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Real deformations and complex topology of plane curve singularities(*)

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RÉSUMÉ. — On associe à l'aide d'une construction hodographique à une immersion générique et relative d'un nombre fini de copies de l'intervalle $[0, 1]$ dans le disque unité D un entrelât dans la sphère de dimension 3. Un partage d'une singularité de courbes planes complexes, dont les branches locales sont réelles, est une telle immersion. On peut obtenir le partage en déformant les paramétrisations des branches réelles. Le résultat clef est le théorème 2, qui affirme que l'entrelât d'un partage d'une singularité de courbes planes complexes est en fait isotope à l'entrelât local de la singularité. L'algorithme graphique de la section 4 permet de déduire d'un partage d'une singularité la fibre de Milnor orientée avec un système ordonné de cycles évanescentes. La monodromie géométrique est le produit ordonné des twists de Dehn droits de ce système. Comme illustration, on calcule la monodromie globale géométrique d'un polynôme particulier à deux variables ayant trois singularités de deux valeurs critiques. L'exemple montre comment passer à l'aide d'un seul croisement de nœud torique itéré $(2,3)(2,3)$ au nœud torique $(4,7)$.

ABSTRACT. — We introduce a hodographic construction, which transforms a generic relative immersion of r copies of the interval $[0, 1]$ in the unit disk into a classical link in the three sphere with r components. For instance, the constuction gives the local link of the plane curve singularity if we apply it to a generic deformation with the maximal possible number of crossings of the real local parametrizations of branches of the singularity. Keywords are: plane curve, singularity, divide, monodromy, knot, link.

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1. Introduction

The geometric monodromy T of a curve singularity in the complex plane is a diffeomorphism of a compact surface with boundary $(F, \partial F)$ inducing the identity on the boundary, which is well defined up to isotopy relative to the boundary. The geometric monodromy of a curve singularity in the complex plane determines the local topology of the singularity. As element of the mapping class group of the surface $(F, \partial F)$, the diffeomorphism T can be written as a composition of Dehn twists. In Section 3 of this paper, the geometric monodromy of an isolated plane curve singularity is written explicitly as a composition of right Dehn twists. In fact, a global graphical algorithm for the construction of the surface $(F, \partial F)$ with a system of simply closed curves on it is given in Section 4, such that the curves of this system are the vanishing cycles of a real morsification of the singularity. In Section 5, as an illustration, the global geometric monodromy of the polynomial $y^4 - 2y^2x^3 + x^6 - x^7 - 4yx^5$, which has two critical fibers, is computed.

The germ of a curve singularity in $\mathbb{C}^2$ is a finite union of parametrized local branches $b_i : \mathbb{C} \to \mathbb{C}^2$, $1 \leq i \leq r$. First observe, that without loss of generality for the local topology, we can assume that the branches have a real polynomial parameterization. The combinatorial data used to describe the geometric monodromy of a curve singularity come from generic real polynomial deformations of the parameterizations of the local branches $b_{i,t} : \mathbb{C} \to \mathbb{C}^2$, $1 \leq i \leq r$, $t \in [0, 1]$, such that:

(i) $b_{i,0} = b_i$, $1 \leq i \leq r$,

(ii) for some $\rho > 0$ the intersection of the union of the branches with the ρ−ball B at the singular point of curve in $\mathbb{C}^2$ is a representative of the germ of the curve and B is a Milnor ball for the germ,

(iii) the images of $b_{i,t}$, $1 \leq i \leq r$, $t \in [0, 1]$, intersect the boundary of the ball B transversally,

(iv) the union of the images $b_{i,t}(\mathbb{R})$, $1 \leq i \leq r$, has for every $t \in (0, 1]$ the maximal possible number of double points in the interior of B.

Such deformations correspond to real morsifications of the defining equation of the singularity and were used to study the local monodromy in [AC2], [AC3] and [G-Z]. Real deformations of singularities of plane algebraic curves with the maximal possible number of double points in the real plane were discovered by Charlotte Angas Scott ([S1], [S2]).

In Section 6, we start with a connected divide, which defines as explained in Section 3 a classical link. We will construct a map from the complement of the link of a connected divide to the circle and prove that this map is a fibration. This fibration is for a divide of a plane curve singularity a model for the Milnor fibration of the singularity. The link of most connected divides are hyperbolic. In a forth coming paper we will study the geometry of a link of a divide.

We used MAPLE for the drawings of parametrized curves and for the computation of suitable deformations of the polynomial equations. Of great help for the investigation of topological changes in families of polynomial equations is the mathematical software SURF which has been developed by Stefan Endrass.

2. Real deformations of plane curve singularities

Let $f : \mathbb{C}^2 \to \mathbb{C}$ be the germ at $0 \in \mathbb{C}^2$ of an holomorphic map with $f(0) = 0$ and having an isolated singularity S at 0. We are mainly interested in the study of topological properties of singularities, therefore we can assume without loss of generality that the germ f is a product of locally irreducible real polynomials. Having chosen a Milnor ball $B(0, \rho)$ for f, there exists a real polynomial deformation family f_t, $t \in [\,0\,,\,1\,]$, of f such that for all t the 0-level of f_t is transversal to the boundary of the ball $B(0, \rho)$ and such that for all $t \in \,]\,0\,,\,1\,]$ the 0-level of f_t has δ transversal double points in the interior of the disk $D(0, \rho) := B(0, \rho) \cap \mathbb{R}^2$, where the Milnor number μ and the number r of local branches of f satisfy $\mu = 2\delta - r + 1$. In particular, the 0-level of f_t, $t \in \,]\,0\,,\,1\,]$, has in $D(0, \rho)$ no self tangencies or triple intersections. It is possible to choose for f_t, $t \in [\,0\,,\,1\,]$, a family of defining equations for the union of the images of $b_{i,t}$, $1 \leq i \leq r$. The deformation f_t, $t \in [\,0\,,\,1\,]$ is called a real morsification with respect to the Milnor ball $B(0, \rho)$ of f. So, the 0-level of the restriction of f_t, $t \in]\,0\,,\,1\,]$, to $D(0, \rho)$ is an immersion without self-tangencies and having only transversal self-intersections of r copies of an interval (see [AC2], [AC3], [G-Z]). The 0-level of the restriction of f_t, $t \in \,]\,0\,,\,1\,]$, to $D(0, \rho)$ is up to a diffeomorphism independent of t, it is called a divide ("partage" in [AC2]) and it is shown that for instance the divide determines the homological monodromy group of the versal deformation of the singularity. Figure 1 represents a divide for the singularity at $0 \in \mathbb{C}^2$ of the curve $(y^5 - x^3)(x^5 - y^3) = 0$.

