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The equivariant fundamental group, uniformization of real algebraic curves, and global complex analytic coordinates on Teichmüller spaces (*)

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RÉSUMÉ. — Soit X une courbe algébrique réelle de genre supérieur ou égal à 2. On construit ici un système global de coordonnées analytiques complexes sur l'espace de Teichmüller $T(X)$ des courbes algébriques complexes de même genre que X. Celui-ci induit un système global de coordonnées analytiques réelles sur l'espace de Teichmüller réel de X. La construction consiste à uniformiser la courbe algébrique réelle X par le double demi-plan $\mathbb{C} \setminus \mathbb{R}$. Ceci donne lieu à une $\mathrm{PGL}_2(\mathbb{R})$-représentation ρ du groupe fondamental équivariant de X. On détermine une présentation explicite du groupe fondamental équivariant. On s'en sert pour construire un système global de coordonnées analytiques complexes sur l'espace des déformations complexes de ρ. Cela nous mène au système de coordonnées recherché sur $T(X)$.

ABSTRACT. — Let X be a real algebraic curve of genus at least 2. We construct a global system of complex analytic coordinates on the Teichmüller space $T(X)$ of complex algebraic curves of the same genus as X. It induces a global system of real analytic coordinates on the real Teichmüller space of X. The method of construction consists of uniformizing the real algebraic curve X by the double half-plane $\mathbb{C}\setminus\mathbb{R}$. This gives rise to a $\mathrm{PGL}_2(\mathbb{R})$-representation ρ of the equivariant fundamental group of X. We determine an explicit presentation of the equivariant fundamental group, that is used to construct a global system of complex analytic coordinates on the complex deformation space of ρ. This will give rise to the desired coordinate system on $T(X)$.

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1. Introduction

Let X be a complex algebraic curve of genus at least 2. We denote by $T(X)$ the Teichmüller space of complex algebraic curves homeomorphic to X. Fricke constructed a global system of real analytic coordinates on $T(X)$ as follows. Choose a uniformization of X by the upper half-plane $\mathbb{U}$. One obtains a representation

$$\rho: \pi_1(X) \to \mathrm{PSL}_2(\mathbb{R})$$

of the fundamental group of X into the group $\mathrm{PSL}_2(\mathbb{R})$ of holomorphic automorphisms of $\mathbb{U}$. The space $\mathrm{Def}_{\mathbb{R}}(\rho)$ of real quasiconformal deformations of ρ parametrizes real bianalytically the Teichmüller space $T(X)$. After having chosen a presentation of the fundamental group $\pi_1(X)$, one easily constructs a real analytic open embedding of $\mathrm{Def}_{\mathbb{R}}(\rho)$ into real Euclidean space. Hence, one gets a global system of real analytic coordinates on $T(X)$.

Fricke's coordinate system may be perceived as unsatisfactory in the sense that it is only real analytic. Several authors have constructed global systems of complex analytic coordinates on $T(X)$ (e.g. [2, 4, 10]). When one wants to construct such a coordinate system, one is tempted to adapt Fricke's construction by considering the space $\mathrm{Def}_{\mathbb{C}}(\rho)$ of complex quasiconformal deformations of ρ. This space, however, is far too big to parametrize $T(X)$ complex bianalytically; by considering complex deformations of ρ, one can deform ρ independently on the upper half-plane and on the lower half-plane. It follows that $\mathrm{Def}_{\mathbb{C}}(\rho)$ parametrizes the product $T(X) \times T(\overline{X})$ of $T(X)$ and the Teichmüller space $T(\overline{X})$ of the complex conjugate algebraic curve $\overline{X}$. This is Bers' simultaneous uniformization [1].

All above-mentioned constructions of complex analytic coordinate systems on $T(X)$ consider $T(X)$ embedded in $T(X) \times T(\overline{X})$ in some way and, in doing so, single out a subspace of complex quasiconformal deformations of ρ that do parametrize $T(X)$ complex bianalytically. Then, by ad hoc arguments, it is shown that this subspace can be embedded complex bianalytically as an open subset of $\mathbb{C}^n$. Hence, one gets global systems of complex analytic coordinates on $T(X)$ [2, 4, 10].

It is clear that the difficulties one encounters when constructing complex analytic coordinates on $T(X)$ come from the uniformization of X by the upper half-plane $\mathbb{U}$. In order to avoid these difficulties one should uniformize X by the double half-plane $\mathbb{D} = \mathbb{C}\backslash\mathbb{R}$. Such a uniformization only exists when X is a real algebraic curve, i.e. $\overline{X} = X$ for some embedding of X into complex projective n-space. This is the idea of the present paper and will lead to a, in my opinion, neat construction of a global system of complex analytic coordinates on the Teichmüller space $T(X)$ for any real algebraic curve X of genus at least 2. We explain briefly our construction.

Assume that X is a real algebraic curve. Let Σ be the Galois group of $\mathbb{C}/\mathbb{R}$. Then Σ acts continuously on X. Choose an equivariant uniformization of X by the double half-plane $\mathbb{D}$. One obtains a representation

$$\tau\colon \pi_1(X,\Sigma) \longrightarrow \mathrm{PGL}_2(\mathbb{R})$$

of the equivariant fundamental group $\pi_1(X,\Sigma)$ of X into the group $\mathrm{PGL}_2(\mathbb{R})$ of equivariant holomorphic automorphisms of $\mathbb{D}$. The equivariant fundamental group is a group whose subgroups classify Σ-equivariant coverings of X; it is naturally isomorphic to the fundamental group of the orbifold quotient of X by the action of Σ [7]. For our purposes, it will be more convenient, however, to stick to the notion of equivariant fundamental group.

It follows immediately from the general theory of quasiconformal deformations of Kleinian groups that the space $\mathrm{Def}_{\mathbb{C}}(\tau)$ of complex quasiconformal deformations of τ parametrizes $T(X)$ complex bianalytically. A suitable presentation of the group $\pi_1(X,\Sigma)$ allows to embed $\mathrm{Def}_{\mathbb{C}}(\tau)$ complex bianalytically as an open subset of $\mathbb{C}^n$. Hence, one gets a global system of complex analytic coordinates on $T(X)$.

The coordinate system on $T(X)$ that we construct has the following property. If X is a real algebraic curve, the Galois group Σ acts naturally on $T(X)$. The set of fixed points $T(X)^\Sigma$ is the real Teichmüller space of X [3, 13]; it depends not only on the genus of X but also on the topological equivalence class of the action of Σ on X. Now, the coordinates on $T(X)$ that we construct are Σ-equivariant. Therefore, they induce a global system of real analytic coordinates on the real Teichmüller space $T(X)^\Sigma$ of the real algebraic curve X.

We should note that in the case that the real algebraic curve X is an M-curve, i.e. the number of connected components of the set of real points of X is maximal for given genus, the complex analytic coordinates on $T(X)$ that we construct coincide with the coordinates constructed by Earle [4]. In fact, the idea of uniformizing the real algebraic curve X by the double half-plane $\mathbb{D}$ is implicitly present in that paper.

The paper is organized as follows. In Sections 2 and 3, we outline the theory of the equivariant fundamental group as automorphism group of a universal equivariant covering. Since this fundamental group is essentially the same as the orbifold fundamental group of the quotient, we omit the proofs. A reader interested in details may consult [8]. Section 4 is devoted to uniformization of real algebraic curves, in fact more generally, of Riemann surfaces equipped with certain actions of Σ. In Section 5, we construct a global system of complex analytic coordinates on the Teichmüller space $T(X)$ for any real algebraic curve X of genus at least 2 that does not have real points. This case is easier to deal with since the action of Σ on X is fixed point-free, and, therefore, the equivariant fundamental group is isomorphic to the ordinary fundamental group of X/Σ. In Section 6, we give a description of the equivariant fundamental group in terms of equivariant loops, again omitting details. We determine, in Section 7, an explicit presentation of the equivariant fundamental group of a real algebraic curve having real points. Such a presentation can also be found in [7]. Our treatment differs in that respect that we fully exploit the global group action of Σ on the real algebraic curve. In Sections 8 and 9 we construct a global complex analytic coordinate system on the Teichmüller space $T(X)$ for any real algebraic curve X of genus at least 2 that has real points.

CONVENTION. — In the rest of the paper, Riemann surfaces—in particular, real and complex algebraic curves—are not necessarily connected or compact.

