_{\Sigma_*(x^0)} \rightharpoonup \chi \text{ in } L^2\Big(0,T;\ H^1(\Gamma_*(x^0))\Big) \text{ when } \mu \text{ goes to infinity}. \tag{7.12}$$

On the other hand, from (7.3) we have

$$\lim_{\mu\to\infty} \phi_\mu\big|_{\Sigma_*(x^0)} = \phi\big|_{\Sigma_*(x^0)} \in L^2\Big(0,T;\ H^{1/2}(\Gamma_*(x^0))\Big) \tag{7.13}$$

and from (7.12) and (7.13) we have $\phi = \chi$ which proves (5.7).

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