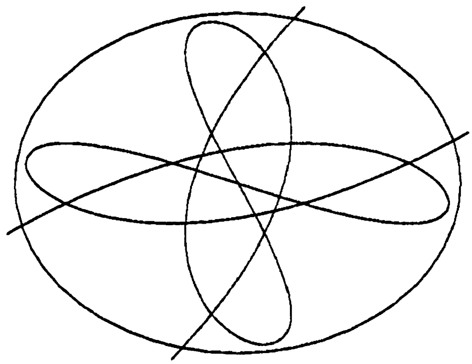

Fig. 1 A divide for $(y^5 - x^3)(x^5 - y^3) = 0$.

Remark.— The transversal isotopy class of the divide of a singularity with real branches is not a topological invariant of the singularity. The singularities of

$$y^4 - 2y^2x^3 + x^6 + x^7 \quad \text{and} \quad y^4 - 2y^2x^3 + x^6 - x^7$$

have congruent but not transversal isotopic divides. The singularities

$$(x^2 - y^2)(x^2 - y^3)(y^2 - x^3) \quad \text{and} \quad (x^2 - 3xy + 2y^2)(x^2 - y^3)(y^2 - x^3)$$

are topologically equivalent but can not have congruent divides. The singularity $y^3 - x^5$ admits two divides, which give a model for the smallest possible transition, according to the mod 4 congruence of V. Arnold [A] and its celebrated strengthening to a mod 8 congruence of V. A. Rohlin ([R1], [R2]), of odd ovals to even ovals for projective real M-curves of even degree owe this remark to Oleg Viro [V]. More precisely, there exist a polynomial family $f_s(x,y)$, $s \in \mathbb{R}$, of polynomials of degree 6, having the central symmetry $f_s(x,y) = -f_{-s}(-x,-y)$ such that the levels $f_s(x,y) = 0$, $s \neq 0$, are divides for the singularity $f_0(x,y) = y^3 - x^5$. Moreover, for $s \in \mathbb{R}$, $s \neq 0$, the divide $f_s(x,y) = 0$ has for regions on which the function has the sign of the parameter s. Therefor, at $s = 0$ four regions of $f_s(x,y) = 0$ collapse and hence, four ovals of $f_s(x,y) = \left(s/2\right)^E$ collapse and change parity at $s = 0$ if the exponent E is big and odd.

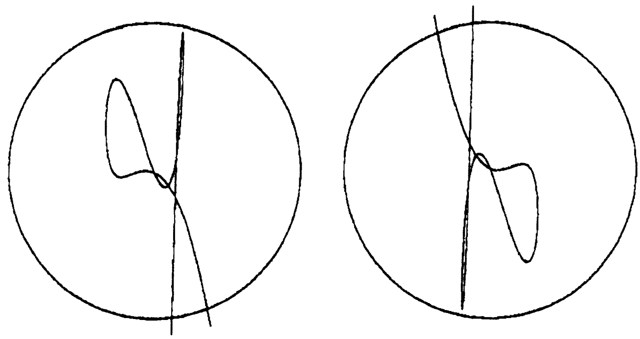

Fig. 2(a) The divides $f_{\pm 1}(x,y)=0$.

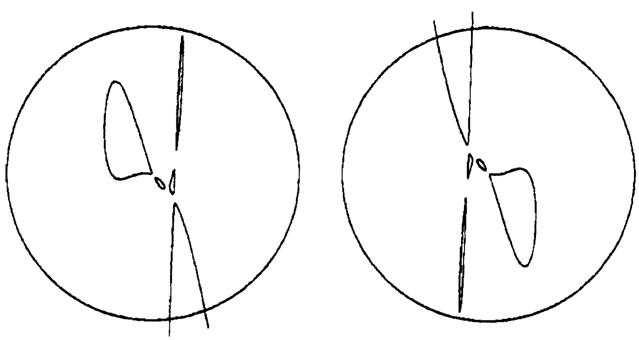

Fig. 2(b) Four ovals change parity.

