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RÉSUMÉ. — Nous allons établir des indices homologiques des sections des espaces linéaires fibrés. Ensuite nous allons démontrer que la somme totale des indices homologiques d'une section est indépendante du choix de la section.

ABSTRACT. — In this paper we establish a homological index for sections of linear fibre spaces, and we prove that the total sum of the homological indices of a section is independent of the choice of the section.

The purpose of this article is to define a "homological index" in the general setting of linear fibre spaces over complex analytic spaces with isolated singularities, provided that the linear fibre space is a bundle away from a finite number of points, and to show that this index is well defined. The homological index was first defined by X. Gomez-Mont for holomorphic vector fields ([G]). When the singular variety is a hypersurface, this provides a homological interpretation of the index previously defined in [GSV].

Let X be an n-dimensional compact reduced analytic space with isolated singularities, and $E \to X$ be a linear fibre space, which is an n-plane holomorphic bundle on X away from a finite number of points; we call this set of points $sing(E)$. Take a section s of $E \to X$ with isolated zeros. We define

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$sing(s) = zero(s) \cup sing(E)$. Now consider the following Koszul complex:

$$0 \to \wedge^n(E) \xrightarrow{i_s} \wedge^{n-1}(E) \xrightarrow{i_s} \cdots \xrightarrow{i_s} \wedge^1(E) \xrightarrow{i_s} \mathcal{O}_X \to 0$$

where $\wedge^i(E)$ is the sheaf of i-forms on E, $\mathcal{O}_X$ is the sheaf of analytic maps on X, and i_s is the contraction map by s. The homology sheaves H^i of the above complex are coherent skyscraper sheaves on X. The homological index of s at the point $p \in sing(s)$ is defined by

$$Ind_{hom}(s,p) = \sum_{i=0}^{n}(-1)^i dim_{\mathbb{C}} H_p^i,$$

where H_p^i is the germ of the sheaf H^i at p. The main properties of the homological index are that it coincides with the topological index and satisfies a law of conservation of number, i.e. if V is a relatively compact neighborhood of p such that s is non-vanishing on $\overline{V}\backslash\{p\}$, E is a vector bundle on $\overline{V}\backslash\{p\}$, and s' is a section of $E|_{\overline{V}}$ close to s, then

$$Ind_{hom}(s,p) - Ind_{hom}(s',p) = \sum_{\substack{q \in sing(s') \cap V \\ q \neq p}} Ind_{hom}(s',q).$$

The main theorem that we establish in this paper, is that the total sum of homological indices of s is independent of the choice of s. The idea of proof is that we compare the homological index with another index, that we call the differential index, which is defined by localization of the top Chern classes of the coherent sheaves of the local sections of linear fibre spaces. It will be shown that the total sum of differential indices of s is independent of the choice of s. Also the differential index coincides with the usual index, i.e. the intersection number of the zero section and s, and satisfies a law of rigidity

$$Ind_{dif}(s,p) - Ind_{dif}(s',p) = \sum_{\substack{q \in sing(s') \cap V \\ q \neq p}} Ind_{dif}(s',q).$$

Both of indices coincide with the usual index, so

$$Ind_{hom}(s,p) - Ind_{dif}(s,p) = Ind_{hom}(s',p) - Ind_{dif}(s',p),$$

when s' is close to s. We show that the space of sections of $E|_{\overline{V}}$ with an isolated zero at p is connected, dense, and open in the space of sections of $E|_{\overline{V}}$. This fact shows that the difference of homological index and differential index is independent of the choice of s. Using this, and the fact that the total sum of the differential indices is independent of the choice of s, we deduce the main theorem.

In section 1, we recall the definition of linear fibre spaces, and some of its properties. For more results concerning linear fibre spaces one may consult [F1].

In section 2, we shall define the homological index using the notion of Koszul complex for sections of linear fibre spaces which coincides with the topological index. We then study some of its properties.

In the final section, we introduce a differential index for sections of linear fibre spaces by localization of the top Chern classes of the coherent sheaves of the local sections of linear fibre spaces. Using this we prove the total sum of homological indices of a section is independent of the choice of sections. In particular the total sum of homological indices of a vector field on a compact analytic space is independent of the choice of vector fields.

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1. Background on linear fibre spaces

We give the necessary background on linear fibre spaces essentially from [F1]. By definition, a linear fibre space over a complex space X is a unitary $X \times \mathbf{C}$-module in the category of complex spaces over X, i.e. a complex space $E \to X$ over X together with composition $+ : E \times_X E \to E$ and $\cdot : (X \times \mathbf{C}) \times_X E = \mathbf{C} \times E \to E$ such that the module axioms hold. The axioms imply that every fibre E_x $(x \in X)$ is a $\mathbf{C}$-module in the category of complex spaces. By a theorem of Cartier ([O]), E_x is a reduced complex space and hence isomorphic to some $\mathbf{C}^n$, where n depends on x.

LEMMA 1.1 ([P]). — *Let $E \to X$ be a linear fibre space. Then for any point $x \in X$ there is an open neighborhood U of x in X such that $E|_U$ is isomorphic to a linear subspace of $U \times \mathbf{C}^n$ for some n.*

We clearly have a canonical isomorphism $Hom_X(X \times \mathbf{C}^n, X \times \mathbf{C}^m) \to M(m \times n, \mathcal{O}_X(X))$, i.e. every homomorphism $\zeta : X \times \mathbf{C}^n \to X \times \mathbf{C}^m$ is given by a holomorphic $m \times n$-matrix ζ_{ij} on X. We define the kernel of ζ, denoted by $ker\zeta$, as the complex subspace of $X \times \mathbf{C}^n$ generated by the holomorphic functions $\zeta_{i1}z_1 + \cdots + \zeta_{in}z_n \in \mathcal{O}(X \times \mathbf{C}^n)$ where $i = 1, \cdots, m$ and $z_1, \cdots, z_n$ denote coordinate functions in $\mathbf{C}^n$. Since they are linear in $z_1, \cdots, z_n$, $ker\zeta$ is a linear subspace of $X \times \mathbf{C}^n$.

