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# On the radial solutions of a degenerate quasilinear elliptic equation in $\mathbf{R}^N$

# On the radial solutions of a degenerate quasilinear elliptic equation in $\mathbb{R}^{N \, (*)}$

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Résumé. — On étudie l'existence, l'unicité ainsi que les propriétés qualitatives des solutions radiales de l'équation $\triangle_p u + \dfrac{q+1-p}{p(q-1)} x \nabla u + \dfrac{u}{q-1} + |u|^{q-1} u = 0$ dans $\mathbb{R}^N$. Si $q > p-1 > 1$, alors pour tout $\gamma \in \mathbb{R}^*$, il existe une unique radiale solution $u(r,\gamma)$ telle que $u(0,\gamma) = \gamma$. On montre aussi que si $2 < p < N$, $q \geqslant \dfrac{Np}{N-p} - 1$ et $\gamma \in \mathbb{R}^*_+$ ou bien si $q > \dfrac{p}{N} + p - 1$ et $\gamma$ assez petit, alors $u(r,\gamma) > 0$ et $\lim\limits_{r \to +\infty} r^{\frac{p}{q+1-p}} u(r,\gamma) = c > 0$. D'autre part si $p-1 < q \leqslant \dfrac{p}{N} + p - 1$ il n'y a pas de solutions strictement positives.

De plus si $2 < p < N$, et $p - 1 < q < \dfrac{Np}{N-p} - 1$ alors il existe $\gamma^* > 0$ tel que pour tout $\gamma \geqslant \gamma^*$, $u(r,\gamma)$ change de signe.

Abstract. — Existence, uniqueness and qualitative behaviour of radial solutions of $\triangle_p u + \dfrac{q+1-p}{p(q-1)} x \nabla u + \dfrac{u}{q-1} + |u|^{q-1} u = 0$ for $x \in \mathbb{R}^N$ are presented. If $q > p - 1 > 1$, for any $\gamma \in \mathbb{R}^*$ there exists a unique radial solution $u(r,\gamma)$ such that $u(0,\gamma) = \gamma$. Also it is proved that if $2 < p < N$, $q \geqslant \dfrac{Np}{N-p} - 1$ and $\gamma \in \mathbb{R}^*$ or if $q > \dfrac{p}{N} + p - 1$ and $\gamma$ small, then $u(r,\gamma) > 0$ and $\lim\limits_{r \to +\infty} r^{\frac{p}{q+1-p}} u(r,\gamma) = c > 0$. On the other hand if $p - 1 < q \leqslant \dfrac{p}{N} + p - 1$ there are no positive solutions.

Moreover if $2 < p < N$ and $p - 1 < q < \dfrac{Np}{N-p} - 1$ then there exists $\gamma^* > 0$ such that for any $\gamma > \gamma^*$, the solution $u(r,\gamma)$ can not be positive.

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# 0. Introduction

This paper is devoted to the study of the quasilinear elliptic equation

$$\triangle_p u + \frac{q+1-p}{p(q-1)}\, x\nabla u + \frac{1}{q-1}u + |u|^{q-1}\,u = 0, \qquad x \in I\!\!R^N. \qquad (0.1)$$

where, as usual $\triangle_p u = div\left(|\nabla u|^{p-2}\nabla u\right)$ is the $p$-laplacian and $\nabla u$ is the gradient of $u$. We are concerned with existence, uniqueness and qualitative behavior of radially-symmetric solutions whenever $1 < p - 1 < q$.

The above equation arises naturally in the study of self-similar solutions of the parabolic equation

$$w_t = \triangle_p w + |w|^{q-1}\,w \qquad \text{in } I\!\!R^N \times (0,+\infty) \qquad (0.2)$$

i.e. solutions $w$ of (0.2) which are of the form

$$w(x,t) = t^{-\frac{1}{q-1}}\,u\left(t^{-\frac{q+1-p}{p(q-1)}}\,x\right) \qquad (0.3)$$

where $u$ satisfies the equation (0.1). If we make $p = 2$ in (0.2) we get the semilinear equation which has been thoroughly studied by Haraux, Weissler, Terman, Peletier ([4], [7], [10]) and others. They noticed the existence of two critical values for $q$ in $(1,+\infty)$ which are $\dfrac{N+2}{N}$ and $\dfrac{N+2}{N-2}$ if $N \geqslant 3$ ( 2 if $N = 2$ and 3 if $N = 1$ ) and they give the precise asymptotic behavior for solutions. Now if we drop the gradient term in (0.1) we obtain a particular case of a large class of equations which have been studied by J. Serrin and P. Pucci [9]. Our purpose here is to extend some results of [4] to the case $p > 2$.

It is worth mentioning that if we change the sign in front of the absorption term in (0.2) we get the following equation

$$w_t = \triangle_p w - w^q \qquad \text{in } I\!\!R^N \times (0,+\infty) \qquad (0.4)$$

which have been intensively studied. A special attention has been devoted to the study of singular solutions i.e. non negative functions $w$ satisfying (0.4) in $I\!\!R^N \times (0,+\infty)$ and such that $w(x,0) = 0$ if $x \in I\!\!R^N \setminus\{0\}$. In fact it was proved by the second author, in [2], that (0.4) has a non trivial singular solution if and only if $0 < q < p-1+\frac{P}{N}$ . The uniqueness of the fundamental solution i.e. $u(x,0) = c\delta(x)$ for some positive constant $c$, is proved by Kamin and Vàzquez in [5]. On the other hand, whenever $p - 1 < q < p-1+\frac{P}{N}$ , Peletier and Wang [8] proved the existence of a singular solution satisfying

$\lim_{t\to\infty} \int_{|x|<r} u(x,t)dx = +\infty$ for any $r > 0$; which called "very singular solution". Finally Diaz and Saa in [1] studied the uniqueness of a radial solution of following problem

$$\begin{cases} \Delta_p(u^m) + \dfrac{q - m(p-1)}{p(q-1)} \, x\nabla u + \dfrac{1}{q-1} \, u - |u|^{q-1} u = 0, & x \in I\!R^N . \\ \lim_{|x|\to\infty} u(x) = 0 \end{cases} \tag{0.5}$$

where $G(u) = \dfrac{1}{q-1} \, u - |u|^{q-1} u$ and $N \geqslant 1$, $m > 0$, $q > 0$, $p > 1$ such that $m(p-1) > 1$ and $(p-1)m < q < (p-1)m + \frac{p}{N}$ . More precisely they prove that problem (0.5) has at most one solution and this solution has compact support. As a consequence they deduce the uniqueness of the very singular solution of (0.4). From the point of view of the techniques involved in [8], the concrete expression of function $G$ is fundamental. so, we shall not use the shooting method in the phase plane but a different technique.

Since our attention is focused on radial solutions $u(r)$, where $r = |x|$, then the problem (0.1) can be written in the following initial value problem

$$(P) \begin{cases} \left( \left| u' \right|^{p-2} u' \right)' + \dfrac{N-1}{r} \left| u' \right|^{p-2} u' + \dfrac{q+1-p}{p(q-1)} r u' + f(u) = 0, \ r > 0. \\ \hspace{9cm} (0.6) \\ u(0) = \gamma, \ u'(0) = 0, \hspace{5cm} (0.7) \end{cases}$$

where $\gamma$ is a constant and

$$f(u) = \dfrac{u}{q-1} + |u|^{q-1} u . \tag{0.8}$$

By a solution of $(P)$ we mean a function $u \in C^1([0, +\infty))$ such that $\left| u' \right|^{p-2} u' \in C^1([0, +\infty))$ which satisfies (0.6) and (0.7) for a fixed real $\gamma$.

The following results are obtained.

THEOREM 1. — Let $q \geqslant p - 1 > 1$. Then for any $\gamma \in I\!R^*$, problem $(P)$ has a unique solution $u(r, \gamma)$.

It is important to remark that $u(r, -\gamma) = -u(r, \gamma)$. Therefore, from now on we shall state the results only for $\gamma$ positive.