In Figure 2(b) are drawn the smoothings of the divides $f_{-1}(x,y)=0$ and $f_{+1}(x,y)=0$ of the singularity of $f_0(x,y)=y^3-x^5$ at 0. Each of the smoothings $\{f_{\pm 1}(x,y)=\pm\epsilon\}\cap D$ consists of four ovals and a chord, such that the ovals lie on the positive, respectively on the negative side, of the chord. Such a family $f_s(x,y)$ is for instance given by:

$$y^3-x^5-\frac{125}{8}s^3x^6+\left(\frac{375}{64}s^6+\frac{245}{16}s^4-\frac{25}{4}s^2\right)x^5+$$
$$+\frac{75}{4}s^2x^4y+\left(\frac{2\,695}{128}s^7+\frac{21\,625}{256}s^9-\frac{35}{8}s^3-\frac{4\,847}{160}s^5\right)x^4+$$
$$-\left(5\,s+\frac{159}{4}s^3-\frac{75}{4}s^5\right)x^3y+$$

$$- \frac{15}{2} s x^2 y^2 - \left(\frac{2\,703}{500} s^6 + \frac{1\,281}{32} s^8 - \frac{29\,625}{512} s^{12} + \frac{2\,345}{128} s^{10} \right) x^3 +$$
$$- \left(\frac{17\,625}{256} s^8 + \frac{3\,583}{32} s^6 + \frac{5\,793}{400} s^4 \right) x^2 y - \left(\frac{95}{2} s^4 + \frac{53}{10} s^2 \right) x y^2 +$$
$$- \left(\frac{997}{4\,000} s^9 + \frac{4\,2875}{2\,048} s^{15} + \frac{4\,575}{256} s^{13} + \frac{857}{320} s^{11} \right) x^2 +$$
$$- \left(\frac{177\,325}{2\,048} s^{11} + \frac{1\,803}{200} s^7 + \frac{35\,441}{512} s^9 \right) xy - \left(\frac{6\,395}{128} s^7 + \frac{317}{40} s^5 \right) y^2 +$$
$$+ \left(\frac{19\,871}{1\,280} s^{14} + \frac{10\,165}{1\,024} s^{16} - \frac{59\,125}{4\,096} s^{18} + \frac{4\,171}{2\,000} s^{12} \right) x +$$
$$+ \left(\frac{51\,025}{4\,096} s^{12} + \frac{54\,223}{25\,600} s^{10} - \frac{153\,725}{16\,384} s^{14} \right) y \,.$$

Problem.— Classify up to transversal isotopy, i.e., isotopy through immersions with only transversal double and triple point crossings, the divides for an isolated real plane curve singularity.

3. Complex topology of plane curve singularities

In this section we wish to explain how one can read off from the divide of a plane curve singularity S the local link L, the Milnor fiber and the geometric monodromy group of the singularity. In particular, we will give the geometric monodromy of the singularity explicitly as a product of Dehn twists.

Let $P \subset D(0,\rho)$ be the divide of the singularity f. For a tangent vector $v \in TD(0,\rho) = D(0,\rho) \times \mathbb{R}^2$ of D at the point $p \in D(0,\rho)$, let $J(v) \in \mathbb{C}^2$ be the point $p + iv$. The Milnor ball B can be viewed as

$$B(0,\rho) = \left\{ J(v) \;\middle|\; v \in T\big(D(0,\rho)\big) \text{ and } \|J(v)\| \le \rho \right\}.$$

Observe that

$$L(P) := \left\{ J(v) \;\middle|\; v \in T(P) \text{ and } \|J(v)\| = \rho \right\}$$

is a closed submanifold of dimension one in the boundary of the Milnor ball $B(0,\rho)$. We call $L(P)$ the link of the divide P. Note further that

$$R(P) := \left\{ J(v) \;\middle|\; v \in T(P) \text{ and } \|J(v)\| \leq \rho \right\}$$

is an immersed surface in $B(0,\rho)$ with boundary $L(P)$ having only transversal double point singularities. Let $F(P)$ be the surface obtained from $R(P)$ by replacing the local links of its singularities by cylinders. The differential model of those replacements is as follows: let $\phi : \mathbb{C}^2 \to \mathbb{R}$ be a smooth bump function at $0 \in \mathbb{C}^2$; replace the immersed surface $\{(x,y) \in \mathbb{C}^2 \mid xy = 0\}$ by the smooth surface $\{(x,y) \in \mathbb{C}^2 \mid xy = \tau^2\phi(x/\tau, y/\tau)\}$, where τ is a sufficiently small positive real number. We call $R(P)$ the singular and $F(P)$ the regular ribbon surface of the divide P. The connected, compact surface $F(P)$ has genus $g := \delta - r + 1$ and r boundary components. Note, that g is the number of regions of the divide P. A region of P is a connected component of the complement of P in $D(0,\rho)$, which lies in the interior of $D(0,\rho)$. For the example drawn in Figure 1, we have $r = 2$, $\delta = 17$, $g = 16$.

The ribbon surface $R(P)$ carries a natural orientation, since parametrized by an open subset of the tangent space $T(\mathbb{R})$. Hence the surface $F(P)$ and the link $L(P)$ are also naturally oriented. We orient B as a submanifold of $-T\mathbb{R}^2$, which is the orientation of B as a submanifold in $\mathbb{C}^2$.

THEOREM 1.— *Let P be the divide for an isolated plane curve singularity S. The submanifold $\big(F(P), L(P)\big)$ is up to isotopy a model for the Milnor fiber of the singularity S.*

Proof.— Choose $0 < \rho_- < \rho$ such that $P \cap D(0,\rho_-)$ is still a divide for the singularity S. Along the divide the singular level $F_{t,0} := \{(x,y) \in B \mid f_t(x,y) = 0\}$ is up to order 1 tangent to the immersed surface $R(P)$. Hence, for

$$B'_- := \{u + iv \in B(0,\rho_-) \mid u,v \in \mathbb{R}^2, \|v\| \leq \rho'\}$$

with $0 < \rho' \ll \rho$, the intersections $R'(P) := \partial B'_- \cap R(P)$ and $F'_{t,0} := \partial B'_- \cap F_{t,0}$ are transversal and are regular collar neighbourhoods of the divide in $R(P)$ and in $F_{t,0}$. Therefore the nonsingular level

$$F_{t,\eta} := \{(x,y) \in B(0,\rho) \mid f_t(x,y) = \eta\},$$

where $\eta \in \mathbb{R}$ is sufficiently small, contains in its interior $F'_{t,\eta} := B'_- \cap F_{t,\eta}$, which is a diffeomorphic copy of the surface with boundary $F(P)$. Since