LEMMA 1.2 ([F2]).— *Let E be a linear subspace of $X \times \mathbf{C}^n$. Then for any $x \in X$ there is an open neighborhood U of x and a homomorphism $\zeta : U \times \mathbf{C}^n \to U \times \mathbf{C}^m$ such that $E|_U = ker\zeta$.*

It is clear that the sheaf of local sections of a linear fibre space is coherent. Let $\underline{E}$ be the sheaf of local sections of a linear fibre space $E \to X$. If $x \in X$ the lemmas 1.1 and 1.2 gives us an open neighborhood U of x in X and an exact sequence

$$0 \to E \to U \times \mathbf{C}^n \to U \times \mathbf{C}^m$$

of homomorphisms of linear fibre spaces. Then we obtain

$$0 \to \underline{E}|_U \to (\mathcal{O}_X|U)^n \to (\mathcal{O}_X|U)^m$$

which shows the coherence of $\underline{E}$. Also, if we apply the functor $Hom_{\mathcal{O}_U}(-, \mathcal{O}_U)$ to the above sequence we deduce that the sheaf of linear forms of a linear fibre space is a coherent sheaf.

Example 1.3.— As an important example, the tangent sheaf of an analytic space is a linear fibre space.

Example 1.4.— The tangent space of a singular holomorphic foliation on a complex manifold is a linear fibre space.

Remark 1.5.— We know that the sheaf of linear forms of a linear fibre space is a coherent sheaf. One can prove that the functor from category of linear fibre spaces to category of coherent sheaves, which assign to a linear fibre space its sheaf of linear forms, is an antiequivalence ([F1]).

Let $E \to X$ be a linear fibre space. By the previous lemmas, $E|U$ is $ker\zeta$, where $\zeta : U \times \mathbf{C}^n \to U \times \mathbf{C}^m$ is given by a matrix (ζ_{ij}). We projectivize every fibre of E_x, $x \in U$, and we may define a projective variety over U, $P(E|_U) \hookrightarrow U \times P_{n-1}(\mathbf{C})$. This is the subspace determined by the homogeneous system of m linear equations with coefficient ζ_{ij}. Given a covering of X with open sets U as above it is obvious how to glue together the local pieces $P(E|_U)$ and we obtain $P(E)$ together with a canonical projection map $P(\pi) : P(E) \to X$. It is called the projective variety over X associated to E. We use this fact in the proof of Proposition 2.2.

2. Homological index

Let X be an n-dimensional compact reduced analytic space with isolated singularities, and $E \to X$ be a linear fibre space, which is an n-plane holomorphic bundle on X away from a finite number of points; we call this set of points $sing(E)$. Let s be a section of $E \to X$ with isolated zeros.

We define $sing(s) = zero(s) \cup sing(E)$. Now consider the following Koszul complex:

$$0 \to \wedge^n(E) \xrightarrow{i_s} \wedge^{n-1}(E) \xrightarrow{i_s} \cdots \xrightarrow{i_s} \wedge^1(E) \xrightarrow{i_s} \mathcal{O}_X \to 0$$

where $\wedge^i(E)$ is the sheaf of i-forms on E, $\mathcal{O}_X$ is the sheaf of analytic map on X, and i_s is the contraction map by s. We fix $x_0 \in sing(s)$. The i-homology of the above complex H^i has its support at x_0 in a small neighborhood of x_0 ([GH], p. 688, Lemma). Since H^i is a coherent sheaf, we have the following exact sequence in a neighborhood U of x_0 :

$$\oplus_{k=0}^m \mathcal{O}_U \xrightarrow{\varphi} H^i|_U \to 0.$$

We denote by φ_k the restriction of φ on the k-component. $(ker\varphi_k)_{x_0}$ is an ideal of $\mathcal{O}_{x_0}$ and $zer(ker(\varphi_k)_{x_0}) = \{x_0\}$. According to [D2], $dim_{\mathbf{C}} \frac{\mathcal{O}_{x_0}}{<ker\varphi_k>_{x_0}} < \infty$. We have the following natural exact sequence :

$$\oplus_{k=0}^m \frac{\mathcal{O}_{x_0}}{< ker\varphi_k >_{x_0}} \to H^i_{x_0} \to 0,$$

and we obtain $dim_{\mathbf{C}} H^i_{x_0} < \infty$. Define the homological index by (see [G])

$$Ind_{hom}(s, x_0) = \sum_{i=0}^n (-1)^i dim_{\mathbf{C}} H^i_{x_0}.$$

Let $(X, 0) \subset (\mathbf{C}^{n+k}, 0)$ be a complete intersection of dimension n with an isolated singularity at 0. $s : (X, 0) \to (\mathbf{C}^n, 0)$ is a holomorphic map, and $s^{-1}(0) = \{0\}$. We know that $(X, 0)$ has a conic structure at 0 ([D1], [M]), and let K be the link of X at 0. We consider the following map $\frac{s}{||s||} : K \to S^{2n-1}$ and define

$$Ind_0(s) = degree(\frac{s}{||s||}).$$

It is obvious that this is independent of the choice of K.

PROPOSITION. 2.1.— *Using the above notations, we have*

$$Ind_{hom}(s, 0) = Ind_0(s) = dim_{\mathbf{C}} \frac{\mathcal{O}(X, 0)}{< s_1, s_2, \cdots, s_n >}.$$

Proof.— Since X is Cohen-Macaulay at 0 and $s^{-1}(0) = \{0\}$, then $(s_1, s_2, \cdots, s_n)$ is a regular sequence in the ring $\mathcal{O}(X, 0)$. We have $H^i_0 = 0$ for $i > 0$ and $H^0_0 = \frac{\mathcal{O}(X,0)}{<s_1,s_2,\cdots,s_n>}$ ([GH], p. 688, Lemma). Thus,

$$Ind_{hom}(s, 0) = dim_{\mathbf{C}} \frac{\mathcal{O}(X, 0)}{< s_1, s_2, \cdots, s_n >},$$

where $\mathcal{O}(X,0)$ is the germ of $\mathcal{O}_X$ at 0. On the other hand we know that there exists a holomorphic map $f : (\mathbf{C}^{n+k},0) \to (\mathbf{C}^k,0)$ such that $f^{-1}(0) = X$ and if $f = (f_1, f_2, \cdots, f_k)$, then $\{f_i\}$ is a set of generator for the ideal defining $(X,0)$. Now, put $K_t = f^{-1}(t) \cap S_\epsilon^{2(n+k)-1}$, where $S_\epsilon^{2(n+k)-1}$ is the sphere of radius ϵ centered at 0, and $0 < \epsilon \ll 1$. K_t is a deformation of K_0, so we have $Ind_0(s) = degree(\frac{s}{||s||}|_{K_t})$ for $0 < |t| \ll 1$. If D_f is the critical value of f then D_f is an analytic subspace. According to curve selection lemma ([M]), there is a real analytic curve $\gamma : [0,\delta] \to \mathbf{C}^k$ such that $\gamma(0) = 0$ and $\gamma((0,\delta)) \cap D_f = \emptyset$. Put $t = \gamma(s)$. For $t \neq 0$ we have ([GH], p. 666, (e)),