THEOREM 2. — (i) Assume $q > \dfrac{p}{N} + p - 1$. Then for any

$$\gamma \in \left( 0, \left( \dfrac{N(q+1-p)}{p(q-1)} - \dfrac{1}{q-1} \right)^{\frac{1}{q-1}} \right),$$

*the solution $u$ of $(P)$ is strictly positive. Moreover, there is a constant $c$ such that*

$$\lim_{r \to +\infty} r^{\frac{p}{q+1-p}} u(r, \gamma) = c > 0. \tag{0.9}$$

(ii)  *Assume $2 < p < N$ and $q > \dfrac{Np}{N-1} - 1$. Then the conclusion (i) holds for any $\gamma \in I\!R^+$.*

THEOREM 3. — (i)  *Assume $p - 1 < q \leqslant \dfrac{p}{N} + p - 1$. Then there are no positive solutions of $(P)$.*

(ii)  *Assume $2 < p < N$ and $p - 1 < q < \dfrac{Np}{N-p} - 1$. Then there exists $\gamma^* > 0$ such that for any $\gamma > \gamma^*$, the solution $u$ of $(P)$ can not be positive.*

The organization of this paper is as follows.

Section 1 is devoted to the proof of theorem1 which is based on the Banach fixed point theorem.

Section 2 collects some properties of solutions of problem $(P)$. In particular, using an energy function, we prove that any solution $u$ of (0.6) and its derivative $u'$ tend to zero as $r$ tends to infinity.

The existence and the asymptotic behavior of strictly positive solutions are studied in section 3.

Finally, section 4 is devoted to the proof of theorem 3. In part (i) of this theorem we us an estimate of $u'$ and part (ii) is proved by using a scaling argument. We also present some important information for non positive solutions. The study of positive rapidly decreasing solutions of $(P)$ remains an open problem.

## 1. Existence and uniqueness of solutions

In this section we investigate the existence and uniqueness of a solution $u(., \gamma)$ of the initial value problem $(P)$ for a fixed $\gamma \in I\!R^*$. Recall that this solution, which we continue denoting by $u$, is a $\mathcal{C}^1$ function such that $\left|u'\right|^{p-2} u' \in \mathcal{C}^1([0, +\infty))$.

We start with a local uniqueness and existence result which is analogous to a result due to Serrin and Veron (see $[3]$, Theorem 5.2).

PROPOSITION 1.1. — *Assume $q \geqslant p-1 > 1$. Then for any $\gamma \in \mathbb{R}^*$, there exists a constant $r_\gamma > 0$ such that, problem $(P)$ admits a unique solution $u$ in $[0, r_\gamma)$.*

*Proof.* — The proof is divided in three steps.

**Step 1:** *Local uniqueness.*

If $u$ is a solution of problem $(P)$ in $[0, r_\gamma)$. Then $u$ satisfies the following equation

$$( r^{N-1} \left| u' \right|^{p-2} u' )' + \frac{q+1-p}{p(q-1)} r^N u' + r^{N-1} f(u) = 0 \tag{1.1}$$

where $f$ is given in (0.8). Then

$$u(r) = \gamma - \int_0^r G \left\{ \frac{q+1-p}{p(q-1)} su(s) + s^{1-N} \int_0^s \sigma^{N-1} F(u(\sigma)) d\sigma \right\} ds \tag{1.2}$$

where

$$G(s) = |s|^{-\frac{p-2}{p-1}} s \tag{1.3}$$

and

$$F(s) = \frac{1}{q-1} \left( 1 - \frac{N(q+1-p)}{p} \right) s + |s|^{q-1} s. \tag{1.4}$$

Note that $F$ is a locally Lipschitz continuous function.

Let $v$ be another solution of $(P)$ in $[0, r_\gamma)$. Hence, for any $r \in (0, r_\gamma)$

$$|u(r) - v(r)| \leqslant \int_0^r |G(A(s)) - G(B(s))| \, ds \tag{1.5}$$

where

$$A(s) = \frac{q+1-p}{p(q-1)} su(s) + s^{1-N} \int_0^s \sigma^{N-1} F(u(\sigma)) d\sigma \tag{1.6}$$

and

$$B(s) = \frac{q+1-p}{p(q-1)} sv(s) + s^{1-N} \int_0^s \sigma^{N-1} F(v(\sigma)) d\sigma \tag{1.7}$$

for $0 < s \leqslant r < r_\gamma$.

Next, let

$$\phi(s) = Min \left( |A(s)|, |B(s)| \right). \tag{1.8}$$

It is not difficult to see that

$$|G(A(s)) - G(B(s))| \leqslant \frac{G(\phi(s))}{\phi(s)} |A(s) - B(s)|. \tag{1.9}$$

Substituting this last inequality into (1.5) we obtain

$$|u(r) - v(r)| \leqslant \int_0^r \frac{G(\phi(s))}{\phi(s)} |A(s) - B(s)| \, ds \, . \tag{1.10}$$

Let $m > 0$ and $K_m$ be the Lipschitz constant of $F$ on $[\gamma - m, \gamma + m]$. In view of the continuity of $u$ and $v$, there exists $r_0 > 0$ such that

$$\underset{0 \leqslant r \leqslant r_0}{\mathrm{Max}} \, (|u(r) - \gamma|, |v(r) - \gamma|) \leqslant m \, . \tag{1.11}$$

Let

$$\alpha = \frac{|\gamma| \left( \dfrac{1}{q-1} + |\gamma|^{q-1} \right)}{2 \left( K_m + \dfrac{N(q+1-p)}{p(q-1)} \right)}; \tag{1.12}$$

Notice that, since

$$K_m \leqslant \sup_{[\gamma-m,\gamma+m]} \left| F'(s) \right| \leqslant \frac{1}{q-1} \left| 1 - \frac{N(q+1-p)}{p} \right| + q \left( |\gamma| + m \right)^{q-1}, \tag{1.13}$$

it is possible to choose $m$ such that $m \leqslant \alpha$. By using (1.11), we obtain easily the following inequalities on $[0, r_0]$

$$|F(u(s)) - F(\gamma)| \leqslant K_m |u(s) - \gamma| \leqslant mK_m \, , \tag{1.14}$$

$$\frac{F(\gamma) - mK_m}{N} s \leqslant s^{1-N} \int_0^s \sigma^{N-1} F(u(\sigma)) d\sigma \leqslant \frac{F(\gamma) + mK_m}{N} s \tag{1.15}$$

and

$$(\gamma - m)s \leqslant su(s) \leqslant (\gamma + m)s. \tag{1.16}$$

Combining (1.15)) and (1.16) we deduce that $A(s)$ has a constant sign on $[0, r_0]$. On the other hand, since $\gamma \neq 0$ we have

$$\frac{q+1-p}{p(q-1)} \gamma + \frac{F(\gamma)}{N} \neq 0. \tag{1.17}$$

Moreover, if we set

$$\beta = \frac{1}{2} \left| \frac{F(\gamma)}{N} + \frac{q+1-p}{p(q-1)} \gamma \right| = \frac{|\gamma|}{2N} \left( \frac{1}{q-1} + |\gamma|^{q-1} \right) \tag{1.18}$$

we deduce, from the choice of $m$, that

$$|A(s)| \geqslant \beta s \qquad \text{for } 0 \leqslant s \leqslant r \leqslant r_0. \tag{1.19}$$

Of course the same estimate is also valid for $B(s)$. As a consequence

$$\phi(s) \geqslant \beta s \qquad \text{for any } 0 \leqslant s \leqslant r \leqslant r_0. \tag{1.20}$$

This last inequality, together with (1.10) and the fact that the function $r \longrightarrow \dfrac{G(r)}{r}$ is decreasing implies that

$$|u(r) - v(r)| \leqslant \int_0^r \frac{G(\beta s)}{\beta s} |A(s) - B(s)| \, ds, \tag{1.21}$$

for every $r \in (0, r_0)$. But

$$|A(s) - B(s)| \leqslant \frac{q + 1 - p}{p(q - 1)} s \, |u(s) - v(s)|$$

$$+ K_m s^{1-N} \int_0^s \sigma^{N-1} |u(\sigma) - v(\sigma)| \, d\sigma. \tag{1.22}$$

So that

$$|u(r) - v(r)| \leqslant \frac{q + 1 - p}{p(q - 1)} \int_0^r \frac{G(\beta s)}{\beta s} s \, |u(s) - v(s)| \, ds \tag{1.23}$$

$$+ K_m \int_0^r \frac{G(\beta s)}{\beta s} s^{1-N} \left( \int_0^s \sigma^{N-1} |u(\sigma) - v(\sigma)| \, d\sigma \right) ds.$$