$F'_{t,\eta}$ and $F_{t,\eta}$ are connected surfaces both with r boundary components and the intersection forms on the first homology are isomorphic, the difference $F_{t,\eta} \setminus F'_{t,\eta}$ is a union of open collar tubular neighbourhoods of the boundary components of the surface $F_{t,\eta}$. So, the surfaces $F_{t,\eta}$, $F'_{t,\eta}$ and $F(P)$ are diffeomorphic. We conclude by observing that the nonsingular levels $F_{t,\eta}$ and the Milnor fiber are connected in the local unfolding through nonsingular levels. □

From this proof it follows also that the local link $L(S)$ of the singularity S in ∂B is cobordant to the submanifold $\partial F'_{t,0}$ in $\partial B'_{-}$. The cobordism is given by the pair $\big(B \setminus \mathrm{int}(B'_{-}),\, F_{t,\eta} \setminus \mathrm{int}(F'_{t,\eta})\big)$. It is clear, that the pairs $\big(\partial B, L(P)\big)$, $(\partial B'_{-}, \partial F'_{t,0})$ and $(\partial B'_{-}, \partial F'_{t,\eta})$ are diffeomorphic. One can prove even the following result.

THEOREM 2.— *Let P be the divide for an isolated plane curve singularity S. The pairs $\big(\partial B, L(S)\big)$, where $L(S)$ is the local link of the singularity S and $\big(\partial B, L(P)\big)$ are diffeomorphic.*

The proof is given in Section 6.

Remark.— The signed planar Dynkin diagram of the divide determines up to isotopy the divide of the singularity. It follows from Theorem 2, that the signed planar Dynkin diagram determines geometrically the topology of the singularity. Using the theorem of Burau and Zariski stating that the topological type of a plane curve singularity is determined by the mutual intersection numbers of the branches and the Alexander polynomial of each branch, the authors Ludwig Balke and Rainer Kaenders [BK] have proved that the signed Dynkin diagram, without its planar embedding, determines the topology of the singularity.

We need a combinatorial description of the surface $F(P)$. For a divide P we define: a vertex of P is double point of P, and an edge of P is the closure of a connected component of the complement of the vertices in P. Now we choose an orientation of $\mathbb{R}^2$, and a small deformation $\overline{f}$ of the polynomial f such that the 0-level of $\overline{f}$ is the divide P. We call a region of the divide positive or negative according to the sign of $\overline{f}$. We orient the boundaries of the positive regions such that the outer normal and the oriented tangents of the boundary agree in this order with the chosen orientation of $\mathbb{R}^2$. We choose a midpoint on each edge, which connects two vertices. The link P_v of a vertex v is the closure of the connected component of the complement of the midpoints in P containing the given vertex v.

For each vertex v of P we will construct a piece of surface F_v, such that those pieces glue together and build $F(P)$. Let P_v be the link of the vertex v. Call c_v, c'_v the endpoints of the branches of P_v, which are oriented towards v, and d_v, d'_v the endpoints of the branches of P_v, which are oriented away from v. Thus, c_v, c'_v, d_v and d'_v are midpoints or endpoints of the divide P (Fig. 3) Using an orientation of the divide P we label c_v, c'_v such that c'_v comes after c_v, and we label d_v, d'_v such that the sector c_v, d_v is in a positive region. Then F_v is the surface with boundary and corners drawn in Figure 4. There are 8 corners and there are 8 boundary components in

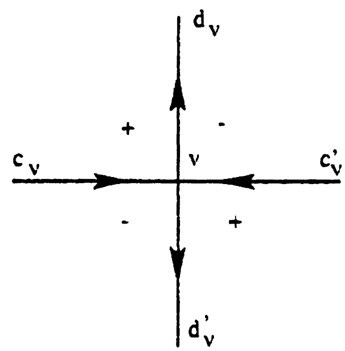

Fig. 3 The link P_v.

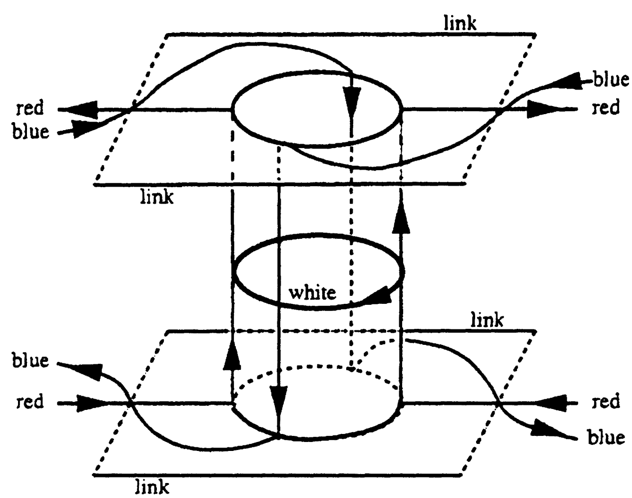

Fig. 4 A piece of surface F_v.

between the corners, 4 of them will get a marking by c_v, c'_v, d_v, d'_v, which will determine the gluing with the piece of the next vertex and 4 do not have a marking. The gluing of the pieces F_v along the marked boundary components according to the gluing scheme given by the divide P yields the surface $F(P)$. On the pieces F_v we have drawn oriented curves colored red, white, and blue. The white curves are simple closed pairwise disjoint curves. The surface $F(P)$ will be oriented such that the curves taken in the order red-white-blue have nonnegative intersections. The remaining red curves glue together and build a red graph on $F(P)$. The remaining blue curves build a blue graph. After deleting each contractible component of the red or blue graph, each of the remaining components contains a simple closed red or blue curve. All together, we have constructed on $F(P)$ a system of μ simple closed curves $\delta_1, \delta_2, \ldots, \delta_{\mu-1}, \delta_\mu$, which we list by first taking red, then white and finally blue. We denote by n_+ the number of red curves which is also the number of positive regions, by $n_.$ the number of crossing points and by n_- the number of blue which equals the number of negative regions of the divide P.

