ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

F. D. ARARUNA
G. O. ANTUNES
L. A. MEDEIROS

Semilinear wave equation on manifolds

Annales de la faculté des sciences de Toulouse 6e série, tome 11, nº 1 (2002), p. 7-18
<http://www.numdam.org/item?id=AFST_2002_6_11_1_7_0>

Annales de la Faculté des Sciences de Toulouse Vol. XI, n° 1, 2002
pp. 7–18

Semilinear wave equation on manifolds (*)

F. D. Araruna, G. O. Antunes and L. A. Medeiros (1)

Dedicated to M. Milla Miranda in the occasion of his 60th. anniversary.

Résumé. — Dans ce travail nous étudions un problème pour les équations des ondes non linéaire définies dans une varieté. Ce problème a été motivé par J.L.Lions [8], p. 134. Pour l'existence de solutions nous avons appliqué la méthode de Galerkin. Le comportement asymptotique des solutions a été examiné aussi.

Abstract. — In this paper, we study a type of second order evolution equation on the lateral boundary Σ of the cylinder $Q = \Omega \times]0, T[$, with Ω an open bounded set of $\mathbb{R}^n$. In this problem is fundamental that the unknown function solves an elliptic problem on Ω. This results are motivated by Lions [8], pg. 134 where he works with another type of nonlinearity.

1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^n$ $(n \geqslant 1)$ with smooth boundary Γ. Let ν be the outward normal unit vector to Γ and $T > 0$ a real number. We consider the cylinder $Q = \Omega \times]0, T[$ with lateral boundary $\Sigma = \Gamma \times]0, T[$.

We investigate existence and asymptotic behaviour of weak solution for the problem

$$\left| \begin{array}{l} w'' + \dfrac{\partial w}{\partial \nu} + F(w) + \beta(x) w' = 0 \quad \text{on} \quad \Sigma, \\ w(0) = w_0, \quad w'(0) = w_1 \quad \text{on} \quad \Gamma, \end{array} \right. \tag{1.1}$$

(*) Reçu le 24 janvier 2002, accepté le 4 septembre 2002
(1) Partially supported by PCI-LNCC-MCT-2001.
Instituto de Matemática, Universidade Federal do Rio de Janeiro, Caixa Postal 68530, 21945-970, Rio de Janeiro-RJ, Brasil.
E-mail: araruna@pg.im.ufrj.br, gladson@email.com.br, lmedeiros@abc.org.br

where the prime means the derivative with respect to t, $\dfrac{\partial w}{\partial \nu}$ normal derivative of w and $F : \mathbb{R} \to \mathbb{R}$ is a function that satisfies

$$F \text{ continuous and } sF(s) \geqslant 0,\ \forall s \in \mathbb{R}. \tag{1.2}$$

It is important to call the attention to the reader that the idea employed in this work comes from Lions [8], pg. 134. The main point consists in adding to (1.1) an elliptic equation in Ω to reduce the problem to a canonical model of Mathematical Physics, but in this case on a manifold which is the lateral boundary Σ of the cylinder Q . A Similar type of problem, also motivated by Lions [8], can be seen in Cavalvanti and Domingos Cavalcanti [2].

The plan of this article is the following: In the section 2, we give notations, terminology and we treat the linear case associated to (1.1). In the section 3, we prove existence for weak solution when F satisfies the condition (1.2), approximating F by Lipschtz functions. In this Lipschitz case, we employ Picard's successive approximations and then we apply the Strauss' method [9]. Finally in the section 4, we obtain the asymptotic behaviour by the method of pertubation of energy as in Zuazua [10].

2. Notations, Assumptions and Results

Denote by $|\cdot|$, $(\cdot,\cdot)$ and $\|\cdot\|$, $((\cdot,\cdot))$ the inner product and norm, respectively, of $L^2(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$.

For

$$G(s) = \int_0^s F(\sigma)d\sigma$$

we will denote a primitive of F.