We have

$$K_m \int_0^r \frac{G(\beta s)}{\beta s} s^{1-N} \left( \int_0^s \sigma^{N-1} |u(\sigma) - v(\sigma)| \, d\sigma \right) ds \tag{1.24}$$

$$= K_m \int_0^r \sigma^{N-1} |u(\sigma) - v(\sigma)| \left( \int_\sigma^r s^{1-N} \frac{G(\beta s)}{\beta s} ds \right) d\sigma \; .$$

On the other hand

$$\int_\sigma^r s^{1-N} \frac{G(\beta s)}{\beta s} ds \leqslant \frac{p - 1}{N(p - 1) - p} \beta^{-\frac{p-2}{p-1}} \sigma^{\frac{p}{p-1} - N}. \tag{1.25}$$

Hence (1.23) implies that

$$|u(r) - v(r)| \leqslant \int_0^r |u(s) - v(s)| \, a(s) ds, \quad \text{for } 0 \leqslant r \leqslant r_0, \tag{1.26}$$

where

$$a(s) = \left( \frac{q + 1 - p}{p(q - 1)} + \frac{p - 1}{N(p - 1) - p} K_m \right) \beta^{-\frac{p-2}{p-1}} s^{\frac{1}{p-1}}. \tag{1.27}$$

Finally using Gronwall's lemma we conclude that $u = v$ on $[0, r_0]$.

**Step 2:** *Local existence*

Let $m$, $K_m$, $\alpha$ and $\beta$ be the previously chosen constants. Set

$$\delta = \frac{\dfrac{q+1-p}{p(q-1)}|\gamma| + \dfrac{|F(\gamma)|}{N}}{2\dfrac{K_m}{N}\dfrac{p}{N(p-1)-p}} \tag{1.28}$$

and

$$m_1 = \inf(m, \alpha, \delta). \tag{1.29}$$

Let $I = [0, r_\gamma]$, where $r_\gamma$ is given by

$$r_\gamma = \max\left\{ r \geqslant 0 : \int_0^r G(\beta s)ds \leqslant \frac{\beta m_1}{\dfrac{q+1-p}{p(q-1)}(|\gamma| + m_1) + \dfrac{|F(\gamma)| + m_1 K_m}{N}} \right\} \tag{1.30}$$

$$= \left(\frac{p}{p-1}\right)^{\frac{p-1}{p}} \left( \frac{\beta^{\frac{p-2}{p-1}} m_1}{\dfrac{q+1-p}{p(q-1)}(|\gamma| + m_1) + \dfrac{|F(\gamma)| + m_1 K_m}{N}} \right)^{\frac{p-1}{p}}. \tag{1.31}$$

For the sequel, let $C(I, \mathbb{R})$ denote the Banach space of continuous functions on $I$ and consider the following complete metric space

$$E_\gamma = \{\varphi \in C(I, \mathbb{R}) : \|\varphi - \gamma\| \leqslant m_1\}. \tag{1.32}$$

Define the mapping $T$ on $E_\gamma$, by

$$T(\varphi)(r) = \gamma - \int_0^r G\left\{ \frac{q+1-p}{p(q-1)}s\varphi(s) + s^{1-N}\int_0^s \sigma^{N-1}F(\varphi(\sigma))d\sigma \right\} ds \tag{1.33}$$

for $\varphi \in E_\gamma$ and $r \in [0, r_\gamma)$. Here $G$ and $F$ have the same meaning as in the previous formulas (1.3) and (1.4).

We claim that $T$ maps $E_\gamma$ into itself. To see this fix $\varphi \in E_\gamma$. Using (1.29) we can prove, as in step 1, that $\dfrac{q+1-p}{p(q-1)}s\varphi(s) + s^{1-N}\displaystyle\int_0^s \sigma^{N-1}F(\varphi(\sigma))d\sigma$ has a constant sign on $[0, r_\gamma)$ and consequently we have the following estimates

$$\left| \frac{q+1-p}{p(q-1)}s\varphi(s) + s^{1-N}\int_0^s \sigma^{N-1}F(\varphi(\sigma))d\sigma \right| \geqslant \beta s, \quad \text{for } 0 < s < r_\gamma \tag{1.34}$$

and

$$\frac{q+1-p}{p(q-1)}s\,|\varphi(s)| + s^{1-N}\int_0^s \sigma^{N-1}\,|F(\varphi(\sigma))|\,d\sigma \leqslant \qquad (1.35)$$

$$\left\{\frac{q+1-p}{p(q-1)}\,(|\gamma|+m_1) + \frac{|F(\gamma)|+m_1 K_m}{N}\right\}s.$$

Moreover

$$\left|G\left\{\frac{q+1-p}{p(q-1)}s\varphi(s) + s^{1-N}\int_0^s \sigma^{N-1}F(\varphi(\sigma))d\sigma\right\}\right| \qquad (1.36)$$

$$\leqslant \frac{G\left\{\dfrac{q+1-p}{p(q-1)}s\varphi(s) + s^{1-N}\displaystyle\int_0^s \sigma^{N-1}F(\varphi(\sigma))d\sigma\right\}}{\dfrac{q+1-p}{p(q-1)}s\varphi(s) + s^{1-N}\displaystyle\int_0^s \sigma^{N-1}\,|F(\varphi(\sigma))|\,d\sigma}$$

$$\left\{\frac{q+1-p}{p(q-1)}s\,|\varphi(s)| + s^{1-N}\int_0^s \sigma^{N-1}\,|F(\varphi(\sigma))|\,d\sigma\right\}.$$

From these estimates we also obtain

$$|T(\varphi)(r)-\gamma| \leqslant \int_0^r \frac{G(\beta s)}{\beta s}\left\{\frac{q+1-p}{p(q-1)}(|\gamma|+m_1) + \frac{|F(\gamma)|+m_1 K_m}{N}\right\}s\,ds.$$
$$(1.37)$$

Then, from the choice of $r_\gamma$, we deduce that

$$|T(\varphi)(r)-\gamma| \leqslant m_1; \qquad \text{for } \varphi \in E_\gamma \text{ and } r \in (0, r_\gamma). \qquad (1.38)$$

Now it is clear that $T$ is a contraction. In fact, as in step 1, we can show that for any $\varphi_1,\varphi_2 \in E_\gamma$ and $r \in (0, r_\gamma)$

$$|T(\varphi_1)(r) - T(\varphi_2)(r)| \leqslant \int_0^r |\varphi_1(s) - \varphi_2(s)|\,a(s)ds\upsilon \qquad (1.39)$$

where a is defined by (1.27).

Putting together (1.29) and (1.30), we get

$$\int_0^{r_\gamma} a(s)ds < 1. \qquad (1.40)$$

The Banach contraction theorem implies that there exists a unique function solving the problem $(P)$ in $[0, r_\gamma)$.

**Step 3:** $\left|u'\right|^{p-2} u' \in C^1\left([0,r_\gamma)\right).$

We have just to prove that $\left|u'\right|^{p-2} u'$ is a $C^1$ function near $r = 0$. Integrating equation (1.1) from 0 to $r$, we obtain

$$\left|u'\right|^{p-2} u'(r) = -\frac{q+1-p}{p(q-1)}ru(r)- \tag{1.41}$$

$$r^{1-N}\int_0^r s^{N-1}u(s)\left\{\frac{1}{q-1} - \frac{N(q+1-p)}{p(q-1)} + |u|^{q-1}(s)\right\}ds$$

which yields

$$\lim_{r\to 0}\frac{\left|u'\right|^{p-2} u'(r)}{r} = -\frac{\gamma}{N}\left(\frac{1}{q-1} + |\gamma|^{q-1}\right). \tag{1.42}$$

On the other hand, since $u$ satisfies equation (0.6) then

$$\lim_{r\to 0}\left(\left|u'\right|^{p-2} u'\right)'(r) = -\frac{\gamma}{N}\left(\frac{1}{q-1} + |\gamma|^{q-1}\right). \tag{1.43}$$

Hence

$$\left|u'\right|^{p-2} u' \in C^1\left([0,r_\gamma)\right). \tag{1.44}$$

The proof is complete. $\square$

REMARK 1.1. — *It is possible to see that for $N \geqslant 4$ and $q \geqslant p-1 > 1$ we have $\delta \geqslant \alpha$. Thus, in this case we can choose $m_1 = \inf(m,\alpha)$ in the proof of the previous proposition.*

PROPOSITION 1.2. — *For any $\gamma \in \mathbb{R}^*$, there exists a unique solution $u$ of problem $(P)$ in $[0,+\infty)$, which satisfies*

$$|u(r)| \leqslant |\gamma|, \qquad \forall r \geqslant 0, \tag{1.45}$$

*and*

$$\left|u'(r)\right| \leqslant \left(\frac{p}{p-1}\right)^{1/p}\left(\frac{\gamma^2}{2(q-1)} + \frac{|\gamma|^{q+1}}{q+1}\right)^{1/p}, \qquad \forall r \geqslant 0. \tag{1.46}$$

*Proof.* — The proof is divided in two steps.