2. The universal equivariant covering

Let Σ be a group. A *Σ-space* is a topological space X endowed with an action of Σ such that the law $\Sigma \times X \longrightarrow X$ is continuous, when Σ is given the discrete topology.

Let X and Y be Σ-spaces. A map $f: X \to Y$ is called *equivariant* if $f(\sigma x) = \sigma f(x)$ for all $x \in X$ and $\sigma \in \Sigma$.

Let X be a Σ-space. A subset U of X is *stable* if $\sigma x \in U$ for all $x \in U$ and all $\sigma \in \Sigma$. The Σ-space X is said to be *equivariantly connected* if $\emptyset$ and X are the only open and closed stable subsets of X.

Recall that for a locally connected topological space the notion of connected component makes sense. In fact, such a topological space is the disjoint union of its connected components. If X is a locally connected Σ-space, then X is equivariantly connected if and only if the induced action of Σ on the set of connected components of X is transitive.

Let $f: Y \to X$ be an equivariant map of Σ-spaces. The map f is an *equivariant covering of X* if f is a covering in the usual sense, i.e., f is surjective and for all $x \in X$ there is an open neighborhood U of x in X such that $f^{-1}U$ is the disjoint union of open subsets V_i of Y, for $i \in I$, such that the restriction $f_{|V_i}$ of f to V_i is a homeomorphism onto U.

Let X be a Σ-space. Consider the group Σ itself as a discrete Σ-space. An *equivariant base point* of X is an equivariant map $b\colon \Sigma \to X$. If b is an equivariant base point, we denote by b_σ the image $b(\sigma)$ of $\sigma \in \Sigma$. Let Y also be a Σ-space, and let $c\colon \Sigma \to Y$ be an equivariant base point of Y. An equivariant map $f\colon Y \to X$ is *base point-preserving* if $f(c_\sigma) = b_\sigma$ for all $\sigma \in \Sigma$. Such a map is denoted by $f\colon (Y,c) \to (X,b)$.

Let $p\colon (\hat{X},\hat{b}) \to (X,b)$ be an equivariant covering map. The map p is a *universal equivariant covering* of (X,b) if for all equivariant coverings $q\colon (Y,c) \to (X,b)$, there is a unique equivariant continuous map $f\colon (\hat{X},\hat{b}) \to (Y,c)$ such that the diagram

$$\begin{array}{ccc} \hat{X} & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \ \swarrow{\scriptstyle q} & \\ & X & \end{array}$$

commutes. By abuse of language, we will also say that $p\colon \hat{X} \to X$ is a universal covering of X.

PROPOSITION 2.1.— *Let X be a locally and equivariantly connected Σ-space, and let b be an equivariant base point of X. Suppose that $p\colon (\hat{X},\hat{b}) \to (X,b)$ is an equivariant covering of X. Let X_σ be the connected component of X containing b_σ, and let $\hat{X}_\sigma$ be the connected component of $\hat{X}$ containing $\hat{b}_\sigma$. Let p_σ be the restriction of p to $\hat{X}_\sigma$, considered as a map into X_σ. Then, p is a universal equivariant covering of X if and only if the following two conditions hold.*

1. *The map $p_\sigma\colon (\hat{X}_\sigma,\hat{b}_\sigma) \to (X_\sigma,b_\sigma)$ is a universal covering of X_σ for all $\sigma \in \Sigma$.*
2. *The group Σ acts freely and transitively on the set of connected components of $\hat{X}$.* □

COROLLARY 2.2.— *Let X be a locally and equivariantly connected Σ-space, and let b be an equivariant base point of X. Let X_σ be the connected component of X containing b_σ. Suppose that (X_σ,b_σ) has a universal covering for all $\sigma \in \Sigma$. Then, (X,b) has a universal equivariant covering.* □

COROLLARY 2.3.— *Let X be a locally and equivariantly connected Σ-space, and let b be an equivariant base point of X. Suppose that (X,b) has a universal equivariant covering $p\colon (\hat{X},\hat{b}) \to (X,b)$. Let $\bar{b}$ be the induced base point of the quotient X/Σ, and let $\bar{\hat{b}}$ be the induced base point of the quotient $\hat{X}/\Sigma$. Suppose that Σ acts discontinuously and freely on X. Then, the action* *of Σ on $\hat{X}$ is discontinuous and free, and the induced map on the quotient spaces*

$$\bar{p}\colon (\hat{X}/\Sigma, \bar{\hat{b}}) \longrightarrow (X/\Sigma, \bar{b})$$

is a universal covering of the topological space X/Σ. □

3. The equivariant fundamental group

Let $q\colon Y \to X$ and $r\colon Z \to X$ be equivariant coverings of a Σ-space X. An equivariant continuous map $f\colon Y \to Z$ is a *morphism of equivariant coverings* of X if the diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ & {\scriptstyle q}\searrow \; \swarrow{\scriptstyle r} & \\ & X & \end{array}$$

commutes. It is then clear what is understood by an automorphism of an equivariant covering. The group of automorphisms of an equivariant covering $q\colon Y \to X$ is denoted by $\mathrm{Aut}(Y/X)$.

An equivariant covering $q\colon Y \to X$ is said to be *Galois* if the map q is a quotient of Y for the action of the group $\mathrm{Aut}(Y/X)$ on Y.

Let X be a Σ-space, and let b be an equivariant base point of X. Suppose that (X, b) has a universal equivariant covering $p\colon (\hat{X}, \hat{b}) \to (X, b)$. By the universal property of such a covering, the group $\mathrm{Aut}(\hat{X}/X)$ is uniquely determined by X, up to unique isomorphism.

DEFINITION 3.1.— *The group $\mathrm{Aut}(\hat{X}/X)$ of automorphisms of the equivariant covering $\hat{X}$ over X is called the* equivariant fundamental group *of X, and is denoted by $\pi_1(X, \Sigma; b)$, or simply by $\pi(X, \Sigma)$.*

Of course, if Σ is the trivial group, then the equivariant fundamental group $\pi_1(X, \Sigma; b)$ of the Σ-space X is nothing but the ordinary fundamental group $\pi_1(X; b_1)$ of the topological space X with base point b_1.

PROPOSITION 3.2.— *Let X be a locally and equivariantly connected Σ-space, and let b be an equivariant base point of X. Suppose that (X, b) has a universal equivariant covering. Let X_1 be the connected component of X containing b_1. Let Σ_1 be the subgroup of Σ consisting of all σ such that $\sigma \cdot X_1 = X_1$. Then, there is an exact sequence*

$$0 \longrightarrow \pi_1(X_1; b_1) \longrightarrow \pi_1(X, \Sigma; b) \longrightarrow \Sigma_1 \longrightarrow 0.$$

This sequence is split, if $b_\sigma = b_1$ *for all* $\sigma \in \Sigma_1$. *In particular, the equivariant fundamental group* $\pi_1(X, \Sigma; b)$ *of* X *is isomorphic to the semidirect product of* $\pi_1(X_1; b_1)$ *and* Σ_1, *if* $b_\sigma = b_1$ *for all* $\sigma \in \Sigma_1$. □

Using Propositions 2.1 and 3.2, one easily shows the following statement.

PROPOSITION 3.3. — *Let* X *be a locally and equivariantly connected* Σ-*space, and let* b *be an equivariant base point of* X. *Suppose that* $p: (\hat{X}, \hat{b}) \to (X, b)$ *is a universal equivariant covering of* X. *Then, the covering* p *is Galois, i.e., the map* p *is the quotient of* $\hat{X}$ *by the action of* $\pi_1(X, \Sigma; b)$. □

The following proposition follows directly from Corollary 2.3.

PROPOSITION 3.4. — *Let* X *be a locally and equivariantly connected* Σ-*space, and let* b *be an equivariant base point of* X. *Suppose that* X *has a universal covering. Suppose, moreover, that* Σ *acts discontinuously and freely on* X. *Let* $\bar{b}$ *be the induced base point of the quotient* X/Σ. *Then, one has an isomorphism*

$$\pi_1(X, \Sigma; b) \cong \pi_1(X/\Sigma; \bar{b}). \qquad \square$$

4. Uniformization of real algebraic curves

Let Σ be the Galois group of $\mathbb{C}$ over $\mathbb{R}$, i.e., $\Sigma = \{1, \sigma\}$, where σ is complex conjugation. Let X be a Riemann surface. A *real structure* on X is an action of Σ on X such that σ acts antiholomorphically. We will also say that the Riemann surface is *defined over* $\mathbb{R}$.

