Annales de la faculté des sciences de Toulouse

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Annales de la faculté des sciences de Toulouse 6e série, tome 10, nº 1 (2001), p. 185-215
<http://www.numdam.org/item?id=AFST_2001_6_10_1_185_0>

Annales de la Faculté des Sciences de Toulouse Vol. X, n° 1, 2001
pp. 185–215

Nearly ordinary deformations of irreducible residual representations [(*)]

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RÉSUMÉ. — Nous prouvons dans cet article la modularité de certaines représentations p-adiques de $\mathrm{Gal}(\overline{F}/F)$, où F est un corps totalement réel. Les conditions principales sont que p soit impair, que la représentation soit irréductible et impaire, et que la représentation résiduelle ait un relèvement modulaire convenable. La méthode est une adaptation de celle employée par les auteurs dans le cas où la représentation résiduelle est réductible.

ABSTRACT. — In this paper we establish the modularity of certain p-adic representations of $\mathrm{Gal}(\overline{F}/F)$, where F is a totally real field. The main conditions imposed are that p be odd, the representation be irreducible and odd, and that the residual representation have a suitable modular lift. The methods are an adaptation of those used in our work on the case where the residual representation is reducible.

(*) Reçu le 8 février 2000, accepté le 8 novembre 2000
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1. Introduction

In this paper we give criteria for the modularity of certain two-dimensional Galois representations. To say that $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(E)$, E either a finite extension of $\mathbf{Q}_p$ or a finite field of characteristic p, is modular or that it comes from a modular form is to mean that there exists a modular form f with the property that if $T(\ell)f = c(\ell) \cdot f$, then

$$c(\ell) = \mathrm{trace}\rho(\mathrm{Frob}_\ell) \tag{1.1}$$

for all ℓ at which ρ is unramified. Here $T(\ell)$ is the ℓ^{th} Hecke operator. To make sense of (1.1) in the case where E has characteristic zero an embedding of E into $\mathbf{C}$ is chosen so that $\mathrm{trace}\rho(\mathrm{Frob}_\ell)$ can be viewed as an element of $\mathbf{C}$. When E is finite then a prime of $\overline{\mathbf{Q}}$ over p is chosen. Reducing modulo this prime permits each $c(\ell)$ to be viewed as an element of $\overline{\mathbf{F}}_p$. Finally, an embedding of E into $\overline{\mathbf{F}}_p$ is chosen so that $\mathrm{trace}\rho(\mathrm{Frob}_\ell)$ can be viewed as being in $\overline{\mathbf{F}}_p$ as well.

Suppose that E has characteristic zero. If we pick a stable lattice in E^2 and reduce ρ modulo a uniformizer λ of $\mathcal{O}_E$, the ring of integers of E, then we get a representation $\overline{\rho}$ of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ into $\mathrm{GL}_2(\mathcal{O}_E/\lambda)$. If $\overline{\rho}$ is irreducible, then it is uniquely determined by ρ. In general we write $\overline{\rho}^{ss}$ for the semisimplification of $\overline{\rho}$, and this is uniquely determined by ρ in all cases.

In this paper we consider the case where $\overline{\rho}^{ss}$ is irreducible, and we prove the following theorem. Here and throughout the rest of this paper p is an odd prime.

THEOREM.— *Suppose that $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}_2(E)$ is a continuous representation, irreducible and unramified outside a finite set of primes, where E is a finite extension of $\mathbf{Q}_p$. Suppose also that*

(i) *$\overline{\rho}^{ss}$ is irreducible,*

(ii) *$\overline{\rho}^{ss}|_{D_p} \simeq \begin{pmatrix} \chi_1 & * \\ & \chi_2 \end{pmatrix}$ where $(\chi_1/\chi_2)|_{D_p} \neq 1$,*

(iii) *$\overline{\rho}^{ss}$ comes from a modular form,*

(iv) *$\rho|_{I_p} \simeq \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$,*

(v) *$\det\rho = \psi\varepsilon^{k-1}$ for some $k \geqslant 2$ and is odd,*

where ε is the cyclotomic character and ψ is of finite order. Then ρ comes from a modular form.

This is not included in the results of [W] and [D1] as this theorem also covers the cases where $\overline{\rho}^{ss}$ is reducible over $\mathbf{Q}(\zeta_p)$. These are in fact precisely the cases not covered in [W] and [D1]. We also prove a similar theorem with $\mathbf{Q}$ replaced by a general totally real number field F, including the cases where $\overline{\rho}^{ss}$ is reducible over $F(\zeta_p)$: see §5. Results in the totally real case similar to the main results of [W] and [D1] have been obtained by Fujiwara [F], again excluding the cases where $\overline{\rho}^{ss}$ is reducible over $F(\zeta_p)$. Fujiwara also assumes the existence of a minimal modular lift, but this can be bypassed thanks to the the base change technique of [SW2].

In a previous paper [SW1] we considered the case where $\overline{\rho}^{ss}$ is reducible, and our proof of the above theorem follows the proof of the main theorems of that paper. The techniques we use were developed for the reducible case. However they also work in general, and our purpose in this paper is to show how they can be applied to the case where $\overline{\rho}^{ss}$ is reducible over $F(\zeta_p)$, a case that eludes the more direct approaches of [W], [D1], and [F]. Hypothesis (iv) of the theorem, which we refer to as the condition that ρ be *ordinary*, is essential to these techniques.

We now give an outline of this paper. In §2 we introduce certain deformation problems over totally real fields and their associated deformation rings $R_{\mathcal{D}}$. In §3 we introduce certain Hecke rings $\mathbf{T}_{\mathcal{D}}$ associated to these deformation problems and recall some of their important properties. These rings were studied in detail in [SW1] and the reader is often referred there for definitions and proofs. What sets apart the case where $\overline{\rho}^{ss}$ is irreducible from the case where it is reducible is that in the former there exists a deformation into $\mathrm{GL}_2(\mathbf{T}_{\mathcal{D}})$. This considerably simplifies the proofs, as a comparison of the next section, §4, with [SW1,§4] shows. One consequence of the existence of a deformation into $\mathrm{GL}_2(\mathbf{T}_{\mathcal{D}})$ is the existence of a canonical surjection $R_{\mathcal{D}} \to \mathbf{T}_{\mathcal{D}}$. We say that a prime of $R_{\mathcal{D}}$ is pro-modular if it is the inverse image under this surjection of a prime of $\mathbf{T}_{\mathcal{D}}$. The main result of §4 establishes the pro-modularity of all primes of $R_{\mathcal{D}}$ under certain hypotheses on $\mathcal{D}$ and F. The proof of this result is modelled on the proof of the main result of [SW1,§4], though, as mentioned before, it is considerably easier. The main ideas behind this proof are discussed in the introduction to [SW1], and the interested reader should consult there. In §5 we apply the result of §4 to deduce the theorem above as well as its generalization to totally real fields. In doing so we find it necessary to change the base field so that our residual representation $\overline{\rho}^{ss}$ satisfies the hypotheses of §4 and so that the results of [SW2] on the existence of minimal modular lifts apply. The remaining sections, §§6-8, contain the proof of the remaining assumption made in §4, namely that the property (P) defined therein holds for $R_{\mathcal{D}}$.

2. A deformation problem

Let F be a totally real number field of degree d. For any finite set of finite places Σ let F_Σ be the maximal extension of F unramified outside of Σ and all $v|\infty$. For each place v fix once and for all an embedding of $\overline{F}$ into $\overline{F}_v$. Doing so fixes a choice of decomposition group D_v and inertia group I_v for each finite place v and a choice of complex conjugation for each infinite place. Let $z_1, \dots, z_d$ be the d complex conjugations so chosen, and let $v_1, \dots, v_t$ be the places dividing p. Write D_i and I_i for the decomposition group and inertia group chosen for the place v_i. Let d_i be the degree of F_{v_i} over $\mathbf{Q}_p$. Normalize the reciprocity maps of Class Field Theory so that uniformizers correspond to arithmetic Frobenii.

Suppose that k is a finite field of characteristic p and that

$$\rho_0 : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(k)$$

is a representation such that

(i) ρ_0 is absolutely irreducible,

(ii) $\det \rho_0(z_i) = -1$ for $i = 1, ..., d$, (2.1)

(iii) $\rho_0|_{D_i} \simeq \begin{pmatrix} \chi_1^{(i)} & * \\ & \chi_2^{(i)} \end{pmatrix}$, $\chi_1^{(i)} \neq \chi_2^{(i)}$, for $i = 1, ..., t$, and

(iv) if ρ_0 is ramified at some $v \nmid p\infty$ then $\rho_0|_{D_v} \simeq \begin{pmatrix} 1 & * \\ & \omega^{-1} \end{pmatrix}$.

If $\rho_0|_{D_i}$ is split, then we fix once and for all a labelling of the characters in (2.1iii). As usual, ω is the character giving the action of $\mathrm{Gal}(\overline{F}/F)$ on the pth roots of unity.

A *deformation datum* relative to F and ρ_0 is a 3-tuple $\mathcal{D} = (\mathcal{O}, \Sigma, \mathcal{M})$ consisting of the ring of integers $\mathcal{O}$ of a local field with residue field k, a finite set of finite places Σ containing all those at which ρ_0 is ramified together with $\mathcal{P} = \{v_1, \dots, v_t\}$, and a set of places $\mathcal{M} \subseteq \Sigma \backslash \mathcal{P}$ at which ρ_0 is ramified. For future reference we set Σ_0 to be the set of finite places consisting of the places in $\mathcal{P}$ and the places at which ρ_0 is ramified, and we set $\mathcal{M}_0$ to be the set $\Sigma_0 \backslash \mathcal{P}$.

A *deformation* of ρ_0 is a local complete Noetherian ring A with residue field k and maximal ideal $\mathfrak{m}_A$ together with a strict equivalence class of continuous representations $\rho : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_2(A)$ satisfying $\rho_0 = \rho \bmod \mathfrak{m}_A$. Such a deformation is *of type-$\mathcal{D}$* if

- A is an $\mathcal{O}$ - algebra,
- ρ is unramified outside of Σ and the places above ∞,
- $\rho|_{D_i} \simeq \begin{pmatrix} \psi_1^{(i)} & * \\ & \psi_2^{(i)} \end{pmatrix}$ with $\chi_1^{(i)} = \psi_1^{(i)} \bmod \mathfrak{m}_A$ for each $i = 1, \ldots, t$, and
- if $w \in \mathcal{M}$ then $\rho|_{I_w} \simeq \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$.

We usually denote a deformation by a member of its equivalence class.

For any deformation datum $\mathcal{D} = (\mathcal{O}, \Sigma, \mathcal{M})$ there is a universal deformation of type-$\mathcal{D}$:

$$\rho_{\mathcal{D}} : \mathrm{Gal}(F_\Sigma/F) \longrightarrow \mathrm{GL}_2(R_{\mathcal{D}}).$$

This is well known (see [M1] and [M2] for a precise formulation of the universal property as well as a proof of existence).

We now state a preliminary result on the structure of $R_{\mathcal{D}}$ as an abstract ring. We omit the proof of this result since it is essentially the same as that of [SW1, Proposition 2.4]. The only difference is that it may be that $\rho_0|_{D_i}$ is non-split. In this case one needs to analyze the universal (nearly ordinary) deformation ring associated to $\rho_0|_{D_i}$. The necessary analysis can be done just as in [SW1, Corollary 2.3] (the split case). Let δ_F be the $\mathbf{Z}_p$-rank of the Galois group of the maximal abelian pro-p-extension of F.

Proposition 2.1. — *Suppose that $\mathcal{D} = (\mathcal{O}, \Sigma, \mathcal{M})$ is a deformation datum. There exist integers g and r, depending on $\mathcal{D}$, such that*

$$R_{\mathcal{D}} \simeq \mathcal{O}[\![x_1, \ldots, x_g]\!]/(f_1, \ldots, f_r)$$

and

$$g - r \geqslant d + \delta_F - 2t - 3 \cdot \#\mathcal{M}.$$

In the course of our analysis of the rings $R_{\mathcal{D}}$ we shall sometimes have to consider some augmented deformation problems. Here we introduce these deformations and their universal deformation rings.

Let $\mathcal{D} = (\mathcal{O}, \Sigma, \mathcal{M})$ be a deformation datum and Q a finite set of finite places $Q = \{w_1, \ldots, w_r\}$ disjoint from Σ. A deformation ρ of type-$(\mathcal{O}, \Sigma \cup Q, \mathcal{M})$ is *of type-$\mathcal{D}_Q$* if

- $\det\rho$ is unramified at each $w_i \in Q$.