We consider the following assumption on β in (1.1) :

$$\beta \in L^\infty(\Gamma) \text{ such that } \beta(x) \geqslant \beta_0 > 0, \text{ a.e. on } \Gamma. \tag{2.1}$$

As was said in the introduction, for $\lambda > 0$, let us consider the problem

$$\left|\begin{array}{l} -\Delta w + \lambda w = 0 \quad \text{in} \quad Q, \\ w'' + \dfrac{\partial w}{\partial \nu} + F(w) + \beta(x) w' = 0 \quad \text{on} \quad \Sigma, \\ w(0) = w_0, \quad w'(0) = w_1 \quad \text{on} \quad \Gamma. \end{array}\right. \tag{2.2}$$

From elliptic theory, we know that for $\varphi \in H^{\frac{1}{2}}(\Gamma)$, the solution Φ of the boundary value problem

$$\left|\begin{array}{ll} -\Delta\Phi + \lambda\Phi = 0 & \text{in} \quad \Omega, \\ \Phi = \varphi & \text{on} \quad \Gamma, \end{array}\right. \tag{2.3}$$

belongs to $H^1(\Omega,\Delta)=\left\{u\in H^1(\Omega)\,;\Delta u\in L^2(\Omega)\right\}$. By the trace theorem, it follows that $\dfrac{\partial\Phi}{\partial\nu}\in H^{-\frac{1}{2}}(\Gamma)$.

Formally, we have by (2.3) that

$$0=\int_\Omega\nabla\Phi\nabla\Psi dx+\lambda\int_\Omega\Phi\Psi dx-\int_\Gamma\frac{\partial\Phi}{\partial\nu}\Psi d\Gamma.$$

We take $\Psi\in H^1(\Omega,\Delta)$ and we define

$$a(\Phi,\Psi)=\int_\Omega\nabla\Phi\nabla\Psi dx+\lambda\int_\Omega\Phi\Psi dx \tag{2.4}$$

Thus, by (2.4)

$$a(\Phi,\Psi)=\langle\gamma_1\Phi,\gamma_0\Psi\rangle,$$

where γ_0 and γ_1 are the traces of order zero and one, respectively, and $\langle\cdot,\cdot\rangle$ represents the duality pairing between $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$.

We consider the scheme

$$\begin{array}{ccccc}\varphi\in H^{\frac{1}{2}}(\Gamma) & & \xrightarrow{\gamma_0^{-1}} & & \Phi\in H^1(\Omega,\Delta)\\ & \searrow_A & & \swarrow_{\gamma_1} & \\ & & \dfrac{\partial\Phi}{\partial\nu}\in H^{-\frac{1}{2}}(\Gamma) & & \end{array}$$

Thus

$$A=\gamma_1\circ\gamma_0^{-1}:H^{\frac{1}{2}}(\Gamma)\to H^{-\frac{1}{2}}(\Gamma),\ A\varphi=\frac{\partial\Phi}{\partial\nu}.$$

Therefore A is self-adjoint and $A\in\mathcal{L}\left(H^{\frac{1}{2}}(\Gamma),H^{-\frac{1}{2}}(\Gamma)\right)$.

Moreover, we have

$$\langle A\varphi,\varphi\rangle=a(\Phi,\Phi) \tag{2.5}$$

and so by (2.4) we get

$$\langle A\varphi,\varphi\rangle=\int_\Omega|\nabla\Phi|^2dx+\lambda\int_\Omega|\Phi|^2dx\geqslant\min\{1,\lambda\}\|\Phi\|^2_{H^1(\Omega)}\geqslant$$
$$\geqslant\alpha\|\gamma_0\Phi\|^2=\alpha\|\varphi\|^2,$$

proving that A is positive.