Recall that a Riemann surface X is of *finite type* if X is isomorphic to the complement of a finite set in a compact Riemann surface. A Riemann surface of finite type is essentially a complex algebraic curve. Similarly, a Riemann surface X of finite type endowed with a real structure is essentially a real algebraic curve. Therefore, in what follows, by a complex algebraic curve (resp. a real algebraic curve) is meant a Riemann surface of finite type (resp. a Riemann surface of finite type defined over $\mathbb{R}$).

Let X be a Riemann surface defined over $\mathbb{R}$. Its subset X^Σ of fixed points for the action of Σ is called its *set of real points*.

We will call a connected Riemann surface *hyperbolic* if it is universally covered, in the holomorphic sense, by the upper half-plane $\mathbb{U}$. An equivariantly connected Riemann surface defined over $\mathbb{R}$ will be said to be *hyperbolic* if each of its connected components is a hyperbolic Riemann surface.

We denote the lower half-plane by $\mathbb{L}$, and we let $\mathbb{D}$ be the double half-plane $\mathbb{U} \cup \mathbb{L}$. The uniformization of Riemann surfaces over $\mathbb{R}$ is then merely a consequence of the classical uniformization of Riemann surfaces [5], Theorem IV.4.1.

THEOREM (Uniformization of Riemann surfaces over $\mathbb{R}$). — *Let X be a hyperbolic equivariantly connected Riemann surface defined over $\mathbb{R}$. Then, there is a universal equivariant holomorphic covering $p: \mathbb{D} \to X$ of X by the double half-plane $\mathbb{D}$.* □

In case X is a hyperbolic equivariantly connected real algebraic curve, a uniformization as a Riemann surface defined over $\mathbb{R}$ will be called a *uniformization of X as a real algebraic curve.*

If $p: \mathbb{D} \to X$ is a uniformization of a Riemann surface X over $\mathbb{R}$, then the group G of automorphisms of p acts holomorphically on $\mathbb{D}$, i.e., G is a subgroup of the group $\mathrm{Aut}_\Sigma(\mathbb{D})$ of equivariant automorphisms of $\mathbb{D}$. Using the fact that the automorphism group of $\mathbb{U}$ can be identified with $\mathrm{PSL}_2(\mathbb{R})$, one easily shows the following statement.

PROPOSITION 4.1. — *The group $\mathrm{Aut}_\Sigma(\mathbb{D})$ is equal to the group $\mathrm{PGL}_2(\mathbb{R})$.* □

Let $p: \mathbb{D} \to X$ be a uniformization of a Riemann surface X defined over $\mathbb{R}$, and let G be the group of automorphisms of the covering p. Then, according to Proposition 4.1, G is a subgroup of the group $\mathrm{PGL}_2(\mathbb{R})$. Since G acts discontinuously on $\mathbb{D}$, the group G is Kleinian. (We refer to [11] for definitions and facts concerning Kleinian groups.)

We will say that a Kleinian subgroup G of $\mathrm{PGL}_2(\mathbb{R})$ is *of the first kind* if its region of discontinuity is equal to $\mathbb{D}$. Otherwise, G is said to be *of the second kind.* In that case, the domain of discontinuity of G contains $\mathbb{D}$ as a proper subset, and the limit set of G is a nowhere dense subset of $\mathbb{P}^1(\mathbb{R})$. Note that the definition the kind of a Kleinian subgroup of $\mathrm{PGL}_2(\mathbb{R})$ extends the classical definition of its kind in case G is contained in $\mathrm{PSL}_2(\mathbb{R})$, i.e., in case G is Fuchsian.

PROPOSITION 4.2. — *Let X be a hyperbolic equivariantly connected Riemann surface defined over $\mathbb{R}$. Let $p: \mathbb{D} \to X$ be a universal equivariant holomorphic covering of X. Let G be the group of automorphisms of the covering p. Then,*

1. *the group G is isomorphic to the equivariant fundamental group $\pi_1(X, \Sigma)$ of X;*

2. *the group G is a Kleinian subgroup of* $\mathrm{PGL}_2(\mathbb{R})$*, acting discontinuously on* $\mathbb{D}$*;*
3. *the quotient Riemann surface* $\mathbb{D}/G$ *is equivariantly isomorphic to* X*.*

Moreover, the following equivalences hold.

4. *The group G is of the second kind if and only if the Riemann surface X has a nonempty ideal boundary.*
5. *The group G is Fuchsian if and only if X is not connected.*
6. *The group G contains parabolic elements if and only if X has punctures.*
7. *The group G contains elliptic elements if and only if X has real points.*

Proof. — Statement 1 is clear. Statement 2 follows from Proposition 4.1. Statement 3 follows from Proposition 3.3.

In order to show the equivalences 4, 5, 6 and 7, let $G_1 \subseteq G$ be the subgroup of $\alpha \in G$ such that $\alpha{\cdot}\mathbb{U} \subseteq \mathbb{U}$, i.e., $G_1 = G \cap \mathrm{PSL}_2(\mathbb{R})$. Then, G_1 is at most of index 2 in G, and G_1 is the group of automorphisms of the restriction p_1 of p to $\mathbb{U}$. This restriction is nothing but the ordinary uniformization of the Riemann surface $X_1 = p_1(\mathbb{U})$. Clearly, X_1 is a connected component of X.

It is well known that X_1 has a nonempty ideal boundary if and only if G_1 is of the second kind. Using the action of Σ and the fact that X is equivariantly connected, X has a nonempty ideal boundary if and only if X_1 has one. Since G_1 is of finite index in G, the group G is of the second kind if and only if G_1 is so. This proves equivalence 4.

The Riemann surface X is not connected if and only if $G_1 = G$. Since $G_1 = G \cap \mathrm{PSL}_2(\mathbb{R})$, the subgroup G_1 of G is the largest Fuchsian subgroup of G. Equivalence 5 follows.

Since the group G_1 is of finite index in G, the group G contains parabolic elements if and only if G_1 contains such elements. It is well known that G_1 contains parabolic elements if and only if X_1 has punctures. But X_1 has punctures if and only if X has punctures. This proves equivalence 6.

Since elliptic elements of $\mathrm{PSL}_2(\mathbb{R})$ have fixed points in $\mathbb{U}$, G_1 does not contain such elements. An elliptic element α of G is therefore necessarily of order 2. After conjugating G in $\mathrm{PGL}_2(\mathbb{R})$, we may assume that $\alpha(z) = -z$ for all $z \in \mathbb{P}^1(\mathbb{C})$. Then, $\sigma(\sqrt{-1}) = -\sqrt{-1} = \alpha(\sqrt{-1})$, i.e., the image of $\sqrt{-1}$ is a real point in X.

Conversely, suppose that X has a real point. Since X is, moreover, equivariantly connected, X is connected, i.e., $X_1 = X$. Then according to Proposition 3.2, the exact sequence

$$0 \longrightarrow \pi_1(X) \longrightarrow \pi_1(X,\Sigma) \longrightarrow \Sigma \longrightarrow 0$$

is split. But this sequence is isomorphic to the exact sequence

$$0 \longrightarrow G_1 \longrightarrow G \longrightarrow G/G_1 \longrightarrow 0$$

which is, therefore, split too. Hence, G contains elliptic elements. This shows equivalence 7. $\square$

5. Complex coordinates on Teichmüller space; the case of a real algebraic curve without real points

Fix an integer $g \geqslant 2$. Let, throughout this section, X be a compact connected real algebraic curve of genus g having no real points. Let $\pi\colon \mathbb{D} \to X$ be a uniformization of X as a real algebraic curve, G the group of automorphisms of π, and Σ the Galois group of $\mathbb{C}$ over $\mathbb{R}$.

LEMMA 5.1.— *Assume that $X^\Sigma = \emptyset$. Then, there are elements $\gamma_1, \ldots, \gamma_{g+1}$ of G satisfying $\gamma_1^2 \cdots \gamma_{g+1}^2 = 1$ such that the induced morphism*

$$< \gamma_1, \ldots, \gamma_{g+1} \,|\, \gamma_1^2 \cdots \gamma_{g+1}^2 = 1 > \longrightarrow G$$

is an isomorphism.