Let D_i be the right Dehn twist along the curve δ_i. A model for the right Dehn twist is the linear action $(x, y) \mapsto (x + y, y)$ on the cylinder $\{(x, y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R} \mid 0 \leq y \leq 1\}$ with as orientation the product of the natural orientations of the factors. A right Dehn twist around a simply closed curve δ on an oriented surface is obtained by embedding the model as an oriented bicollar neighbourhood of δ such that δ and $\mathbb{R}/\mathbb{Z} \times \{1/2\}$ of the model match. The local geometric monodromy of the singularity of $xy = 0$ is as diffeomorphism a right Dehn twist (see the " Théorème Fondamental " of [L, p. 23] and [PS, p. 95]). Using as in [AC2] a local version of a theorem of Lefschetz, one obtains the following result.

THEOREM 3.— *Let P be the divide for an isolated plane curve singularity S. The Dehn twists D_i are generators for the geometric monodromy group of the unfolding of the singularity S. The product $T := D_\mu D_{\mu-1} \cdots D_2 D_1$ is the local geometric monodromy of the singularity S.*

4. The singularity D_5 and a graphical algorithm in general

We will work out the picture for the singularity D_5 with the equation $x(x^3 - y^2)$ and the divide given by the deformation $(x - s)(x^3 + 5sx^2 - y^2)$,

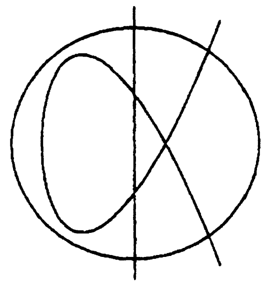

Fig. 5 A divide for the singularity D_5.

$s \in [0, 1]$, which is shown for $s = 1$ in Figure 5. There are one positive triangular region, one negative region and three crossings.

By gluing three pieces together, one gets the Milnor fiber with a system of vanishing cycles as depicted in Figure 6.

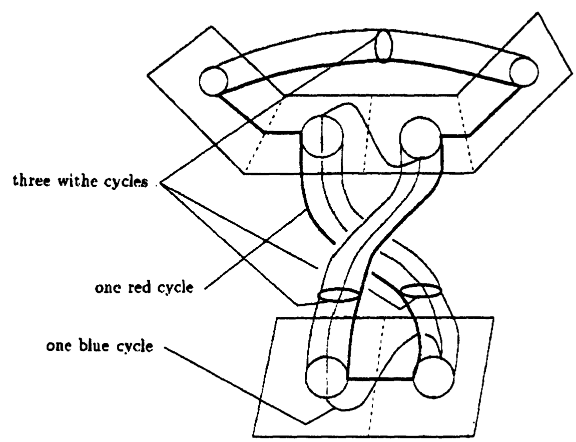

Fig. 6 Milnor fiber with vanishing cycles for D_5.

An easy and fast graphical algorithm of visualizing the Milnor fiber with a system of vanishing cycles directly from the divide is as follows: think the divide as a road network which has δ junctions, and replace every junction by a roundabout, which leads you to a new road network with 4δ T-junctions. Realize now every road section in between two T-junctions by a strip with a half twist. Do the same for every road section in between a T-junction and the boundary of the divide. Altogether you will need $6\delta + r$ strips. The core line of the four strips of a roundabout is a white vanishing cycle, the strips corresponding to boundary edges and corners of a positive or negative region have as core line a red or blue vanishing cycle.

In Figure 7 is worked out the singularity with two Puiseux pairs and $\mu = 16$, where we used the divide from Figure 9.

We have drawn for convenience in Figure 7 only one red, white, or blue cycle. We have also indicated the position of the arc α, which will play a role in the next section.

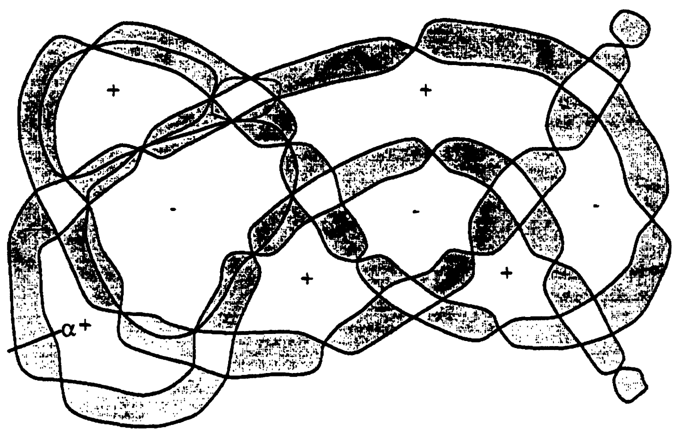

Fig. 7 Milnor fiber with vanishing cycles for $y^4 - 2y^2x^3 + x^6 - x^7 - 4yx^5$.

