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Optimal Lipschitz estimates for the $\overline{\partial}$ equation on a class of convex domains [*]

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RÉSUMÉ. — Dans ce travail, nous considérons l'équation de Cauchy-Riemann $\overline{\partial}u = f$ dans une nouvelle classe de domaines convexes de $\mathbb{C}^n$. Nous prouvons que si la donnée f est dans l'espace L^p, alors il existe une solution u dans un espace de Lipschitz Λ_α, où le nombre $\alpha > 0$ donné explicitement en fonction de p est optimal.

ABSTRACT. — In this paper, we consider the Cauchy-Riemann equation $\overline{\partial}u = f$ in a new class of convex domains in $\mathbb{C}^n$. We prove that under L^p data, we can choose a solution in the Lipschitz space Λ_α, where α is an optimal positive number given explicitly in terms of p.

1. Introduction and statement of the main results

For every m-uplet of positive integers $N := (n_1, \ldots, n_m)$, we consider the following domain:

$$\Omega_N := \left\{ Z = (Z_1, \ldots, Z_m) \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_m} : \sum_{j=1}^{m} \left(|Z_j|^2 + |Z_j \bullet Z_j| \right) < 1 \right\},$$

(1.1)

where $z \bullet w := \sum_{j=1}^{k} z_j w_j$ and $|z| := \sqrt{z \bullet \bar{z}}$, for all elements $z := (z_1, \ldots, z_k)$ and $w := (w_1, \ldots, w_k)$ of $\mathbb{C}^k$.

The euclidean ball of radius $\frac{\sqrt{2}}{2}$ in $\mathbb{C}^m$ and the minimal ball in $\mathbb{C}^{n_1}$ correspond respectively to the cases $n_1 = \cdots = n_m = 1$ and $m = 1$. The domains Ω_N were introduced by the second author in [?] where he computed their Bergman and Szegö kernels. We should point out that these domains are convex but they are neither strictly pseudoconvex nor piecewise smooth except for the case of the euclidean balls.

Optimal estimates for the $\bar{\partial}$-equation were considered for the category of smooth domains by several authors. In [?], Krantz obtained the optimal Lipschitz and L^p estimates for smooth strongly pseudoconvex domains. Later in [?], Chen, Krantz and Ma established that this kind of regularity holds for smooth complex ellipsoids. The general case of smooth convex domains of finite type was considered only recently in the works of Cumenge ([?],[?]), Diederich-Fischer-Fornæss [?], Fischer [?] and Hefer [?]. The aim of the present paper is to study the optimal Lipschitz regularity for the $\bar{\partial}$-equation in the class of convex domains Ω_N.

To state the main results, we fix some notations and suppose without loss of generality that $n_1 \leqslant \cdots \leqslant n_m$. Since the case of the euclidean balls is well-known, we shall assume that $N \neq (1, \ldots, 1)$ and let l denote the smallest nonnegative integer such that $n_{l+1} > 1$. We set $|N| := \sum\limits_{j=1}^{m} n_j$.

The Lipschitz spaces we use herein are the classical ones and those given for $0 < \alpha \leqslant 1$, by

$$\Lambda_{\widetilde{\alpha}}(\Omega_N) := \left\{ f : \|f\|_{L^\infty(\Omega_N)} + \sup_{\substack{z, z+h \in \Omega_N \\ 0 < |h| < \frac{1}{2}}} \frac{|f(z+h) - f(z)|}{|h|^\alpha |\log|h||} \equiv \|f\|_{\Lambda_{\widetilde{\alpha}}(\Omega_N)} < \infty \right\}.$$

The first main result is the following. It generalizes our previous result [?]:

THEOREM 1.1.— *Suppose that $N := (n_1, \ldots, n_m)$ is as above and the domain Ω_N is given by (1.1). Let*

$$\alpha = \alpha(N, p) := \begin{cases} \frac{1}{2} - \frac{|N|+m-l+1}{p}, & \text{if } N \neq (2, \ldots, 2) \text{ and } p > 2(|N|+m-l+1); \\ \frac{1}{2} - \frac{3m}{p}, & \text{if } N = (2, \ldots, 2) \text{ and } p > 6m. \end{cases}$$

Then for every $\overline{\partial}$-closed $(0,1)$-form f with coefficients in $L^p(\Omega_N)$, there exists a function u defined on Ω_N that satisfies $\overline{\partial} u = f$ (in the distribution sense) and the estimate

$$\begin{cases} \|u\|_{\Lambda_\alpha(\Omega_N)} \leqslant C_p \|f\|_{L^p(\Omega_N)}, & \text{if } p < \infty; \\ \|u\|_{\Lambda_{\widetilde{\frac{1}{2}}}(\Omega_N)} \leqslant C_\infty \|f\|_{L^\infty(\Omega_N)}, & \text{if } p = \infty. \end{cases}$$

The following result asserts that the regularity in Theorem 1.1 is sharp.

THEOREM 1.2.— *Let N, Ω_N, p, and $\alpha := \alpha(N, p)$ be as in the statement of Theorem 1.1. Then there exists a $\overline{\partial}$-closed $(0,1)$-form f with coefficients in $\mathcal{C}^\infty(\Omega_N)$ that satisfies*

$$\begin{cases} f \in L^s(\Omega_N), \ \forall s < p, & \text{if } p < \infty; \\ f \in L^\infty(\Omega_N), & \text{if } p = \infty; \end{cases}$$

and if u is a function satisfying $\overline{\partial} u = f$, then $u \notin \Lambda_{\alpha+\epsilon}(\Omega_N), \ \forall \epsilon > 0$.

These results have been announced in [?].

Theorem 1.2 implies that if $N \neq (2, \dots, 2)$ and $p \leqslant 2(|N| + m - l + 1)$ or if $N = (2, \dots, 2)$ and $p \leqslant 6m$, then we can not solve the $\bar{\partial}$-equation on Ω_N under L^p data with the Lipschitz regularity given above.

We observe that for $N = (2)$, the domain $\Omega_{(2)}$ is linearly biholomorphic to the Reinhardt triangle $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| + |z_2| < 1\}$. The reduction of our Theorems 1.1 and 1.2 to this case, compared with the results obtained for domains of finite type ([?],[?],[?],[?],[?],[?]), shows that domain $\Omega_{(2)}$ has the same gain of smoothness for the $\bar{\partial}$-equation as strictly pseudoconvex smooth domains in $\mathbb{C}^2$. Our results show also that there exist smooth domains of finite type for which the gain of smoothness for the $\bar{\partial}$-equation is worse than that of the singular domains Ω_N.

The paper is organised as follows.

In Section 2 we introduce the main tools and prove preliminary results. The objects used are a complex manifold $\mathbb{H}_N$, its intersection $\mathbb{M}_N$ with the euclidean unit ball and a proper holomorphic mapping F_N relating the $\bar{\partial}$-equation on $\mathbb{M}_N$ to that on Ω_N. We establish in this section Proposition 2.2 which gives an integral representation formula of Berndtsson type for the complex manifold $\mathbb{M}_N$. From this result we derive in Section 3 a formula of Martinelli-Bochner type (Theorem 3.1) and two formulas of Cauchy type (Theorems 3.2 and 3.3) for the complex manifold $\mathbb{M}_N$. These integral representations play a peculiar role in the construction of the $\bar{\partial}$-solving operators on $\mathbb{M}_N$ and Ω_N.

In Section 4 we give appropriate local coordinates on the complex manifold $\mathbb{H}_N$ which permit us to prove Theorem 5.1 in Section 5. The latter result will be called Theorem of reduction of estimates since from broad outlines, it reduces certain integral estimates on $\mathbb{M}_N$ to analogous integrals, but simpler, which are taken on some balls of $\mathbb{C}^{|N|}$. This result, combined with Section 6, allows us to establish integral estimates in Section 7.