There exists a universal deformation of type-$\mathcal{D}_Q$:

$$\rho_{\mathcal{D}_Q} : \mathrm{Gal}\,(F_{\Sigma\cup Q}/F) \longrightarrow \mathrm{GL}_2(R_{\mathcal{D}_Q}).$$

For a deformation datum $\mathcal{D} = (\mathcal{O}, \Sigma, \mathcal{M})$ let L_Σ/F be the maximal abelian pro-p-extension of F unramified away from Σ, and let N_Σ be the torsion subgroup of $\mathrm{Gal}(L_\Sigma/F)$. A deformation $\rho : \mathrm{Gal}(F_\Sigma/F) \longrightarrow \mathrm{GL}_2(A)$ of type- $\mathcal{D}_Q$ is *$\mathcal{D}_Q$-minimal ($\mathcal{D}$-minimal* if $Q = \emptyset$) if $\det \rho$ is trivial on N_Σ. Let

$$\rho^{\min}_{\mathcal{D}_Q} : \mathrm{Gal}\,(F_{\Sigma\cup Q}/F) \longrightarrow \mathrm{GL}_2(R^{\min}_{\mathcal{D}_Q})$$

be the universal $\mathcal{D}_Q$-minimal deformation. If $Q = \emptyset$, then we just write $\rho^{\min}_{\mathcal{D}}$ and $R^{\min}_{\mathcal{D}}$. There is a simple relation between $R_{\mathcal{D}_Q}$ and $R^{\min}_{\mathcal{D}_Q}$. We fix for each Σ a free $\mathbf{Z}_p$-summand $H_\Sigma \subseteq \mathrm{Gal}(L_\Sigma/F)$ such that $\mathrm{Gal}(L_\Sigma/F) \simeq H_\Sigma \oplus N_\Sigma$. We choose the H_Σ's to be compatible with varying Σ. Let $\Psi_\Sigma : \mathrm{Gal}(L_\Sigma/F) \twoheadrightarrow N_\Sigma$ denote the character obtained by projecting modulo H_Σ. The representation $\rho^{\min}_{\mathcal{D}_Q} \otimes \Psi_\Sigma : \mathrm{Gal}(F_\Sigma/F) \longrightarrow \mathrm{GL}_2(R^{\min}_{\mathcal{D}_Q} \otimes_{\mathcal{O}} \mathcal{O}[N_\Sigma])$ is easily seen to be a deformation of type-$\mathcal{D}_Q$. It follows from the universal properties of $R_{\mathcal{D}_Q}$ and $R^{\min}_{\mathcal{D}_Q}$ that

$$R_{\mathcal{D}_Q} \simeq R^{\min}_{\mathcal{D}_Q} \otimes_{\mathcal{O}} \mathcal{O}[N_\Sigma] \quad \text{and} \quad \rho_{\mathcal{D}_Q} \simeq \rho^{\min}_{\mathcal{D}_Q} \otimes \Psi_\Sigma. \tag{2.2}$$

Next we consider a special class of deformations. A deformation ρ of ρ_0 is *dihedral* if there exists a quadratic extension F' of F such that the restriction of ρ to $\mathrm{Gal}(\overline{F}/F')$ factors through an abelian quotient. The following properties of dihedral deformations are trivialities:

(i) ρ is dihedral if and only if there exists an abelian extension L of F such that the restriction of ρ to $\mathrm{Gal}(\overline{F}/L)$ factors through an abelian quotient; (2.3)

(ii) ρ is dihedral only if ρ_0 is dihedral.

From property (ii) we see that the field F' is uniquely determined by ρ_0. If ρ_0 is dihedral, then we shall assume that

$$\rho_0 = \mathrm{Ind}_F^{F'} \psi_0 \tag{2.4}$$

where $\psi_0 : \mathrm{Gal}(\overline{F}/F') \to k^\times$ is a character and $\mathrm{Ind}_F^{F'} \psi_0$ is the usual two-dimensional representation induced from ψ_0. This is not a real restriction since if ρ_0 is dihedral then it always has a basis satisfying (2.4), possibly upon replacing k by a quadratic extension. Let ψ_0' be the conjugate of ψ_0 by an element restricting to the non-trivial automorphism of F' over F. Since

ρ_0 is irreducible it must be that the characters ψ_0 and ψ_0' are distinct. If $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(A)$ is a dihedral deformation, then $\rho \simeq \mathrm{Ind}_F^{F'} \Psi$ where $\Psi : \mathrm{Gal}(\overline{F}/F') \to A^\times$ is a character such that $\psi_0 = \Psi \bmod \mathfrak{m}_A$.

Assume for the moment that $\rho_0 = \mathrm{Ind}_F^{F'} \psi_0$ (i.e., that ρ_0 is dihedral). Often one can describe the universal dihedral deformation of type-$(\mathcal{O}, \Sigma, \mathcal{M})$. Let L'_Σ be the maximal abelian pro-p-extension of F' unramified away from Σ, and let $\Gamma_\Sigma = \mathrm{Gal}(L'_\Sigma/F')$. Denote by ψ the canonical projection of $\mathrm{Gal}(F_\Sigma/F')$ onto Γ_Σ. Define a character $\Psi_\Sigma : \mathrm{Gal}(F_\Sigma/F') \to \mathcal{O}[\![\Gamma_\Sigma]\!]^\times$ by $\Psi_\Sigma(\sigma) = \widetilde{\psi}_0(\sigma)\psi(\sigma)$. Here the 'tilde' denotes the Teichmüller lift. If every place of F above p splits in F' and if $\mathcal{M} = \emptyset$ then the desired universal deformation of type-$(\mathcal{O}, \Sigma, \mathcal{M})$ is just

$$\mathrm{Ind}_F^{F'} \Psi_\Sigma : \mathrm{Gal}(F_\Sigma/F) \to \mathrm{GL}_2(\mathcal{O}[\![\Gamma_\Sigma]\!]).$$

In the remaining cases (i.e., if some prime of F above p does not split in F' or if $\mathcal{M} \neq \emptyset$) it is a quotient of this deformation.

Let $\delta_{F'}^-$ be the $\mathbf{Z}_p$-rank of the summand of Γ_Σ on which $\mathrm{Gal}(F'/F) \simeq \mathbf{Z}/2$ acts non-trivially. By [Wal] we know that $\delta_{F'}^- \leqslant d/2$. The following lemma is a simple consequence of this and the preceding analysis of the universal dihedral deformations.

Lemma 2.2. — *If $\mathfrak{q} \subseteq R_\mathcal{D}$ is a prime containing p such that $\rho_\mathcal{D} \bmod \mathfrak{q}$ is dihedral and its determinant has finite order, then $\dim R_\mathcal{D}/\mathfrak{q} \leqslant \delta_{F'}^- \leqslant d/2$.*

Finally, we describe how each of the deformation rings $R_\mathcal{D}$ and $R_{\mathcal{D}_Q}$ is an algebra over a certain multivariate "Iwasawa algebra". Let L_0 be the maximal abelian pro-p extension of F unramified away from $\mathcal{P}$. Let $I \subseteq \mathrm{Gal}(L_0/F)$ be the subgroup generated by the images of the inertia groups I_i, $i = 1, \ldots, t$. We fix once and for all a maximal free $\mathbf{Z}_p$-summand I_0 of I (necessarily of rank δ_F). Fix also a free $\mathbf{Z}_p$-summand G_0 of $\mathrm{Gal}(L_0/F)$ containing I_0 (this also has rank δ_F). Finally, fix elements $\gamma_1, \ldots, \gamma_{\delta_F} \in \mathrm{Gal}(\overline{F}/F)$ whose images in $\mathrm{Gal}(L_0/F)$ generate G_0 and for which there exist integers $r_1, \ldots, r_{\delta_F}$ such that $\gamma_1^{p^{r_1}}, \ldots, \gamma_{\delta_F}^{p^{r_{\delta_F}}}$ generate I_0. For each $0 \leqslant i \leqslant t$ fix once and for all $y_1^{(i)}, \ldots, y_{d_i}^{(i)} \in \mathcal{U}_i$ (the units of F_{v_i}) generating a free $\mathbf{Z}_p$-summand of rank d_i. Put

$$\Lambda_\mathcal{O} = \mathcal{O}[\![T_1, \ldots, T_{\delta_F},\ Y_1^{(1)}, \ldots, Y_{d_t}^{(t)}]\!].$$

The rings $R_\mathcal{D}$ (and hence the $R_{\mathcal{D}_Q}$) are algebras over $\Lambda_\mathcal{O}$ via

- $T_i \longmapsto \det \rho_\mathcal{D}(\gamma_i) - 1, \quad i = 1, \ldots, \delta_F.$

• $Y_j^{(i)} \longmapsto \psi_2^{(i)}(y_j^{(i)}) - 1$, where $\rho_{\mathcal{D}}|_{D_i} \simeq \begin{pmatrix} \psi_1^{(i)} & * \\ & \psi_2^{(i)} \end{pmatrix}$ and we have identified $\mathcal{U}_i$ with the inertia subgroup of D_i^{ab} via local reciprocity.

Suppose that $\mathcal{D} = (\mathcal{O}, \Sigma, \mathcal{M})$ and $\mathcal{D}' = (\mathcal{O}, \Sigma', \mathcal{M}')$ are deformation data with $\Sigma \subseteq \Sigma'$ and $\mathcal{M}' \subseteq \mathcal{M}$. The natural map $R_{\mathcal{D}'} \twoheadrightarrow R_{\mathcal{D}}$ is a map of $\Lambda_{\mathcal{O}}$-algebras.

The following lemma is immediate from [C] and the universality of $R_{\mathcal{D}_Q}$.

LEMMA 2.3.— *If $\mathcal{D}_Q = (\mathcal{O}, \Sigma, \mathcal{M})_Q$ is any augmented deformation datum, and if S is any finite set of places of F containing $\Sigma \cup Q$, then $R_{\mathcal{D}_Q}$ is generated as a $\Lambda_{\mathcal{O}}$-algebra by the set $\{\mathrm{trace}\rho_{\mathcal{D}_Q}(\mathrm{Frob}_\ell) \ : \ \ell \notin S\}$.*

3. Hecke rings

In this section we introduce the Hecke rings that will play prominent roles in our study of deformations of ρ_0. We keep the conventions of the previous section. We shall borrow freely from the definitions, notations, and results of [SW1,§3]. As in the just-referenced paper, we fix an embedding of $\overline{F}$ into $\overline{\mathbf{Q}}_p$.

Let $\mathbf{A}$ and $\mathbf{A}_f$ denote the adeles and finite adeles of F, respectively. Let $\mathcal{O}_{\mathcal{F}}$ be the ring of integers of F, and let $\mathcal{O}$ be the ring of integers of some local field having residue field k. Let $\mathfrak{p}_1, ..., \mathfrak{p}_t$ be the prime ideals of F dividing p. For each compact subgroup such that

$$\begin{aligned} &\bullet\ U = \textstyle\prod_{w \nmid \infty} U_w, \quad U_w \subseteq \mathrm{GL}_2(\mathcal{O}_{F,w}), \\ &\bullet\ U_v = \mathrm{GL}_2(\mathcal{O}_{F,v}) \ \ \forall v | p, \\ &\bullet\ U_0(\mathfrak{n}) \subseteq U \subseteq U(\mathfrak{n}) \quad \text{for some ideal } \mathfrak{n}, \end{aligned} \tag{3.1}$$

each $\kappa = \kappa_1\tau_1 + \cdots + \kappa_d\tau_d \in \mathbf{Z}[I]$ with all $\kappa_i \geqslant 2$ ($I = \{\tau_1, ..., \tau_d\}$ are the d embeddings of F into $\mathbf{R}$), and each integer $a > 0$, let $\mathbf{T}_\kappa(U_a, \mathcal{O})$ be the Hecke ring defined and so denoted in [SW1,§3.2]. Let $\mathbf{G}(U)$ be the group defined there as well. In [SW1, §§3.1, 3.2] we defined Hecke operators $T(\ell)$ and $S(\ell)$ for each $\ell \nmid \mathfrak{n}$, $T_0(\mathfrak{p}_i)$ for each $i = 1, ..., t$, and $T_0(p)$, all acting on the space of modular forms of weight κ and level U_a and commuting one with another. (The definition of $T_0(\mathfrak{p}_i)$ depended on a choice of a uniformizer $\lambda_{\mathfrak{p}_i}^{(\mathfrak{p}_i)}$ which we now choose so that $(\chi_1^{(i)}/\chi_2^{(i)})(\lambda_{\mathfrak{p}_i}^{(\mathfrak{p}_i)}) \neq 1$.) We also defined an action of $\mathbf{G}(U)$ on this space of forms via certain Hecke operators denoted by T_y and S_x, where y runs over the elements of $(\mathcal{O}_f \otimes \mathbf{Z}_p)^\times$ and x runs over the elements of

$$Z(U) = \varprojlim_a (U \cap \mathbf{A}_f) \cdot \mathcal{O}_F^\times / (U_a \cap \mathbf{A}_f) \cdot \mathcal{O}_F^\times.$$

The ring $\mathbf{T}_\kappa(U_a, \mathcal{O})$ is just the ring (over $\mathcal{O}$) obtained by restricting the action of these operators to certain spaces of modular forms. These Hecke rings are finite, flat, commutative, reduced $\mathcal{O}$-algebras. If $V \supseteq U$ is another open compact subgroup satisfying (3.1) and if $b \geqslant a$, then there is a canonical homomorphism $\mathbf{T}_\kappa(V_b, \mathcal{O}) \to \mathbf{T}_\kappa(U_a, \mathcal{O})$. Put

$$\mathbf{T}_\infty(U, \mathcal{O}) = \varprojlim_a \mathbf{T}_2(U_a, \mathcal{O}).$$

(The subscript 2 indicates the parallel weight $2 \cdot (\tau_1 + \cdots + \tau_d)$.)

Let $\Lambda'_\mathcal{O} = \mathcal{O}[\![X_1, ..., X_{\delta_F}, Y_1^{(1)}, ..., Y_{d_t}^{(t)}]\!]$. In [SW1,§3.2] we defined a homomorphism $\Lambda'_\mathcal{O} \to \mathbf{T}_\infty(U, \mathcal{O})$ which we now recall. Let $\mathcal{U} = \prod_{v_i} \mathcal{U}_{v_i} \subseteq (\mathcal{O}_F \otimes \mathbf{Z}_p)^\times = \prod_{v_i} \mathcal{O}_{F,v_i}^\times$ where $\mathcal{U}_{v_i} \subseteq \mathcal{O}_{F,v_i}^\times$ is the subgroup of units congruent to one modulo v_i. Let $y_j^{(i)} \in \mathcal{U}$ be as at then end of §2. Let $x_1, \dots, x_{\delta_F} \in Z(U)$ be the images of $\gamma_1^{p^{r_1}}, \dots, \gamma_{\delta_F}^{p^{r_{\delta_F}}}$, respectively, via the global reciprocity map (for the definition of γ_i and r_i see the end of §2). The x_i's generate a maximal $\mathbf{Z}_p$-free direct summand of $Z(U)$. The ring $\mathbf{T}_\infty(U, \mathcal{O})$ is an algebra over the ring $\Lambda'_\mathcal{O}$ via $X_i \longmapsto S_{x_i} - 1$ and $Y_j^{(i)} \longmapsto T_{y_j^{(i)}} - 1$.

We now make the added hypothesis that

$$\text{the degree of } F \text{ over } \mathbf{Q} \text{ is even.} \qquad (\mathrm{H_{even}})$$

Let $M_\infty(U)$ and $M_\infty^\pm(U)$ be the $\mathbf{T}_\infty(U, \mathcal{O})$-modules defined in [SW1,§3.2]. We now recall some important properties of these modules and rings. We will say that a compact open subgroup U is *sufficiently small* if $U \subseteq U_1(\ell_1 \cdots \ell_s)$ where $\{\ell_1, ..., \ell_s\}$ is a set of unramified primes satisfying the hypotheses of [SW1, Corollary 3.6].