$$degree(\frac{s}{||s||}|_{K_t})$$
$$= \sum_{p\in s^{-1}(0)\cap f^{-1}(t)} (s_1^{-1}(0) \cap f^{-1}(t), s_2^{-1}(0) \cap f^{-1}(t), \cdots, s_n^{-1}(0) \cap f^{-1}(t))_p,$$

i.e., the intersection of the divisors $\{(s_i^{-1}(0) \cap f^{-1}(t)\}$ in the space $f^{-1}(t)$ at point p. Since $f^{-1}(t)$ is smooth (for $t \neq 0$), then using (d) of ([GH], p. 665)

$$\sum_{p\in s^{-1}(0)\cap f^{-1}(t)} (s_1^{-1}(0) \cap f^{-1}(t), \cdots, s_n^{-1}(0) \cap f^{-1}(t))_p =$$
$$\sum_{p\in s^{-1}(0)\cap f^{-1}(t)} (f_1^{-1}(t), \cdots, f_k^{-1}(t), s_1^{-1}(0), \cdots, s_n^{-1}(0))_p.$$

Since $f_i^{-1}(t)$ is the deformation of the divisor $f_i^{-1}(0)$, then by (c) of ([GH], p. 664)

$$(f_1^{-1}(0), \cdots, f_k^{-1}(0), s_1^{-1}(0), \cdots, s_n^{-1}(0))_0 =$$
$$\sum_{p\in s^{-1}(0)\cap f^{-1}(t)} (f_1^{-1}(t), \cdots, f_k^{-1}(t), s_1^{-1}(0), \cdots, s_n^{-1}(0))_p$$

and by (f) of ([GH], p. 669)

$$(f_1^{-1}(0), \cdots, f_k^{-1}(0), s_1^{-1}(0), \cdots, s_n^{-1}(0))_0 = dim_{\mathbf{C}} \frac{\mathcal{O}(\mathbf{C},0)}{< f, s >}$$
$$= dim_{\mathbf{C}} \frac{\mathcal{O}(X,0)}{< s_1, s_2, \cdots, s_n >}. \qquad \square$$

Let $(X,x) \subset B_1 \subset (\mathbf{C}^m,0)$ be a local embedding of X into the unit ball B_1. We will denote $X \cap B_r$ by X_r, where B_r is the ball around 0 and radius $r \ll 1$. We shall denote the space of continuous section defined on $\overline{X}_r$ and holomorphic on X_r by Γ_r. The following proposition and its proof mimic those of a proposition 2.1 in [BG] for the case $E = TX$.

PROPOSITION 2.2. — *The subset $\Gamma'_r \subset \Gamma_r$ consisting of holomorphic sections that have an isolated zero at 0 is a connected dense open subset of* Γ_r.

Proof. — Let $E|_{\overline{X}_r} \subset \overline{X} \times \mathbf{C}^k$ be a local embedding of the linear fibre space E. Assume that s has an isolated zero at 0. For $r' < r$ small, s restricted to $\partial X_{r'}$ does not vanish. Let 2ϵ be the minimum value of $||s||$ on $\partial X_{r'}$. If $\overline{s} \in \Gamma_r$ with $||\overline{s}|| < \epsilon$, then $s + \overline{s}$ cannot vanish on $\partial X_{r'}$. This implies that $s + \overline{s}$ will have an isolated zero at 0. For if it vanishes on a set of positive dimension passing through 0, this set has an intersection with $\partial X_{r'}$ (Otherwise, one would have a compact analytic subspace in $X_{r'}$ of positive dimension). This shows that Γ'_r is open in Γ_r.

Now, given $s_0 \in \Gamma_r$ and $\epsilon > 0$. To each section s in Γ_r such that $||s - s_0|| < \epsilon$, we may associate the dimension of its critical set at 0, i.e. $dim_0(\{s = 0\})$. Let $\overline{s}$ be a section where this minimum is attained. We claim that $\overline{s}$ has an isolated zero at 0. Let P be the projectivized space of linear fibre space E, $\pi : P \to X$ be the projection map ([F1]), and $P_r = P|_{X_r}$. Let $A = A_1 \cup \cdots \cup A_m$ be the decomposition into irreducible components of $\{\overline{s} = 0\} \subset X_r$ passing through 0. By assumption A does not reduce to 0. Let $\Gamma_{\overline{s}} \subset P_r$ be the closure of the graph of $proj(\overline{s})$ on $X_r \backslash A$. $\Gamma_{\overline{s}}$ has dimension $n = dim(X_r)$. The intersection of $\Gamma_{\overline{s}}$ with $\pi^{-1}(A)$ has dimension at most n-1, since it is contained in the boundary of the graph of $proj(\overline{s})$ which has dimension n. Since $\pi^{-1}(A_j)$ has dimension $n - 1 + dim(A_j) > n - 1$, we may choose points in $\pi^{-1}(A_j) \backslash \Gamma_{\overline{s}}$. That is, there are points p_j arbitrarily close to 0 and vectors $s_j \in E_{p_j}$ such that $proj(s_j)$ is disjoint from $\Gamma_{\overline{s}}$. Since X_r is a Stein space, there is a section $\overline{\overline{s}}$ of $E|_{X_r}$ such that $\overline{\overline{s}}\,(p_j) = s_j$. We claim that for small values of $t \neq 0$, $\overline{s} + t\,\overline{\overline{s}}$ has critical set at 0 of dimension smaller than the critical set of $\overline{s}$, contradicting the choice of $\overline{s}$. To see this, let $r' < r$ so that $A \cap X_{r'} = \{\overline{s} = 0\} \cap X_{r'}$. Without loss of generality, we may assume that $p_j \in X_{r'}$. Let C be the set of points of $X_{r'} \times \mathbf{C}$ where $\overline{s} + t\,\overline{\overline{s}}$ vanishes and let $\rho : X_{r'} \times \mathbf{C} \to \mathbf{C}$ be the projection to the second factor. We claim that the A_j's are irreducible components of C. To see this consider $(\overline{s} + t\,\overline{\overline{s}})(p) = 0$ for p near p_j. By our choice of $\overline{\overline{s}}\,(p_j)$, one may conclude that $\overline{\overline{s}}\,(p)$ is linearly independent from $\overline{s}(p)$ if $\overline{s}(p) \neq 0$. Hence $(\overline{s} + t\,\overline{\overline{s}})(p) \neq 0$. If $\overline{s}(p) = 0$, then for $t \neq 0$ we have $(\overline{s} + t\,\overline{\overline{s}})(p) = t\,\overline{\overline{s}}\,(p) \neq 0$. This implies that the decomposition into irreducible components of C in a neighborhood of $(0,0)$ is of form $C = A_1 \cup \cdots A_m \cup C_1 \cup \cdots C_r$. Hence the irreducible components C_k are not contained in $\rho^{-1}(0)$ and its intersection with $\rho^{-1}(0)$ does not contain any of A_j. By the theorem of upper semi continuity of the dimension of fibres of a holomorphic map, we conclude that $(C_1 \cup \cdots \cup C_r) \cap \rho^{-1}(t_0)$ has dimension smaller than the dimension of A, for $t_0 \neq 0$. But this set is exactly the critical set of $\overline{s} + t_0\,\overline{\overline{s}}$. This contradicts the hypothesis that minimum dimension of its critical set is attained at $\overline{s}$. Hence $\overline{s}$ has isolated zeros. This shows that Γ_r' is dense in Γ_r.