Proof.— According to Proposition 4.2, G is isomorphic to the equivariant fundamental group $\pi_1(X,\Sigma)$. Since Σ acts freely on X, the equivariant fundamental group $\pi_1(X,\Sigma)$ is isomorphic to the ordinary fundamental group $\pi_1(X/\Sigma)$ of the quotient X/Σ, by Proposition 3.4.

Since the group Σ acts freely on X, the quotient X/Σ is a nonorientable topological surface, and its Euler characteristic is equal to

$$\chi(X/\Sigma) = \tfrac{1}{2}\chi(X) = \tfrac{1}{2}(2-2g) = 1-g.$$

It follows that the topological surface X/Σ is homeomorphic to the connected sum of $g+1$ real projective planes. Since G is isomorphic to $\pi_1(X/\Sigma)$, the statement follows. $\square$

We choose once and for all elements $\gamma_1, \ldots, \gamma_{g+1}$ of G satisfying the statement of the preceding lemma.

By Proposition 4.2, the elements γ_1 and γ_2 of G are loxodromic Möbius transformations. We claim that γ_1 and γ_2 do not have any fixed points in common. Indeed, if they would have a fixed point in common, then they would have both of their fixed points in common by [11], Proposition I.D.4. This would imply that there are nonzero integers $m, n \in \mathbb{Z}$ such that $\gamma_1^m = \gamma_2^n$, which contradicts Lemma 5.1, given the fact that $g \geqslant 2$. Therefore, γ_1 and γ_2 do not have a fixed point in common. Hence, after conjugating the subgroup G of $\mathrm{PGL}_2(\mathbb{R})$ by an element of $\mathrm{PGL}_2(\mathbb{R})$, we may assume that γ_1 has 0 as attracting and ∞ as repelling fixed point, and that γ_2 has 1 as attracting fixed point.

Recall that a *quasiconformal deformation* of G is a homomorphism

$$\iota: G \longrightarrow \mathrm{PGL}_2(\mathbb{C})$$

satisfying the following property. There is a quasiconformal orientation-preserving homeomorphism h of $\mathbb{P}^1(\mathbb{C})$ having 0, 1 and ∞ as fixed points, and such that $\iota(\alpha) = h \circ \alpha \circ h^{-1}$ for all $\alpha \in G$ [12], Section 1.3.3.

Note that all connected components of the region of discontinuity of G are simply connected. Therefore, the *Teichmüller space* $T(G)$ of G is in bijection, in a natural way, with the set of all quasiconformal deformations of G [9], §7, Corollary 2. This means that we may identify $T(G)$ with the latter set, i.e.,

$$T(G) = \{\iota: G \to \mathrm{PGL}_2(\mathbb{C}) \mid \iota \text{ is a quasiconformal deformation of } G\},$$

and one has a natural biholomorphic map

$$\Phi: T(G) \longrightarrow T(X).$$

Hence, in order to define a global system of complex analytic coordinates on $T(X)$, it suffices to define one on $T(G)$. We do this by embedding $T(G)$ into the space S_g of normalized marked Schottky groups of rank g.

Recall that a *Schottky group of rank* g is a free loxodromic Kleinian subgroup of $\mathrm{PGL}_2(\mathbb{C})$ on g generators [11], Section X.H. A *marked Schottky group of rank* g is a pair $(H, (\alpha_1, \dots, \alpha_g))$, where H is a Schottky group of rank g and $\alpha_1, \dots, \alpha_g$ are free generators of H. The group H being redundant in the notation $(H, (\alpha_1, \dots, \alpha_g))$, we simply write $(\alpha_1, \dots, \alpha_g)$ for this marked Schottky group. A marked Schottky group $(\alpha_1, \dots, \alpha_g)$ of rank g is called *normalized* if α_1 has 0 as attracting and ∞ as repelling fixed point, and α_2 has 1 as attracting fixed point. The set of all normalized marked Schottky groups of rank g is denoted by S_g. This set acquires the structure of a connected complex analytic manifold of dimension $3g-3$ by considering it as an open subset of $\mathrm{PGL}_2(\mathbb{C})^g$ [9].

Observe that the g-tuple $(\gamma_1,\ldots,\gamma_g)$ of elements of G is a normalized marked Schottky group of rank g. It follows that $(\iota(\gamma_1),\ldots,\iota(\gamma_g))$ is a normalized marked Schottky group of rank g for any quasiconformal deformation ι of G. Define

$$\Psi\colon T(G) \longrightarrow S_g$$

by letting $\Psi(\iota)$ be $(\iota(\gamma_1),\ldots,\iota(\gamma_g))$.

THEOREM 5.2.— *The map Ψ is an open holomorphic embedding of $T(G)$ into S_g.*

Proof.— One can show that Ψ is holomorphic by standard techniques of Teichmüller theory [12]. Hence, it suffices to show that Ψ is injective, since $T(G)$ and S_g are of the same dimension.

Let sq be the map from $\mathrm{PGL}_2(\mathbb{C})$ into itself that associates to an element α of $\mathrm{PGL}_2(\mathbb{C})$ its square α^2. Let $L \subseteq \mathrm{PGL}_2(\mathbb{C})$ be the open subset of loxodromic Möbius transformations. Then, $\mathrm{sq}(L) \subseteq L$. We denote again by sq the restriction of sq to L. For any $\beta \in L$ there are exactly 2 solutions $\alpha \in \mathrm{PGL}_2(\mathbb{C})$ to the equation $\alpha^2 = \beta$. Moreover, such a solution α is automatically loxodromic, i.e., $\alpha \in L$. The map $\mathrm{sq}\colon L \to L$ is therefore a 2-to-1 covering of L.

Let m be the the map from $\mathrm{PGL}_2(\mathbb{C})^g$ into $\mathrm{PGL}_2(\mathbb{C})$ that associates to $(\alpha_1,\ldots,\alpha_g)$ the element $\alpha_g^{-2}\cdots\alpha_1^{-2}$. Since all nontrivial elements of a Schottky group are loxodromic, $m(S_g) \subseteq L$. We denote again by m the restriction of m to the subset S_g of normalized marked Schottky groups of rank g.

Now we define V as the fiber product of L and S_g over L, i.e., V is defined as to make the square

$$\begin{array}{ccc} V & \xrightarrow{p} & S_g \\ \downarrow & & \downarrow \\ L & \xrightarrow{\mathrm{sq}} & L \end{array}$$

Cartesian. It follows that V is a complex analytic manifold, and that p is a 2-to-1 covering map. In fact, one has

$$V = \{(\alpha_1,\ldots,\alpha_{g+1}) \in \mathrm{PGL}_2(\mathbb{C})^{g+1} \mid (\alpha_1,\ldots,\alpha_g) \in S_g \text{ and } \alpha_1^2\cdots\alpha_{g+1}^2 = 1\},$$

and the map $p: V \longrightarrow S_g$ is the restriction to V of the projection from $\mathrm{PGL}_2(\mathbb{C})^{g+1}$ onto the product of its first g factors.

Let $\tilde{\Psi}: T(G) \to V$ be the map defined by

$$\tilde{\Psi}(\iota) = (\iota(\gamma_1), \ldots, \iota(\gamma_{g+1})).$$

Then, $p \circ \tilde{\Psi} = \Psi$. Since a homomorphism ι from G into $\mathrm{PGL}_2(\mathbb{C})$ is uniquely determined by the images of a system of generators, the map $\tilde{\Psi}$ is injective.

We show that the 2-to-1 covering map $p: V \to S_g$ is in fact a trivial covering map. We show this by showing that V has at least 2 connected components. It will then follow that the restriction of p to any of the connected components of V is injective. Since $T(G)$ is connected, it will follow that $\Psi = p \circ \tilde{\Psi}$ is injective.

Let $\rho: \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$ be the natural homomorphism. Let μ_2 be the subgroup $\{\pm 1\}$ of $\mathrm{SL}_2(\mathbb{C})$. The map ρ is in fact a quotient of $\mathrm{SL}_2(\mathbb{C})$ by the subgroup μ_2. We will denote elements of $\mathrm{SL}_2(\mathbb{C})$ by Roman capitals, like A_i, and their images in $\mathrm{PGL}_2(\mathbb{C})$ by the corresponding small Greek letters, like α_i.

The $(g+1)$-st power of ρ^{g+1} of ρ is a homomorphism from $\mathrm{SL}_2(\mathbb{C})^{g+1}$ into $\mathrm{PGL}_2(\mathbb{C})^{g+1}$. Of course, ρ^{g+1} is a quotient of $\mathrm{SL}_2(\mathbb{C})^{g+1}$ by the subgroup μ_2^{g+1}. We will denote ρ^{g+1} again by ρ since no confusion is likely to occur.