We formulate now the problem on Σ. For this, we define

$$w(t)\,|_\Gamma = u(t) \quad \text{and} \quad \frac{\partial w(t)}{\partial \nu}\,|_\Gamma = Au(t).$$

In this way, the problem (1.2) is reduced to find a function $u : \Sigma \to \mathbb{R}$ such that

$$\left| \begin{array}{l} u'' + Au + F(u) + \beta(x)\,u' = 0 \quad \text{on} \quad \Sigma, \\ u(0) = u_0, \quad u'(0) = u_1 \quad \text{on} \quad \Gamma, \end{array} \right.$$

which will be investigated in the section 3.

Firstly we will state a result that guarantees the existence and uniqueness of solution for the linear problem associated the (1.1).

THEOREM 2.1.— *Given* $(u_0, u_1, f) \in H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma) \times L^2(\Sigma)$, *there exists a unique function* $u : \Sigma \to \mathbb{R}$ *such that*

$$u \in C^0\left(0, T; H^{\frac{1}{2}}(\Gamma)\right) \cap C^1\left(0, T; L^2(\Gamma)\right), \tag{2.6}$$

$$u'' + Au + \beta u' = f \quad in \quad L^2\left(0, T; H^{-\frac{1}{2}}(\Gamma)\right), \tag{2.7}$$

$$u(0) = u_0, \quad u'(0) = u_1 \quad on \quad \Gamma. \tag{2.8}$$

Moreover we have the energy inequality

$$\frac{1}{2}\left|u'(t)\right|^2 + \frac{\alpha}{2}\left\|u(t)\right\|^2 \leqslant \frac{1}{2}\left|u_1\right|^2 + \frac{\alpha}{2}\left\|u_0\right\|^2 + \int_0^T \left(f(s), u'(s)\right) ds, \ a.e \ in \ [0,T]. \tag{2.9}$$

Proof. — In the proof of this linear case, we employ the Faedo-Galerkin's method. □

3. Existence of Solution

The goal of this section is to obtain existence of solutions for the problem (1.1).

THEOREM 3.1.— *Consider* F *satisfying* (1.2) *and suppose*

$$(u_0, u_1) \in H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma) \ \ and \ G(u_0) \in L^1(\Gamma).$$

Then there exists a function $u : \Sigma \to \mathbb{R}$ *such that*

$$u \in L^\infty\left(0, T; H^{\frac{1}{2}}(\Gamma)\right), \tag{3.1}$$

$$u' \in L^\infty\left(0, T; L^2(\Gamma)\right), \tag{3.2}$$

$$u'' + Au + F(u) + \beta u' = 0 \quad in \quad L^1\left(0,T;H^{-\frac{1}{2}}(\Gamma) + L^1(\Gamma)\right), \tag{3.3}$$

$$u(0) = u_0, \quad u'(0) = u_1 \quad on \quad \Gamma. \tag{3.4}$$

To prove the Theorem 3.1, the following Lemma will be used:

LEMMA 3.1.— *Assume that* $(u_0, u_1) \in H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma)$ *and suppose that the function* F *satisfies*

$$F : \mathbb{R} \to \mathbb{R} \text{ be Lipschitz function such that } sF(s) \geqslant 0, \ \forall s \in \mathbb{R}. \tag{3.5}$$

Then there exists only one function $u : \Sigma \to \mathbb{R}$ *satisfying the conditions*

$$u \in L^\infty\left(0,T;H^{\frac{1}{2}}(\Gamma)\right), \tag{3.6}$$

$$u' \in L^\infty\left(0,T;L^2(\Gamma)\right), \tag{3.7}$$

$$u'' + Au + F(u) + \beta u' = 0 \quad in \quad L^2\left(0,T;H^{-\frac{1}{2}}(\Gamma)\right), \tag{3.8}$$

$$u(0) = u_0, \quad u'(0) = u_1 \quad on \quad \Gamma. \tag{3.9}$$

Furthermore

$$\begin{aligned} &\frac{1}{2}|u'(t)|^2 + \frac{\alpha}{2}\|u(t)\|^2 + \int_\Gamma G(u(x,t))\,d\Gamma \leqslant \frac{1}{2}|u_1|^2 + \\ &+\frac{\alpha}{2}\|u_0\|^2 + \int_\Gamma G(u_0(x))\,d\Gamma, \ a.e \ in \ [0,T]. \end{aligned} \tag{3.10}$$