**Step 1.** *Global existence.*

On the radial solutions of a degenerate quasilinear elliptic equation in $I\!R^N$

Thanks to proposition 1.1, for any $\gamma \in I\!R_*^+$ we know the local existence and uniqueness of a solution $u$ of problem $(P)$. In order to establish global existence, we define the energy function

$$E(r) = \frac{p-1}{p} \left|u^{'}\right|^p (r) + \frac{|u|^2 (r)}{2(q-1)} + \frac{|u|^{q+1} (r)}{q+1}. \tag{1.47}$$

According to (1.1), $E$ satisfies that

$$E^{'}(r) = -\left\{ \frac{N-1}{r} \left|u^{'}\right|^p (r) + \frac{q+1-p}{p(q-1)} r \left|u^{'}\right|^2 (r) \right\} \leqslant 0. \tag{1.48}$$

Hence $u$ and $u^{'}$ are bounded, and then $u$ exists for all $r \geqslant 0$.

Moreover, since $u^{'}(0) = 0$, $E(r) \leqslant E(0)$ and the fact that the function $x \longrightarrow \dfrac{x^2}{2(q-1)} + \dfrac{|x|^{q+1}}{q+1}$ is increasing on $[0, +\infty)$, we get (1.45) and (1.46).

**Step 2.** *Global uniqueness*

Since the energy function $E$ is non increasing, we see that if $u = 0$ and $u^{'} = 0$ at some point $r_0 > 0$, then $u(r) = 0$ for all $r \geqslant r_0$. Hence, we can suppose that $(u(r_0), u^{'}(r_0)) \neq (0,0)$ and the uniqueness is treated similarly as in proposition 1.1 . $\square$

REMARK 1.2. — *It is not difficult to see that the solution $u$ of $(P)$ is a $C^\infty$ function at any point $r > 0$ whenever $u^{'}(r) \neq 0$.*

## 2. Asymptotic behaviour of solutions

In this section we give some preliminary results for radial solutions of (0.1). Here and in the rest of the paper, we assume $N \geqslant 1$, $q > p - 1 > 1$ and $\gamma > 0$. Using equation (1.1) we immediately have the following lemma

LEMMA 2.1. — *Suppose $u$ is a solution of $(P)$. Then any critical point $r$ of $u$ must be a relative minimum if $u(r) < 0$ or a relative maximum if $u(r) > 0$.*

Now, proceeding as in [6], we shall prove the following result

LEMMA 2.2. — *Suppose $u$ is a solution of $(P)$. Then the energy function $E(r)$ given by (1.49) goes to zero as $r \longrightarrow \infty$.*

*Proof.* — Since $E'(r) \leqslant 0$ and $E(r) \geqslant 0$ for all $r > 0$, then there is a constant $l \geqslant 0$ such that $\lim_{r \to \infty} E(r) = l$. Suppose for contradiction that $l > 0$. Hence there exists $r_1 > 0$ such that

$$E(r) \geqslant \frac{l}{2}, \qquad \text{for all } r \geqslant r_1. \tag{2.1}$$

Let us define disjoints subsets $I_1$ and $I_2$ of $[r_1, +\infty)$ by

$$I_1 = \left\{ r \geqslant r_1 : \frac{u^2(r)}{2(q-1)} + \frac{|u|^{q+1}(r)}{q+1} < \frac{1}{4}l \right\}$$

and

$$I_2 = \left\{ r \geqslant r_1 : \frac{u^2(r)}{2(q-1)} + \frac{|u|^{q+1}(r)}{q+1} \geqslant \frac{1}{4}l \right\}.$$

Let $r \in [r_1, +\infty[$. We distinguish two cases.

(i) $r \in I_1$, from (2.1), we obtain

$$\left| u' \right|^p (r) \geqslant c_0 = \frac{p}{4(p-1)}l. \tag{2.2}$$

(ii) $r \in I_2$. Remembering that the function $x \longrightarrow \dfrac{x^2}{2(q-1)} + \dfrac{|x|^{q+1}}{q+1}$ is increasing on $[0, +\infty)$, we obtain the existence of a positive constant $c_1$ such that $|u(r)| \geqslant c_1$. Thus, for any $r \in I_2$ we have

$$\frac{u^2(r)}{q-1} + |u|^{q+1}(r) \geqslant c_2 \tag{2.3}$$

where $c_2 = \dfrac{c_1^2}{q-1} + c_1^{q+1}$. Putting together (2.2) and (2.3) the following inequality holds for any $r \geqslant r_1$

$$\left| u' \right|^p (r) + \frac{u^2(r)}{q-1} + |u|^{q+1}(r) \geqslant c = \min\{c_0, c_2\}. \tag{2.4}$$

Now consider the function

$$D(r) = E(r) + \frac{N-1}{2r} \left| u' \right|^{p-2} u' u(r) + \frac{N-1}{4} \frac{q+1-p}{p(q-1)} |u|^2(r) + \tag{2.5}$$

$$+ \frac{q+1-p}{p(q-1)} \int_0^r s \left| u' \right|^2 (s)ds.$$

Applying equation (0.6), we obtain the following equality

$$D'(r) = -\frac{N-1}{2r}\left\{\left|u'\right|^p(r) + \frac{u^2(r)}{q-1} + |u|^{q+1}(r) + \frac{N}{r}\left|u'\right|^{p-2}u'u(r)\right\}.$$
(2.6)

Recall that $u$ and $u'$ are bounded, so that there exists $r_2(\geqslant r_1)$ such that

$$\frac{N}{r}\left|u'\right|^{p-2}u'u(r) \geqslant -\frac{c}{2}, \quad \text{for } r \geqslant r_2$$
(2.7)

Combining (2.4) and (2.7) yields

$$\left|u'\right|^p(r) + \frac{u^2(r)}{q-1} + |u|^{q+1}(r) + \frac{N}{r}\left|u'\right|^{p-2}u'u(r) \geqslant \frac{c}{2}, \text{ for } r \geqslant r_2.$$
(2.8)

Integrating equation (2.6) between $r_2$ and $r$ we get

$$D(r) \leqslant D(r_2) - \frac{c(N-1)}{4}\log\frac{r}{r_2}, \quad \text{for } r \geqslant r_2.$$
(2.9)

But, $E(r) + \dfrac{N-1}{2r}\left|u'\right|^{p-2}u'u(r) \leqslant D(r)$, and using again the boundedness of $u$ and $u'$, we get from (2.9) that $\lim\limits_{r\to+\infty}E(r) = -\infty$. This is impossible. Hence $l = 0$ and the proof is ended . □

As a consequence of the above lemma, we derive the following result

COROLLARY 2.1. — *Suppose $u$ is a solution of $(P)$. Then necessarily $u(r)$ and $u'(r)$ go to 0 as $r \longrightarrow +\infty$.*

## 3. Strictly positive solutions

The goal of this section is to investigate the existence of strictly positive solutions of $(P)$. But first of all, we establish some results on the behavior of such solutions.