5. An example of global geometric monodromy

Let $b : \mathbb{C} \to \mathbb{C}^2$, $b(t) := (t^6 + t^7, t^4)$ be the parametrized curve C having at $b(0) = (0,0)$ the singularity with two essential Puiseux pairs and with local link the compound cable knot $(2,3)(2,3)$. The polynomial

$$f(x,y) := y^4 - 2y^2x^3 + x^6 - x^7 - 4yx^5$$

is the equation of C. The function $f : \mathbb{C}^2 \to \mathbb{C}$ has besides 0 the only other critical value

$$c := \frac{14\,130\,940\,973\,168\,155\,968}{558\,545\,864\,083\,284\,007}.$$

The fiber of 0 has besides its singularity at $(0,0)$ a nodal singularity at $(-8,-4)$, which corresponds to the node $b(-1+i) = b(-1-i) = (-8,-4)$.

The geometric monodromy of the singularity at $(0,0)$, which is up to isotopy piecewise of finite order, is described in [AC1]. The fiber of c has a nodal singularity at $(1\,014/343\,,\,16\,807/79\,092)$. The singularity at infinity of the curve C is at the point $(0:1:0)$ and its local equation is $z^3 - 2z^2x^4 + zx^6 - x^7 - 4zx^5$, whose singularity is topologically equivalent to the singularity $u^3 - v^7$ with Milnor number 12. The function f has no critical values coming from infinity. We aim at a description of the global geometric monodromy of the function f. Working with the distance on $\mathbb{C}^2$ given by $\|(x,y)\|^2 := |x|^2 + 4|y|^2$, we have that the parametrized curve b is transversal to the spheres

$$S_r := \left\{(x,y) \in \mathbb{C}^2 \;\middle|\; |x|^2 + 4|y|^2 = r^2\right\}$$

with center $0 \in \mathbb{C}^2$ and radius $r > 0$. So for $0 < r < 8\sqrt{2}$, the intersection $K_r := C \cap S_r$ is the local knot in S_r of the singularity at $0 \in \mathbb{C}^2$ (Fig. 8), at $r = 8\sqrt{2}$ the knot K_r is singular with one transversal crossing, and for $8\sqrt{2} < r$ the knot K_r is the so called knot at infinity of the curve C. By making one extra total twist in a braid presentation of the knot K_r one gets the local knot of the singularity at infinity of the projective completion of the curve C. The crossing at the bottom of Figure 8 flips for $r = 8\sqrt{2}$ and the knot K_r, $r > 8\sqrt{2}$, becomes the $(4,7)$ torus knot.

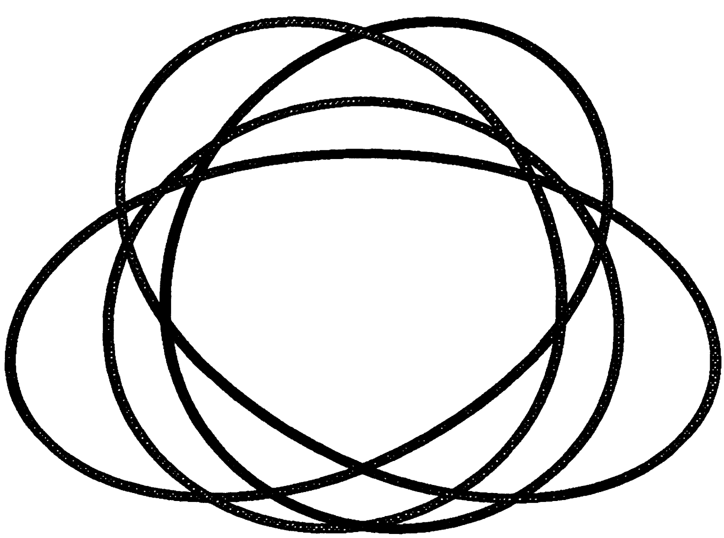

Fig. 8 The torus cable knot (2,3)(2,3).

From the above we get the following partial description of the global geometric monodromy. The typical regular fiber $F := f^{-1}(c/2)$ is the interior of the oriented surface obtained as the union of two pieces A and B, where A is a surface of genus 8 with one boundary component and B is a cylinder. The pieces are glued together in the following way: in each boundary component of B there is an arc, which is glued to an arc in the boundary of A. The interior of A or B can be thought of as a Milnor fiber of the singularity at 0 or $(-8,-4)$. So, the geometric monodromy around 0 is a diffeomorphism with support in the interior of A and B, given for instance by a construction as in Section 2. The piece A can be constructed from the divide in Figure 9.

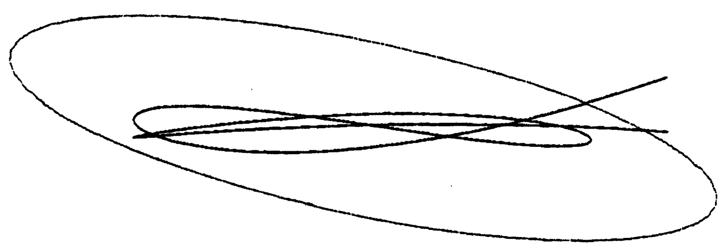

Fig. 9 The curve $(x_s(t), y_s(t))$, $s := 1$, as divide for the singularity of C.

Clearly, the monodromy in B is a positive Dehn twist around the simple essential closed curve δ_{17} in B, whereas the monodromy in A is a product of positive Dehn twists around a system $(\delta_1, \ldots, \delta_{16})$ of 16 red, white or blue curves. The monodromy around the critical point c is a positive Dehn twist around a simple curve δ_{18}, in F, which is the union of two simple arcs $\alpha \subset A$ and $\beta \subset B$. The arcs α and β have their endpoints p, $q \in A \cap B$ in common, and moreover the points p and q lie in different components of $A \cap B$. The arc β cuts the curve δ_{17} transversally in one point. The arc α intersects the curves $(\delta_1, \ldots, \delta_{16})$ transversally in some way. For the position of the system (δ_{17}, β) in B there is up to a diffeomorphism of the pair $(B, A \cap B)$ only one possibility. To obtain a complete description of the global monodromy it remains to describe the position of the system $(\delta_1, \ldots, \delta_{16}, \alpha)$ in $(A, A \cap B)$.