LEMMA 3.1. — ([SW1, Prop. 3.3, Cor. 3.4, 3.6])

(i) *If U is sufficiently small, then $M_\infty(U)$ and $M_\infty^\pm(U)$ are free $\Lambda'_\mathcal{O}$-modules of equal rank.*

(ii) *$M_\infty(U)$ and $M_\infty^\pm(U)$ are faithful $\mathbf{T}_\infty(U, \mathcal{O})$-modules.*

(iii) *$\mathbf{T}_\infty(U, \mathcal{O})$ is a finite, torsion-free $\Lambda'_\mathcal{O}$-module. In particular, it is a semilocal ring complete with respect to its radical.*

An $\mathcal{O}$-algebra homomorphism $\lambda : \mathbf{T}_\infty(U,\mathcal{O}) \to \overline{\mathbf{Q}}_p$ is *algebraic* if $\lambda(1+Y_i^{(j)}) = $ (root of unity) for $1 \leqslant j \leqslant t, 1 \leqslant i \leqslant d_j$ and there is an intger $\mu \geqslant 0$ such that $\lambda(1+X_i) = $ (root of unity)$\varepsilon^\mu(\gamma_i^{p^{r_i}})$ for $1 \leqslant i \leqslant \delta_F$. An *algebraic prime* is a prime of $\mathbf{T}_\infty(U,\mathcal{O})$ that is the kernel of an algebraic homomorphism. There are only finitely many algebaic homomorphisms whose kernel is a given algebraic prime P. We denote this finite set of homomorphisms by $\mathcal{H}(P)$. The set of algebraic primes is Zariski dense in spec$(\mathbf{T}_\infty(U,\mathcal{O}))$. Hida [H1] has established the following remarkable connection between algebraic homomorphisms and automorphic representations.

PROPOSITION 3.2. — *If* $\lambda : \mathbf{T}_\infty(U,\mathcal{O}) \longrightarrow \overline{\mathbf{Q}}_p$ *is an algebraic homomorphism, then there exists a nearly ordinary automorphic representation* π *of weight* $\kappa = \kappa_1\tau_1 + \cdots + \kappa_d\tau_d$, $\kappa_i \geqslant 2$, *containing a vector fixed by* U_a *for some* a *and for which* $\lambda(T(\ell))$ *and* $\lambda(S(\ell))$ *equal, respectively, the eigenvalues of* $T(\ell)$ *and* $S(\ell)$ *acting on the newform associated to* π *for all prime ideals* $\ell \nmid p$ *for which* $U_\ell = \mathrm{GL}_2(\mathcal{O}_{F,\ell})$.

Suppose that Q is a prime of $\mathbf{T}_\infty(U,\mathcal{O})$. Let $R = \mathbf{T}_\infty(U,\mathcal{O})/Q$ and let L be the field of fractions of R. Note that R is a complete local domain. There is a continuous, semi-simple representation $\rho_Q : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_2(\overline{L})$ such that

$$
\begin{aligned}
&\text{(i)}\ \ \rho_Q(z_1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\\
&\text{(ii)}\ \ \rho_Q \text{ is unramified at all primes } \ell \nmid \mathfrak{n}p\\
&\text{(iii)}\ \ \mathrm{trace}\rho_Q(\mathrm{Frob}_\ell) = T(\ell) \bmod Q \text{ for all } \ell \nmid \mathfrak{n}p\\
&\text{(iv)}\ \ \det\rho_Q(\mathrm{Frob}_\ell) = S(\ell)\ \varepsilon(\ell) \bmod Q \text{ for all } \ell \nmid \mathfrak{n}p\\
&\text{(v)}\ \ \det\rho_Q(x) = S_x\ \varepsilon(x) \bmod Q \text{ for all } x \in Z(U)\\
&\text{(vi)}\ \ \rho_Q|_{D_i} \cong \begin{pmatrix} \psi_1^{(i)} & * \\ & \psi_2^{(i)} \end{pmatrix} \text{ with } \psi_2^{(i)}(y) = T_y \bmod Q \text{ forall } y \in \mathcal{O}_{F,v_i}^\times
\end{aligned}
\tag{3.2}
$$

and $\psi_2^{(i)}(\lambda_{\mathfrak{p}_i}^{(\mathfrak{p}_i)}) = T_0(\mathfrak{p}_i) \bmod Q$ forall $i = 1,\ldots,t$.

By ρ_Q being continuous we mean that there is a finitely generated $\mathrm{Gal}(\overline{F}/F)$-stable R-module $\mathcal{M}$ in the underlying representation space of ρ_Q such that $\mathrm{Gal}(\overline{F}/F)$ acts continuously on $\mathcal{M}$. The existence of ρ_Q is essentially due to Hida (cf. [H2]), but is also established in [SW1, §3.3].

If Q is an algebraic prime, then fixing an identification of $\overline{L}$ with $\overline{\mathbf{Q}}_p$ (as $\mathcal{O}$-algebras) amounts to choosing an algebraic homomorphism $\lambda \in \mathcal{H}(Q)$. Let π be the automorphic representation corresponding to λ as in Proposition 3.2. Under the chosen identification $\overline{L} = \overline{\mathbf{Q}}_p$ we have $\rho_Q \simeq \rho_\pi$. Here, $\rho_\pi : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ is the representation associated to π and the fixed embedding $\overline{F} \hookrightarrow \overline{\mathbf{Q}}_p$ (cf. [SW1,§3.3]).

Suppose that $\mathfrak{m}$ is a maximal ideal of $\mathbf{T}_\infty(U, \mathcal{O})$. From the existence of the ρ_Q's it follows easily that there is a representation

$$\rho_{U,\mathfrak{m}} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbf{T}_\infty(U, \mathcal{O})_\mathfrak{m} \otimes_{\Lambda'_\mathcal{O}} F_{\Lambda'_\mathcal{O}})$$

satisfying the list of properties (3.2) but with ρ_Q replaced by $\rho_{U,\mathfrak{m}}$ and with the 'mod Q' omitted. (Here $F_{\Lambda'_\mathcal{O}}$) is the field of fractions of $\Lambda'_\mathcal{O}$.) If the representation $\rho_\mathfrak{m}$ is irreducible, then for a suitable choice of basis $\rho_{U,\mathfrak{m}}$ takes values in $\mathrm{GL}_2(\mathbf{T}_\infty(U, \mathcal{O})_\mathfrak{m})$. This last fact can be proven by the arguments on the bottom of page 482 and top of page 483 of [W].

If $\mathfrak{m}$ is a maximal ideal of $\mathbf{T}_\infty(U, \mathcal{O})$ such that $\rho_\mathfrak{m}$ is irreducible, then $\mathbf{T}_\infty(U, \mathcal{O})$ is a $\Lambda_\mathcal{O}$-algebra via $1 + T_i \mapsto \det \rho_{U,\mathfrak{m}}(\gamma_i)$ for $i = 1, ..., \delta_F$ and $1 + Y_j^{(i)} \mapsto T_{y_j^{(i)}}$ for $1 \leqslant i \leqslant d$, $1 \leqslant j \leqslant d_i$. (Here $\Lambda_\mathcal{O}$ is the ring introduced at the end of §2.) The natural inclusion of $\Lambda'_\mathcal{O}$ into $\Lambda_\mathcal{O}$ makes the latter a finite, free module over the former.

Let $\Sigma(U)$ be the set of places w such that $U_w \neq \mathrm{GL}_2(\mathcal{O}_{F,w})$ together with the places in $\mathcal{P}$. Part (i) of the following lemma can be proven as was [SW1, Lemma 3.11], while part (ii) follows from part (i) and Lemma 3.1.

LEMMA 3.3.

(i) *Let S be any finite set of places containing $\Sigma(U)$. If $\mathfrak{m}$ is a maximal ideal of $\mathbf{T}_\infty(U, \mathcal{O})_\mathfrak{m}$ such that $\rho_\mathfrak{m}$ is irreducible, then $\mathbf{T}_\infty(U, \mathcal{O})_\mathfrak{m}$ is generated over $\Lambda_\mathcal{O}$ by the set $\{T(\ell) \ : \ \ell \notin S\}$.*

(ii) *If U is sufficiently small and if $\mathfrak{m}$ is as in part* (i), *then $M_\infty(U)_\mathfrak{m}$ and $M_\infty^{\pm}(U)_\mathfrak{m}$ are free $\Lambda_\mathcal{O}$-modules of equal rank.*

If $\mathfrak{m}$ is a maximal ideal of $\mathbf{T}_\infty(U, \mathcal{O})$ such that $\rho_\mathfrak{m} \simeq \rho_0$ and $\{T_0(\mathfrak{p}_i) - \chi_2^{(i)}(\lambda_{\mathfrak{p}_i}^{(\mathfrak{p}_i)}) \ : \ 1 \leqslant i \leqslant t\} \subseteq \mathfrak{m}$, then we say that $\mathfrak{m}$ is *permissible*. Such a maximal ideal is clearly unique.

Next we associate Hecke rings to various deformation data. Essentially this is done by first defining a suitable open compact subgroup of $\mathrm{GL}_2(\mathcal{O}_F \otimes \widehat{\mathbf{Z}})$ and then localizing the corresponding Hecke ring at a permissible maximal ideal. Obviously, this requires the existence of a permissible maximal ideal of the prescribed level. This is a consequence of the hypothesis denoted (H_{def}) below.

Suppose that $\mathcal{D}_Q = (\mathcal{O}, \Sigma, \mathcal{M})_Q$ is an (augmented) deformation datum. For each finite place w we write ℓ_w for the prime ideal of F corresponding to w and we write Δ_w for the Sylow p-subgroup of $(\mathcal{O}_F/\ell_w)^\times$ which we identify with a subgroup of $(\mathcal{O}_F/\ell_w^r)^\times$ for any $r \geqslant 1$. We also write Δ'_w for a complementary subgroup of $(\mathcal{O}_F/\ell_w)^\times$ (so $(\mathcal{O}_F/\ell_w)^\times \simeq \Delta_w \times \Delta'_w$). We define a subgroup $U_{\mathcal{D}_Q} = \prod_{w \nmid \infty} U_{\mathcal{D}_Q,w} \subseteq \mathrm{GL}_2(\mathcal{O}_F \otimes \widehat{\mathbf{Z}})$ by putting

$$U_{\mathcal{D}_Q,w} = \begin{cases} \mathrm{GL}_2(\mathcal{O}_{F,w}) & \text{if } \ w \notin \Sigma\backslash\mathcal{P} \cup Q \\ \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F,w}) : c, a-1 \in \ell_w^2\right\} & \text{if } w \in \Sigma\backslash(\mathcal{P} \cup \mathcal{M}) \\ \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F,w)} : c \in \ell_w, a \bmod \ell_w \in \Delta_w\right\} & \text{if } w \in \mathcal{M}. \\ \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F,w}) : c \in \ell_w,\ ad^{-1} \bmod \ell_w \in \Delta'_w\right\} & \text{if } w \in Q. \end{cases}$$

If $Q = \emptyset$, then we write $U_\mathcal{D}$ for $U_{\mathcal{D}_\emptyset}$. Let $\mathcal{D}_0$ be the minimal deformation datum $(\mathcal{O}, \Sigma_0, \mathcal{M}_0)$. We assume that

$$\mathbf{T}_\infty(U_{\mathcal{D}_0}, \mathcal{O}) \text{ has a permissible maximal ideal.} \qquad (\mathrm{H}_{def})$$

As $U_{\mathcal{D}_Q} \subseteq U_{\mathcal{D}_0}$ for any $\mathcal{D}_Q$, it follows that the hypothesis (H_{def}) implies that every $\mathbf{T}_\infty(U_{\mathcal{D}_Q}, \mathcal{O})$ has a permissible maximal ideal.

Let $\mathfrak{m}$ be a permissible maximal ideal of $\mathbf{T}_\infty(U_{\mathcal{D}_Q}, \mathcal{O})$. Put

$$\mathbf{T}_{\mathcal{D}_Q} = \mathbf{T}_\infty(U_{\mathcal{D}_Q}, \mathcal{O})_\mathfrak{m}.$$

We define $\mathbf{T}_\mathcal{D}$ to be $\mathbf{T}_{\mathcal{D}_\emptyset}$ We write $\rho^{\mathrm{mod}}_{\mathcal{D}_Q}$ for the representation into $\mathrm{GL}_2(\mathbf{T}_{\mathcal{D}_Q})$ described above. We assume that we have chosen a basis for $\rho^{\mathrm{mod}}_{\mathcal{D}_Q}$ so that $\rho^{\mathrm{mod}}_{\mathcal{D}_Q} \bmod \mathfrak{m} = \rho_0$.

Proposition 3.4. — *$\rho^{\mathrm{mod}}_{\mathcal{D}_Q}$ is a deformation of ρ_0 of type-$\mathcal{D}_Q$, and the corresponding map $r_{\mathcal{D}_Q} : R_{\mathcal{D}_Q} \to \mathbf{T}_{\mathcal{D}_Q}$ is surjective.*

Proof. — Let P be a minimal prime of $\mathbf{T}_{\mathcal{D}_Q}$ Let $\rho_P = \rho^{\mathrm{mod}}_{\mathcal{D}_Q} \bmod P$. (This is a slight change from the notation in (3.2).) We first prove that ρ_P is a deformation of type-$\mathcal{D}_Q$. That $\det \rho_P$ is unramified at each $w \in Q$ can be seen by reducing ρ_P modulo algebraic primes and then invoking the compatibility of these reduced representations with the local Langlands' correspondence (i.e., invoking Proposition 3.2 and [SW1, (3.3)]).