The inverse image $\tilde{V} = \rho^{-1}(V)$ of V is the union of $\tilde{V}_{-1}$ and $\tilde{V}_1$, where, for $\epsilon = \pm 1$,

$$\tilde{V}_\epsilon = \{(A_1, \ldots, A_{g+1}) \in \mathrm{SL}_2(\mathbb{C})^{g+1} \mid (\alpha_1, \ldots, \alpha_g) \in S_g \text{ and } A_1^2 \cdots A_{g+1}^2 = \epsilon\}.$$

It is clear that $\tilde{V}_{-1}$ and $\tilde{V}_1$ are disjoint, open and closed subsets of $\tilde{V}$. Note that both subset $\tilde{V}_{-1}$ and $\tilde{V}_1$ are nonempty since the square map from $\mathrm{SL}_2(\mathbb{C})$ into itself is surjective. Moreover, $\tilde{V}_{-1}$ and $\tilde{V}_1$ are both stable for the action of μ_2^{g+1}. Therefore, their ρ-images V_{-1} and V_1 are nonempty, open and closed, disjoint subsets of V. Moreover, $V = V_{-1} \cup V_1$. It follows that V is not connected. □

We define a map

$$\Xi: S_g \longrightarrow \mathbb{C}^{3g-3}$$

as follows. Given $(\alpha_1, \ldots, \alpha_g) \in S_g$, let a_i and b_i respectively be the attracting and repelling fixed points of α_i, and let c_i, $|c_i| < 1$, be the multiplier of α_i. Since α_1 has ∞ as fixed point, the points a_i and b_i are different from

∞, for $i = 2, \ldots, g$. This can be shown by an argument similar to the one employed to show that γ_1 and γ_2 have no fixed point in common. We define

$$\Xi(\alpha_1, \ldots, \alpha_g) = (a_3, \ldots, a_g, b_2, \ldots, b_g, c_1, \ldots, c_g).$$

It follows from the normalization of normalized marked Schottky groups that Ξ is an open holomorphic embedding of S_g into $\mathbb{C}^{3g-3}$.

COROLLARY 5.3.— *Let X be a connected compact real algebraic curve of genus g having no real points, where $g \geqslant 2$. Then, the map*

$$\Xi \circ \Psi \circ \Phi^{-1} : T(X) \to \mathbb{C}^{3g-3}$$

is a Σ-equivariant global system of complex analytic coordinates on the Teichmüller space $T(X)$ of complex algebraic curves of genus g.

6. Equivariant loops

Let Σ be a group and let X be a Σ-space Denote by I the unit interval $[0, 1]$. Consider Σ with the discrete topology. Then, $\Sigma \times I$ is a Σ-space when we define the action of Σ on $\Sigma \times I$ by $\sigma \cdot (\tau, x) = (\sigma\tau, x)$ for all $\sigma \in \Sigma$ and for all $(\tau, x) \in \Sigma \times I$.

An *equivariant path in X* is an equivariant map $\gamma : \Sigma \times I \to X$. Let γ be an equivariant path in X. Define $\gamma_0 : \Sigma \to X$ by $\gamma_0(\sigma) = \gamma(\sigma, 0)$. Then, γ_0 is an equivariant base point of X. We call γ_0 the *begin point of* γ. Similarly, one defines the *end point* γ_1 *of* γ by $\gamma_1(\sigma) = \gamma(\sigma, 1)$ for all $\sigma \in \Sigma$.

Let b and c be equivariant base points of X. An equivariant path γ in X is a path from b to c if $b = \gamma_0$ and $c = \gamma_1$.

Let γ and δ be two equivariant paths in X. Then, γ and δ are said to be *homotopic relative end points* if there is an equivariant map

$$F : I \times \Sigma \times I \longrightarrow X,$$

Σ acting trivially on the first and the third factor of $I \times X \times I$, such that

1. $F(0, \sigma, s) = \gamma(\sigma, s)$ and $F(1, \sigma, s) = \delta(\sigma, s)$ for all $(\sigma, s) \in \Sigma \times I$, and
2. $F(t, \sigma, 0) = F(t', \sigma, 0)$ and $F(t, \sigma, 1) = F(t', \sigma, 1)$ for all $t, t' \in I$ and $\sigma \in \Sigma$.

In particular, if γ and δ are homotopic relative end points, γ and δ have the same begin point as well as the same end point.

Let b and c be equivariant base points of X. Observe that homotopic relative end points is an equivalence relation on the set of all equivariant paths in X from b to c.

One can define the composition $\delta\gamma$ of two equivariant paths in X in case $\gamma_1(\Sigma) = \delta_0(\Sigma)$ and $\gamma_1: \Sigma \to X$ and $\delta_0: \Sigma \to X$ are injective. Indeed, there is a unique bijection $\alpha: \Sigma \to \Sigma$ such that $\gamma_1(\sigma) = \delta_0(\alpha(\sigma))$ for all $\sigma \in \Sigma$. Define $\delta\gamma: \Sigma \times I \to X$ by

$$(\delta\gamma)(\sigma, t) = \begin{cases} \gamma(\sigma, 2t) & \text{if } t \in [0, \frac{1}{2}], \\ \delta(\alpha(\sigma), 2t-1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

The map $\delta\gamma$ is equivariant since α is equivariant. Therefore, $\delta\gamma$ is an equivariant path of X with begin point γ_0 and end point δ_1.

Let $\gamma, \gamma', \delta, \delta'$ be equivariant paths such that γ and γ' (resp. δ and δ') are homotopic relative end points and such that the compositions $\delta\gamma$ and $\delta'\gamma'$ are defined. Then, $\delta\gamma$ and $\delta'\gamma'$ are homotopic relative end points.

Let $b: \Sigma \to X$ be an equivariant base point of X. An *equivariant loop in* (X, b) is an equivariant path γ in X such that $\gamma_0 = b$ and $\gamma_1(\Sigma) = b(\Sigma)$. Two equivariant loops γ and δ in (X, b) are *homotopic* if γ and δ are homotopic relative end points as equivariant paths. Let $\lambda_1(X, \Sigma; b)$ be the set of homotopy classes of all equivariant loops in (X, b). If $b: \Sigma \to X$ is injective, composition of equivariant paths induces the structure of a group on $\lambda_1(X, \Sigma; b)$.

PROPOSITION 6.1.— *Let X be a Σ-space. Assume that X is path connected and semilocally simply connected ([6], p. 27) as a topological space. Let b be an injective equivariant base point of X. Let $\pi_1(X, \Sigma; b)$ be the equivariant fundamental group of X. Then, there is a canonical isomorphism of groups*

$$\pi_1(X, \Sigma; b) \cong \lambda_1(X, \Sigma; b)^\circ,$$

where G° denotes the opposite group of a group G.

Proof.— Let $\Pi_1(X; b)$ be the groupoid of homotopy classes relative end points of ordinary paths in X having begin and end points in the set $b(\Sigma)$. Recall that a groupoid is a category in which all morphisms are automorphisms. Here, the objects are the elements of $b(\Sigma)$, and the morphisms from b_σ into b_τ are the homotopy classes relative end points of ordinary paths in X from b_σ to b_τ, for $\sigma, \tau \in \Sigma$. If $\gamma: I \to X$ is such a path and ν is in Σ, the path γ^ν defined by $\gamma^\nu(t) = \nu \cdot \gamma(t)$, for $t \in I$, has also begin and end points in $b(\Sigma)$. Hence, one gets an induced action of Σ on the groupoid $\Pi_1(X; b)$.

Thinking of $\Pi_1(X;b)$ as its set of morphisms, the quotient $\Pi_1(X;b)/\Sigma$ is canonically isomorphic to the set $\lambda_1(X,\Sigma;b)$. Indeed, define

$$\varphi\colon \Pi_1(X;b) \longrightarrow \lambda_1(X,\Sigma;b)$$

as follows. If γ is a path from b_σ to b_τ, there is a unique equivariant loop γ' in (X,b) such that its restriction to $\{\sigma\}\times I$ is equal to γ. Define the φ-image of the homotopy class of γ to be the homotopy class of γ'. It is clear that φ is a quotient map for the action of Σ on $\Pi_1(X;b)$. In fact, φ is a morphism of groupoids and the group $\lambda_1(X,\Sigma;b)$ is the quotient of the groupoid $\Pi_1(X;b)$ by the action of Σ.