Proof of Lemma 3.1.— The proof will be done employing the Picard successive approximations method. Let us consider the sequence of successives approximations

$$u_0, \ u_1, \ u_2, \ \ldots, \ u_n, \ \ldots \tag{3.11}$$

defined as the solutions of the linear problems

$$\left| \begin{array}{l} u_n'' + Au_n + F(u_{n-1}) + \beta u_n' = 0 \quad \text{on} \quad \Sigma, \\ u_n(0) = u_0, \quad u_n'(0) = u_1 \quad \text{on} \quad \Gamma. \end{array} \right. \tag{3.12}$$

Using that F is Lipschitz and from Theorem 2.1, one can prove, using induction, that (3.12) has a solution for each $n \in \mathbb{N}$ with the regularity claimed in the Theorem 2.1. We will prove now that the sequence (3.11) converges to a function $u : \Sigma \to \mathbb{R}$ in the conditions of the Lemma 3.1.

For this end, we define $v_n = u_n - u_{n-1}$ which is the unique solution of the problem

$$\left| \begin{array}{l} v_n'' + Av_n + F(u_{n-1}) - F(u_{n-2}) + \beta v_n' = 0 \quad \text{on} \quad \Sigma, \\ v_n(0) = 0, \quad v_n'(0) = 0 \quad \text{on} \quad \Gamma. \end{array} \right. \tag{3.13}$$

By the energy inequality (2.9), we have

$$\frac{1}{2}|v_n'(t)|^2 + \frac{\alpha}{2}\|v_n(t)\|^2 \leqslant -\int_0^t (F(u_{n-1}) - F(u_{n-2}), v_n'(s))\, ds. \tag{3.14}$$

Set

$$e_n(t) = \underset{s\in]0,t[}{ess\sup}\left\{\frac{1}{2}|v_n'(s)|^2 + \frac{\alpha}{2}\|v_n(s)\|^2\right\}. \tag{3.15}$$

Thus, since F is Lipschitz, we have

$$-\int_0^t (F(u_{n-1}) - F(u_{n-2}), v_n'(s))\, ds \leqslant C\int_0^t |v_{n-1}(s)|^2\, ds + \frac{1}{2}e_n(t). \tag{3.16}$$

We have also

$$|v_{n-1}(s)|^2 \leqslant Ce_{n-1}(s). \tag{3.17}$$

Combining $(3.14) - (3.17)$, we get

$$e_n(t) \leqslant C\int_0^t e_{n-1}(s)\, ds,$$

and, by interation, we obtain, for $n = 1, 2, ...,$ that

$$e_n(t) \leqslant e_0 C_T \frac{(Ct)^n}{n!},$$

hence, we conclude that the series $\sum_{n=1}^{\infty} e_n(t)$ is uniformly convergent on $]0,T[$. By the definition of $e_n(t)$, see (3.15), it follows that the series $\sum_{n=1}^{\infty}(u_n' - u_{n-1}')$ and $\sum_{n=1}^{\infty}(u_n - u_{n-1})$ are convergents in the norms of $L^\infty(0,T;L^2(\Gamma))$ and $L^\infty\left(0,T;H^{\frac{1}{2}}(\Gamma)\right)$, respectively. Therefore, there exists $u : \Sigma \to \mathbb{R}$ such that

$$u_n \to u \text{ strong in } L^\infty\left(0,T;H^{\frac{1}{2}}(\Gamma)\right), \tag{3.18}$$

$$u_n' \to u' \text{ strong in } L^\infty\left(0,T;L^2(\Gamma)\right). \tag{3.19}$$

Since F is Lipschitz, we have by (3.18) that

$$F(u_n) \to F(u) \text{ strong in } L^\infty\left(0,T;L^2(\Gamma)\right). \tag{3.20}$$

Then, by the convergences (3.18) – (3.20), we can pass to the limit in (3.12) and we obtain, by standard procedure, a unique function u satisfying (3.6) – (3.10). □

We will prove now the main result.