PROPOSITION 3.1. — *If $u$ is a strictly positive solution of $(P)$ then*

$$u'(r) < 0, \quad \text{for all } r > 0.$$
(3.1)

*Proof.* — Suppose for contradiction that there exists $r_0 > 0$ such that $u'(r_0) \geqslant 0$. As

$$\left(\left|u'\right|^{p-2}u'\right)'(0) = -\frac{\gamma}{N}\left(\frac{1}{q-1} + \gamma^{q-1}\right) < 0$$
(3.2)

then there exists $r_1 \in (0, r_0]$ such that $u(r_1) > 0$, $u'(r_1) = 0$ and $\left(\left|u'\right|^{p-2} u'\right)'(r_1) \geqslant 0$. This contradicts the fact that

$$\left(\left|u'\right|^{p-2} u'\right)'(r_1) + \frac{N-1}{r_1} \left|u'\right|^{p-2} u'(r_1) + \frac{q+1-p}{p(q-1)} r_1 u'(r_1) \qquad (3.3)$$

$$+ \frac{u(r_1)}{q-1} + u^q(r_1) = 0.$$

Thus, $u'(r) < 0$, for all $r > 0$ . □

PROPOSITION 3.2. — *Let $u$ be a strictly positive solution of $(P)$. Then there exists $r_0 > 0$ such that the function $\dfrac{u'(r)}{u(r)}$ is monotone in $[r_0, +\infty)$.*

As a consequence of this result, we have

COROLLARY 3.1. — *Let $u$ be a strictly positive solution of $(P)$. Then* $\lim\limits_{r \to \infty} \dfrac{u'(r)}{u(r)}$ *exists in $[-\infty, 0]$.*

One of the main tools for the proof of proposition 3.2. is the following lemma.

LEMMA 3.1. — *Let $u$ be a strictly positive solution of $(P)$. Then there exists $r_1 > 0$ such that*

$$a'(r)u'(r) + b'(r)u(r) < 0 \qquad (3.4)$$

*for any $r > r_1$ satisfying*

$$(u')^2(r) + a(r)u(r)u'(r) + b(r)u^2(r) = 0 \qquad (3.5)$$

*where*

$$a(r) = \frac{1}{p-1} \left( \frac{N-1}{r} + \frac{q+1-p}{p(q-1)} \frac{r}{|u'|^{p-2}(r)} \right) \qquad (3.6)$$

*and*

$$b(r) = \frac{1}{(p-1) |u'|^{p-2}(r)} \left( \frac{1}{q-1} + u^{q-1}(r) \right). \qquad (3.7)$$

*Proof.* — Proposition 3.1 together with Remark 1.2 and equation (0.6) imply that $u$ satisfies the following equation

$$u''(r) + a(r)u'(r) + b(r)u(r) = 0, \text{ for } r > 0. \tag{3.8}$$

Define the set

$$J_u = \left\{ r > 0 : (u')^2(r) + a(r)u(r)u'(r) + b(r)u^2(r) = 0 \right\}. \tag{3.9}$$

It is clear that for any $r \in J_u$, the following equations hold

$$uu'' = (u')^2 \tag{3.10}$$

and

$$\frac{u}{(p-1)|u'|^{p-2}} \left\{ \frac{q+1-p}{p(q-1)} ru' + \frac{u}{q-1} + u^q \right\} = -(u')^2 - \frac{N-1}{(p-1)} \frac{uu'}{r}. \tag{3.11}$$

This leads to

$$(p-1)(a'u' + b'u) = |u'| \left\{ \frac{N-1}{r^2} + (p-2)(N-1)\frac{|u'|}{ru} - \right.$$

$$\left. (p-1)(p-2)\frac{|u'|^2}{u^2} - \frac{q+1-p}{p(q-1)} \frac{1}{|u'|^{p-2}} - (q-1)\frac{u^{q-1}}{|u'|^{p-2}} \right\}. \tag{3.12}$$

for any $r \in J_u$.

On the other hand equation (3.11) is equivalent to

$$u^2 \left( \frac{1}{q-1} + u^{q-1} \right) + \frac{q+1-p}{p(q-1)} ruu' + (p-1)|u'|^p \tag{3.13}$$

$$+ \frac{N-1}{r} |u'|^{p-2} u'u = 0.$$

Set

$$A(r) = \frac{N-1}{r} |u'|^{p-2}(r) + \frac{q+1-p}{p(q-1)} r \tag{3.14}$$

and

$$B(r) = \frac{1}{q-1} + u^{q-1}(r). \tag{3.15}$$

Then, $u$ satisfies

$$Bu^2 - A\left|u'\right|u + (p-1)\left|u'\right|^p = 0, \text{ on } J_u. \tag{3.16}$$

so that

$$u = \frac{A \pm \sqrt{A^2 - 4(p-1)B\,|u'|^{p-2}}}{2B}\left|u'\right|, \quad \text{on } J_u. \tag{3.17}$$

Set

$$J_u^{\pm} = \left\{r > 0 : u(r) = \frac{A \pm \sqrt{A^2 - 4(p-1)B\,|u'|^{p-2}}}{2B}\left|u'\right|(r)\right\}. \tag{3.18}$$

It is clear that $J_u = J_u^+ \cup J_u^-$.

First, observe that from corollary 2.1, we have

$$A^2(r) >> 4(p-1)B\left|u'\right|^{p-2}(r), \quad \text{for large } r. \tag{3.19}$$

Next, we consider two cases.

(i) $r \in J_u^+$ is large enough. Then we have the following behaviour

$$u(r) \sim \frac{q+1-p}{p}r\left|u'\right|(r). \tag{3.20}$$

Carrying this last result in (3.12), we obtain

$$(p-1)(a'u' + b'u) \sim \left|u'\right|\left\{\frac{N-1}{r^2} + (N-1)(p-2)\frac{p}{(q+1-p)r^2}\right. \tag{3.21}$$

$$\left. -(q-1)\frac{u^{q-1}}{|u'|^{p-2}} - \frac{p^2(p-1)(p-2)}{(q+1-p)^2r^2} - \frac{q+1-p}{p(q-1)\,|u'|^{p-3}}\right\}.$$

Using again corollary 2.1, we deduce that

$$(p-1)(a'u' + b'u) \sim -\frac{q+1-p}{p(q-1)}\frac{1}{|u'|^{p-2}} < 0. \tag{3.22}$$

(ii) $r \in J_u^-$ is large enough, then

$$u(r) \sim \frac{p(p-1)(q-1)}{q+1-p}\frac{\left|u'\right|^{p-1}}{r}. \tag{3.23}$$

Hence, from (3.12), we get

$$(p-1)(a'u' + b'u) \sim -\frac{(p-2)(q+1-p)^2}{p^2(p-1)(q-1)^2} \frac{r^2}{|u'|^{2p-3}} < 0. \qquad (3.24)$$

So (3.4) holds and the proof is ended . $\square$

*Proof of Proposition 3.2.* — Introducing the functions

$$\widetilde{a}(r) = a(r) + k(r)u(r), \quad \text{for } r > 0 \qquad (3.25)$$

and

$$\widetilde{b}(r) = b(r) - k(r)u'(r), \quad \text{for } r > 0 \qquad (3.26)$$

where $a(r)$ and $b(r)$ are given in lemma 3.1, and function $k(r)$ will be chosen later. As $u$ satisfies (3.8), we have

$$u'' + \widetilde{a}u' + \widetilde{b}u = 0. \qquad (3.27)$$

Let $w = u'$, then $w$ satisfies

$$w'' + \widetilde{a}w' + \widetilde{b}w = -(a'u' + b'u) - k\left[(u')^2 + auu' + bu^2\right]. \qquad (3.28)$$

Applying lemma 3.1, it is possible to choose $r_1 > 0$ and $k(r)$ such that

$$-(a'u' + b'u) - k\left[(u')^2 + auu' + bu^2\right] > 0, \text{ for } r > r_1. \qquad (3.29)$$

Therefore

$$w'' + \widetilde{a}w' + \widetilde{b}w > 0, \quad \text{on } (r_1, +\infty). \qquad (3.30)$$

If we put $y = \dfrac{w}{u} = \dfrac{u'}{u}$, then we obtain

$$y'' + (\widetilde{a} + 2\frac{u'}{u})\, y' > 0, \quad \text{on } (r_1, +\infty).$$

So, we have either $y'(r) < 0$, for all $r > r_1$ or there exists $r_2 \geqslant r_1$ such that $y'(r) > 0$, for all $r > r_2$, which ends the proof. $\square$

In the next two propositions, we will give more precise asymptotic behavior for strictly positive solutions.