To see that Γ_r' is connected, let $\overline{s}$ and $\overline{\overline{s}}$ belong to Γ_r', then consider the family $\{(1-t)\overline{s}+t\ \overline{\overline{s}}\}_{t\in\mathbf{C}}$. The critical set C of the family consists of $(t,p)\in\mathbf{C}\times X_r$ such that $((1-t)\overline{s}+t\ \overline{\overline{s}})(p)=0$. C is an analytic subspace containing the line $\mathcal{L}_0=\mathbf{C}\times\{0\}$. By hypothesis $(0,0)$ and $(1,0)$ lie on $\mathcal{L}_0$ and not in any other irreducible component of C. Hence $\mathcal{L}_0$ is an irreducible component of C. The other irreducible components of C intersect $\mathcal{L}_0$ on a finite number of points. Hence all points of $\mathcal{L}_0$, except a finite number, represent sections with isolated zeros. Hence Γ_r' is connected. □

Let T and V be complex analytic spaces, with T reduced and locally irreducible, and $\pi:T\times V\to T$ the projection map. Let $\mathcal{K}^*$ be a complex

$$0\to\mathcal{K}^n\xrightarrow{X_n}\mathcal{K}^{n-1}\xrightarrow{X_{n-1}}\cdots\xrightarrow{X_2}\mathcal{K}^1\xrightarrow{X_1}\mathcal{K}^0\to 0$$

of $\mathcal{O}_{T\times V}$-sheaves (i.e., $\mathcal{K}^j$ is $\mathcal{O}_{T\times V}$-linear), where the sheaves $\mathcal{K}^j$ are $\mathcal{O}_T$-flat and such that the support of the homology sheaves $\mathcal{H}^j(\mathcal{K}^*)$ is π-finite. For every $t\in T$ denote $\{t\}\times V\simeq V$ by V_t and by $\mathcal{K}_t^*$ the restriction of complex $\mathcal{K}^*$ on V_t. $\mathcal{K}_{t,p}^*$ denotes the complex formed by the germs at p of $\mathcal{K}_t^*$ and $H_j(\mathcal{K}_{t,p}^*)$ is the j^{th}-homology group of $\mathcal{K}_{t,p}^*$. At a point (t,p) where $\mathcal{K}^*$ is exact, we have that $\mathcal{K}_t^*$ is exact at point p, and so the hypothesis that the support of the homology sheaves $\mathcal{H}^j(\mathcal{K}^*)$ is π-finite implies that the homology sheaves $\mathcal{H}^j(\mathcal{K}_t^*)$ are coherent skyscraper sheaves on V_t. Define the Euler characteristic of the complex of sheaves $\mathcal{K}_t^*$ at a point $p\in V_t$ by

$$\chi(\mathcal{K}_{t,p}^*)=\sum_{j=0}^{n}(-1)^j dim_{\mathbf{C}}H_j(\mathcal{K}_{t,p}^*)$$

Now we recall the following theorem from [GK].

THEOREM 2.3. ([GK]).— *Under the above hypothesis, for every $(t_0,p_0)\in T\times V$, there are neighborhood T' and V' of t_0 and p_0 respectively such that for every $t\in T'$ we have*

$$\chi(\mathcal{K}_{t_0,p_0}^*)=\sum_{p\in V'}\chi(\mathcal{K}_{t,p}^*).$$

Now we return to the previous situation. Let V be a relatively compact neighborhood of x_0 in X such that $sing(s)\cap(V-\{x_0\})=\emptyset$ and s_t is a one parameter family of sections $E|_V$ which has isolated zeros on V, $sing(s_t)\cap\partial V=\emptyset$, and $s_0=s$.

COROLLARY 2.4. (Law of conservation). — *In the above situation we have*

$$Ind_{hom}(s, x_0) = \sum_{x \in sing(s_t) \cap V} Ind_{hom}(s_t, x)$$

Proof. — Let $\pi_2 : \mathbf{C} \times V \to V$ be the projection map on the second component. We consider the following complex:

$$0 \to \pi_2^* \wedge^n (E) \xrightarrow{i_{s_t}} \pi_2^* \wedge^{n-1} (E) \xrightarrow{i_{s_t}} \cdots \xrightarrow{i_{s_t}} \pi_2^* \wedge^1 (E) \xrightarrow{i_{s_t}} \mathcal{O}_{\mathbf{C} \times V} \to 0$$

According to the hypothesis the support of homology sheaves is π_1-finite. It is sufficient to show that each $\pi_2^* \wedge^i (E)$ is $\mathcal{O}_{\mathbf{C}}$-flat. This is a general fact, i.e. if $\mathcal{F}$ is a coherent sheaf on V then $\pi_2^* \mathcal{F}$ is $\mathcal{O}_T$-flat. To see this, we use the following algebraic result. If M is a A-module where A is a local neotherian ring then M is A-flat if and only if $Tor_1^A(A/m, M) = 0$, where m is the maximal ideal of A. Since $\mathcal{F}$ is a coherent sheaf on V, for $p_0 \in V$, we may obtain a neighborhood $U \subset V$ of p and the exact sequence

$$\mathcal{O}_U^{n_2} \to \mathcal{O}_U^{n_1} \to \mathcal{O}_U^{n_0} \to \mathcal{F}|_U \to 0.$$