Next we establish the required properties of $\rho_P|_{D_i}$. By hypothesis $\chi_1^{(i)}(\lambda_{\mathfrak{p}_i}^{(\mathfrak{p}_i)}) \neq \chi_2^{(i)}(\lambda_{\mathfrak{p}_i}^{(\mathfrak{p}_i)})$. Choose $\sigma_i \in D_i$ mapping to $\lambda_{\mathfrak{p}_i}^{(\mathfrak{p}_i)}$ via local reciprocity. Choose a basis for ρ_P such that $\rho_P(\sigma_i) = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$ with α

mod $\mathfrak{m} = \chi_1^{(i)}(\lambda_{\mathfrak{p}_i}^{(\mathfrak{p}_i)})$. It follows easily from (3.2vi) and from the fact that $T_0(\mathfrak{p}_i) - \chi_2^{(i)}(\lambda_{\mathfrak{p}_i}^{(\mathfrak{p}_i)}) \in \mathfrak{m}$ that with respect to this basis $\rho_P|_{D_i} = \begin{pmatrix} \psi_1^{(i)} & * \\ & \psi_2^{(i)} \end{pmatrix}$ with $\psi_2^{(i)}$ mod $\mathfrak{m} = \chi_2^{(i)}$ as desired.

It remains to verify the required properties of $\rho_P|_{I_w}$ for each place $w \in \mathcal{M}$. Let $w \in \mathcal{M}$. Let $\mathfrak{p}$ be any algebraic homomorphism containing P. Choose $\lambda \in \mathcal{H}(P)$ and let π be the automorphic representation associated to λ by Proposition 3.2. Since π has a non-zero vector fixed by $U_{\mathcal{D}_Q}$ it follows from the definition of $U_{\mathcal{D}_Q,w}$ and from the compatibility of $\rho_\pi|_{D_w}$ with the local Langlands' correspondence for π_w (see [SW1, (3.3)]) that $\rho_\pi|_{D_w}$ is either unramified or Type A in the sense of [SW1, §2.3]. Now let $R = \mathbf{T}_{\mathcal{D}_Q}/P$ and let L be the field of fractions of R. The arguments used to prove [SW1, Proposition 3.13] then show that $(\rho \otimes \overline{L})|_{D_w}$ is either unramified or of Type A. Since $\rho_0|_{D_w}$ is ramified, the first alternative cannot be. Thus $(\rho \otimes \overline{L})|_{D_w}$ is Type A. This means that $(\rho \otimes \overline{L})|_{I_w} \simeq \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$ with $*$ not identically zero. Let M be a free R-module of rank two on which $\mathrm{Gal}(\overline{F}/F)$ acts via ρ_P. Let $\sigma \in I_w$ map to a (topological) generator of the pro-p-part of tame inertia at w. Clearly $(\sigma - 1)^2$ annihilates M. Thus $\sigma - 1$ annihilates $N = M/M[\sigma - 1]$. The reduction $\overline{N} = N$ mod $\mathfrak{m}$ is then a (non-zero) quotient of ρ_0 annihilated by $(\sigma - 1)$ and thus is one-dimensional over k. It follows from this that $N \simeq R$ and $M[\sigma - 1] \simeq R$. Therefore M has a basis with respect to which $\rho_P|_{I_w} = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$, as was to be proven.

We have shown that ρ_P is a deformation of type-$\mathcal{D}_Q$. Let $r_P : R_{\mathcal{D}_Q} \to \mathbf{T}_{\mathcal{D}_Q}/P$ be the corresponding map coming from the universality of $R_{\mathcal{D}_Q}$. The ring $\mathbf{T}_{\mathcal{D}_Q}/P$ obtains a $\Lambda_{\mathcal{O}}$-algebra structure from that of $R_{\mathcal{D}_Q}$ via r_P. This agrees with the $\Lambda_{\mathcal{O}}$-structure inherited from $\mathbf{T}_{\mathcal{D}_Q}$, so r_P is a map of $\Lambda_{\mathcal{O}}$-algebras. We have thus a map of $\Lambda_{\mathcal{O}}$-algebras

$$r_{\mathcal{D}_Q} : R_{\mathcal{D}_Q} \to \prod \mathbf{T}_{\mathcal{D}_Q}/P, \quad r_{\mathcal{D}_Q} = \prod r_P \tag{3.3}$$

where the product is over the minimal primes of $\mathbf{T}_{\mathcal{D}_Q}$. Since $r_{\mathcal{D}_Q}(\mathrm{trace}\rho_{\mathcal{D}_Q}$ $(\mathrm{Frob}_\ell)) = \prod(T(\ell) \bmod P)$ for all $\ell \notin \Sigma \cup Q$, it follows from Lemmas 2.3 and 3.3 that the image of $r_{\mathcal{D}_Q}$ is just $\mathbf{T}_{\mathcal{D}_Q}$. We have thus obtained a surjection $r_{\mathcal{D}_Q} : R_{\mathcal{D}_Q} \twoheadrightarrow \mathbf{T}_{\mathcal{D}_Q}$ of $\Lambda_{\mathcal{O}}$-algebras. Let ρ^{mod} be the induced representation into $\mathrm{GL}_2(\mathbf{T}_{\mathcal{D}_Q})$. Since ρ^{mod} mod $P = \rho^{\mathrm{mod}}_{\mathcal{D}_Q}$ mod P for all minimal primes P it must be that $\rho^{\mathrm{mod}} = \rho^{\mathrm{mod}}_{\mathcal{D}_Q}$. This proves that $\rho_{\mathcal{D}_Q}$ is a deformation of type-$\mathcal{D}_Q$, completing the proof of the proposition. □

For each datum $\mathcal{D}_Q$ we also define $\mathbf{T}_{\mathcal{D}_Q}$-modules $M_{\mathcal{D}_Q}$ and $M^+_{\mathcal{D}_Q}$. These are defined exactly as in the paragraph following [SW1, Lemma 3.25] but with $\mathcal{M}_c$ replaced by $\mathcal{M}_0$ and $r(w) = 1$. These modules are just the localizations of $M_\infty(U_{\mathcal{D}_Q}^{\min})$ and $M_\infty^+(U_{\mathcal{D}_Q}^{\min})$ at a permissible maximal ideal, where $U_{\mathcal{D}_Q}^{\min}$ is a suitable subgroup of $U_{\mathcal{D}_0}$ containing $U_{\mathcal{D}_Q}$.

Suppose $w \in Q$. Recall that Δ_w is the Sylow-p-subgroup of $(\mathcal{O}_F/\ell_w)^\times$, where ℓ_w is the prime ideal of $\mathcal{O}_F$ corresponding to w. In [SW1,§3.5] we defined an action of Δ_w on $M_{\mathcal{D}_Q}$ and $M_{\mathcal{D}_Q}^+$ commuting with that of $\mathbf{T}_{\mathcal{D}_Q}$. For varying w these actions commute and give rise to an action of $\Delta_Q := \prod_{w\in Q}\Delta_w$ on $M_{\mathcal{D}_Q}$ and $M_{\mathcal{D}_Q}^+$. For each $w \in Q$ fix once and for all a generator δ_w of Δ_w and a $\sigma_w \in I_w$ mapping to δ_w via the local reciprocity map.

As in [SW1] we shall find it necessary to work with a quotient $\mathbf{T}_{\mathcal{D}_Q}^{\min}$ of $\mathbf{T}_{\mathcal{D}_Q}$. This is essentially the largest reduced quotient of $\mathbf{T}_{\mathcal{D}_Q}$ such that the induced map from $R_{\mathcal{D}_Q}$ factors through $R_{\mathcal{D}_Q}^{\min}$. We will not define $\mathbf{T}_{\mathcal{D}_Q}^{\min}$ here, referring the reader instead to [SW1, §3.6] and contenting ourselves with recalling the important properties of $\mathbf{T}_{\mathcal{D}_Q}^{\min}$. Write $\rho_{\mathcal{D}_Q^{\min}}^{\mathrm{mod}}$ for the deformation into $\mathrm{GL}_2(\mathbf{T}_{\mathcal{D}_Q}^{\min})$ induced from $\rho_{\mathcal{D}_Q}^{\mathrm{mod}}$. The deformation $\rho_{\mathcal{D}_Q^{\min}}^{\mathrm{mod}}$ is $\mathcal{D}_Q$-minimal. Write $r_{\mathcal{D}_Q}^{\min} : R_{\mathcal{D}_Q}^{\min} \to \mathbf{T}_{\mathcal{D}_Q}^{\min}$ for the corresponding map. It is surjective by Proposition 3.4. The following follows from the definition of $\mathbf{T}_{\mathcal{D}_Q}^{\min}$ and the arguments used to prove [SW1, Prop. 3.23, Lemmas 3.21, 3.25].

LEMMA 3.5. — *Let N_Σ be as in* (2.2).

(i) *There is an isomorphism $\mathbf{T}_{\mathcal{D}_Q} \simeq \mathbf{T}_{\mathcal{D}_Q}^{\min} \otimes_{\mathcal{O}} \mathcal{O}[N_\Sigma]$ compatible with the maps $r_{\mathcal{D}_Q}$ and $r_{\mathcal{D}_Q}^{\min}$ and with the isomorphism* (2.2).

(ii) *$\mathbf{T}_{\mathcal{D}_Q}^{\min}$ is a finite, reduced, torsion-free $\Lambda_{\mathcal{O}}$-algebra.*

(iii) *For each $w \in Q$,* $\mathrm{trace}\,\rho_{\mathcal{D}_Q^{\min}}^{\mathrm{mod}}(\sigma_w)$ *acts on $M_{\mathcal{D}_Q}$ and $M_{\mathcal{D}_Q}^+$ as $\delta_w + \delta_w^{-1}$.*

4. The key result

We keep the conventions of §§2 and 3. We say that a pair (F, ρ_0) is *good* if

(i) $d/2 > 2 + 2t + 7 \cdot \#\mathcal{M}_0$,

(ii) $d_i > 2 + 2t + 7 \cdot \#\mathcal{M}_0, \quad i = 1, ..., t,$ (4.1)

(iii) hypotheses (H_{even}) and (H_{def}) of §3 hold.

Let $\mathcal{D} = (\mathcal{O}, \Sigma, \mathcal{M})$ be a deformation datum. A prime $\mathfrak{q}$ of $R_{\mathcal{D}}$ is *promodular* if it is the inverse image under $r_{\mathcal{D}}$ of a prime of $\mathbf{T}_{\mathcal{D}}$. In other words, $\mathfrak{q}$

is pro-modular if there is a map $\theta_{\mathfrak{q}} : \mathbf{T}_{\mathcal{D}} \to R_{\mathcal{D}}/\mathfrak{q}$ such that the composition $\theta_{\mathfrak{q}} \circ r_{\mathcal{D}}$ is just reduction modulo $\mathfrak{q}$. A deformation $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(A)$ of type-$\mathcal{D}$ with A a domain is said to be pro-modular if the kernel of the corresponding map $R_{\mathcal{D}} \to A$ is a pro-modular prime.

A prime $\mathfrak{p}$ of $R_{\mathcal{D}}$ is *nice for* $\mathcal{D}$ if

(i) $\mathfrak{p}$ is a dimension one prime containing p,

(ii) $\rho_{\mathcal{D}}$ mod $\mathfrak{p}$ is not dihedral, and $\det \rho_{\mathcal{D}}$ mod $\mathfrak{p}$ has finite order, (4.2)

(iii) for each $v_i|p$, $(\rho_{\mathcal{D}} \bmod \mathfrak{p})|_{D_i} \simeq \begin{pmatrix} \psi_1^{(i)} & * \\ & \psi_2^{(i)} \end{pmatrix}$ with $\psi_1^{(i)}/\psi_2^{(i)}$ having infinite order,

(iv) $\mathfrak{p}$ is the inverse image of a prime of $\mathbf{T}_{\mathcal{D}_0}$ (so in particular $\mathfrak{p}$ is pro-modular).

Recall that $\mathcal{D}_0 = (\mathcal{O}, \Sigma_0, \mathcal{M}_0)$. A prime satisfying (i), (ii), (iii), and such that $\rho_{\mathcal{D}}$ mod $\mathfrak{p}$ is of type-$\mathcal{D}_0$ will be called merely *nice.*

Our preliminaries are completed by stating the following property of primes that are nice for $\mathcal{D}$.

If $\mathfrak{p}$ is a prime that is nice for $\mathcal{D}$, then
any prime $\mathfrak{q} \subseteq \mathfrak{p}$ is pro-modular. (P)

Sections 6, 7, and 8 of this paper are devoted to establishing property (P) (under hypotheses (H_{even}) and (H_{def})).

PROPOSITION 4.1. — *Suppose* (F, ρ_0) *is a good pair and that* $\mathcal{D} = (\mathcal{O}, \Sigma, \mathcal{M})$ *is a deformation datum. If* (P) *holds for every deformation datum* $(\mathcal{O}, \Sigma', \mathcal{M}')$ *with* $\mathcal{M} \subseteq \mathcal{M}'$ *and* $\Sigma' \subseteq \Sigma$, *then every prime of* $R_{\mathcal{D}}$ *is pro-modular.*

Proof. — The proof of this Proposition is essentially the same as that of [SW1, Proposition 4.1].

Let $\mathcal{C}_{\mathcal{D}}$ be the set of irreducible components of $\mathrm{spec}(R_{\mathcal{D}})$. An irreducible component is said to be pro-modular if the corresponding minimal prime is. Clearly, if an irreducible component is pro-modular then *all* primes on that component are pro-modular. Let $\mathcal{C}_{\mathcal{D}}^{\mathrm{mod}} \subseteq \mathcal{C}_{\mathcal{D}}$ be the subset consisting of pro-modular components. The assertion of the proposition is equivalent to $\mathcal{C}_{\mathcal{D}} = \mathcal{C}_{\mathcal{D}}^{\mathrm{mod}}$.

We begin by proving the proposition for the case $\mathcal{D} = \mathcal{D}_0$. The proof consists of three steps. In the first we show that $R_{\mathcal{D}_0}$ has a prime that is nice for $\mathcal{D}_0$. In the second we show that any component of spec$(R_{\mathcal{D}_0})$ containing a prime that is nice for $\mathcal{D}_0$ is itself pro-modular. As a consequence of this and step one we have that $\mathcal{C}_{\mathcal{D}_0}^{\mathrm{mod}} \neq \emptyset$. In the third step we combine step two with our analysis of the structure of the ring $R_{\mathcal{D}_0}$ to conclude that $\mathcal{C}_{\mathcal{D}_0}^{\mathrm{mod}} = \mathcal{C}_{\mathcal{D}_0}$.