Proof of Theorem 3.1.— By Strauss [9], there exists a sequence of functions $(F_\nu)_{\nu\in\mathbb{N}}$, such that each $F_\nu : \mathbb{R} \to \mathbb{R}$ is Lipschitz and $(F_\nu)_{\nu\in\mathbb{N}}$ approximates F uniformly on bounded sets of $\mathbb{R}$. Since the initial data u_0 is not necessarily bounded, we have to approximate u_0 by bounded functions of $H^{\frac{1}{2}}(\Gamma)$. We consider the functions $\xi_j : \mathbb{R} \to \mathbb{R}$ defined by

$$\xi_j(s) = \left| \begin{array}{lll} -j, & \text{if} & s < -j, \\ s, & \text{if} & |s| \leqslant j, \\ j, & \text{if} & s > j. \end{array} \right.$$

Considering $\xi_j(u_0) = u_{0j}$, we have by Kinderlehrer and Stampacchia [5] that the sequence $(u_{0j})_{j\in\mathbb{N}} \subset H^{\frac{1}{2}}(\Gamma)$ is bounded a.e. in Γ and

$$u_{0j} \to u_0 \text{ strong in } H^{\frac{1}{2}}(\Gamma). \tag{3.21}$$

Thus, for $(u_{0j}, u_1) \in H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma)$, the Lemma 3.1 says that there exists only one solution $u_{j\nu} : \Sigma \to \mathbb{R}$ satifying (3.6) – (3.9) and the energy inequality

$$\begin{aligned} &\frac{1}{2}\left|u'_{j\nu}(t)\right|^2 + \frac{\alpha}{2}\left\|u_{j\nu}(t)\right\|^2 + \int_\Gamma G_\nu(u_{j\nu}(x,t))\,d\Gamma \leqslant \frac{1}{2}|u_1|^2 + \\ &+\frac{\alpha}{2}\left\|u_{0j}\right\|^2 + \int_\Gamma G_\nu(u_{0j}(x))\,d\Gamma. \end{aligned} \tag{3.22}$$

We need an estimate for the term $\int_\Gamma G_\nu(u_{0j}(x))\,d\Gamma$. Since u_{0j} is bounded a.e. in Γ, $\forall j \in \mathbb{N}$, it follows that

$$F_\nu(u_{0j}) \to F(u_{0j}) \text{ uniform in } \Gamma.$$

So

$$\int_\Gamma G_\nu(u_{0j}(x))\,d\Gamma \to \int_\Gamma G(u_{0j}(x))\,d\Gamma \text{ uniform in } \mathbb{R}. \tag{3.23}$$

From (3.21), there exists a subsequence of $(u_{0j})_{j\in\mathbb{N}}$, which will still be denoted by $(u_{0j})_{j\in\mathbb{N}}$, such that

$$u_{0j} \longrightarrow u_0 \text{ a.e. in } \Gamma.$$

Hence, by continuity of G, we have that $G(u_{0j}) \to G(u_0)$ a.e. in Γ. We also have that $G(u_{0j}) \leqslant G(u_0) \in L^1(\Gamma)$. Thus, by the Lebesgue's dominated convergence theorem, we get

$$G(u_{0j}) \longrightarrow G(u_0) \text{ strong in } L^1(\Gamma). \tag{3.24}$$

Then, by (3.23) and (3.24), we obtain that

$$\int_\Gamma G_\nu(u_{0j}(x))\, d\Gamma \leqslant C, \tag{3.25}$$

where C is independent of j and ν. In this way, using (3.21) and (3.25), we have from (3.22) that

$$|u'_{j\nu}|^2 + \|u_{j\nu}\|^2 + \int_\Gamma G(u_{j\nu}(x,t))\, d\Gamma \leqslant C, \tag{3.26}$$

where C is independent of j, ν and t.