An operator solution T_1 of the $\bar{\partial}$-equation on $\mathbb{M}_N$ is constructed in Section 8 and related Lipschitz estimates are established there. The formula for T_1 is explicit and contains an integral term taken over the boundary $\partial \mathbb{M}_N$ of $\mathbb{M}_N$. In order to handle this term, we prove a sort of Stokes theorem in Section 9 which allows us to transform these integral estimates into analogous ones taken over $\mathbb{M}_N$ and then apply the Theorem of reduction of estimates.

Theorem 1.1 is proved in Section 10. By means of the operator T_1 and the proper holomorphic mapping F_N, we define an operator T, solution of the $\bar{\partial}$-equation on the domain Ω_N and transfer the Lipschitz regularity for T_1

to that of the operator T. Finally, we prove Theorem 1.2 by giving concrete examples to show the sharpness of the results of Theorem 1.1. Then we conclude the paper by some remarks and open questions.

Throughout the paper, the letter C denotes a finite constant that is not necessarily the same at each occurence and that depends on N and eventually other parameters.

2. The complex manifolds $\mathbb{H}_N$ and $\mathbb{M}_N$

In this section we fix the notations and prove some preliminary results. For the simplicity of calculations we only consider, without loss of generality, the case of the domain Ω_N with $N = (\underbrace{1, \ldots, 1}_{l}, n, m)$, where l, m, n are positive integers and $n, m > 1$. In this case we have $|N| = l + n + m$ and Ω_N can be written in the form

$$\Omega_N := \left\{ Z = (x, z, w) \in \mathbb{C}^l \times \mathbb{C}^n \times \mathbb{C}^m : 2|x|^2 + |z|^2 + |z \bullet z| + |w|^2 + |w \bullet w| < 1 \right\}.$$

Consider the complex manifold $\mathbb{H}_N$ given by

$$\mathbb{H}_N := \left\{ \mathbb{Z} = (x, z, w) \in \mathbb{C}^l \times \mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}^{m+1} \setminus \{0\} : z \bullet z = w \bullet w = 0 \right\}.$$

Let $\mathbb{B}_N$ be the euclidean open unit ball in $\mathbb{C}^{|N|+2}$ and $\partial \mathbb{B}_N$ its boundary. We set $\mathbb{M}_N := \mathbb{H}_N \cap \mathbb{B}_N$ and $\partial \mathbb{M}_N := \mathbb{H}_N \cap \partial \mathbb{B}_N$. We first point out that $\mathbb{H}_N$ and $\partial \mathbb{B}_N$ are transverse while the variety $\{Z = (x, z, w) \in \mathbb{C}^l \times \mathbb{C}^{n+1} \times \mathbb{C}^{m+1} : z \bullet z = w \bullet w = 0\}$ does not meet $\partial \mathbb{B}_N$ transversally. Denote by dV, dV_l, dV_n and dV_m the respective canonical measures on the complex manifolds $\mathbb{H}_N, \mathbb{C}^l, \mathbb{H}_n$ and $\mathbb{H}_m$. These measures are related by the following

PROPOSITION 2.1.— *For all compactly supported continuous functions f on $\mathbb{H}_N$, we have*

$$\int_{\mathbb{H}_N} f(Z) dV(Z) = C \int_{\mathbb{C}^l} \int_{\mathbb{H}_n} \int_{\mathbb{H}_m} f(x, z, w) dV_l(x) dV_n(z) dV_m(w).$$

Proof.— Observe that

$$dV(Z) := C \left(\sum_{p=1}^{l} dx_p \wedge d\overline{x}_p + \sum_{j=1}^{n+1} dz_j \wedge d\overline{z}_j + \sum_{k=1}^{m+1} dw_k \wedge d\overline{w}_k \right)^{l+n+m} \Bigg|_{\mathbb{H}_N} .$$