By hypothesis (F, ρ_0) is a good pair. In particular hypothesis (H_{def}) holds, and the ring $\mathbf{T}_{\mathcal{D}_0}$ exists. As the deformation $\rho_{\mathcal{D}_0}^{\mathrm{mod}}$ is of type-$\mathcal{D}_0$ by Proposition 3.4 and since the corresponding map $r_{\mathcal{D}_0} : R_{\mathcal{D}_0} \to \mathbf{T}_{\mathcal{D}_0}$ is surjective, any prime $\mathfrak{p}$ of $\mathbf{T}_{\mathcal{D}_0}$ satisfying

(i) $\mathbf{T}_{\mathcal{D}_0}/\mathfrak{p}$ is one-dimensional and contains p, (4.3)

(ii) $\rho_{\mathcal{D}_0}^{\mathrm{mod}}$ mod $\mathfrak{p}$ satisfies (4.2ii,iii).

will be nice for $\mathcal{D}_0$. We now prove the existence of such a $\mathfrak{p}$.

Let Q be a minimal prime of $R = \mathbf{T}_{\mathcal{D}_0}/(p, T_1, ..., T_{\delta_F})$. Since $\mathbf{T}_{\mathcal{D}_0}$ is a finite, torsion free $\Lambda_{\mathcal{O}}$-algebra by Lemma 3.1, it follows that $R_1 = R/Q$ is an integral extension of $k[\![Y_1^{(1)}, ..., Y_{d_t}^{(t)}]\!]$. Thus the dimension of R_1 is d. We claim that $\rho_1 = \rho_{\mathcal{D}_0}^{\mathrm{mod}}$ mod Q is not dihedral. This is clear if ρ_0 is not dihedral. Suppose then that $\rho_0 = \mathrm{Ind}_F^{F'} \psi_0$ (notation as in §2). Note that $\det \rho_1$ has finite order since Q contains $T_1, ..., T_{\delta_F}$. Since R_1 is a quotient of $R_{\mathcal{D}_0}$ (see Proposition 3.4), if ρ_1 were dihedral, then the dimension of R_1 would be at most $d/2$ by Lemma 2.2, contradicting what we just proved about the dimension of R_1. It then follows easily that there are infinitely many dimension one primes $\mathfrak{p}$ of R_1 such that ρ_1 mod $\mathfrak{p}$ is not dihedral and $\mathfrak{p}$ does not contain $Y_1^{(1)}, ..., Y_1^{(t-1)}$, or $Y_1^{(t)}$. It now follows easily from (3.2vi) that $\mathfrak{p}$ and $\rho_{\mathcal{D}_0}^{\mathrm{mod}}$ mod $\mathfrak{p} = \rho_1$ mod $\mathfrak{p}$ satisfy (4.3). This completes step one.

Suppose that $\mathfrak{p}$ is nice for $\mathcal{D}_0$. It follows from (P) that if $Q \subseteq \mathfrak{p}$ is any prime of $R_{\mathcal{D}_0}$ then Q is pro-modular. In particular, any minimal prime of $R_{\mathcal{D}_0}$ contained in $\mathfrak{p}$ is pro-modular. This completes step two. Combining this with step one, which asserts the existence of a prime nice for $\mathcal{D}_0$, yields $\mathcal{C}_{\mathcal{D}_0}^{\mathrm{mod}} \neq \emptyset$.

The last step is to prove that $\mathcal{C}_{\mathcal{D}_0} = \mathcal{C}_{\mathcal{D}_0}^{\mathrm{mod}}$. Put $\mathcal{C}'_{\mathcal{D}_0} = \mathcal{C}_{\mathcal{D}_0} \backslash \mathcal{C}_{\mathcal{D}_0}^{\mathrm{mod}}$. If $\mathcal{C}'_{\mathcal{D}_0} = \emptyset$, then there is nothing to prove, so assume otherwise. It follows from Proposition 2.1 and [SW1, Corollary A.2] that there are components $C_1 \in \mathcal{C}_{\mathcal{D}_0}^{\mathrm{mod}}$ and $C_2 \in \mathcal{C}'_{\mathcal{D}_0}$ such that $C_1 \cap C_2$ contains a prime Q of dimension $d - 2t + \delta_F - 3 \cdot \#\mathcal{M}_0$. Let I_1 be the ideal generated by the set $\{p; \det \rho_{\mathcal{D}_0}(\gamma_i) - 1 \mid i = 1, \ldots, \delta_F\}$. Let Q_1 be a minimal prime of $R_{\mathcal{D}_0}/(Q, I_1)$. The dimension of Q_1 is at least $d - 2t - 3 \cdot \#\mathcal{M}_c - 1 > 1 + d/2$, the inequality by (4.1). It follows from Lemma 2.2 that $\rho_{\mathcal{D}_c} \mathrm{mod}\, Q_1$ is not dihedral.

Since $Q_1 \in C_1$, Q_1 is pro-modular. The prime Q_1 determines a prime Q_1^{mod} of $\mathbf{T}_{\mathcal{D}_0}$. The prime Q_1^{mod} is the kernel of $\theta_{Q_1} : \mathbf{T}_{\mathcal{D}_0} \longrightarrow R_{\mathcal{D}_0}/Q_1$. Note that dim $\mathbf{T}_{\mathcal{D}_0}/Q_1^{\mathrm{mod}} =$ dim $R_{\mathcal{D}_0}/Q_1$. Recall that $\mathbf{T}_{\mathcal{D}_0}$ is an integral extension of $\Lambda_{\mathcal{O}} = \mathcal{O}[\![Y_1^{(1)}, \ldots, Y_{d_t}^{(t)}, T_1, \ldots, T_{\delta_F}]\!]$ (cf. Lemma 3.1). By construction $Q_1^{\mathrm{mod}} \cap \Lambda_{\mathcal{O}}$ contains $p, T_1, \ldots, T_{\delta_F}$. If $Q_1^{\mathrm{mod}} \cap \Lambda_{\mathcal{O}}$ also contained $Y_1^{(i)}, \ldots, Y_{d_i}^{(i)}$ it would follow that the dimension of Q_1^{mod} would be at most $d - d_i$. Comparing this with the lower bound for the dimension of Q_1 obtained earlier and recalling that the dimension of Q_1 equals that of Q_1^{mod}, one finds that $d_i \leqslant 2t + 3 \cdot \#\mathcal{M}_0 + 1$ which contradicts (4.1). Thus, after possibly reordering the $Y_j^{(i)}$'s we may assume that $Y_1^{(i)} \notin Q_1$ for each $i = 1, \ldots, t$.

Let $\mathfrak{p} \supseteq Q_1$ be a prime of dimension one not containing $Y_1^{(1)}, \ldots, Y_1^{(t)}$ and such that $\rho_{\mathcal{D}_0}$ mod $\mathfrak{p}$ is not dihedral. Such a $\mathfrak{p}$ always exists. As $\mathfrak{p} \in C_1$ it is pro-modular. We claim that it is also nice. By construction $\mathfrak{p}$ contains p, and it is, of course, a prime of $R_{\mathcal{D}_0}$. As $\mathfrak{p}$ contains $T_1, \ldots, T_{\delta_F}$ as well, det $\rho_{\mathcal{D}_0}$ mod $\mathfrak{p}$ has finite order. So it remains to check the conditions at each D_i. Let $A = R_{\mathcal{D}_0}/\mathfrak{p}$ and let $\rho : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_2(A)$ be the deformation $\rho_{\mathcal{D}_0} \mathrm{mod}\, \mathfrak{p}$. Consider $\rho|_{D_i} \simeq \begin{pmatrix} \psi_1^{(i)} & * \\ & \psi_2^{(i)} \end{pmatrix}$. By definition $\psi_2^{(i)}(y_1^{(i)})$ equals $1 + Y_1^{(i)}$, which has infinite order in A. Thus $\psi_2^{(i)}$ is a character of infinite order. Since det ρ has finite order, it follows that $\psi_1^{(i)}/\psi_2^{(i)}$ has infinite order. Therefore $\mathfrak{p}$ is a prime of $R_{\mathcal{D}_0}$ that is nice for $\mathcal{D}_0$. As $\mathfrak{p} \in C_2$ it follows from step two that $C_2 \in \mathcal{C}_{\mathcal{D}_0}^{\mathrm{mod}}$ contradicting the assumption that $C_2 \in \mathcal{C}'_{\mathcal{D}_0}$. This proves that $\mathcal{C}_{\mathcal{D}_0} = \mathcal{C}_{\mathcal{D}_0}^{\mathrm{mod}}$.

We now prove the proposition in its full generality. We first show that any component of spec$(R_{\mathcal{D}})$ containing a nice prime is pro-modular. For this we use the proposition in the case $\mathcal{D} = \mathcal{D}_0$. We then combine this with our previous analysis of $R_{\mathcal{D}}$ to conclude that $\mathcal{C}_{\mathcal{D}} = \mathcal{C}_{\mathcal{D}}^{\mathrm{mod}}$.

Suppose that $\mathfrak{p}$ is a nice prime of $R_{\mathcal{D}}$. It follows that $\mathfrak{p}$ is the inverse image of a prime $\mathfrak{p}_1$ of $R_{\mathcal{D}_0}$ under the canonical map $R_{\mathcal{D}} \twoheadrightarrow R_{\mathcal{D}_0}$. By the proposition in the case $\mathcal{D} = \mathcal{D}_0$, $\mathfrak{p}_1$ is a pro-modular prime. Thus there is a map $\theta_{\mathfrak{p}_1} : \mathbf{T}_{\mathcal{D}_0} \longrightarrow R_{\mathcal{D}_0}/\mathfrak{p}_1 = R_{\mathcal{D}}/\mathfrak{p}$ inducing the deformation $\rho_{\mathcal{D}_0}$ mod $\mathfrak{p}_1 = \rho_{\mathcal{D}}$ mod $\mathfrak{p}$. Composing $\theta_{\mathfrak{p}_1}$ with the canonical map $\mathbf{T}_{\mathcal{D}} \twoheadrightarrow \mathbf{T}_{\mathcal{D}_0}$ yields a map $\theta_{\mathfrak{p}} : \mathbf{T}_{\mathcal{D}} \longrightarrow R_{\mathcal{D}}/\mathfrak{p}$ inducing the deformation $\rho_{\mathcal{D}}$ mod $\mathfrak{p}$. It follows that $\mathfrak{p}$ is nice for $\mathcal{D}$, whence by (P) any prime $Q \subseteq \mathfrak{p} \subseteq R_{\mathcal{D}}$ is pro-modular. Therefore any component of spec$(R_{\mathcal{D}})$ containing $\mathfrak{p}$ is also pro-modular.

In our final step we complete the proof of the proposition in its full generality. Let Q be a minimal prime of $R_{\mathcal{D}}$. For each place $v \in \Sigma \backslash \mathcal{P}$ fix a generator $\tau_v \in I_v$ of the pro-p-part of tame inertia at v and choose a basis for $\rho_{\mathcal{D}}$ such that $\rho_{\mathcal{D}}(\tau_v) \bmod \mathfrak{m}_{\mathcal{D}} = \begin{pmatrix} 1 & u_v \\ & 1 \end{pmatrix}$ ($\simeq \rho_0(\tau_v)$). With respect to this basis write $\rho_{\mathcal{D}}(\tau_v) = \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix}$. Let I_2 be the ideal generated by the set

$$\{p; a_v - 1,\, b_v - u_v,\, c_v,\, d_v - 1; \det \rho_{\mathcal{D}}(\gamma_j) - 1 \mid v \in \Sigma \backslash \mathcal{P},\ \ j = 1, \ldots, \delta_F\}.$$

Let Q_2 be a minimal prime of $R_{\mathcal{D}}/(Q, I_2)$. By Proposition 2.1 the dimension of Q_2 is at least $d - 7 \cdot \#\Sigma - 1$. It follows from this and from (4.1) that the dimension of Q_2 is at least $d/2 + 1$, whence it follows from Lemma 2.2 that $\rho_{\mathcal{D}} \bmod Q_2$ is not dihedral. Moreover, it is clear from the fact that $Q_2 \supseteq I_2$ that $\rho_{\mathcal{D}} \bmod Q_2$ is a deformation of type-$\mathcal{D}_0$. It follows from the proposition in the case $\mathcal{D} = \mathcal{D}_0$ that Q_2 is pro-modular. Arguing as in step two of the proof in the case $\mathcal{D} = \mathcal{D}_0$ shows that Q_2 is contained in a nice prime. As $Q \subseteq Q_2$, the same is true of Q. The conclusion of the preceding paragraph now implies that Q is pro-modular. Therefore, every minimal prime of $R_{\mathcal{D}}$ is pro-modular. This completes the proof of the proposition. $\square$

5. An application of the key result

In this section we drop all of our preceding conventions except those listed in the first paragraph of §2. Suppose now that

$$\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$$

is a representation such that

(i) ρ is continuous and irreducible,
(ii) ρ is unramified at all finite places outside of some finite set Σ,
(iii) $\det \rho(\tau) = -1$ for all complex conjugations τ, (5.1)
(iv) $\det \rho = \psi \varepsilon^{\mu}$ for some integer $\mu \geqslant 1$ and some character ψ of finite order,
(v) for each $i = 1, \ldots, t$, $\rho|_{D_i} \simeq \begin{pmatrix} \psi_1^{(i)} & * \\ & \psi_2^{(i)} \end{pmatrix}$ with $\psi_2^{(i)}|_{I_i}$ having finite order.

It is widely believed that such a ρ is a representation associated to some (nearly ordinary) automorphic representation of $\mathrm{GL}_{2/F}$. This is a natural generalization to a totally real field of a special case of [FM, Conjecture 3c].