PROPOSITION 3.3.— *Let $u$ be a strictly positive solution of $(P)$ then*

$$\lim_{r \to +\infty} \frac{ru'(r)}{u(r)} = -\frac{p}{q+1-p}. \qquad (3.32)$$

*Proof.* — Set $L = \lim\limits_{r \to +\infty} \dfrac{u'(r)}{u(r)}$. Thanks to corollary 3.1, we have $L \in [-\infty, 0]$. We claim that $L \neq -\infty$. Indeed, suppose that $L = -\infty$. As we have $u(r) > 0$ and $u'(r) < 0$, for all $r > 0$, equation (0.6) is equivalent to

$$(p-1)\left|u'\right|^{p-2}\frac{u''}{ru'} + \frac{N-1}{r^2}\left|u'\right|^{p-2} + \frac{q+1-p}{p(q-1)} \tag{3.33}$$

$$+\frac{u}{ru'}\left(\frac{1}{q-1} + u^{q-1}\right) = 0, \quad \text{for } r > 0.$$

Therefore, since $u$ and $u'$ are bounded, by letting $r \longrightarrow +\infty$

$$\lim_{r \longrightarrow +\infty} u'' \frac{\left|u'\right|^{p-3}}{r}(r) = \frac{q+1-p}{p(p-1)(q-1)} \tag{3.34}$$

we deduce that, for large $r$, we have

$$\frac{1}{(p-2)}\left(\left|u'\right|^{p-2}\right)'(r) = -\frac{q+1-p}{p(p-1)(q-1)}r + O(r) \tag{3.35}$$

which contradicts the fact that $\lim\limits_{r \to +\infty} u'(r) = 0$.

We turn now to prove (3.32). By l'Hôpital's rule, we get

$$L = \lim_{r \to +\infty} \frac{u'(r)}{u(r)} = \lim_{r \to +\infty} \frac{u''(r)}{u'(r)}. \tag{3.36}$$

On the other hand, dividing equation (0.6) by $u(r)$, we obtain the following equality

$$(p-1)\frac{\left|u'\right|^{p-2}u''}{u} + \frac{N-1}{r}\frac{\left|u'\right|^{p-2}u'}{u} + \frac{q+1-p}{p(q-1)}\frac{ru'}{u} \tag{3.37}$$

$$+\left(\frac{1}{q-1} + u^{q-1}\right) = 0, \quad \text{for } r > 0.$$

Letting $r \longrightarrow +\infty$ ; the proposition follows by (3.36) and corollary 2.1. $\quad\square$

COROLLARY 3.2. — *Let $u$ be a strictly positive solution of $(P)$. Then,* $\lim\limits_{r \to +\infty} r^{\frac{p}{q+1-p}}u(r)$ *exists and is a positive number.*

REMARK 3.1. — *By using the previous result, we deduce that if there exist two constants $C$ and $m$ such that $u(r) \leqslant Cr^{-m}$, for large $r$. Then $\left|u'(r)\right| \leqslant \widetilde{C} r^{-m-1}$, for large $r$, where $\widetilde{C}$ is another constant.*

PROPOSITION 3.4. — *Let $u$ be a strictly positive solution of $(P)$ such that $\lim\limits_{r \to +\infty} r^{\frac{p}{q+1-p}} u(r) = 0$. Then, for all $m > 0$, we have $\lim\limits_{r \to +\infty} r^m u(r) = 0$ and $\lim\limits_{r \to +\infty} r^{m+1} u'(r) = 0$.*

Proof. — Let $\lambda = \dfrac{p}{q+1-p}$. Multiplying equation (0.6) by $r^{\lambda-1}$, then the following identity holds

$$\left(r^{\lambda-1}\left|u'\right|^{p-2} u'\right)' + (N-\lambda)r^{\lambda-2}\left|u'\right|^{p-2} u' + \qquad (3.38)$$

$$\frac{q+1-p}{p(q-1)}\left(r^\lambda u\right)' + r^{\lambda-1} u^q = 0, \quad \text{for } r > 0.$$

Integrating from $x$ to $y$ $\quad(0 < x < y < +\infty)$, we get

$$y^{\lambda-1}\left|u'\right|^{p-2} u'(y) - x^{\lambda-1}\left|u'\right|^{p-2} u'(x) + (N-\lambda)\int_x^y r^{\lambda-2}\left|u'\right|^{p-2} u'(r)dr$$
$$\qquad (3.39)$$
$$+\frac{q+1-p}{p(q-1)}y^\lambda u(y) - \frac{q+1-p}{p(q-1)}x^\lambda u(x) + \int_x^y r^{\lambda-1} u^q(r)dr = 0.$$

Letting $y \longrightarrow +\infty$, we obtain

$$x^{\lambda-1}\left|u'\right|^{p-2} u'(x) + \frac{q+1-p}{p(q-1)}x^\lambda u(x) = \int_x^{+\infty} r^{\lambda-1} u^q(r)dr \qquad (3.40)$$

$$+(N-\lambda)\int_x^{+\infty} r^{\lambda-2}\left|u'\right|^{p-2} u'(r)dr.$$

Thus

$$\frac{q+1-p}{p(q-1)}x^\lambda u(x) \leqslant x^{\lambda-1}\left|u'\right|^{p-1}(x) + \int_x^{+\infty} r^{\lambda-1} u^q(r)dr \qquad (3.41)$$

$$+|N-\lambda|\int_x^{+\infty} r^{\lambda-2}\left|u'\right|^{p-1}(r)dr.$$

It follows that if $u(x) \leqslant Cx^{-m_1}$, (with $m_1 \geqslant \lambda$ ), for large $x$ (consequently $\left| u'(x) \right| \leqslant Cx^{-m_1-1}$, by Remark 3.1) and we get

$$\int_x^{+\infty} r^{\lambda-1} u^q(r) dr \leqslant Cx^{\lambda-m_1 q} \qquad (3.42)$$

$$x^{\lambda-1} \left| u' \right|^{p-1} (x) \leqslant Cx^{\lambda-1-(p-1)(m_1+1)} \qquad (3.43)$$

and

$$\int_x^{+\infty} r^{\lambda-2} \left| u' \right|^{p-1} (r) dr \leqslant Cx^{\lambda-1-(p-1)(m_1+1)} \qquad (3.44)$$

where we denote by $C$ different constants.

Plugging all these estimates in (3.41), we deduce that

$$u(x) \leqslant C(x^{-m_1 q} + x^{-(p-1)(m_1+1)-1}), \text{ for large } x. \qquad (3.45)$$

Then

$$u(x) \leqslant Cx^{-m_2}, \text{ for large } x \qquad (3.46)$$

where

$$m_2 = \min(m_1 q, (p-1)(m_1+1)+1). \qquad (3.47)$$

Since $q > 1$ and $p > 2$, we have $m_2 > m_1$. The proposition follows by induction starting with $m_1 = \lambda$ and setting $m_{j+1} = \min(m_j q, (m_j+1)(p-1)+1)$. $\square$

We may now turn to the main result of this section: the existence of strictly positive solutions of $(P)$ for sufficiently small initial values $\gamma$ and under suitable hypothesis on $p, q$ and $N$. More precisely, we have

PROPOSITION 3.5. — *Assume* $q > \dfrac{p}{N} + p - 1$ *and*

$$0 < \gamma < \left( \frac{N(q+1-p)}{p(q-1)} - \frac{1}{q-1} \right)^{\frac{1}{q-1}} . \quad \textit{Then the solution } u \textit{ of } (P) \textit{ is}$$

*strictly positive and* $\lim\limits_{r \to +\infty} r^{\frac{p}{q+1-p}} u(r) > 0.$

*Proof.* — Suppose $u$ is not strictly positive and let $r_0$ be the first positive zero of $u$. Then, $u'(r_0) \leqslant 0$ and $u(r) > 0$, for all $r \in [0, r_0)$. Integrating equation (1.1) from 0 to $r_0$ gives the following formula

$$r_0^{N-1} \left| u' \right|^{p-2} u'(r_0) = \int_0^{r_0} s^{N-1} u(s) \left[ \frac{N(q+1-p)}{p(q-1)} - \frac{1}{q-1} - u^{q-1}(s) \right] ds. \qquad (3.48)$$

Since $|u(s)| < \gamma$ and from the choice of $\gamma$, we deduce that the right side of (3.48) is strictly positive but the left side is negative. This contradiction allows to confirm that $u$ is strictly positive and so $\lim\limits_{r \to +\infty} r^{\frac{p}{q+1-p}} u(r) \geqslant 0$.