In this formula the constant C is equal to $\frac{1}{(l+n+m)!} \left(\frac{i}{2}\right)^{l+n+m}$. Therefore, a direct computing shows that

$$
\begin{aligned}
dV(Z) &= C \left(\sum_{p=1}^{l} dx_p \wedge d\overline{x}_p\right)^l \left(\sum_{j=1}^{n+1} dz_j \wedge d\overline{z}_j\right)^n \left(\sum_{k=1}^{m+1} dw_k \wedge d\overline{w}_k\right)^m \Bigg|_{\mathbb{H}_N} . \\
&= dV_l(x) dV_n(z) dV_m(w).
\end{aligned}
$$

This completes the proof. $\quad\square$

Let $\mathbb{E} := \{t = (t_1, t_2, t_3) \in]0, 1[^3 : t_1^2 + t_2^2 + t_3^2 < 1\}$ and $\partial\mathbb{E} := \{t \in]0, 1[^3 : t_1^2 + t_2^2 + t_3^2 = 1\}$ its boundary. Then the mapping $F : \mathbb{E} \times \partial\mathbb{B}_l \times \partial\mathbb{M}_n \times \partial\mathbb{M}_m \longrightarrow \mathbb{M}_N$ given by $F(t, x, z, w) := tZ = (t_1 x, t_2 z, t_3 w)$, where $t = (t_1, t_2, t_3)$ and $Z = (x, z, w)$, is a diffeomorphism. Moreover, it maps $\partial\mathbb{E} \times \partial\mathbb{B}_l \times \partial\mathbb{M}_n \times \partial\mathbb{M}_m$ onto $\partial\mathbb{M}_N$.

Let $d\sigma_n$ be the unique probability measure, $SO(n+1, \mathbb{R})$-invariant on $\partial\mathbb{M}_n$. Similarly, let $d\sigma_m$ be the unique probability measure, $SO(m+1, \mathbb{R})$-invariant on $\partial\mathbb{M}_m$. Finally, let $d\sigma_l$ be the surface measure on $\partial\mathbb{B}_l$. Combining Proposition 2.1 of [?] and Lemma 2.1 of [?], we obtain

COROLLARY 2.2.— *For all compactly supported continuous functions f on $\mathbb{H}_n$, we have*

$$
\int_{\mathbb{H}_n} f(z) dV_n(z) = C \int_0^{+\infty} t^{2n-1} \int_{\partial\mathbb{M}_n} f(t\zeta) d\sigma_n(\zeta) dt.
$$

There are obviously analogous integral formulas in polar coordinates with $\mathbb{M}_m$ and $\mathbb{B}_l$ in place of $\mathbb{H}_n$. We now define a natural measure $d\sigma$ on $\partial\mathbb{M}_N$ by setting $d\sigma := (F_*) (d\phi \wedge d\sigma_l \wedge d\sigma_n \wedge d\sigma_m)$, where $d\phi$ is the surface measure of the unit sphere $\partial\mathbb{E}$. Using this, Corollary 2.2 and integration in polar coordinates, one can establish the following

LEMMA 2.3.— *For all compactly supported continuous functions f on $\mathbb{H}_N$, we have*

$$
\int_{\mathbb{H}_N} f(Z) dV(Z) = C(N) \int_0^{+\infty} t^{2|N|-1} \int_{\partial\mathbb{M}_N} f(t\Theta) d\sigma(\Theta) dt.
$$

In what follows we shall establish some integral formulas on $\mathbb{M}_N$. To do so, we shall approximate $\mathbb{M}_N$ by appropriate regular varieties which are complete intersections. Then we apply to each of these varieties the results of Berndtsson in [?].