As ρ satisfies (4.1i,ii) it is not difficult to see that there exists a basis for ρ with respect to which ρ takes values in $\mathrm{GL}_2(\mathcal{O})$ with $\mathcal{O}$ the ring of integers of a finite extension of $\mathbf{Q}_p$. Let λ be a uniformizer of $\mathcal{O}$ and let $k = \mathcal{O}/\lambda$. Let $\overline{\rho}^{ss} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(k)$ be the semisimplification of the reduction $\rho \bmod \lambda$. The representation $\overline{\rho}^{ss}$ is unique up to equivalence and extension of the scalar field k. We say that ρ is *D_i-distinguished* if $\overline{\rho}^{ss}|_{D_i} \simeq \begin{pmatrix} \chi_1^{(i)} & * \\ & \chi_2^{(i)} \end{pmatrix}$ with $\chi_1^{(i)} \neq \chi_2^{(i)}$. In this case we fix $\chi_1^{(i)}$ and $\chi_2^{(i)}$ so that $\chi_2^{(i)} = \psi_2^{(i)} \bmod \lambda$.

Suppose that $\overline{\rho}^{ss}$ is D_i-distinguished for each $i = 1, ..., t$. If $\rho' : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ is another representation satisfying (5.1) then we say that ρ' is a *χ_2-good lift* of $\overline{\rho}^{ss}$, where $\chi_2 = (\chi_2^{(1)}, ..., \chi_2^{(t)})$, if $(\overline{\rho}')^{ss} \simeq \overline{\rho}^{ss}$ and if for each $i = 1, ..., t$, $\rho'|_{D_i} \simeq \begin{pmatrix} \phi_1^{(i)} & * \\ & \phi_2^{(i)} \end{pmatrix}$ and the reduction of $\phi_2^{(i)}$ is $\chi_2^{(i)}$.

Recall that for a holomorphic automorphic representation π of $\mathrm{GL}_{2/F}$ we write ρ_π for the representation $\rho_\pi : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ associated to π and the fixed embedding $\overline{F} \hookrightarrow \overline{\mathbf{Q}}_p$.

THEOREM 5.1.— *Suppose $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ is a representation satisfying* (5.1). *If*

(i) *$\overline{\rho}^{ss}$ is irreducible and D_i-distinguished for all $i = 1, ..., t$,*

(ii) *there exists a nearly-ordinary automorphic representation π_0 of $\mathrm{GL}_{2/F}$ such that ρ_{π_0} is a χ_2-good lift of $\overline{\rho}^{ss}$,*

then $\rho \simeq \rho_\pi$ for some automorphic representation π.

Remark.— Generalizations of the main results of [W] and [D1] to totally real fields have also been obtained by Fujiwara [F]. (A restricted version of his results has been available for a few years.) Like those in [W] and [D1], his results exclude some cases where $\overline{\rho}^{ss}$ is dihedral. His proofs closely follow those in [W] and [D1] but with the crucial difference that he avoids having to establish "multiplicity one." This he achieves by patching both Hecke rings and modules, a technique which was also developed, but independently, by Diamond in [D3].

Proof.— Fix $\mathcal{O}$, λ, and k as in the discussion preceding the statement of the theorem, but so that if $\overline{\rho}^{ss}$ is dihedral then $\overline{\rho}^{ss} = \mathrm{Ind}_F^{F'} \psi_0$ (notation as in §2). We then have $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathcal{O})$. The hypotheses of the theorem, together with the main result of [SW2], implies that there exists some totally real extension L of F such that the Galois closure of L over F is solvable and

- $\rho_0 = \overline{\rho}^{ss}|_{\mathrm{Gal}(\overline{L}/L)}$ satisfies (2.1)
- (H_{even}) and (H_{def}) hold for L and ρ_0.

Let Σ_L be the set of places of L over those in Σ. By further requiring L to contain a large real subfield of a suitable cyclotomic extension we can also ensure that

- $d_L/2 > 2 + 7 \cdot \#\Sigma_L$,
- $d_v > 2 + 7 \cdot \#\Sigma_L$ for all $v|p$.

Here d_L is the degree of L over $\mathbf{Q}$, and d_v is the degree of L_v over $\mathbf{Q}_p$ for $v|p$. It follows that the pair (L, ρ_0) is good in the sense of §4.

The representation $\rho_1 = \rho|_{\mathrm{Gal}(\overline{L}/L)}$ is a deformation of ρ_0 of type-$\mathcal{D}_L$, with $\mathcal{D}_L = (\mathcal{O}, \Sigma_L, \emptyset)$. As (L, ρ_0) is a good pair, and since property (P) holds for primes good for all data relative to L and ρ_0 by Proposition 8.2, it follows from Proposition 4.1 that ρ_1 is pro-modular. The corresponding homomorphism $\mathbf{T}_{\mathcal{D}_L} \to \mathcal{O}$ is clearly algebraic. Thus it follows from Proposition 3.2 that $\rho_1 \simeq \rho_{\pi_1}$ for some nearly ordinary automorphic representation of $\mathrm{GL}_{2/L}$. As the Galois closure of L over F is solvable, the conclusion of the theorem now follows from known cases of base change for GL_2. □

If $F = \mathbf{Q}$ then one can weaken hypothesis (ii) of the above theorem.

THEOREM 5.2. — *Suppose $F = \mathbf{Q}$ and suppose $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ is a representation satisfying* (5.1). *If*

(i) *$\overline{\rho}^{ss}$ is irreducible and D_p-distinguished,*

(ii) *there exists an automorphic representation π_0 of $\mathrm{GL}_{2/\mathbf{Q}}$ such that $\overline{\rho}^{ss}_{\pi_0} \simeq \overline{\rho}^{ss}$,*

then $\rho \simeq \rho_\pi$ for some automorphic representation π.

Proof. — The key is to prove that hypothesis (ii) implies the existence of a nearly ordinary automorphic representation π_1 such that ρ_{π_1} is a χ_2-good lift of $\overline{\rho}^{ss}$, for then the theorem follows from Theorem 5.1. If $p \neq 3$ or if $\overline{\rho}^{ss}|_{\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}(\sqrt{-3}))}$ is irreducible, then this follows from [D2, Theorem 6.4], for instance. The existence of π_1 in the remaining cases is obvious. □

6. Complete intersections and free modules

In the remaining sections we give the proof of property (P) (see Proposition 8.2). These sections do not make use of any results from §§4 or 5.

In this section we describe a criterion for a ring to be a complete intersection and for a module over that ring to be free. This criterion is a special case of that in [SW1,§5], whose proof is a variant on the patching argument in [TW] and its refinement in [D3] and [F]. In section §8 we verify this criterion for certain localizations of deformation rings and associated modules.

Let k be a finite field of characteristic p, and let $A = k[\![T]\!]$. Let K be the field of fractions of A. Let $\mathcal{L} = \{N\}$ be a sequence of strictly increasing odd integers together with zero. Let n be a fixed positive integer.

We introduce rings A_N, B_N (for each $N \in \mathcal{L}$) given by

$$\begin{aligned} A_N &= A[\![s_1,\dots,s_n]\!] \,/\, (s_1^{N+1},\dots,s_n^{N+1}), \quad A_0 = A \\ B_N &= A[\![t_1,\dots,t_n]\!] \,/\, (t_1^{(N+1)/2},\dots,t_n^{(N+1)/2}), \quad B_0 = A. \end{aligned}$$

There is a homomorphism $B_N \longrightarrow A_N$ given by $t_i \longmapsto (1+s_i) + (1+s_i)^{-1} - 2$ which we use to identify B_N as a subring of A_N. We assume that we are given a ring $R^{(N)}$ for each $N \in \mathcal{L}$ of the form

$$R^{(N)} = A[\![x_1,\dots,x_m]\!]/\mathfrak{a}^{(N)} \tag{6.1}$$

with m independent of N. Furthermore we assume that $R^{(N)}$ has the following properties:

(i) $R^{(N)}$ is finite and free as an A-module.

(ii) $\mathfrak{a}^{(N)} \subseteq (x_1,\dots,x_m)$.

(iii) $\exists$ a surjective map $R^{(N)} \twoheadrightarrow R^{(0)}$ of A-algebras. (6.2)

(iv) $R^{(N)}$ is a B_N-algebra for $N > 0$.

Now letting $\mathfrak{p}^{(N)}$ be the prime of $R^{(N)}$ corresponding to $(x_1,\dots,x_m)$ (which we usually abbreviate to $\mathfrak{p}$ if the N is clear from the context) we assume two further (and less formal) properties of $R^{(N)}$:

(i) $\exists\ d(0) > 0$ such that $\mathfrak{p}^{d(0)} = 0$ in $R^{(0)}$,

(ii) $\mathfrak{p}^{(N)}/(\mathfrak{p}^{(N)})^2 \cong A^n \oplus \mathrm{Tor}_{(N)}$, (6.3)

where the free summand A^n is spanned by $x_1,\dots,x_n$ and $\mathrm{Tor}_{(N)}$ is a finite group whose order is bounded independent of N.

For each $0 \leqslant a \leqslant N$ (a odd or $a = 0$) we assume given a ring $R_a^{(N)}$ which has the following properties:

(i) $R_a^{(N)}$ is finite and free as an A-module.

(ii) $R_N^{(N)} = R^{(N)}$, $R_0^{(N)} = R^{(0)}$.

(iii) There are surjective maps of B_N-algebras

$$R_0^{(N)} \twoheadleftarrow R_1^{(N)} \twoheadleftarrow R_3^{(N)} \twoheadleftarrow R_5^{(N)} \ldots \ . \tag{6.4}$$

(iv) $R_a^{(N)}$ is a B_a-algebra (compatible with $B_N \twoheadrightarrow B_a$) such that if $a > 1$, then

$$(R_a^{(N)} \otimes_A K)/(t_1^{\frac{a-1}{2}}, \ldots, t_n^{\frac{a-1}{2}}) \simeq R_{a-1}^{(N)} \otimes_A K.$$

(v) $R_a^{(N)} \otimes_A K$ is an $A_a \otimes_A K$–algebra satisfying (via the map in (iii))

$$R_a^{(N)} \otimes_A K/(s_1, \ldots, s_n) \twoheadrightarrow R_0^{(N)} \otimes_A K.$$

Letting $\mathfrak{p}_a^{(N)}$ denote the prime corresponding to $(x_1, \ldots, x_m)$ (which we again write as $\mathfrak{p}$ if a and N are clear from the context) we assume two further properties:

(i) $\exists\ d(a) > 0$ independent of N such that $\mathfrak{p}^{d(a)} = 0$ in $R_a^{(N)}$,

(ii) $\mathfrak{p}_a^{(N)}/(\mathfrak{p}_a^{(N)})^2 \cong A^n \oplus \mathrm{Tor}_{(N,a)}$, (6.5)

where the free summand A^n is spanned by $x_1, \ldots, x_n$ and $\mathrm{Tor}_{(N,a)}$ is a finite group whose order is bounded independent of N and a.

Associated to the rings we have described we assume given a set of modules as follows. First we assume given an integer r, independent of N. Then we assume we are given $M^{(N)}$, a finite $R^{(N)}$-module, satisfying the hypotheses that

(i) $M^{(N)}$ is a free A-module of rank equal to the rank of A_N^r,

(ii) $M^{(N)}$ is an A_N-module compatible with the B_N-structure via $R^{(N)}$,

(iii) There is a map $M^{(N)} \longrightarrow M^{(0)}$ of $R^{(N)}$-modules. (6.6)

For $0 \leqslant a \leqslant N$ (a odd or $a = 0$) we assume that we are given an $R^{(N)}$-module quotient of $M^{(N)}$ denoted $M_a^{(N)}$ and satisfying

(i) $M_a^{(N)}$ is a free A-module of rank equal to the rank of A_a^r.

(ii) $M_N^{(N)} = M^{(N)}, M_0^{(N)} = M^{(0)}$.

(iii) There are surjective maps of $R^{(N)}$-modules $M_0^{(N)} \twoheadleftarrow M_1^{(N)} \twoheadleftarrow M_3^{(N)} \ldots$.

(iv) $M_a^{(N)}$ is an $R_a^{(N)}$-module (compatible with the $R^{(N)}$-structure).(6.7)

(v) $M_a^{(N)}$ is an A_a-module (compatible with the B_a-structure induced in (iv)) in such a way that the maps in (iii) are compatible with $A_0 \twoheadleftarrow A_1 \twoheadleftarrow A_3 \dots$. The actions of $R_a^{(N)}$ and A_a commute on $M_a^{(N)}$.

(vi) $M_a^{(N)} \otimes_A K$ is a free $A_a \otimes_A K$-module and $M_a^{(N)} \otimes_A K/(s_1, \dots, s_n) \simeq M_0^{(0)} \otimes_A K$.

Furthermore, we assume there exists $x^{(N)} \in R^{(N)}$ such that

(i) $x^{(N)}$ annihilates $\ker\{M_a^{(N)}/(s_1, \dots, s_n) \twoheadrightarrow M_0^{(N)}\}$,

(ii) $\mathrm{ord}_T(x^{(N)} \bmod \mathfrak{q}^{(N)}) = t < \infty$ with t independent of N. (6.8)

PROPOSITION 6.1.— *$R^{(0)} \otimes_A K$ is a complete intersection as a K-algebra and $M^{(0)} \otimes_A K$ is a free $R^{(0)} \otimes_A K$-module.*

This is just the main results of [SW1, Propositions 5.8 and 5.9] in the case that $R_a^{tr(N)} = R_a^{(N)}$ for all N and a. The proof proceeds by "patching" various quotients of the $R_a^{(N)}$'s and the $M_a^{(N)}$'s, thereby constructing a power series ring R_∞ and a free R_∞-module M_∞ such that $R^{(0)} \otimes_A K \simeq R_\infty/I$ and $M^{(0)} \otimes_A K \simeq M_\infty/I$, where I is an ideal generated by n elements. A complete proof can be found in [SW1,§5].

7. Selmer groups

In this section we consider a representation

$$\rho : G_\Sigma = \mathrm{Gal}(F_\Sigma/F) \longrightarrow \mathrm{GL}_2(A)$$

where $A \simeq k[\![\lambda]\!]$. Here we are using the notation and assumptions of §2. We let K be the field of fractions of A. Throughout this section we make the following assumptions on ρ:

- $\rho \otimes \overline{K}$ is irreducible and not dihedral.

- $\rho \bmod \lambda = \rho_0$

- Σ contains the primes dividing p together with all primes at which ρ_0 is ramified.