Since the map $\pi : T \times U \to U$ is flat ([F]), by applying π_2^* to the above sequence, we obtain the following exact sequence:

$$\mathcal{O}_{T \times U}^{n_2} \to \mathcal{O}_{T \times U}^{n_1} \to \mathcal{O}_{T \times U}^{n_0} \to \pi_2^*(\mathcal{F}|_U) \to 0.$$

To compute $Tor_1^{\mathcal{O}_{T,t}}(\mathcal{O}_{T,t}/m_t, \pi_2^* \mathcal{F}_p)$, tensor the above exact sequence with $\mathcal{O}_{T,t}/m_t \simeq \mathbf{C}$ over the ring $\mathcal{O}_{T,t}$ and for $p \in U$ we obtain:

$$\mathcal{O}_{U,p}^{n_2} \to \mathcal{O}_{U,p}^{n_1} \to \mathcal{O}_{U,p}^{n_0} \to \mathcal{F}_{U,p} \to 0$$

which is the first complex, but the objects are considered as $\mathcal{O}_{T,t}/m_t \simeq \mathbf{C}$-modules. Since it is an exact sequence we have

$$Tor_1^{\mathcal{O}_{T,t}}(\mathcal{O}_{T,t}/m_t, \pi_2^* \mathcal{F}_p) = 0. \qquad \square$$

We will prove the following theorem in the next section.

THEOREM 2.5. — *Let X be an n-dimensional compact reduced analytic space with isolated singularities, $E \to X$ be a linear fibre space which is an n-plane holomorphic vector bundle on X away from a finite number of points, and s is a section of $E \to X$ with isolated zeros. Then the total sum $\sum_{x \in sing(s)} Ind_{hom}(s, x)$ is independent of the choice of s.*

COROLLARY 2.6. — *Let X be a compact reduced analytic space with isolated singularities, and ν be a holomorphic vector field on X with isolated* *zeros. Then the total sum* $\sum_{x\in zero(\nu)} Ind_{hom}(\nu, x)$ *is independent of the choice of* ν.

Remark 2.7. — Let (X, x) be the germ of a reduced analytic space, and $\nu_1, \nu_2, \cdots, \nu_k$ are germs of vector fields on (X, x) such that the vectors $\nu_1(x), \nu_2(x), \cdots, \nu_k(x)$ are linearly independent. By a theorem of Rossi ([F]), there exist an isomorphism $(X, x) = \mathbf{C}^k \times (X', x')$, where (X', x') is the germ of an analytic space. So if (X, x) has an isolated singularity at x, and ν is a holomorphic vector field on it, then we have $\nu(x) = 0$.

Remark 2.8. — Let M be a complex manifold, and X be an n-dimensional, compact, local complete intersection subspace of M. We know that if N is the normal bundle of X in M away from the singularities of X, then N has a holomorphic extension to a bundle on X ([LS]), also we denote it by N. Suppose N has a smooth extension to a bundle in a neighborhood of X in M. Let ν be a holomorphic vector field on M tangent on X. In [LSS], the LSS-index is defined by localization of the top Chern class of the virtual tangent bundle of X, i.e, $TM|_X - N$ in the neighborhoods of connected components of $zero(\nu) \cap X$. In the case that X is with isolated singularities, and $zero(\nu) \cap X$ is a discrete set, we have ([LSS])

$$\sum_{x\in zero(\nu)\cap X} Ind_{LSS}(\nu, x) = c_n(TM|_X - N) \frown X.$$

The LSS-index coincides with GSV-index, which is defined in [GSV], [SS] by the Milnor fibration ([LSS]), and

$$\sum_{x\in zero(\nu)\cap X} Ind_{GSV}(\nu, x) = \chi(X) + \sum_{x\in sing(X)} (-1)^n \mu(x),$$

where $\mu(x)$ is the Milnor number of germ (X, x) ([SS]).

When X is a hypersurface, X. Gomez-Mont proved the homological index coincides with GSV-index ([G]).

3. Differential index

Let X be an n-dimensional compact reduced analytic space with isolated singularities, and $E \to X$ be a linear fibre space, which is an n-plane holomorphic bundle on X away from a finite number of points; we call this set of points $sing(E)$. Take s a section of $E \to X$ with isolated zeros. We define $sing(s) = zero(s) \cup sing(E)$. Let $\pi : \overline{X} \to X$ be a desingularization ([H]), we consider the linear fibre space π^*E and its section π^*s on $\overline{X}$. We fix $q \in sing(s)$ and we take U a neighborhood of $\pi^{-1}(q)$ such that

$U \cap \pi^{-1}(q') = \emptyset$ for $q' \neq q, q' \in sing(s)$. We denote the sheaf of real analytic maps on $\overline{X}$ by $\mathcal{A}$. Consider the following free resolution on U ([AH]):

$$0 \to \underline{E_k} \to \underline{E_{k-1}} \to \cdots \to \underline{E_1} \to \underline{E_0} \to \underline{\pi^* E} \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{A} \to 0,$$

where E_i 's are vector bundle on U, $\underline{E_i}$ is the sheaf of local sections of E_i and $\underline{\pi^* E}$ is the sheaf of local sections of linear fibre space $\pi^* E$. We obtain the following exact sequence on $U \backslash \pi^{-1}(q)$:

$$0 \to E_k \to \ E_{k-1} \to \cdots \to E_1 \to E_0 \to \pi^* E \to 0$$

Since the section $\pi^* s$ of $\pi^* E|_{U \backslash \pi^{-1}(q)}$ does not have any zero, then there is a connection ∇ for $\pi^* E|_{U\backslash \pi^{-1}(q)}$ such that $\nabla \pi^* s = 0$. Let $\sum$ be a relatively compact subset of U such that $\pi^{-1}(q) \subset \overset{\circ}{\sum}$. By lemma 5.26 of [BB], there exists connections ∇_i for E_i on U such that the following diagrams are commutative on $U \backslash \sum$:

$$\begin{array}{ccc} \Gamma(E_i) & \overset{\nabla_i}{\to} & \Gamma(E_i) \otimes \Omega^1 \\ \downarrow & & \downarrow \\ \Gamma(E_{i-1}) & \overset{\nabla_{i-1}}{\to} & \Gamma(E_{i-1}) \otimes \Omega^1, \end{array}$$

where $E_{-1} = \pi^* E$ and $\nabla_{-1} = \nabla$. If we denote by $K(\nabla_i)$ the curvature tensor of ∇_i on $U \backslash \sum$, then by Lemma 4.22 of [BB]