From (3.26), we obtain that

$$(u_{j\nu}) \text{ is bounded in } L^\infty\left(0,T;H^{\frac{1}{2}}(\Gamma)\right), \tag{3.27}$$

$$(u'_{j\nu}) \text{ is bounded in } L^\infty\left(0,T;L^2(\Gamma)\right). \tag{3.28}$$

We have that (3.27) and (3.28) are true for all pairs $(j,\nu) \in \mathbb{N}^2$, in particular, for $(i,i) \in \mathbb{N}^2$. Thus, there exists a subsequence of (u_{ii}), which we denote by (u_i), and a function $u : \Sigma \to \mathbb{R}$, such that

$$u_i \longrightarrow u \text{ weak star in } L^\infty\left(0,T;H^{\frac{1}{2}}(\Gamma)\right), \tag{3.29}$$

$$u'_i \longrightarrow u' \text{ weak in } L^\infty\left(0,T;L^2(\Gamma)\right). \tag{3.30}$$

We also have by (3.8) that

$$u''_i + Au_i + F_i(u_i) + \beta u'_i = 0 \quad \text{in} \quad L^2\left(0,T;H^{-\frac{1}{2}}(\Gamma)\right). \tag{3.31}$$

From (3.29), (3.30) and observing that the injection of $H^1(\Sigma)$ in $L^2(\Sigma)$ is compact, there exists a subsequence of (u_i), which we still denote by (u_i), such that

$$u_i \longrightarrow u \text{ a.e. in } \Sigma.$$

Since F is continuous

$$F(u_i) \to F(u) \text{ a.e. in } \Sigma.$$

Furthermore, since $u_i(x,t)$ is bounded in $\mathbb{R}$,

$$F_i(u_i) - F(u_i) \to 0 \text{ a.e. in } \Sigma.$$

Therefore, we conclude

$$F_i(u_i) \to F(u) \text{ a.e. in } \Sigma. \tag{3.32}$$

Taking duality between (3.31) and u_i we obtain

$$\begin{aligned} \int_0^T (F_i(u_i), u_i(t))\, dt = \int_0^T |u_i'(t)|^2 dt - \alpha \int_0^T \|u_i(t)\|^2 dt - \\ - (u_i'(T), u_i(T)) + (u_1, u_{0j}) - \int_0^T (\beta u_i'(t), u_i(t))\, dt. \end{aligned} \tag{3.33}$$

Using (2.1), (3.6) and (3.7), we have by (3.33) that

$$\int_0^T (F_i(u_i), u_i(t))\, dt \leqslant C, \tag{3.34}$$

where C is independent of i.

Thus, from (3.32) and (3.34), it follows by Strauss' theorem, see Strauss [9], that

$$F_i(u_i) \to F(u) \text{ strongly in } L^1(\Sigma). \tag{3.35}$$

By (3.29), (3.30) and (3.35) it is permissible to pass to the limit in (3.31) obtaining a function $u : \Sigma \to \mathbb{R}$ satisfying (3.1) – (3.4). □

4. Asymptotic Behaviour

In this section we study the exponential decay for the energy $E(t)$ associated to the weak solution u given by the Theorem 3.1. This energy is given by

$$E(t) = \frac{1}{2}|u'(t)|^2 + \frac{\alpha}{2}\|u(t)\|^2 + \int_\Gamma G(u(x,t))d\Gamma,\ t \geqslant 0. \tag{4.1}$$

We consider the followings additional hypothesis:

$$0 \leqslant G(s) \leqslant sF(s),\ \forall s \in \mathbb{R} \tag{4.2}$$

THEOREM 4.1.— *Let F satisfying* (1.2) *and* (4.2). *Then the energy* (4.1) *satisfies*

$$E(t) \leqslant 4E(0)e^{-\frac{\epsilon}{2}t}, \tag{4.3}$$

where ϵ is a positive constant.