For $0 < r < 1$, let $\mathcal{D}_r$ be the domain of $\mathbb{C}^{|N|+2}$ defined by

$$\mathcal{D}_r := \{Z = (x, z, w) \in \mathbb{B}_N : |z| > r, |w| > r\}.$$

Note that the boundary of $\mathcal{D}_r$ is piecewisely smooth. We put $\mathbb{M}_r := \mathbb{H}_N \cap \mathcal{D}_r$. Let

$$s := (s_1, \ldots, s_{|N|+2}) : \ \overline{\mathcal{D}}_r \times \overline{\mathcal{D}}_r \longrightarrow \mathbb{C}^{|N|+2}$$

be a $\mathcal{C}^1$ function that satisfies

$$|s(\Theta, Z)| \leqslant C|\Theta - Z| \quad \text{and} \quad |s(\Theta, Z) \bullet (\Theta - Z)| \geqslant C|\Theta - Z|^2 \quad (2.1)$$

uniformly for $\Theta \in \overline{\mathcal{D}}_r$ and for Z in any compact subset of $\mathcal{D}_r$. We shall use the same symbol s and set

$$s := \sum_{j=1}^{|N|+2} s_j d\Theta_j.$$

In the sequel, we shall use simultaneously the following notations for $\Theta \in \mathbb{C}^{|N|+2}$:

$$\Theta \equiv \left(\Theta_1, \ldots, \Theta_{|N|+2}\right) \equiv (\xi, \zeta, \eta) \in \mathbb{C}^l \times \mathbb{C}^{n+1} \times \mathbb{C}^{m+1}.$$

We next set

$$\Phi := \left(\sum_{j=1}^{n+1}(\zeta_j + z_j)d\zeta_j\right) \wedge \left(\sum_{k=1}^{m+1}(\eta_k + w_k)d\eta_k\right).$$

For every $\epsilon > 0$, consider the differential form of bidegree $(|N| + 2, |N| + 1)$

$$K_s^\epsilon := \frac{s \wedge (\overline{\partial}s)^{|N|-1} \wedge (\overline{\partial}Q_\epsilon)^2}{[s(\Theta, Z) \bullet (\Theta - Z)]^{|N|}}, \quad (2.2)$$

where Q_ϵ is the differential form of bidegree $(1, 0)$ given by

$$Q_\epsilon := \frac{\overline{\zeta \bullet \zeta}\left(\sum_{j=1}^{n+1}(\zeta_j + z_j)d\zeta_j\right) + \overline{\eta \bullet \eta}\left(\sum_{k=1}^{m+1}(\eta_k + w_k)d\eta_k\right)}{|\zeta \bullet \zeta|^2 + |\eta \bullet \eta|^2 + \epsilon}. \quad (2.3)$$

Denote by $d\Theta$ the canonical holomorphic form of $\mathbb{C}^{|N|+2}$ given by

$$d\Theta_1 \wedge \ldots \wedge d\Theta_{|N|+2} \equiv d\xi_1 \wedge \ldots \wedge d\xi_l \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_{n+1} \wedge d\eta_1 \wedge \ldots \wedge d\eta_{m+1}.$$

LEMMA 2.4. — *Suppose that $0 < r < 1$.*

1) If $u \in \mathcal{C}^1(\overline{\mathcal{D}}_r)$ and $Z \in \overline{\mathbb{M}}_r$, then

$$u(Z) = C(N) \lim_{\epsilon \to 0} \left(\int_{\partial \mathcal{D}_r} u K_s^\epsilon - \int_{\mathcal{D}_r} \overline{\partial} u \wedge K_s^\epsilon \right).$$

2) If $u \in \mathcal{C}(\mathcal{D}_r)$, then

$$\lim_{\epsilon \to 0} \int_{\mathcal{D}_r} \frac{\epsilon |\zeta|^2 |\eta|^2 u(\Theta)}{(|\zeta \bullet \zeta|^2 + |\eta \bullet \eta|^2 + \epsilon)^3} d\overline{\Theta} \wedge d\Theta = C(N) \int_{\mathbb{M}_r} u(\Theta) dV(\Theta).$$