We consider the family with parameter s of parametrized curves with parameter t:

$$x_s(t) = \frac{T(4,t)}{8} = t^4 - t^2 + \frac{1}{8},$$

$$
\begin{aligned}
y_s(t) &= \frac{sT(6,t)}{32} + \frac{T(7,t)}{64} \\
&= st^6 + t^7 - \frac{3}{2}\, st^4 - \frac{7}{4}\, t^5 + \frac{9}{16}\, st^2 + \frac{7}{8}\, t^3 - \frac{1}{32}\, s - \frac{7}{64}\, t\,,
\end{aligned}
$$

where $T(d,t)$ is the Chebychev polynomial of degree d. Let

$$
\begin{aligned}
f_s(x,y) := {}& s^4x^6 - \frac{3}{128}\, s^4x^4 + \frac{1}{1\,024}\, s^4x^3 - 2\, s^2y^2x^3 - 4\, syx^5 + \\
& - x^7 + \frac{9}{65\,536}\, s^4x^2 - \frac{3}{262\,144}\, s^4x \frac{3}{128}\, s^2y^2x - \frac{1}{4\,096}\, s^2x^3 + \\
& + \frac{5}{64}\, syx^3 + \frac{7}{256}\, x^5 + \frac{1}{4\,194\,304}\, s^4 - \frac{1}{1\,024}\, s^2y^2 - \frac{1}{1\,024}\, syx^2 + \\
& + y^4 + \frac{3}{1\,048\,576}\, s^2x - \frac{5}{16\,384}\, syx - \frac{7}{32\,768}\, x^3 - \frac{1}{8\,388\,608}\, s^2 + \\
& + \frac{1}{131\,072}\, sy - \frac{1}{4\,096}\, y^2 + \frac{7}{16\,777\,216}\, x + \frac{1}{134\,217\,728}
\end{aligned}
$$

be the equation, monic in y, for the curve $\big(x_s(t),\, y_s(t)\big)$, whose real image is for $s = 1$ a divide (Fig. 9) for the singularity of C at $(0,0)$. The 0-level of f_s for $s = 1$ consists of this divide and an isolated minimum not in one of its regions, which corresponds to the minimum of the restriction of f to $\mathbb{R}^2$ at $(-8,-4)$. For a small, we call $\delta_{17,a,s} \subset \{f_s = a\}$ the vanishing cycle of the local minimum of f_s, $s = 1$, which does not belong to a region. The curve $\big(x_s(t), y_s(t)\big)$, $t \in \mathbb{R}^2$, $s = 7\sqrt{2}/24$, has 8 nodes, a cusp at $t = -\sqrt{2}/2$, (Fig. 10) and at infinity a singularity with Milnor number 12. We now vary the parameter $s \in [\,7\sqrt{2}/24 - \sigma\,,\, 1\,]$ from 1 to $7\sqrt{2}/24 - \sigma$ for a very small $\sigma > 0$.

The value of the local minimum, which does not belong to a region, becomes smaller and by adjusting the parameter a we can keep the cycle

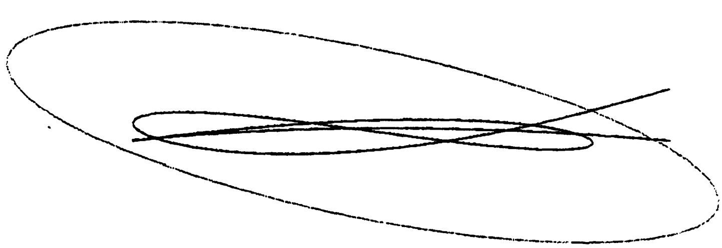

Fig. 10 The curves $\big(x_s(t), y_s(t)\big)$ for $s := 7\sqrt{2}/24$.

$\delta_{17,a,s}$ in the new region which emerges from the cusp at $s = 7\sqrt{2}/24$. Since the total Milnor number of f_s is

$$\big(\mathrm{degree}(f_s) - 1\big)\big(\mathrm{degree}(f_s) - 2\big) - 12 = 18\,,$$

it follows that all its singularities have Milnor number 1 for $s = 7\sqrt{2}/24 - \sigma$. The vanishing cycle of the node, which appears when deforming the cusp singularity, will be called $\delta_{18,s}$ and the vanishing cycle in the region of the divide of Figure 9, in whose boundary the cusp appeared, will be called $\delta_{16,s}$.

We label the vanishing cycles on the regular ribbon surface F_- of the divide of Figure 11 by $\delta_1, \ldots, \delta_{15}$. The cycles $\delta_{17,a,s}$, $\delta_{18,s}$, $\delta_{16,s}$ deform without changing their intersection pattern and $\delta_{16,s}$ becomes the cycle δ_{16} of the regular ribbon surface F_+ of the divide of Figure 9. Observe that the regular ribbon surface F_- of the divide of Figure 11 is naturally a subset of the regular ribbon surface F_+ of the divide of Figure 9. The description of the position of the system $(\delta_1, \ldots, \delta_{16}, \alpha)$ in $(A, A \cap B)$, for which we are looking, is the system $(\delta_1, \ldots, \delta_{16})$ on F_+, where the relative cycle α is a simple arc on $F_+ - F_-$ with endpoints on the boundary of F_+ and cutting the cycle δ_{16} transversally in one point. Observe that $F_+ - F_-$ is a strip with core δ_{16} (Fig. 7).