Proof.— For an arbitrary $\epsilon > 0$, we define the perturbed energy

$$E_{\nu\epsilon}(t) = E_\nu(t) + \epsilon\eta(t) \tag{4.4}$$

where $E_\nu(t)$ is the energy similar to (4.1) associated to the solution obtained in the Lemma 3.1 and

$$\eta(t) = (u_\nu(t), u'_\nu(t)).$$

Note that

$$|\eta(t)| \leqslant C_2 E_\nu(t),$$

where $C_2 = \max\left\{C_1, \frac{1}{\alpha}\right\}$, and C_1 is the immersion constant of $H^{\frac{1}{2}}(\Gamma)$ into $L^2(\Gamma)$.

Then,

$$|E_{\nu\epsilon}(t) - E_\nu(t)| \leqslant \epsilon C_2 E_\nu(t),$$

or

$$(1 - \epsilon C_2) E_\nu(t) \leqslant E_{v\epsilon}(t) \leqslant (1 + \epsilon C_2) E_v(t).$$

Taking $0 < \epsilon \leqslant \frac{1}{2C_2}$, we get

$$\frac{E_\nu(t)}{2} \leqslant E_{\nu\epsilon}(t) \leqslant 2E_\nu(t), \; \forall t \geqslant 0. \tag{4.5}$$

Multiplying the equation in (3.8) for u'_ν, using (2.1) and the fact of A to be positive, we obtain

$$E'_\nu(t) \leqslant -\beta_0 \left|u'_\nu(t)\right|^2 \leqslant 0. \tag{4.6}$$

Differentiating the function $\eta(t)$ and using (3.8), (4.2) and the fact of A to be positive comes that

$$\eta'(t) \leqslant \left(1 + \frac{\beta_1}{2\mu}\right)\left|u'_\nu(t)\right|^2 + \left(\frac{\beta_1\mu C_1}{2} - \alpha\right)\|u_\nu(t)\|^2 - \int_\Gamma G_\nu(u_\nu)d\Gamma, \tag{4.7}$$

where $\beta_1 = \|\beta\|_{L^\infty(\Gamma)}$ and $\mu > 0$ to be chosen.

It follows by (4.4), (4.6) and (4.7) that

$$E'_{\nu\epsilon}(t) \leqslant \left[\epsilon\left(1+\frac{\beta_1}{2\mu}\right)-\beta_0\right]\left|u'_\nu(t)\right|^2-\epsilon\left(\alpha-\frac{\beta_1\mu C_1}{2}\right)\|u_\nu(t)\|^2 - \\ -\epsilon\int_\Gamma G_\nu(u_\nu)d\Gamma. \tag{4.8}$$

Taking $\mu=\dfrac{\alpha}{\beta_1 C_1}$ and $0<\epsilon\leqslant\dfrac{2\alpha\beta_0}{3\alpha+\beta_1^2 C_1}$ we get

$$E'_{\nu\epsilon}(t)\leqslant -E_\nu(t). \tag{4.9}$$

Choosing $\epsilon\leqslant\min\left\{\dfrac{1}{2C_2},\dfrac{2\alpha\beta_0}{3\alpha+\beta_1^2C_1}\right\}$ then (4.5) and (4.9) occur simultaneously, therefore

$$E'_{\nu\epsilon}(t)+\frac{\epsilon}{2}E_{\nu\epsilon}(t)\leqslant 0,$$

that is,

$$E_\nu(t)\leqslant 4E_\nu(0)e^{-\frac{\epsilon}{2}t}. \tag{4.10}$$

From (3.29) , (3.30) and since G_ν is continuous, we have

$$G_\nu(u_\nu(\cdot,t))-G_\nu(u(\cdot,t))\to 0 \text{ a.e. in } \Gamma,\ \forall t\geqslant 0. \tag{4.11}$$