$$\Pi_{i=0}^k det(1 + \frac{\sqrt{-1}}{2\pi} K(\nabla_i))^{\epsilon(i)} = det(1 + \frac{\sqrt{-1}}{2\pi} K(\nabla)),$$

where $\epsilon(i) = (-1)^i$. If we take

$$\Pi_{i=0}^k det(1 + \frac{\sqrt{-1}}{2\pi} K(\nabla_i))^{\epsilon(i)} = 1 + \sigma_1(\nabla_0, \nabla_1, \cdots, \nabla_k) + \sigma_2(\nabla_0, \nabla_1, \cdots, \nabla_k)$$
$$+ \cdots + \sigma_n(\nabla_0, \nabla_1, \cdots, \nabla_k)$$
$$det(1 + \frac{\sqrt{-1}}{2\pi} K(\nabla)) = 1 + \sigma_1(\nabla) + \sigma_2(\nabla) + \cdots + \sigma_n(\nabla),$$

then $\sigma_n(\nabla_0, \nabla_1, \cdots, \nabla_k) = \sigma_n(\nabla) = 0$ on $U \backslash \sum$.

Now, define the differential index by

$$Ind_{dif}(s, q) = \int_U \sigma_n(\nabla_0, \nabla_1, \cdots, \nabla_k).$$

It must be shown that this is well defined.

Recall the following lemma which is implicit in the proof of Proposition 5.62 in [BB].

LEMMA 3.1. ([BB]). — *Consider the following diagram of complex smooth vector bundles :*

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & L_k & \to & L_{k-1} & \to & \cdots & \to & L_0 & \to & 0 & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & I_k & \to & I_{k-1} & \to & \cdots & \to & I_0 & \to & \nu & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & E_k & \to & E_{k-1} & \to & \cdots & \to & E_0 & \to & \nu & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & & & 0 & & 0 & &
\end{array}$$

with properties

i) *L_i, I_i, E_i 's are smooth complex vector bundles on a smooth manifold M and ν is a complex bundle on $M\backslash Z$, where $Z \subset M$ is a compact subset.*

ii) *The morphism $\nu \to \nu$ is identity on $M\backslash Z$.*

iii) *The first row is exact on M and the other row are exact on $M\backslash Z$.*

iv) *Each column is exact on M.*

v) *The diagram is commutative on $M\backslash Z$.*

Assume that we have a connection ∇ for $\nu|_{M\backslash Z}$, connections ∇_i for E_i on M, and $\sum$ is a relative compact neighborhood of Z such that the following diagrams are commutative on $M\backslash \sum$:

$$\begin{array}{ccc}
\Gamma(E_i) & \xrightarrow{\nabla_i} & \Gamma(E_i) \otimes \Omega^1 \\
\downarrow & & \downarrow \\
\Gamma(E_{i-1}) & \xrightarrow{\nabla_{i-1}} & \Gamma(E_{i-1}) \otimes \Omega^1,
\end{array}$$

where $E_{-1} = \nu$ and $\nabla_{-1} = \nabla$. Then there are connections ∇_i' for I_i on M such that the following diagrams are commutative on $M\backslash \sum$:

$$\begin{array}{ccc}
\Gamma(I_i) & \xrightarrow{\nabla_i'} & \Gamma(I_i) \otimes \Omega^1 \\
\downarrow & & \downarrow \\
\Gamma(I_{i-1}) & \xrightarrow{\nabla_{i-1}'} & \Gamma(I_{i-1}) \otimes \Omega^1,
\end{array}$$

where $I_{-1} = \nu$, $\nabla'_{-1} = \nabla$ *and we have:*

$$\Pi_{i=0}^{k} det(1 + \frac{\sqrt{-1}}{2\pi} K(\nabla_i))^{\epsilon(i)} = \Pi_{i=0}^{k} det(1 + \frac{\sqrt{-1}}{2\pi} K(\nabla'_i))^{\epsilon(i)}.$$

Now we show that the differential index is well defined. To see this, first we show that it is independent of the choice of connections. Let ∇' and ∇'_i 's be other connections which satisfy the similar properties of ∇ and ∇_i 's, take $t : U \times [0,1] \to [0,1]$ and $\rho : U \times [0,1] \to U$ the projective maps. $\overline{\nabla}_i = t\rho^*\nabla'_i + (1-t)\rho^*\nabla_i$ is a connection for the bundle $\rho^* E_i$ on $U \times [0,1]$, and $\overline{\nabla} = t\rho^*\nabla' + (1-t)\rho^*\nabla$ is a connection for the bundle $\rho^*\pi^* E$ on $(U \backslash \pi^{-1}(q)) \times [0,1]$. Consider the following exact sequence on $(U \backslash \pi^{-1}(q)) \times [0,1]$:

$$0 \to \rho^* E_k \to \rho^* E_{k-1} \to \cdots \to \rho^* E_0 \to \rho^* \pi^* E \to 0,$$

and the following commutative diagrams on $(U \backslash \sum) \times [0,1]$:

$$\begin{array}{ccc} \Gamma(\rho^* E_i) & \overset{\overline{\nabla}_i}{\to} & \Gamma(\rho^* E_i) \otimes \Omega^1 \\ \downarrow & & \downarrow \\ \Gamma(\rho^* E_{i-1}) & \overset{\overline{\nabla}_{i-1}}{\to} & \Gamma(\rho^* E_{i-1}) \otimes \Omega^1, \end{array}$$

where $\rho^* E_{-1} = \rho^*\pi^* E$ and $\overline{\nabla}_{-1} = \overline{\nabla}$. Then we have on $(U \backslash \sum) \times [0,1]$: $\sigma_n(\overline{\nabla}_0, \overline{\nabla}_1, \cdots, \overline{\nabla}_k) = \sigma_n(\overline{\nabla}) = 0$ (since $\overline{\nabla}\pi^* s = 0$). Define maps $i_t : U \to U \times [0,1]$ by $i_t(p) = (p,t)$. There is a form ω with compact support in U such that

$$i_1^*\sigma_n(\overline{\nabla}_0, \overline{\nabla}_1, \cdots, \overline{\nabla}_k) - i_0^*\sigma_n(\overline{\nabla}_0, \overline{\nabla}_1, \cdots, \overline{\nabla}_k) = d\omega.$$

It is obvious that

$$i_0^*\sigma_n(\overline{\nabla}_0, \overline{\nabla}_1, \cdots, \overline{\nabla}_k) = \sigma_n(\nabla_0, \nabla_1, \cdots, \nabla_k),$$

and $i_1^*\sigma_n(\overline{\nabla}_0, \overline{\nabla}_1, \cdots, \overline{\nabla}_k) = \sigma_n(\nabla'_0, \nabla'_1, \cdots, \nabla'_k)$. Stokes theorem shows that the definition of differential index is independent of ∇ and ∇_i 's.