- If $w \nmid p$, then ρ is ramified at w if and only if ρ_0 is. For all $w \nmid p$,

$$\rho|_{I_w} \simeq \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}.$$

- $\det\rho = \det\rho_0$.
- $\rho|_{D_v} \simeq \begin{pmatrix} \chi_1^{(v)} & * \\ & \chi_2^{(v)} \end{pmatrix}$ with $\chi_1^{(v)}/\chi_2^{(v)}$ of infinite order for each $v|p$.

Let F' be the splitting field of $\det\rho$ adjoin all p-power roots of unity, and let F^+ be the subfield of F' fixed by the complex conjugation z_1.

LEMMA 7.1

(i) There exists $\sigma \in \mathrm{Gal}(F_\Sigma/F')$ such that the eigenvalues of $\rho(\sigma)$ have infinite order and are in A.

(ii) There exists $\sigma \in \mathrm{Gal}(F_\Sigma/F^+)\backslash\mathrm{Gal}(F_\Sigma/F')$ such that the eigenvalues of $\rho(\sigma)$ have infinite order and are in A.

This can be proved exactly as [SW1, Lemma 6.3]. The only difference is that while in the former situation we had to first prove that ρ is not dihedral, in the present situation we have ruled that out from the start.

Let $\mathcal{U}$ be the representation space for ρ. This is a free A-module of rank 2 having for each $v_i|p$ a filtration $0 \subset \mathcal{U}_{1,i} \subset \mathcal{U}$ such that $\mathcal{U}_{1,i}$ is a free A-module on which D_i acts via a character reducing to $\chi_1^{(i)}$ modulo λ. The quotient $\mathcal{U}_{2,i} = \mathcal{U}/\mathcal{U}_{1,i}$ is a free A-module on which D_i acts via a character reducing to $\chi_2^{(i)}$ modulo λ. If ρ_0 is ramified at $w \nmid p$, then there is a filtration $0 \subset \mathcal{U}_1^w \subset \mathcal{U}$ such that both $\mathcal{U}_1^w$ and the quotient $\mathcal{U}_2^w = \mathcal{U}/\mathcal{U}_1^w$ are free A-modules on which I_w acts trivially. Let $\mathcal{T} = \{f \in \mathrm{ad}\rho : \mathrm{trace}(f) = 0\}$. Let $\mathcal{T}_{v_i}^{\mathrm{ord}} = \{f \in \mathcal{T} : f(\mathcal{U}) \subseteq \mathcal{U}_{1,i}\}$. Similarly, if ρ_0 is ramified at $w \nmid p$, then let $\mathcal{T}^w = \{f \in \mathcal{T} : f(\mathcal{U}) \subseteq \mathcal{U}_1^w\}$. We write $\mathcal{T}_n, \mathcal{T}_{n,v_i}^{\mathrm{ord}}$, and $\mathcal{T}_n^w$ for $\mathcal{T}/\lambda^n, \mathcal{T}_{v_i}^{\mathrm{ord}}/\lambda^n$, and $\mathcal{T}^w/\lambda^n$, respectively. Let

$$H_{v_i}(\mathcal{T}_n) = H^1(I_i, \mathcal{T}_n/\mathcal{T}_{n,i}^{\mathrm{ord}}),$$

and let

$$H_w(\mathcal{T}_n) = \begin{cases} H^1(I_w, \mathcal{T}_n/\mathcal{T}_n^w) & \text{if } \rho_0 \text{ is ramified at } w, \\ 0 & \text{otherwise.} \end{cases}$$

For each $w \in \Sigma$, put

$$L_w(\mathcal{T}_n) = \ker\{H^1(D_w, \mathcal{T}_n) \longrightarrow H_w(\mathcal{T}_n)\}.$$

We define a Selmer group for $\mathcal{T}_n$ by

$$H_\Sigma(\mathcal{T}_n) = \{\alpha \in H^1(F_\Sigma/F, \mathcal{T}_n) : \mathrm{res}_w\alpha \in L_w(\mathcal{T}_n) \text{ foreach } w \in \Sigma\}.$$

PROPOSITION 7.2. — *Let $\sigma \in \mathrm{Gal}(F_\Sigma/F')$ be an element such that the eigenvalues of $\rho(\sigma)$ are in A and have infinite order, as in Lemma 7.1(i). Then there exists an integer $r = r(\rho)$ such that for each $m > 0$ there are infinitely many sets $Q = \{w_1, \ldots, w_r\}$ such that*

(i) $Nm(w_i) \equiv 1 \bmod p^m$ *for each* i.

(ii) $\rho_{\mathfrak{p}}(\mathrm{Frob}\, w_i) \equiv \rho_{\mathfrak{p}}(\sigma) \bmod lambda^m$ *for each* i.

(iii) $\varinjlim_n H_{\Sigma_Q}(\mathcal{T}_n) \simeq (K/A)^r \oplus X_{\Sigma_Q}$

with $\Sigma_Q = \Sigma_0 \cup Q$, $\#X_{\Sigma_Q} < C(\sigma, r) < \infty$ for some constant $C(\sigma, r)$ depending only on σ and r.

This proposition is proved exactly as [SW1, Proposition 6.10], to which we refer the dedicated reader. The proof is slightly easier in the present case.

8. Property (P)

We retain the notation and assumptions of §§2 and 3. We further assume that hypotheses (H_{even}) and (H_{def}) hold. In this section we complete the proof that property (P) holds for all primes that are nice for some deformation datum, at least under the present assumptions.

Suppose that $\mathcal{D} = (\mathcal{O}, \Sigma, \mathcal{M})$ is a deformation datum and that $\mathfrak{p} \subseteq R_{\mathcal{D}}$ is a prime that is nice for $\mathcal{D}$. By definition $\mathfrak{p}$ is the inverse image under $r_{\mathcal{D}}$ of a prime of $\mathbf{T}_{\mathcal{D}}$, which we also denote by $\mathfrak{p}$. Since $\mathfrak{p}$ contains p, it is not difficult to see from Lemma 3.5(i) that $\mathfrak{p}$ even comes from a prime of $\mathbf{T}_{\mathcal{D}}^{\min}$. We denote this prime by $\mathfrak{p}$ as well.

Since $\mathfrak{p}$ is nice for $\mathcal{D}$, the integral closure A of $\mathbf{T}_{\mathcal{D}}^{\min}/\mathfrak{p}$ is isomorphic to $k'[\![\lambda]\!]$ for some finite extension k' of k. Furthermore, under the composite map

$$\Lambda_{\mathcal{O}} \to \mathbf{T}_{\mathcal{D}}^{\min}/\mathfrak{p} \hookrightarrow A,$$

A is finite over $\Lambda_{\mathcal{O}}$. (This is because of the assumptions on $(\rho_{\mathcal{D}} \bmod \mathfrak{p})|_{D_i}$.) Writing $\Lambda_{\mathcal{O}} = \mathcal{O}[\![z_1, \ldots, z_m]\!]$ let us suppose that $z_i \longmapsto \lambda^{r_i} u_i \in A$ with u_i a unit or zero for each i. Then we may take $r_i > 0$ for each i and we may assume, after possibly renumbering, that u_1 is a unit. Set

$$\widetilde{\Lambda}_{\mathcal{O}} = \mathcal{O}'[\![W_1, \ldots, W_m]\!]$$

where $\mathcal{O}'$ is the ring of integers of any local field whose residue field is k' and which contains $\mathcal{O}$. There is a map $\widetilde{\Lambda}_{\mathcal{O}} \longrightarrow A$ defined by $W_1 \longmapsto \lambda$ and

$W_i \longmapsto 0$ for $2 \leqslant i \leqslant m$. Define a homomorphism $\Lambda_{\mathcal{O}} \longrightarrow \widetilde{\Lambda}_{\mathcal{O}}$ by

$$z_1 \longmapsto W_1^{r_1}\tilde{u}_1, \quad z_i \longmapsto -W_i + W_1^{r_i}\tilde{u}_i \text{ for } 2 \leqslant i \leqslant m.$$

Here $\tilde{u}_i$ denotes any fixed choice of lift of u_i to $\widetilde{\Lambda}_{\mathcal{O}}$. Then $\widetilde{\Lambda}_{\mathcal{O}}$ is finite and free over $\Lambda_{\mathcal{O}}$ and we have a commutative diagram of rings

$$\begin{tikzcd} & \widetilde{\Lambda}_{\mathcal{O}} \arrow[dr] & \\ \Lambda_{\mathcal{O}} \arrow[ur] \arrow[dr] & & A. \\ & \mathbf{T}_{\mathcal{D}}^{\min} \arrow[ur] & \end{tikzcd}$$

and a similar diagram with $\mathbf{T}_{\mathcal{D}}^{\min}$ replaced by $R_{\mathcal{D}}^{\min}$. From these diagrams we deduce the existence of primes $\widetilde{\mathfrak{p}}$ of $\mathbf{T}_{\mathcal{D}}^{\min} \otimes_{\Lambda_{\mathcal{O}}} \widetilde{\Lambda}_{\mathcal{O}}$ and $R_{\mathcal{D}}^{\min} \otimes_{\Lambda_{\mathcal{O}}} \widetilde{\Lambda}_{\mathcal{O}}$ extending $\mathfrak{p}$. It will be convenient to write $\widetilde{\mathbf{T}}_{\mathcal{D}}^{\min}$ for $(\mathbf{T}_{\mathcal{D}}^{\min} \otimes_{\Lambda_{\mathcal{O}}} \widetilde{\Lambda}_{\mathcal{O}})$ and $\widetilde{R}_{\mathcal{D}}^{\min}$ for $R_{\mathcal{D}}^{\min} \otimes_{\Lambda_{\mathcal{O}}} \widetilde{\Lambda}_{\mathcal{O}}$ from now on. Let $(\widetilde{R}_{\mathcal{D}}^{\min})_{\widetilde{\mathfrak{p}}}$ and $(\widetilde{\mathbf{T}}_{\mathcal{D}}^{\min})_{\widetilde{\mathfrak{p}}}$ denote the localization *and* completions of $\widetilde{R}_{\mathcal{D}}^{\min}$ and $\widetilde{\mathbf{T}}_{\mathcal{D}}^{\min}$, respectively, at $\widetilde{\mathfrak{p}}$. The map $r_{\mathcal{D}}$ induces a surjection

$$\psi(\mathcal{D}, \mathfrak{p}) : (\widetilde{R}_{\mathcal{D}}^{\min})_{\widetilde{\mathfrak{p}}} \longrightarrow (\widetilde{\mathbf{T}}_{\mathcal{D}}^{\min})_{\widetilde{\mathfrak{p}}}.$$

Assume now that $\mathcal{D} = \mathcal{D}_0$ (i.e., that $\Sigma = \Sigma_0$ is the set of primes at which ρ_0 ramifies together with $\mathcal{P} = \{v_i : v_i | p\}$, and that $\mathcal{M} = \mathcal{M}_0 = \Sigma_0 \backslash \mathcal{P}$). We will use the criteria of §6 to prove that $\psi(\mathcal{D}_0, \mathfrak{p})$ is an isomorphism.

Next we define the needed rings and modules. Let $\widetilde{M}_{\mathcal{D}} = M_{\mathcal{D}} \otimes_{\Lambda_{\mathcal{O}}} \widetilde{\Lambda}_{\mathcal{O}}$. Then we define

$$N^{(0)} = \operatorname{im}\left\{\widetilde{M}_{\mathcal{D}} \longrightarrow (\widetilde{M}_{\mathcal{D}})_{\widetilde{\mathfrak{p}}}/P\right\}$$

where $(\widetilde{M}_{\mathcal{D}})_{\widetilde{\mathfrak{p}}}$ is the localization and completion of $\widetilde{M}_{\mathcal{D}}$ with respect to $\widetilde{\mathfrak{p}}$ and $P \subseteq \widetilde{\Lambda}_{\mathcal{O}}$ is the prime $P = (\pi, W_2, ..., W_m)$. We define a ring

$$R^{(0)} = \operatorname{im}\left\{\tilde{R}_{\mathcal{D}}^{\min} \to (\tilde{R}_{\mathcal{D}}^{\min})_{\tilde{\mathfrak{p}}}/(\tilde{\mathfrak{p}} \cdot F_0, P)\right\}$$

where $F_0 = \operatorname{Fitt}((\widetilde{M}_{\mathcal{D}})_{\tilde{\mathfrak{p}}}) \subset (\widetilde{R}_{\mathcal{D}}^{\min})_{\tilde{\mathfrak{p}}}$ is the Fitting ideal of $(\widetilde{M}_{\mathcal{D}})_{\tilde{\mathfrak{p}}}$ as an $(\widetilde{R}_{\mathcal{D}}^{\min})_{\tilde{\mathfrak{p}}}$-module. Note that $N^{(0)}$ is an $R^{(0)}$-module.

Now we introduce auxiliary levels. First we fix a $\sigma \in \operatorname{Gal}(F_\Sigma/F')$ as in Lemma 7.1(i). Then there exists an integer $r = r(\rho)$ as in Proposition 7.2 with the following property. For each odd integer N there is a set of primes

$Q_N = \{w_1^{(N)}, \ldots, w_r^{(N)}\}$ of F satisfying $Nm(w_i^{(N)}) \equiv 1 \pmod{p^N}$ as well as property (ii) and (iii) of Proposition 7.2. We can and do choose the sets Q_N to be disjoint from each other as well as from Σ. For such a set $Q = Q_N$, we earlier introduced a deformation problem $\mathcal{D}_Q$ as well as associated Hecke and deformation rings $\mathbf{T}_{\mathcal{D}_Q}^{\min}$ and $R_{\mathcal{D}_Q}^{\min}$. In particular at the end of §3 we associated a $\mathbf{T}_{\mathcal{D}_Q}^{\min}$-module $M_{\mathcal{D}_Q}$ to $\mathcal{D}_Q$. We now set

$$\widetilde{M}_{\mathcal{D}_Q} = M_{\mathcal{D}_Q} \otimes_{\Lambda_{\mathcal{O}}} \widetilde{\Lambda}_{\mathcal{O}}.$$

To each $w_i = w_i^{(N)} \in Q = Q_N$ we associated in §3 an element $\delta_{w_i} \in \mathrm{End}(\widetilde{M}_{\mathcal{D}_Q})$. We let $s_i = \delta_{w_i} - 1$. We then define, for each odd integer $1 \leqslant a \leqslant N$,

$$M_a^{(N)} = \mathrm{im}\left\{\widetilde{M}_{\mathcal{D}_Q} \longrightarrow (\widetilde{M}_{\mathcal{D}_Q})_{\widetilde{\mathfrak{p}}}/(P, s_1^{a+1}, \ldots, s_r^{a+1})\right\}$$

where the completion is as a $\widetilde{\mathbf{T}}_{\mathcal{D}_Q}^{\min}$-module (with respect to $\widetilde{\mathfrak{p}}$) and $Q = Q_N$. Then $M_a^{(N)}$ is a module over the ring $A_a = A[\![s_1, \ldots, s_r]\!]/(s_1^{a+1}, \ldots, s_r^{a+1})$ by construction. We put $M^{(N)} = M_N^{(N)}$.