To finish the proof, one suppose, that $\lim\limits_{r \to +\infty} r^{\frac{p}{q+1-p}} u(r) = 0$. We have, for any $r > 0$,

$$r^{N-1} \left| u' \right|^{p-2} u'(r) + \frac{q+1-p}{p(q-1)} r^N u(r) = \tag{3.49}$$

$$\int_0^r s^{N-1} u(s) \left[ \frac{N(q+1-p)}{p(q-1)} - \frac{1}{q-1} - u^{q-1}(s) \right] ds.$$

By using proposition 3.4, the left side of the last equality converges to zero as $r \longrightarrow +\infty$. But the right side is strictly positive. Then we have a contradiction and this completes the proof. □

PROPOSITION 3.6. — *Assume* $2 < p < N$ *and* $q \geqslant \dfrac{Np}{N-p} - 1$. *Then for any* $\gamma \in \mathbb{R}_*^+$, *the solution* $u(r, \gamma)$ *of* $(P)$ *is strictly positive and*

$$\lim_{r \to +\infty} r^{\frac{p}{q+1-p}} u(r, \gamma) > 0.$$

*Proof.* — (i) Assume that there exists some positive real $r$ such that $u(r) = 0$. Setting

$$R = \sup \left\{ r > 0 : u > 0 \text{ on } [0, r[ \right\}. \tag{3.50}$$

Then $u$ is a radial solution of the following problem:

$$\begin{cases} \Delta_p u + \dfrac{q+1-p}{p(q-1)} \, x \nabla u + \dfrac{u}{q-1} + u^q = 0 & \text{in } B_R \tag{3.51} \\ u = 0 & \text{on } \partial B_R \tag{3.52} \end{cases}$$

where $B_R$ is the ball of radius $R$ centered at $x = 0$.

Multiplying (3.51) by $u$, integrating over $B_R$ and using Green's formula and (3.52), we get

$$\int_{B_R} |\nabla u|^p \, dx = -\frac{N(q+1-p)}{2p(q-1)} \int_{B_R} u^2 dx \tag{3.53}$$

$$+ \frac{1}{q-1} \int_{B_R} u^2 dx + \int_{B_R} u^{q+1} dx.$$

Next, multiply (3.51) by $x.\nabla u$, we obtain

$$\int_{B_R} (\Delta_p u)(x.\nabla u)dx + \frac{q+1-p}{p(q-1)} \int_{B_R} (x.\nabla u)^2 dx \qquad (3.54)$$

$$+\frac{1}{q-1} \int_{B_R} u(x.\nabla u)dx + \int_{B_R} u^q (x.\nabla u)dx = 0.$$

To estimate the first integral we use Gauss-Ostrogradskii's formula to conclude that

$$\int_{B_R} (\Delta_p u)(x.\nabla u) = -\int_{B_R} |\nabla u|^p - \frac{1}{p} \int_{B_R} x.\nabla \left( |\nabla u|^p \right) dx \qquad (3.55)$$

$$+ \int_{\partial B_R} |\nabla u|^{p-2} (x.\nabla u) \nabla u.\nu d\sigma$$

$$= \left( \frac{N}{p} - 1 \right) \int_{B_R} |\nabla u|^p dx + (1 - \frac{1}{p}) \int_{\partial B_R} |\nabla u|^q (x.\nu)d\sigma.$$

By standard computations we have

$$\int_{B_R} u(x.\nabla u)dx = -\frac{N}{2} \int_{B_R} u^2 dx, \qquad (3.56)$$

and

$$\int_{B_R} u^q (x.\nabla u)dx = -\frac{N}{q+1} \int_{B_R} u^{q+1} dx. \qquad (3.57)$$

Now let $k$ be a positive real such that

$$k(N-k) \geqslant \frac{N(N-p)}{2p} \left( 1 + \frac{N(p-2)}{p} \right). \qquad (3.58)$$

If we define the operator $G$ by

$$Gu = ku + x.\nabla u \qquad (3.59)$$

we obtain

$$\int_{B_R} (x\nabla u)^2 dx = \int_{B_R} (Gu)^2 dx + k(N-k) \int_{B_R} u^2 dx. \qquad (3.60)$$

Plugging all these estimates in (3.54), we obtain

$$\left( \frac{N}{p} - 1 \right) \int_{B_R} |\nabla u|^p dx + \left( 1 - \frac{1}{p} \right) \int_{\partial B_R} |\nabla u|^p (x.\nu)d\sigma+ \qquad (3.61)$$

$$\frac{1}{q-1}\left(\frac{q+1-p}{p}k(N-k)-\frac{N}{2}\right)\int_{B_R}u^2dx-\frac{N}{q+1}\int_{B_R}u^{q+1}dx$$

$$+\frac{q+1-p}{p(q-1)}\int_{B_R}G^2(u)dx=0.$$

Substituting (3.53) in this last equality we obtain

$$\left(\frac{N}{p}-1-\frac{N}{q+1}\right)\int_{B_R}u^{q+1}dx+\frac{q+1-p}{p(q-1)}\int_{B_R}G^2(u)dx \qquad (3.62)$$

$$+\left(1-\frac{1}{p}\right)\int_{\partial B_R}|\nabla u|^p\,(x.\nu)d\sigma+$$

$$\left\{\left(\frac{N}{p}-1-\frac{N}{2}\right)\frac{1}{q-1}+\frac{q+1-p}{p(q-1)}\left(k(N-k)-\frac{N}{2}\right)\right\}\int_{B_R}u^2dx=0.$$

From the choice of $k$, the coefficient in front of the integral $\int_{B_R}u^2dx$ is strictly positive. On the other hand as $q\geqslant\dfrac{NP}{N-p}-1$, $\dfrac{N}{p}-1-\dfrac{N}{q+1}\geqslant 0$.

We deduce a contradiction with (3.62). Hence $u(r,\gamma)>0$ for any $r>0$.

(ii) Assume that $\lim\limits_{r\to+\infty}r^{\frac{p}{q+1-p}}u(r)=0$ we deduce from proposition 3.4. that $\lim\limits_{r\to+\infty}r^mu(r)=\lim\limits_{r\to+\infty}r^{m+1}u'(r)=0$ for any $m$.

Then we can let $R$ to the infinity in (3.62). As in step (i) we get a contradiction and the proof is complete. $\square$

## 4. Non positive solutions

In this section we investigate the existence and some properties of non positive solutions of $(P)$. Set $r_0=\inf\{r>0:u(r)=0\}$, the first positive zero of $u$; when $0<r_0<\infty$, $u$ is called nonpositive solution.

PROPOSITION 4.1. — *Assume $p-1<q\leqslant\dfrac{p}{N}+p-1$. Then there are no positive solutions of $(P)$.*

*Proof.* — Suppose by contrary that $u$ is a positive solution of $(P)$. Then, for any $r>0$, the following equality holds true

$$r^{N-1}\left|u'\right|^{p-2}u'(r)+\frac{q+1-p}{p(q-1)}r^Nu(r)= \qquad (4.1)$$

$$\frac{1}{q-1}\left(\frac{N(q+1-p)}{p}-1\right)\int_0^r s^{N-1}u(s)ds - \int_0^r s^{N-1}u^q(s)ds.$$

This implies

$$r^{N-1}\left|u'\right|^{p-2}u'(r)+\frac{q+1-p}{p(q-1)}r^N u(r) \leqslant 0, \qquad \text{for all } r > 0. \qquad (4.2)$$

which means that

$$u'(r) \leqslant -\left(\frac{q+1-p}{p(q-1)}\right)^{\frac{1}{p-1}}r^{\frac{1}{p-1}}u^{\frac{1}{p-1}}(r), \quad r > 0. \qquad (4.3)$$

Integrating from 0 to $r$, we obtain

$$u^{\frac{p-2}{p-1}}(r) \leqslant u^{\frac{p-2}{p-1}}(0)-\frac{p-2}{p}\left(\frac{q+1-p}{p(q-1)}\right)^{\frac{1}{p-1}}r^{\frac{p}{p-1}}. \qquad (4.4)$$

By letting $r \longrightarrow +\infty$, we get a contradiction. $\square$

PROPOSITION 4.2. — *Assume $2 < p < N$ and $p-1 < q < \dfrac{Np}{N-p} - 1$. Then there exists $\gamma^* > 0$ such that for any $\gamma > \gamma^*$ the solution $u$ of $(P)$ is non positive.*

This proposition is a simple consequence of the previous lemma.