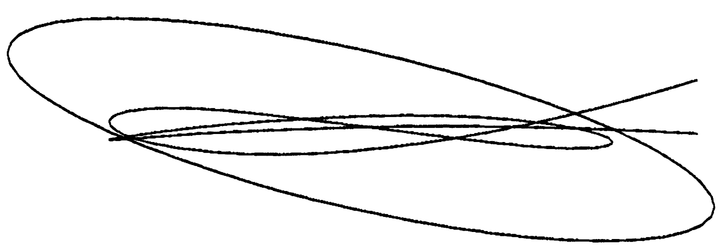

Fig. 11 A divide, which does not come from a singularity.

6. Connected divides and fibered knots. Proof of Theorem 2

In this section we assume, without loss of generality, that a divide is linear and orthogonal near its crossing points. For a connected divide $P \subset D(0, \rho)$, let $f_P : D(0, \rho) \to \mathbb{R}$ be a generic C^∞ function, such that P is its 0-level and that each region has exactly one local maximum or minimum. Such a function exist for a connected divide and is well defined up to sign and isotopy. In particular, there are no critical points of saddle type other then the crossing points of the divide. We assume moreover that the function f_P is quadratic and euclidean in a neighborhood of its critical points, i.e., for euclidean coordinates (X, Y) with center at a critical point c of f_P we have in a neighborhood of c the expression $f_P(X,Y) = f_P(c) + XY$ or $f_P(X,Y) = f_P(c) + X^2 + Y^2$. Let $\chi : D(0,\rho) \to [0\,,1]$ be a C^∞, positive function, which evaluates to zero outside of the neighborhoods where f_P is quadratic and to 1 in some smaller neighborhood of the critical points of f_P. Let $\theta_P : \partial B(0,\rho) \to \mathbb{C}$ be given by

$$\theta_{P,\eta}\big(J(v)\big) := f_P(x) + i\eta\, \mathrm{d}f_P(x)(u) - \frac{1}{2}\,\eta^2\chi(x)H_{f_P}(v)$$

for $J(v) = (x,u) \in TD(0,\rho) = D \times \mathbb{R}^2$ and $\eta \in \mathbb{R}$, $\eta > 0$. Observe that the Hessian H_{f_P} is locally constant in a the neighborhood of the critical points of f_P. The function $\theta_{P,\eta}$ is C^∞. Let $\pi_{P,\eta} : \partial B(0,\rho) \setminus L(P) \to S^1$ be defined by

$$\pi_{P,\eta}\big(J(v)\big) := \frac{\theta_{P,\eta}\big(J(v)\big)}{\Big|\theta_{P,\eta}\big(J(v)\big)\Big|}\,.$$

THEOREM 4.— *Let $P \subset D(0,\rho)$ be a divide, such that the system of immersed curves is connected. The link $L(P)$ in*

$$\partial B(0,\rho) = \Big\{J(v) \;\Big|\; v \in T\big(D(0,\rho)\big) \;\; and \;\; \big\|J(v)\big\| = \rho\Big\}$$

is a fibered link. The map $\pi_P := \pi_{P,\eta}$ is for η sufficiently small, a fibration of the complement of $L(P)$ over S^1. Moreover the fiber of the fibration π_P is $F(P)$ and the geometric monodromy is the product of Dehn twist as in Theorem 3.

The map π_P is compatible with a regular product tubular neighborhood of $L(P)$ in $\partial B(0,\rho)$. The map π_P is a submersion, so, since already a fibration near $L(P)$, it is a fibration by a theorem of Ehresmann. The graphical algorithm, see Figure 7, produces in fact, up to a small isotopy of the image, the projection of the fiber $\pi_P^{-1}(i)$ on $D(0,\rho)$. This projection is except above the twist of the strips a submersion. The proof of Theorem 4 is given in the forthcoming paper [AC4] on generic immersions of curves and knots.

Proof of Theorem 2

The oriented fibered links $L(S)$ and $L(P)$ have the same geometric monodromies according to the Theorem 3 and 4. So, the links $L(S)$ and $L(P)$ are diffeomorphic. □

Remark.— Let $f(x,y)=0$ be a singularity S, such that written in the canonical coordinates of the charts of the embedded resolution the branches of the strict transform have equations of the form $u=a$, $a\in\mathbb{R}$. Let $f_t(x,y)$, $t\in[0\,,\,1]$ be a morsification, with its divide P in $D(0,\rho)$, obtained by blowing down generic real linear translates of the strict transforms, as in [AC2]. We strongly believe that with the use of [BC1] and [BC2] the following transversallity property can be obtained, and which we state as a following problem.

There exists $\rho'_0>0$, such that for all $t\in[0\,,\,1]$ and for all $\rho'\in(0\,,\,\rho'_0]$ the 0-levels of $f_t(x,y)$ in $\mathbb{C}^2$ meet transversally the boundary of

$$B(0,\rho,\rho') := \left\{(x+iu,y+iv)\in\mathbf{C}^2 \mid x^2+y^2+u^2+v^2\le\rho^2\,,\ u^2+v^2\le\rho'^2\right\}.$$

It is easy to deduce from this transversallity statement an isotopy between the links $L(S)$ and $L(P)$.

Remark.— Bernard Perron has given a proof for the triviallity of the cobordism from $L(S)$ to $L(P)$, which uses the holomorphic convexity of the balls $B(0,\rho,\rho')$ of the previous remark [P].

Acknowledgments

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