To show the proposition, one identifies $\pi_1(X,\Sigma;b)$ as the quotient of the opposite groupoid $\Pi_1(X;b)^\circ$ by the action of Σ. This can be done in two steps. Firstly, one identifies $\Pi_1(X;b)^\circ$ with the groupoid $\mathcal{A}ut(X,b)$ of morphisms between universal coverings of (X,b_σ), $\sigma\in\Sigma$. Secondly, one shows that $\pi_1(X,\Sigma;b)$ is a quotient of $\mathcal{A}ut(X,b)$ by the action of Σ. Details are left to the reader. □

7. The equivariant fundamental group of a real algebraic curve having real points

Let Σ be the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and let σ be its nontrivial element. Let X be a compact connected real algebraic curve having real points, i.e., $X^\Sigma \neq \emptyset$. We determine the equivariant fundamental group $\pi_1(X,\Sigma)$ of the Σ-space X.

Since X is a compact connected Riemann surface, the topological space X is a compact connected orientable surface. Moreover, complex conjugation acts orientation-reversingly on X. The following lemma concerning the topological classification of such Σ-actions on compact connected orientable surfaces is well-known and easy to prove.

LEMMA 7.1.— *Let X and Y be compact connected orientable surfaces endowed with Σ-actions such that σ acts orientation-reversingly. Then, the Σ-spaces X and Y are equivariantly homeomorphic if and only if the quotient spaces X/Σ and Y/Σ are homeomorphic.* □

Using Lemma 7.1, we determine a convenient topological model for the underlying Σ-space of the real algebraic curve X. Let T be the tube $[-1,1]\times S^1$. We will consider two Σ-actions on T. The first one is defined by

$$\sigma\cdot(t,p) = (-t,p)$$

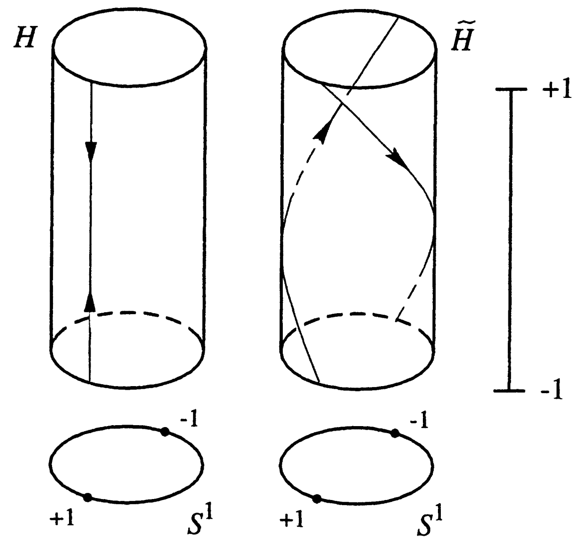

Fig. 1 The two handles H and $\tilde{H}$. Depicted are the equivariant paths α_1 and α_2 from $\Sigma \times [-1, +1]$ into H and $\tilde{H}$, respectively. The path α_1 is defined by $\alpha_1(\tau, t) = \tau \cdot (t, 1)$ for $(\tau, t) \in \Sigma \times [-1, +1]$ and α_2 is defined by $\alpha_2(\tau, t) = \tau \cdot (t, \exp(-\frac{1}{2}(t+1)\pi\sqrt{-1}))$ for $(\tau, t) \in \Sigma \times [-1 + 1]$.

for $(t, p) \in T$. The topological space T considered with this Σ-action will be denoted by H (for handle). The second action of Σ on T is defined by

$$\sigma(t, p) = (-t, e^{(t+1)\pi\sqrt{-1}} \cdot p)$$

for $(t, p) \in T$. The topological space T considered with this Σ-action will be denoted by $\tilde{H}$ (for twisted handle). (See Figure 1.)

Observe that $H^\Sigma = \{0\} \times S^1$ and $\tilde{H}^\Sigma = \emptyset$. Observe also that the quotient H/Σ is homeomorphic to T, whereas $\tilde{H}/\Sigma$ is homeomorphic to a Möbius band.

We will also need the double handle $\mathbb{T} = T \coprod T$ on which Σ acts by permuting the connected components. Observe that $\mathbb{T}^\Sigma = \emptyset$ and $\mathbb{T}/\Sigma$ is homeomorphic to T.

Let g be a natural integer. Choose g disjoint open discs $D_1, \ldots, D_g$ in the upper half-plane $\mathbb{U}$ and let $D_{-i} = \sigma \cdot D_i$, for $i = 1, \ldots, g$. Let C_g be the complement of the union $\bigcup D_i$ of these $2g$ discs in $\mathbb{P}^1(\mathbb{C})$. We consider the topological space C_g with its induced Σ-action.

Let $r, \tilde{r}$ and s be any natural integers such that $r + \tilde{r} + 2s = g$. We are going to glue in, in an equivariant way, r handles $H_1, \ldots, H_r$, $\tilde{r}$ twisted handles $\tilde{H}_1, \ldots, \tilde{H}_{\tilde{r}}$ and s double handles $\mathbb{T}_1, \ldots, \mathbb{T}_s$, making C_g into a compact connected orientable surface $S_{r,\tilde{r},s}$ of genus g on which Σ acts. The quotient $S_{r,\tilde{r},s}/\Sigma$ will be homeomorphic to the connected sum of $\tilde{r}$ real projective planes and a compact connected orientable surface of genus s having $r+1$ connected boundary components.

For $i = 1, \ldots, r$ choose a homeomorphism f_i from ∂D_i onto a boundary component of H_i. Define f_i^σ from ∂D_{-i} into H_i by $f_i^\sigma(p) = \sigma \cdot f_i(\sigma \cdot p)$ for all $p \in \partial D_{-i}$. Then, f_i^σ is a homeomorphism from ∂D_{-i} onto the other connected boundary component of H_i and the map

$$h_i \colon \partial(D_i \cup D_{-i}) \longrightarrow \partial H_i$$

which is the disjoint sum of f_i and f_i^σ is an equivariant homeomorphism. Similarly, one constructs equivariant homeomorphisms

$$h_{r+i} \colon \partial(D_{r+i} \cup D_{-(r+i)}) \longrightarrow \partial \tilde{H}_i$$

for $i = 1, \ldots, \tilde{r}$, and equivariant homeomorphisms

$$h_{r+\tilde{r}+i} \colon \partial(D_{r+\tilde{r}+2i-1} \cup D_{r+\tilde{r}+2i} \cup D_{-(r+\tilde{r}+2i-1)} \cup D_{-(r+\tilde{r}+2i)}) \longrightarrow \partial \mathbb{T}_i$$

for $i = 1, \ldots, s$. For the latter homeomorphisms, we may assume that $\partial(D_{r+\tilde{r}+2i-1} \cup D_{r+\tilde{r}+2i})$ is mapped into a connected component of $\mathbb{T}_i$, for $i = 1, \ldots, s$. Let

$$h \colon \partial C_g \longrightarrow \partial\Big(\big(\bigcup_{i=1}^{r} H_i\big) \cup \big(\bigcup_{i=1}^{\tilde{r}} \tilde{H}_i\big) \cup \big(\bigcup_{i=1}^{s} \mathbb{T}_i\big)\Big)$$

be the disjoint sum of the maps $h_1, \ldots, h_{r+\tilde{r}+s}$. Then, h is an equivariant homeomorphism.

We define $S_{r,\tilde{r},s}$ to be the surface obtained by gluing C_g and $(\bigcup H_i) \cup (\bigcup \tilde{H}_i) \cup (\bigcup \mathbb{T}_i)$ along their boundaries via h. Then, $S_{r,\tilde{r},s}$ is a compact connected orientable surface of genus g with an induced Σ action. In fact, the embeddings of the surfaces $C_g, H_1, \ldots, H_r, \tilde{H}_1, \tilde{H}_{\tilde{r}}, \mathbb{T}_1, \ldots, \mathbb{T}_s$ into $S_{r,\tilde{r},s}$ are equivariant. It follows that the quotient $S_{r,\tilde{r},s}/\Sigma$ is obtained from gluing C_g/Σ and $(\bigcup(H_i/\Sigma)) \cup (\bigcup(\tilde{H}_i/\Sigma)) \cup (\bigcup(\mathbb{T}_i/\Sigma))$ along certain boundary components via the induced homeomorphism $\bar{h}$. From this, it follows that $S_{r,\tilde{r},s}/\Sigma$ is homeomorphic to the connected sum of $\tilde{r}$ real projective planes and a compact connected orientable surface of genus s having $r+1$ connected boundary components.