Let $\sigma_{w_i} \in \mathrm{Gal}(\overline{F}/F)$ be as in Lemma 3.5(iii). There is a map of rings

$$\widetilde{\Lambda}_{\mathcal{O}}\,[\![t_1, \ldots, t_r]\!] \longrightarrow \widetilde{R}_{\mathcal{D}_Q} = R_{\mathcal{D}_Q} \otimes_{\Lambda_{\mathcal{O}}} \widetilde{\Lambda}_{\mathcal{O}}$$

given by $t_i \longmapsto \mathrm{trace}(\rho_{\mathcal{D}_Q}(\sigma_{w_i}) - 2) \otimes 1$, where here $Q = Q_N = \{w_1, \ldots, w_r\}$. We define $R_a^{(N)}$ by

$$R_a^{(N)} = \mathrm{im}\left\{\tilde{R}_{\mathcal{D}_Q} \to (\tilde{R}_{\mathcal{D}_Q})_{\tilde{\mathfrak{p}}}/(P, t_1^{\frac{a+1}{2}}, \ldots, t_r^{\frac{a+1}{2}}, \tilde{\mathfrak{p}}F_N)\right\}$$

where $Q = Q_N$ and $F_N = \mathrm{Fitt}((\widetilde{M}_{\mathcal{D}_Q})_{\widetilde{\mathfrak{p}}})$ is the Fitting ideal of $(\widetilde{M}_{\mathcal{D}_Q})_{\widetilde{\mathfrak{p}}}$ with respect to the ring $(\widetilde{R}_{\mathcal{D}_Q}^{\min})_{\widetilde{\mathfrak{p}}}$. This ring is an algebra over $B_a = A[\![t_1, \ldots, t_r]\!]/(t_1^{\frac{a+1}{2}}, \ldots, t_r^{\frac{a+1}{2}})$ by construction. By Lemma 3.5(iii) the action of $R_a^{(N)}$ on $M_a^{(N)}$ is compatible with the A_a-action of the subring B_a. We define $R_0^{(N)}$ by

$$R_0^{(N)} = R^{(0)}.$$

Now put

$$M^{(0)} = \oplus_{i=1}^{2^r} N^{(0)}.$$

Arguing as in the paragraph following [SW1, (7.14)] shows that there is a natural map $M_{\mathcal{D}_Q} \to M_{\mathcal{D}}^{2^r}$ of $\mathbf{T}_{\mathcal{D}_Q}$-modules. It follows from [SW1, Lemma

3.26] and the assumption that ρ_0 is irreducible that this map is surjective. We define $M_0^{(N)}$ to be

$$M_0^{(N)} = \mathrm{im}\left\{\tilde{M}_{\mathcal{D}_Q} \to (\tilde{M}_{\mathcal{D}})_{\tilde{\mathfrak{p}}}^{2^r}/P\right\}.$$

It follows easily that

$$M_0^{(N)} = M^{(0)}. \tag{8.1}$$

We now verify that these constructions satisfy the properties in §6 needed for the conclusion of Proposition 6.1. A bound for the number of generators of $R^{(N)} = R_N^{(N)}$ is given by $\dim_k(H^1(F_{\Sigma\cup Q}/F, \mathrm{ad}^0\rho_0))$, which is easily bounded independent of N. We can choose the generators in each case so that (6.2ii) holds by subtracting suitable elements of $\mathcal{O}[\![W_1]\!]$. The other properties in (6.2) follow from the definitions.

Next we consider the properties (6.7) of $M_a^{(N)}$. Properties (iii)–(v) are straightforward, and property (ii) follows from (8.1). Property (vi) can be checked using [SW1, Lemma 3.19]. Essentially this is done by showing that one can replace $U_{\mathcal{D}_Q}^{\min}$ in the definition of $M_{\mathcal{D}_Q}$ by a subgroup that is sufficiently small in the sense of §3. For details see [SW1, §7]. Property (6.7i) follows from property (6.7vi).

The properties in (6.6) are consequences of those in (6.7) as well as of the definitions of the $M^{(N)}$'s.

Next we verify properties (6.3i) and (6.5i). Let $d_1(a) = \dim_K(M_a^{(N)} \otimes_A K)$. This is independent of N by (6.7vi). Again using (6.7vi)

$$d_1(1) \geqslant \mu^{(N)}_{R_a\otimes_A K}(M_a^{(N)} \otimes_A K)$$

where $\mu_S(X)$ denotes the minimal number of generators of the S-module X. Now $\tilde{\mathfrak{p}}^{d_1(a)}$ annihilates $M_a^{(N)} \otimes_A K$ and hence

$$\tilde{\mathfrak{p}}^{d_1(a)d_1(1)} \subseteq \mathrm{Fitt}_{R_a^{(N)}\otimes_A K}(M_a^{(N)} \otimes_A K).$$

From the definition of $R_a^{(N)} \otimes_A K$ it follows that $\tilde{\mathfrak{p}}^{d_1(a)d_1(1)+1} = 0$ in this ring so we may take $d(a) = d_1(a)d_1(1) + 1$.

Now we check (6.5ii). Recall that we are given a set $Q_N = \{w_1^{(N)}, \ldots, w_r^{(N)}\}$ of primes satisfying the hypotheses of Proposition 7.2, and that by the same proposition

$$\varinjlim_n H_{\Sigma_Q}(\mathcal{T}_n) \simeq (K/A)^r \oplus X_{\Sigma_Q}$$

with $\Sigma_Q = \Sigma \cup Q_N$ and $\#X_{\Sigma_Q}$ bounded independent of N. We also have the usual isomorphism in the style of [W, Proposition 1.2]

$$\mathfrak{X}_Q = \mathrm{Hom}_A\left((\widetilde{\mathfrak{p}}^{\min}_{\mathcal{D}_Q}/(\widetilde{\mathfrak{p}}^{\min}_{\mathcal{D}_Q})^2, P), \lambda^{-n}A/A\right) \xrightarrow{\sim} H_{\Sigma_Q}(\mathcal{T}_n).$$

The isomorphism is obtained as follows. To an element $\varphi \in \mathfrak{X}_Q$ we associate the representation

$$\rho_\varphi : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_2(\widetilde{R}^{\min}_{\mathcal{D}_Q}/((\widetilde{\mathfrak{p}}^{\min}_{\mathcal{D}_Q})^2, P, \ker\varphi).$$

This is a deformation of $\rho_{\mathfrak{p}}$ with values in $A[\varepsilon]/(\lambda^n\varepsilon, \varepsilon^2)$ and its associated cohomology class lies in $H_{\Sigma_Q}(\mathcal{T}_n)$. Property (6.5ii) follows easily from this. We omit the details and refer to [SW1, §7] for a more detailed argument in a similar situation.

It remains to prove the existence of an element $x^{(N)}$ as in (6.8). This is done just as in [SW1, §7] so we omit the proof.

We have now verified all the hypotheses in §6 and are thus in a position to prove the following result.

PROPOSITION 8.1.— *Suppose that F is a totally real field and that* (H_{even}) *and* (H_{def}) *hold. Suppose that $\mathfrak{p} \subseteq \mathbf{T}_{\mathcal{D}_0}$ is a prime that is nice for $\mathcal{D}_0$ in the sense of* §4.2. *Then*

(i) $\psi(\mathcal{D}_0, \mathfrak{p}) : (\widetilde{R}^{\min}_{\mathcal{D}_0})_{\widetilde{\mathfrak{p}}} \longrightarrow (\widetilde{\mathbf{T}}^{\min}_{\mathcal{D}_0})_{\widetilde{\mathfrak{p}}}$ *is an isomorphism and* $(\widetilde{R}^{\min}_{\mathcal{D}_0})_{\widetilde{\mathfrak{p}}}$ *is a complete intersection over* $\widetilde{\Lambda}_{\mathcal{O},P}$ *and reduced.*

(ii) $\widetilde{M}_{\mathcal{D}_0,\widetilde{\mathfrak{p}}}$ *is a free* $(\widetilde{\mathbf{T}}^{\min}_{\mathcal{D}_0})_{\widetilde{\mathfrak{p}}}$*-module.*

Proof.— Our constructions give the following identifications:

$$\begin{aligned} R^{(0)} \underset{A}{\otimes} K &= (\widetilde{R}^{\min}_{\mathcal{D}_0})_{\widetilde{\mathfrak{p}}}/(\widetilde{\mathfrak{p}}F_0, P) \\ N^{(0)} \underset{A}{\otimes} K &= (\widetilde{M}_{\mathcal{D}_0})_{\widetilde{\mathfrak{p}}}/P, \quad M^{(0)} \underset{A}{\otimes} K = \bigoplus_{i=1}^{2^r} N^{(0)} \underset{A}{\otimes} K. \end{aligned}$$

By Proposition 6.1, $M^{(0)} \otimes_A K$ is free over $R^{(0)} \otimes_A K$. As the action of $R^{(0)} \otimes_A K$ on $M^{(0)} \otimes_A K$ factors through the composite map $R^{(0)} \otimes_A K \twoheadrightarrow (\widetilde{\mathbf{T}}^{\min}_{\mathcal{D}_0})_{\widetilde{\mathfrak{p}}}/P$ we conclude that $M^{(0)} \otimes_A K$ is a free $(\widetilde{\mathbf{T}}^{\min}_{\mathcal{D}_0})_{\widetilde{\mathfrak{p}}}/P$-module and that $\psi(\mathcal{D}_0, \mathfrak{p})$ induces an isomorphism $(\widetilde{R}^{\min}_{\mathcal{D}_0})_{\widetilde{\mathfrak{p}}}/(\widetilde{\mathfrak{p}}F_0, P) \simeq (\widetilde{\mathbf{T}}^{\min}_{\mathcal{D}_0})_{\widetilde{\mathfrak{p}}}/P$. Picking generators of $M^{(0)} \underset{A}{\otimes} K$ as an $R^{(0)} \underset{A}{\otimes} K$-module and lifting them to $(\widetilde{M}_{\mathcal{D}_0})_{\widetilde{\mathfrak{p}}}$ we get a map (for some minimal s)

$$(\widetilde{\mathbf{T}}^{\min}_{\mathcal{D}_0})^s_{\widetilde{\mathfrak{p}}} \twoheadrightarrow (\widetilde{M}_{\mathcal{D}_0})_{\widetilde{\mathfrak{p}}} \tag{8.2}$$

which is an isomorphism modulo P. Since $(\widetilde{M}_{\mathcal{D}_0})_{\tilde{\mathfrak{p}}}$ is free over $\widetilde{\Lambda}_{\mathcal{O},P}$ it follows that (8.2) is an isomorphism. In particular $(\widetilde{\mathbf{T}}^{\min}_{\mathcal{D}_0})_{\tilde{\mathfrak{p}}}$ is free over $\widetilde{\Lambda}_{\mathcal{O},P}$.

As observed, the reduction mod P of the map

$$(\widetilde{R}^{\min}_{\mathcal{D}_0})_{\tilde{\mathfrak{p}}}/(\tilde{\mathfrak{p}}F_0) \longrightarrow (\widetilde{\mathbf{T}}^{\min}_{\mathcal{D}_0})_{\tilde{\mathfrak{p}}} \tag{8.3}$$

induced by $\psi(\mathcal{D}_0, \mathfrak{p})$ is an isomorphism. Using that $(\widetilde{\mathbf{T}}^{\min}_{\mathcal{D}_0})_{\tilde{\mathfrak{p}}}$ is free over $\widetilde{\Lambda}_{\mathcal{O}}$ we now deduce that (8.3) is an isomorphism. Under (8.3) F_0 maps to zero as $\widetilde{M}_{\mathcal{D}_0,\tilde{\mathfrak{p}}}$ is a free $(\widetilde{\mathbf{T}}^{\min}_{\mathcal{D}_0})_{\tilde{\mathfrak{p}}}$-module. So $F_0/\tilde{\mathfrak{p}}F_0 = 0$ whence $F_0 = 0$. Finally $(\widetilde{R}^{\min}_{\mathcal{D}_0})_{\tilde{\mathfrak{p}}}$ is a complete intersection since $(\widetilde{R}^{\min}_{\mathcal{D}_0})_{\tilde{\mathfrak{p}}}/P$ is by Proposition 6.1. (Note that $(\widetilde{\mathbf{T}}^{\min}_{\mathcal{D}_0})_{\tilde{\mathfrak{p}}}$ is reduced as $\mathbf{T}^{\min}_{\mathcal{D}_0}$ is reduced). This completes the proof of the proposition. □

Using this proposition, one can deduce as in [SW1, §8] that $\psi(\mathcal{D}, \mathfrak{p})$ is an isomorphism for any $\mathcal{D}$. This is just an application of the congruences computed in [SW1, §3.8]. From this one can deduce as in [SW1, Proposition 8.4] that property (P) holds for any prime that is nice for some deformation datum. We refer the reader to the aforementioned section of [SW1] for detailed arguments. We thus obtain the following.

PROPOSITION 8.2. — *Suppose that* (H_{even}) *and* (H_{def}) *hold for* F. *Suppose that* $\mathcal{D}$ *is a deformation datum. If* $\mathfrak{p} \subseteq R_{\mathcal{D}}$ *is nice for* $\mathcal{D}$, *then*

(i) $\psi(\mathcal{D}, \mathfrak{p})$ *is an isomorphism.*

(ii) *property* (P) *holds for* $\mathfrak{p}$.

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