LEMMA 4.1. — *Assume $2 < p < N$ and $p-1 < q < \dfrac{Np}{N-p} - 1$. Then there is no positive radial solution of equation*

$$\triangle_p u + u^q = 0, \qquad \text{in } I\!\!R^N. \qquad (4.5)$$

The proof of this lemma is given in [1]

LEMMA 4.2. — *If for any $\gamma > 0$ the solution $u$ of $(P)$ is strictly positive, then there exists a $C^1$ function $v$ which satisfies*

$$\left(\left|v'\right|^{p-2}v'\right)' + \frac{N-1}{r}\left|v'\right|^{p-2}v' + |v|^{q-1}v = 0, \quad r > 0 \qquad (4.6)$$

$$v(0) = 1, \qquad v'(0) = 0 \qquad (4.7)$$

$$v(r) > 0, \qquad \text{for all } r \geqslant 0 \qquad (4.8)$$

*Proof.* — Let $k > 0$ and $u_k$ be a solution of (0.6) with $u_k(0) = k^{\frac{p}{q+1-p}}$ and $u'_k(0) = 0$. Define the function $v_k$ by

$$v_k(r) = k^{-\frac{p}{q+1-p}} u_k\left(\frac{r}{k}\right), \quad \text{for } r \geqslant 0. \tag{4.9}$$

we have $v_k \in C^1\left([0, +\infty)\right)$ and $\left|v'_k\right|^{p-2} v'_k \in C^1\left([0, +\infty)\right)$. Some elementary computation show that $v_k$ satisfies

$$\begin{cases} \left(\left|v'_k\right|^{p-2} v'_k\right)' + \dfrac{N-1}{r}\left|v'_k\right|^{p-2} v'_k + \dfrac{q+1-p}{p(q-1)}k^{-\frac{p(q-1)}{q+1-p}} r v'_k \\ \qquad + \dfrac{1}{q-1}k^{-\frac{p(q-1)}{q+1-p}} v_k + |v_k|^{q-1} v_k = 0, \qquad r > 0 \quad (4.10) \\ \\ v_k(0) = 1 \quad \text{and} \quad v'_k(0) = 0 \,. \end{cases}$$

On the other hand, since

$$|u_k(r)| \leqslant k^{\frac{p}{q+1-p}}, \qquad \text{for all } r \geqslant 0 \tag{4.11}$$

we obtain

$$|v_k(r)| \leqslant 1, \qquad \text{for all } r \geqslant 0. \tag{4.12}$$

Now observe that (4.10) is equivalent to

$$r^{N-1}\left|v'_k\right|^{p-2} v'_k(r) = -\frac{q+1-p}{p(q-1)}k^{-\frac{p(q-1)}{q+1-p}} r^N v_k(r) - \int_0^r s^{N-1}|v_k|^{q-1} v_k(s)ds$$
$$+ \left(\frac{N(q+1-p)}{p(q-1)} - \frac{1}{q-1}\right) k^{-\frac{p(q-1)}{q+1-p}} \int_0^r s^{N-1} v_k(s)ds \tag{4.13}$$

which implies, by using (4.12), that

$$r^{N-1}\left|v'_k\right|^{p-1}(r) \leqslant \frac{q+1-p}{p(q-1)} r^N k^{-\frac{p(q-1)}{q+1-p}} + \frac{r^N}{N}$$
$$+ \frac{1}{q-1}\left|\frac{N(q+1-p)}{p} - 1\right| k^{-\frac{p(q-1)}{q+1-p}} \frac{r^N}{N}. \tag{4.14}$$

Hence, there exists a positive constant $c$ independent of $k$ such that

$$\left|v'_k(r)\right|^{p-1} \leqslant cr(1 + k^{-\frac{p(q-1)}{q+1-p}}), \qquad \text{for all } r \geqslant 0. \tag{4.15}$$

Using (4.12) and (4.15), we see that for $k \geqslant 1$, $v_k$ and $v'_k$ are uniformly bounded on any compact subsets $K$ of $[0, +\infty)$. So, by the Arzela-Ascoli

theorem and a standard diagonal argument, there is a sequence $k_n \longrightarrow +\infty$ and a $\mathcal{C}^1$ continuous function $v : [0, +\infty) \longrightarrow I\!R$ such that $v_{k_n} \underset{k_n \longrightarrow +\infty}{\longrightarrow} v$ uniformly in $C^{1,\alpha}(K)$. It follows from (4.13) that $v$ satisfies

$$
\begin{cases}
r^{N-1} \left| v' \right|^{p-2} v'(r) = - \displaystyle\int_0^r s^{N-1} |v|^{q-1} v(s)ds, & \text{for any } r > 0 \\
v(0) = 1 \text{ and } v'(0) = 0.
\end{cases}
\tag{4.16}
$$

Consequently, we have

$$
r^{1-N}(r^{N-1} \left| v' \right|^{p-2} v'(r))' + |v|^{q-1} v(r) = 0, \quad \text{for } r > 0 .
\tag{4.17}
$$

Now it remains to show that $v$ is strictly positive. Indeed, since $v_k \geqslant 0$ then $v \geqslant 0$. On the other hand (4.16) implies that $v$ is strictly decreasing. Thus, we have $v > 0$. $\square$

PROPOSITION 4.3. — *Assume* $q > \dfrac{p}{N} + p - 1$. *If for some* $\gamma > 0$ *the solution* $u$ *of* $(P)$ *is non positive then, either* $u$ *has a compact support or there is a positive constant* $M$ *such that* $u$ *has at most one zero in* $[M, +\infty)$.

*Proof.* — Since $\lim\limits_{r \longrightarrow +\infty} u(r) = 0$, there exists $M > 0$ such that

$$
|u(r)|^{q-1} < \frac{N(q+1-p)}{p(q-1)} - \frac{1}{q-1}, \text{ for } r \geqslant M.
\tag{4.18}
$$

Suppose that there exist two consecutive zeroes such that $M \leqslant x < y$. Integrating equation (1.1) from $x$ to $y$ therefore gives

$$
y^{N-1} \left| u' \right|^{p-2} u'(y) - x^{N-1} \left| u' \right|^{p-2} u'(x) =
\tag{4.19}
$$

$$
\int_x^y \left[ \frac{N(q+1-p)}{p(q-1)} - \frac{1}{q-1} - |u|^{q-1}(s) \right] s^{N-1} u(s)ds.
$$

Suppose that $u(r) \geqslant 0$ for $x < r < y$ (The same argument can be applied when $u(r) \leqslant 0$) and distinguish two cases.

(i) $u'(x)u'(y) = 0$. Then from the decreasing of the function energy $E$ given by (1.47), $u$ must have a compact support.

(ii) $u'(x)u'(y) \neq 0$. Then, $u'(x) > 0$ and $u'(y) < 0$. Now using (4.18), the right side of (4.19) is strictly positive while the left side is strictly negative, which yields a contradiction. Thus, two such zeroes can not exist. $\square$

As a consequence of the previous propositions, we obtain the following result.

COROLLARY 4.1. — *Assume* $q > \dfrac{p}{N} + p - 1$. *Then any non positive solution of* $(P)$ *has either a compact support or a finite number of zeroes.*

*Proof.* — Let $u$ be a non positive solution of $(P)$. Then, from proposition 4.3, there exists a positive constant $R$ such that all zeroes of $u$ are in $[0, R]$. suppose that there is an increasing sequence $(x_n)$ of zeroes of $u$ and let $x = \lim\limits_{n \to +\infty} x_n$. Then, for any $n \in I\!N$, there exists $z_n \in (x_n, x_{n+1})$ such that $u'(z_n) = 0$. This implies that $\lim\limits_{n \to +\infty} z_n = x$. And so, $u(x) = u'(x) = 0$. Hence $u$ has a compact support. This finishes the proof. $\square$

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