Without loss of generality, we may suppose $k = 2n$ (the length of the sequence), since if $k < 2n$ we have the following exact sequence (with the length $2n$):

$$0 \to \underline{0} \to \cdots \to \underline{0} \to \underline{E_k} \to \underline{E_{k-1}} \to \cdots \to \underline{E_0} \to \underline{\pi^* E} \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{A} \to 0,$$

and if $k > 2n$, according to *syzygy* theorem ([CE]), the kernel of morphism $\underline{E_{2n-1}} \to \underline{E_{2n-2}}$ is a coherent sheaf which is locally free and we denote it by

$\underline{E'_{2n}}$, where E'_{2n} is a vector bundle and $\underline{E'_{2n}}$ is the sheaf of its local sections. Consider the following commutative diagram whose rows and columns are exact:

$$\begin{array}{ccccccccccccccccc}
0 & \to & \underline{E_k} & \to\cdots\to & \underline{E_{2n+1}} & \to & \underline{E_{2n}} & \to & \underline{E_{2n-1}} & \to\cdots\to & \underline{E_0} & \to & \underline{\pi^*E}\otimes_{\mathcal{O}_{\overline{X}}}\mathcal{A} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \underline{0} & \to\cdots\to & \underline{0} & \to & \underline{E'_{2n}} & \to & \underline{E'_{2n-1}} & \to\cdots\to & \underline{E_0} & \to & \underline{\pi^*E}\otimes_{\mathcal{O}_{\overline{X}}}\mathcal{A} & \to & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

Lemma 3.1 shows that the indices defined by the rows of above diagram coincide. Finally, if

$$0 \to \underline{F_{2n}} \to \cdots \to \underline{F_1} \to \underline{F_0} \to \underline{\pi^*E}\otimes_{\mathcal{O}_{\overline{X}}}\mathcal{A} \to 0$$

is another exact sequence on U, we take $W \subset U$ to be a relative compact open subset of U containing $\pi^{-1}(q)$. By the comparison proposition 6.5 in [BB], there are the following exact sequence and commutative diagrams on W:

$$0 \to \underline{I_{2n}} \to \cdots \to \underline{I_0} \to \underline{\pi^*E}\otimes_{\mathcal{O}_{\overline{X}}}\mathcal{A} \to 0$$

$$\begin{array}{ccccccccccc}
0 & \to & \underline{I_{2n}} & \to & \cdots & \to & \underline{I_0} & \to & \underline{\pi^*E}\otimes_{\mathcal{O}_{\overline{X}}}\mathcal{A} & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & \underline{E_{2n}} & \to & \cdots & \to & \underline{E_0} & \to & \underline{\pi^*E}\otimes_{\mathcal{O}_{\overline{X}}}\mathcal{A} & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & & & & 0 & & 0 & &
\end{array}$$

$$\begin{array}{ccccccccccc}
0 & \to & \underline{I_{2n}} & \to & \cdots & \to & \underline{I_0} & \to & \underline{\pi^*E}\otimes_{\mathcal{O}_{\overline{X}}}\mathcal{A} & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & \underline{F_{2n}} & \to & \cdots & \to & \underline{F_0} & \to & \underline{\pi^*E}\otimes_{\mathcal{O}_{\overline{X}}}\mathcal{A} & \to & 0, \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & & & & 0 & & 0 & &
\end{array}$$

where the morphism $\underline{\pi^*E}\otimes_{\mathcal{O}_{\overline{X}}}\mathcal{A} \to \underline{\pi^*E}\otimes_{\mathcal{O}_{\overline{X}}}\mathcal{A}$ is the identity and the columns of above diagrams are exact. Lemma 3.1 shows that the indices defined by the rows of above diagrams coincide. □

LEMMA 3.2. — *When X is smooth at q and E is a vector bundle in a neighborhood of q then $Ind_{dif}(s,q)$ coincides with the ordinary index.*

Proof. — Without loss of generality, we may assume $E = TX$ in a neighborhood of q and s is a vector field in a neighborhood of q and $s(q) = 0$. By the remarks 8.8 of [BB], there is a connection ∇ for TX in a neighborhood U of q such that on $U \setminus \sum$, we have $\nabla s = 0, \nabla\nu = [s,\nu]$ for any holomorphic vector field ν on U, where $\sum$ is a compact subset of U and $q \in \overset{\circ}{\sum}$. By definition, we have $Ind_{dif}(s,q) = \int_U \sigma_n(\nabla)$ and by propositions 8.38, 8.67 in [BB], $\int_U \sigma_n(\nabla)$ coincides with ordinary index. $\square$

THEOREM 3.3. — *Let X be an n-dimensional compact reduced analytic space with isolated singularities, $E \to X$ be a linear fibre space which is an n-plane holomorphic bundle on X away from a finite number of points. If s is a section of $E \to X$ with isolated zeros, then we have:*

$$\sum_{q \in sing(s)} Ind_{dif}(s,q) = \int_{\overline{X}} c_n(\underline{\pi^* E}),$$

where $\pi : \overline{X} \to X$ is a desingularization.

Proof. — There is an exact sequence on $\overline{X}$

$$0 \to \underline{E_k} \to \cdots \to \underline{E_0} \to \underline{\pi^* E} \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{A} \to 0.$$

For each $q \in sing(s)$, take U_q a neighborhood of $\pi^{-1}(q)$ and $\sum_q$ a compact subset of U_q such that $\pi^{-1}(q) \subset \overset{\circ}{\sum_q}$, and $U_q \cap U_{q'} = \emptyset$ for $q \neq q'$. Let ∇ be a connection for the bundle $\pi^* E|_{\overline{X} \setminus \pi^{-1}(sing(s))}$, such that $\nabla \pi^* s = 0$. We have the following exact sequence on $\overline{X} \setminus \pi^{-1}(sing(s))$:

$$0 \to E_k \to \cdots \to E_1 \to E_0 \to \pi^* E \to 0.$$

By lemma 5.26 in [BB], there are connections ∇_i 's for E_i 's on $\overline{X}$ such that the following diagrams are commutative on $\overline{X} \setminus \cup \sum_q$:

$$\begin{array}{ccc} \Gamma(E_i) & \overset{\nabla_i}{\to} & \Gamma(E_i) \otimes \Omega^1 \\ \downarrow & & \downarrow \\ \Gamma(E_{i-1}) & \overset{\nabla_{i-1}}{\to} & \Gamma(E_{i-1}) \otimes \Omega^1, \end{array}$$

where $E_{-1} = \pi^* E$ and $\nabla_{-1} = \nabla$. We know that $c_n(\underline{\pi^* E}) = [\sigma_n(\nabla_0, \nabla_1, \cdots, \nabla_k)]$ and $\sigma_n(\nabla_0, \nabla_1, \cdots, \nabla_k) = \sigma_n(\nabla) = 0$ on $\overline{X} \setminus \cup \sum_q$, so

$$\begin{aligned} \int_{\overline{X}} c_n(\underline{\pi^* E}) &= \sum_{q \in sing(s)} \int_{U_q} \sigma_n(\nabla_0, \nabla_1, \cdots, \nabla_k) \\ &= \sum_{q \in sing(s)} Ind_{dif}(s,q). \quad \square \end{aligned}$$

Fix $q \in sing(s)$. Let U be a neighborhood of q such that $U \cap (sing(s) - \{q\}) = \emptyset$, s_t is a one parameter family of sections with isolated zeros, $sing(s_t) \cap \partial U = \emptyset$ and $s_0 = s$.

PROPOSITION 3.4. (Law of rigidity). — *In the above situation we have*

$$Ind_{dif}(s,q) = \sum_{q'\in sing(s_t)\cap U} Ind_{dif}(s_t,q').$$

Proof. — Let $\pi : \overline{X} \to X$ be a desingularization. Consider the following free resolution on $\pi^{-1}(U)$:

$$0 \to \underline{E_k} \to \cdots \to \underline{E_1} \to \underline{E_0} \to \underline{\pi^* E} \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{A} \to 0.$$

We obtain the following exact sequence on $\pi^{-1}(U-q)$:

$$0 \to E_k \to \cdots \to E_1 \to E_0 \to \pi^* E \to 0.$$

Let $\rho : \pi^{-1}(U) \times [0,1] \to \pi^{-1}(U)$ and $t : \pi^{-1}(U) \times [0,1] \to [0,1]$ be projections. Consider the following exact sequence on $\pi^{-1}(U-q) \times [0,1]$:

$$0 \to \rho^* E_k \to \cdots \to \rho^* E_1 \to \rho^* E_0 \to \rho^* \pi^* E \to 0.$$

$\overline{s}(p,t) = s_t(p)$ is a section of the bundle $\rho^*\pi^*E$ on $\pi^{-1}(U-q) \times [0,1]$. Let $Z = \cup sing(s_t)$, and $\sum$ is a relatively compact subset of U such that $Z \subset \mathring{\sum}$ (note that $Z \cap \partial U = \emptyset$). We take a connection ∇ for the bundle $\rho^*\pi^*E$ on $\pi^{-1}(U\backslash Z) \times [0,1]$ such that $\nabla\overline{s} = 0$ and ∇_i 's are connections for $\rho^* E_i$'s on $\pi^{-1}(U) \times [0,1]$ such that the following diagrams are commutative on $\pi^{-1}(U\backslash \sum) \times [0,1]$:

$$\begin{array}{ccc} \Gamma(\rho^* E_i) & \xrightarrow{\nabla_i} & \Gamma(\rho^* E_i) \otimes \Omega^1 \\ \downarrow & & \downarrow \\ \Gamma(\rho^* E_{i-1}) & \xrightarrow{\nabla_{i-1}} & \Gamma(\rho^* E_{i-1}) \otimes \Omega^1, \end{array}$$

where $\rho^* E_{-1} = \rho^*\pi^*E$ and $\nabla_{-1} = \nabla$. We define maps $i_t : \pi^{-1}(U) \to \pi^{-1}(U) \times [0,1]$ by $i_t(p) = (p,t)$. we have $\sigma_n(\nabla_0, \nabla_1, \cdots, \nabla_k) = \sigma_n(\nabla) = 0$ on $\pi^{-1}(U\backslash \sum) \times [0,1]$. Then there are forms ω_t with compact support such that

$$i_t^*\sigma_n(\nabla_0, \nabla_1, \cdots, \nabla_k) - i_0^*\sigma_n(\nabla_0, \nabla_1, \cdots, \nabla_k) = d\omega_t.$$

By definition $\sum_{q\in sing(s_t)} Ind_{dif}(s_t,q) = \int_U i_t^*\sigma_n(\nabla_0, \nabla_1, \cdots, \nabla_k)$. Finally Stokes theorem shows that $\int_U i_t^*\sigma_n(\nabla_0, \nabla_1, \cdots, \nabla_k) = \int_U i_0^*\sigma_n(\nabla_0, \nabla_1, \cdots, \nabla_k)$. □

LEMMA 3.5.

$$Ind_{dif}(s,q) = Ind_{hom}(s,q) + k,$$

where k is independent of the choice s.

Proof. — We use a technique which was first used in [G]. Let s_t be a deformation of s such that s_t have isolated zeros in U, and $sing(s_t) \cap \partial U = \emptyset$. By the Law of conservation

$$Ind_{hom}(s,q) - Ind_{hom}(s_t,q) = \sum_{\substack{q' \in sing(s_t) \cap U \\ q' \neq q}} Ind_{hom}(s_t,q'),$$

and the Law of rigidity

$$Ind_{dif}(s,q) - Ind_{dif}(s_t,q) = \sum_{\substack{q' \in sing(s_t) \cap U \\ q' \neq q}} Ind_{dif}(s_t,q').$$

Since $U - q$ is smooth and $E|_{U-q}$ is a vector bundle, so both of $Ind_{dif}(s_t,q')$ and $Ind_{hom}(s_t,q')$ coincide with the ordinary index for $q' \neq q$. Then $Ind_{hom}(-,q) - Ind_{dif}(-,q)$ is locally constant on the space of holomorphic sections on $\overline{U}$ which have isolated zero at q. Since this space is connected, $Ind_{hom}(s,q) - Ind_{dif}(s,q)$ is independent of the choice of s. □

Now theorem 2.5 is a consequence of theorem 3.3 and lemma 3.5.

Remark 3.6. — The "differential index" is not an invariant of section of linear fibre space alone, it depends on the choice of resolution as well. One can see this directly or using Theorem 3.3. This is not a problem however, since we are only using this index to show that the homological index is well defined, a different resolution will give a different k in lemma 3.5 but always independent of the selected s, as it is shown in lemma 3.5.

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