But we know that $F_\nu\to F$ uniformly on bounded sets of $\mathbb{R}$. Then

$$G_\nu\left(u\left(\cdot,t\right)\right)\to G\left(u\left(\cdot,t\right)\right) \text{ a.e. in } \Gamma,\ \forall t\geqslant 0. \tag{4.12}$$

Thus, by (4.11) and (4.12)

$$G_\nu\left(u_\nu\left(\cdot,t\right)\right)\to G\left(u\left(\cdot,t\right)\right) \text{ a.e. in } \Gamma,\ \forall t\geqslant 0. \tag{4.13}$$

Moreover, we have, by (4.10) , that

$$\int_\Gamma G_\nu\left(u_\nu\left(x,t\right)\right)d\Gamma\leqslant 4E_\nu\left(0\right).$$

Therefore, using (3.21) , (3.23) and (3.24) , we get

$$\liminf_{\nu\to\infty}\int_\Gamma G_\nu\left(u_\nu\left(x,t\right)\right)d\Gamma\leqslant 4E\left(0\right). \tag{4.14}$$

By (4.13), (4.14) and Fatou's lemma, we have

$$\int_{\Gamma} G(u(x,t))\, d\Gamma \leqslant \liminf_{\nu \to \infty} \int_{\Omega} G_{\nu}(u_{\nu}(x,t))\, d\Gamma.$$

Hence, passing $\liminf_{\nu \to \infty}$ in (4.10), we get (4.3). □

Remark. — In the existence we can take $\lambda = 0$. For this end, we define in $H^1(\Omega)$ the norm

$$[v]^2 = \int_{\Omega} |\nabla v|^2\, dx + \int_{\Gamma} |\gamma_0 v|^2\, d\Gamma, \quad \forall v \in H^1(\Omega),$$

obtaining now the positivity of operator $A + \zeta I$, for $\zeta > 0$ arbitrary, like in Lions [8]. For the asymptotic behaviour, we need the additional hypothesis $\beta_0 > \zeta$.

Bibliography

[1] BREZIS (H.). — Analyse Fonctionelle, Théorie et Applications, Dunod, Paris, 1999.

[2] CAVALCANTI (M. M.) and DOMINGOS CAVALCANTI (V. N.). — On Solvability of Solutions of Degenerate Nonlinear Equations on Manifolds, Differential and Integral Equation, vol. 13 (10-12), (2000), 1445-1458.

[3] DAFERMOS (C. M.). — Almost periodic processes and almost periodic solutions of evolution equations, Proceedings of a university of Florida international symposium, 1977.

[4] HARAUX (A.). — Semilinear Hyperbolic problems in bounded domains, Harwood, Gordon & Breach, 1987.

[5] KINDERLEHRER (D.) and STAMPACCHIA (G.). — An Introduction to Variational Inegualities and their Applications, Academic Press, New York, 1980.

[6] LASALLE. — Some extensions of Liapunov's second method, IRE Trans. Circuit theory CT-7, (1960), 520-527.

[7] LIONS (J. L.) et MAGENES (E.). — Problèmes aux limites non homogènes et applications, Dunod, Paris, 1968-1970.

[8] LIONS (J. L.). — Quelques Méthodes de Résolutions des Problèmes aux Limites Non-Linéaires, Dunod, Paris, 1969.

[9] STRAUSS (W. A.). — On Weak Solutions of Semilinear Hyperbolic Equations, An. Acad. Bras. Ciências, 42(4), (1970), 645-651.

[10] ZUAZUA (E.). — Stabilty and Decay for a class of Nonlinear Hyperbolic Problems. Asymptotic Analysis 1, (1988), 161-185.