Recall that the real algebraic curve X is called *dividing* if $X\backslash X^{\Sigma}$ is not connected. Let g be the genus of X. Let $r+1$ be the number of connected components of X^{Σ}. If X is dividing, then $r \equiv g \bmod 2$ and the quotient space X/Σ is homeomorphic to a compact connected orientable surface of genus $\frac{1}{2}(g-r)$ having $r+1$ connected boundary components. If X is not dividing, then the quotient space X/Σ is homeomorphic to the connected sum of $g-r$ real projective planes and a compact connected orientable surface of genus 0 having $r+1$ connected boundary components. The following lemma then follows from Lemma 7.1.

LEMMA 7.2.— *Let X be a compact connected real algebraic curve having real points. Let g be the genus of X.*

1. *If X is not dividing, there are unique natural integers r et $\tilde{r}$ such that X is equivariantly homeomorphic to $S_{r,\tilde{r},0}$.*
2. *If X is dividing, there are unique natural integers r et s such that X is equivariantly homeomorphic to $S_{r,0,s}$.* □

Now we determine the equivariant fundamental group $\pi_1(S,\Sigma)$ of the Σ-space $S = S_{r,\tilde{r},s}$ for all natural integers $r, \tilde{r}$ and s.

PROPOSITION 7.3.— *Let g be a natural integer. Let r, $\tilde{r}$ and s be natural integers such that $r+\tilde{r}+2s = g$. Then, the equivariant fundamental group $\pi_1(S,\Sigma)$ of $S = S_{r,\tilde{r},s}$ is the group generated by*

$$\alpha_0, \beta_0, \ldots, \alpha_r, \beta_r, \beta_{r+1}, \ldots, \beta_{r+\tilde{r}}, \alpha_{r+\tilde{r}+1}, \beta_{r+\tilde{r}+1}, \ldots, \alpha_{r+\tilde{r}+s}, \beta_{r+\tilde{r}+s},$$

subject to the relations

$$\beta_i^2 = 1 \text{ and } [\alpha_i, \beta_i] = 1 \text{ for } i = 0, \ldots, r, \text{ and}$$
$$\alpha_0 \cdots \alpha_r \cdot \beta_{r+1}^2 \cdots \beta_{r+\tilde{r}}^2 \cdot [\alpha_{r+\tilde{r}+1}, \beta_{r+\tilde{r}+1}] \cdots [\alpha_{r+\tilde{r}+s}, \beta_{r+\tilde{r}+s}] = 1$$

where $[x,y]$ denotes the commutator $xyx^{-1}y^{-1}$ of x and y.

Proof.— Let b be an injective equivariant base point of S such that $b(\Sigma)$ is contained in C_g. Choose, as shown in Figure 2, for each handle H_i, a pair (α_i, β_i) of ordinary paths in S, for each twisted handle $\tilde{H}_i$, a pair of ordinary paths $(\alpha_{r+i}, \beta_{r+i})$ in S, and for each double handle $\mathbb{T}_i$ a pair of ordinary paths $(\alpha_{r+\tilde{r}+i}, \beta_{r+\tilde{r}+i})$ in S. Choose, moreover, a pair of ordinary paths (α_0, β_0) of S contained in C_g as indicated in Figure 2. We assume that the paths α_i, β_i, $i = 0, \ldots, r+\tilde{r}+s$ are chosen in such a way that any two of them only intersect in the base point b_1.

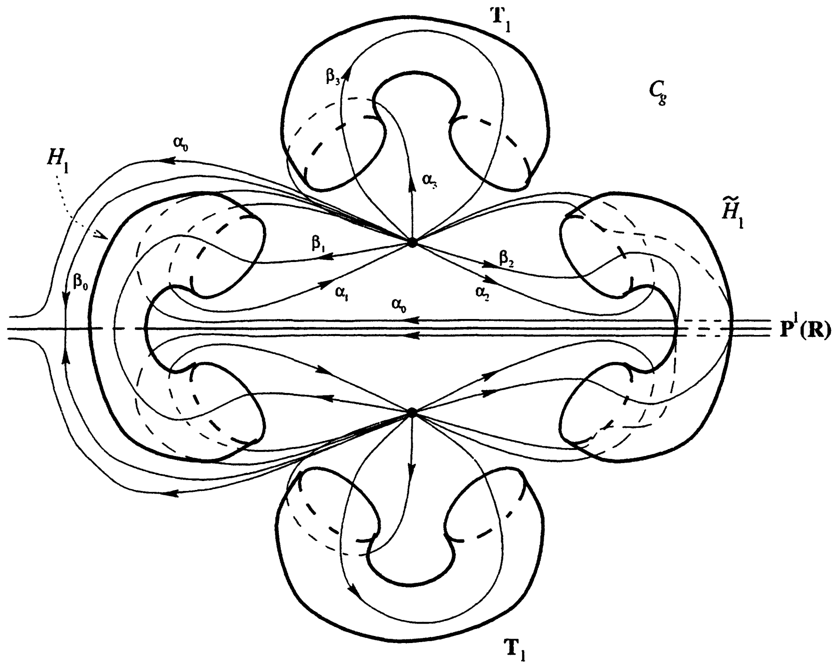

Fig. 2 The surface $S = S_{1,1,1}$. Each arrow represents a path in S. The unlabeled arrows are the complex conjugate paths α_i^σ and β_i^σ of the corresponding paths α_i and β_i, respectively.

Let α_i^σ and β_i^σ be the complex conjugate path of α_i and β_i, respectively, for $i = 0, \ldots, r+\tilde{r}+s$. Then, since the complement of the union of the paths is a disjoint union of open 2-cells, the paths $\alpha_i, \alpha_i^\sigma, \beta_i, \beta_i^\sigma$, $i = 0, \ldots, r+\tilde{r}+s$, generate the fundamental groupoid $\Pi_1(X, b)$. They only satisfy the relations

$$\beta_i^\sigma \beta_i = 1 \text{ and } \alpha_i^\sigma \beta_i \alpha_i^{-1} \beta_i^{-1} = 1, \text{ for } i = 0, \ldots, r$$
$$\beta_i^\sigma (\alpha_i^\sigma)^{-1} \beta_i = 1 \text{ and } \alpha_i^{-1} \beta_i^\sigma \beta_i = 1, \text{ for } i = r+1, \ldots, r+\tilde{r}$$
$$[\alpha_{r+\tilde{r}+s}, \beta_{r+\tilde{r}+s}] \cdots [\alpha_{r+\tilde{r}+1}, \beta_{r+\tilde{r}+1}] \cdot \alpha_{r+\tilde{r}} \cdots \alpha_0 = 1$$
$$[\alpha_{r+\tilde{r}+s}^\sigma, \beta_{r+\tilde{r}+s}^\sigma] \cdots [\alpha_{r+\tilde{r}+1}^\sigma, \beta_{r+\tilde{r}+1}^\sigma] \cdot \alpha_{r+\tilde{r}}^\sigma \cdots \alpha_0^\sigma = 1$$

Since $\lambda_1(X, \Sigma; b)$ is isomorphic to the quotient of the groupoid $\Pi_1(X; b)$ by the action of Σ, the group $\lambda_1(X, \Sigma; b)$ is the group generated by the elements

$$\alpha_0, \beta_0, \ldots, \alpha_{r+\tilde{r}+s}, \beta_{r+\tilde{r}+s}$$

subject to the relations

$$\beta_i^2 = 1 \text{ and } [\alpha_i, \beta_i] = 1 \text{ for } i = 0, \ldots, r$$

$$\alpha_i = \beta_i^2 \text{ for } i = r+1, \dots, r+\tilde{r}$$
$$[\alpha_{r+\tilde{r}+s}, \beta_{r+\tilde{r}+s}] \cdots [\alpha_{r+\tilde{r}+1}, \beta_{r+\tilde{r}+1}] \cdot \alpha_{r+\tilde{r}} \cdots \alpha_0 = 1.$$

The statement follows from the fact that $\pi_1(X, \Sigma; b)$ is isomorphic to the opposite group of $\lambda_1(X, \Sigma; b)$ (Proposition 6.1). □

THEOREM 7.4. — *Let X be a compact connected real algebraic curve. Let g be the genus of X.*

1. *If X does not have real points then the equivariant fundamental group $\pi_1(X, \Sigma)$ is the group generated by $\beta_0, \dots, \beta_g$ subject to the relation*

$$\beta_0^2 \cdots \beta_g^2 = 1.$$

Suppose now that X has real points and let $r+1$ be the number of connected components of its set of real points.

2. *If X is not dividing then the equivariant fundamental group $\pi_1(X, \Sigma)$ of X is the group generated by $\alpha_0, \beta_0, \dots, \alpha_r, \beta_r, \beta_{r+1}, \dots, \beta_g$ subject to the relations*

$$\beta_i^2 = 1 \textit{ and } [\alpha_i, \beta_i] = 1 \textit{ for } i = 0, \dots, r, \textit{ and}$$
$$\alpha_0 \cdots \alpha_r \cdot \beta_{r+1}^2 \cdots \beta_g^2 = 1.$$

3. *If X is dividing, let k be the natural integer $\frac{1}{2}(g+r)$. Then the equivariant fundamental group $\pi_1(X, \Sigma)$ of X is the group generated by $\alpha_0, \beta_0, \dots, \alpha_k, \beta_k$ subject to the relations*

$$\beta_i^2 = 1 \textit{ and } [\alpha_i, \beta_i] = 1 \textit{ for } i = 0, \dots, r, \textit{ and}$$
$$\alpha_0 \cdots \alpha_r \cdot [\alpha_{r+1}, \beta_{r+1}] \cdots [\alpha_k, \beta_k] = 1.$$

Proof. — Statement 1 is Lemma 5.1. The statements 2 and 3 follow from Lemma 7.2 and Proposition 7.3. □

8. Complex coordinates on Teichmüller space; the case of a nondividing real algebraic curve having real points

Fix an integer $g \geqslant 2$. Throughout this section, X is a nondividing compact connected real algebraic genus g curve having real points. Let r be the natural integer such that the number of connected components of the set of real points of X is equal to $r+1$.

Let $\pi: \mathbb{D} \to X$ be a uniformization of X as a real algebraic curve. Let G be the group of automorphisms of π. Then, G is isomorphic to the equivariant fundamental group $\pi_1(X, \Sigma)$ of X. By Theorem 7.4, G is the group generated by $\alpha_0, \beta_0, \ldots, \alpha_r, \beta_r, \beta_{r+1}, \ldots, \beta_g$ subject to the relations

$$\beta_i^2 = 1 \text{ and } [\alpha_i, \beta_i] = 1 \text{ for } i = 0, \ldots, r, \text{ and}$$
$$\alpha_0 \cdots \alpha_r \cdot \beta_{r+1}^2 \cdots \beta_g^2 = 1.$$

Observe that the elements

$$\alpha_1, \ldots, \alpha_r, \beta_{r+1}, \ldots, \beta_g \tag{1}$$

generate a free subgroup H of G of rank g. Therefore, these elements are loxodromic and we may then assume that the first element in the list (1) has 0 as attracting and ∞ as repelling fixed point, and that the second element in the list (1) has 1 as attracting fixed point.

As in Section 5, we define a map

$$\Psi: T(G) \longrightarrow S_g$$

of the Teichmüller space of G into the space of normalized marked Schottky groups of rank g. Let ι be an element of $T(G)$, i.e., $\iota: G \to \mathrm{PGL}_2(\mathbb{C})$ is a quasi-conformal deformation of G. Then, define

$$\Psi(\iota) = (\iota(\alpha_1), \ldots, \iota(\alpha_r), \iota(\beta_{r+1}), \ldots, \iota(\beta_g)).$$

PROPOSITION 8.1.— *The map Ψ is an open holomorphic embedding of $T(G)$ into S_g.*

Proof.— It suffices to prove that Ψ is injective. We do this by showing that a quasi-conformal deformation $\iota: G \to \mathrm{PGL}_2(\mathbb{C})$ of G is uniquely determined by its restriction $\iota_{|H}$ to the subgroup H of G generated by the elements of the list (1). Indeed, since $\alpha_0 \cdots \alpha_r \cdot \beta_{r+1}^2 \cdots \beta_g^2 = 1$,

$$\iota(\alpha_0) = \iota_{|H}(\beta_g^{-2} \cdots \beta_{r+1}^{-2} \cdot \alpha_r^{-1} \cdots \alpha_1^{-1}).$$

Moreover, since $\beta_i^2 = 1$ and β_i commutes with α_i, $\iota(\beta_i)$ is the unique element of $\mathrm{PGL}_2(\mathbb{C})$ of order 2 that commutes with $\iota(\alpha_i)$, for $i = 0, \ldots, r$. □

Remark 8.2.— Note the resemblance with the proof of Theorem 2 in [4].

Let $\Phi: T(G) \to T(X)$ be the natural biholomorphic map, and let $\Xi: S_g \to \mathbb{C}^{3g-3}$ be any equivariant open holomorphic embedding.

COROLLARY 8.3. — *Let X be a nondividing compact connected real algebraic genus g curve having real points, where $g \geqslant 2$. Then, the map*

$$\Xi \circ \Psi \circ \Phi^{-1} : T(X) \longrightarrow \mathbb{C}^{3g-3}$$

is a global system of complex analytic coordinates on the Teichmüller space $T(X)$ of complex algebraic curves of genus g. Moreover, the coordinate system is equivariant with respect to the induced action of Σ on $T(X)$ and the usual action of Σ on $\mathbb{C}^{3g-3}$.

9. Complex coordinates on Teichmüller space; the case of a dividing real algebraic curve

Fix an integer $g \geqslant 2$. Throughout this section, X is a dividing compact connected real algebraic genus g curve. In particular, X has real points. Let, as before, r be the natural integer such that the number of connected components of the set of real points of X is equal to $r+1$. Let k be the natural integer $\frac{1}{2}(g+r)$.

Let $\pi: \mathbb{D} \rightarrow X$ be a uniformization of X as a real algebraic curve. Let G be the group of automorphisms of π. Then, G is isomorphic to the equivariant fundamental group $\pi_1(X, \Sigma)$ of X. By Theorem 7.4, G is the group generated by $\alpha_0, \beta_0, \ldots, \alpha_k, \beta_k$ subject to the relations

$$\beta_i^2 = 1 \text{ and } [\alpha_i, \beta_i] = 1 \text{ for } i = 0, \ldots, r, \text{ and}$$
$$\alpha_0 \cdots \alpha_r \cdot [\alpha_{r+1}, \beta_{r+1}] \cdots [\alpha_k, \beta_k] = 1.$$

Observe that the elements

$$\alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \beta_{r+1}, \ldots, \alpha_k, \beta_k \tag{2}$$

generate again a free subgroup H of G of rank g. Hence, we may assume that the first element in the list (2) has 0 as attracting and ∞ as repelling fixed point, and that the second element in the list (2) has 1 as attracting fixed point.

Now, we define a map

$$\Psi: T(G) \longrightarrow S_g$$

of the Teichmüller space of G into the space of normalized marked Schottky groups of rank g by

$$\Psi(\iota) = (\iota(\alpha_1), \ldots, \iota(\alpha_r), \iota(\alpha_{r+1}), \iota(\beta_{r+1}), \ldots, \iota(\alpha_k), \iota(\beta_k))$$

for $\iota \in T(G)$. Then, one can similarly prove the following statement.

PROPOSITION 9.1. — *The map Ψ is an open holomorphic embedding of $T(G)$ into S_g.* □

Let $\Phi: T(G) \to T(X)$ be the natural biholomorphic map, and let $\Xi: S_g \to \mathbb{C}^{3g-3}$ be any equivariant open holomorphic embedding.

COROLLARY 9.2. — *Let X be a dividing compact connected real algebraic genus g curve having real points, where $g \geqslant 2$. Then, the map*

$$\Xi \circ \Psi \circ \Phi^{-1}: T(X) \longrightarrow \mathbb{C}^{3g-3}$$

is a global system of complex analytic coordinates on the Teichmüller space $T(X)$ of complex algebraic curves of genus g. Moreover, the coordinate system is equivariant with respect to the induced action of Σ on $T(X)$ and the usual action of Σ on $\mathbb{C}^{3g-3}$.

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