3) If $u \in \mathcal{C}(\partial \mathbb{B}_N)$ and ω is the canonical volume form of $\partial \mathbb{B}_N$, then

$$\lim_{\epsilon \to 0} \int_{\partial \mathbb{B}_N} \frac{\epsilon |\zeta|^2 |\eta|^2 u(\Theta)}{(|\zeta \bullet \zeta|^2 + |\eta \bullet \eta|^2 + \epsilon)^3} \omega(\Theta) = C(N) \int_{\partial \mathbb{M}_N} u(\Theta) d\sigma(\Theta).$$

Proof. — Part 1) follows from formulas (23) and (26) in the proof of Theorem 1 in [?]. Also, part 2) is an immediate consequence of identity (25) in [?].

To prove part 3), we may assume without loss of generality that the support of u is contained in a sufficiently small open neighborhood $\mathcal{U} \subset \mathbb{C}^{|N|+2}$ of a point $\Theta_0 \in \partial \mathbb{M}_N$. Using local coordinates and Lelong theory [?], we see that there exists a smooth $(2|N| - 1)$-volume form $d\mu$ defined on $\mathcal{U} \cap \partial \mathbb{M}_N$ such that

$$\lim_{\epsilon \to 0} \int_{\partial \mathbb{B}_N \cap \mathcal{U}} \frac{\epsilon |\zeta|^2 |\eta|^2 u(\Theta)}{(|\zeta \bullet \zeta|^2 + |\eta \bullet \eta|^2 + \epsilon)^3} \omega(\Theta) = C \int_{\partial \mathbb{M}_N \cap \mathcal{U}} u(\Theta) d\mu(\Theta), \quad (2.4)$$

for all $u \in \mathcal{C}_0(\mathcal{U})$. Therefore, part 3) is equivalent to the identity $d\mu = C d\sigma$.

Let ψ be a function of class $\mathcal{C}_0^\infty([0,1])$ supported in $[\frac{1}{2}, 1]$ such that $\int_0^1 \rho^{2|N|-1} \psi(\rho) d\rho = 1$. Consider the $\mathcal{C}_0^\infty$ extension of u given by

$$u(\rho Z) := \psi(\rho) u(Z), \quad \text{for } 0 \leqslant \rho \leqslant 1 \text{ and } Z \in \partial \mathbb{B}_N \cap \mathcal{U}.$$

On the one hand, using (2.4), we have that

$$\lim_{\epsilon \to 0} \int_{\mathbb{B}_N} \frac{\epsilon |\zeta|^2 |\eta|^2 u(\Theta)}{(|\zeta \bullet \zeta|^2 + |\eta \bullet \eta|^2 + \epsilon)^3} d\overline{\Theta} \wedge d\Theta$$

$$= \lim_{\epsilon \to 0} \int_0^1 \int_{\partial \mathbb{B}_N \cap \mathcal{U}} \frac{\epsilon \rho^{2|N|+7} |\zeta|^2 |\eta|^2 u(\rho \Theta)}{(\rho^4 |\zeta \bullet \zeta|^2 + \rho^4 |\eta \bullet \eta|^2 + \epsilon)^3} \omega(\Theta) d\rho$$

$$= \lim_{\epsilon \to 0} \int_0^1 \rho^{2|N|-1} \psi(\rho) d\rho \cdot \int_{\partial \mathbb{B}_N \cap \mathcal{U}} \frac{\epsilon |\zeta|^2 |\eta|^2 u(\Theta)}{(|\zeta \bullet \zeta|^2 + |\eta \bullet \eta|^2 + \epsilon)^3} \omega(\Theta) = \int_{\mathcal{U} \cap \partial \mathbb{M}_N} u \, d\mu.$$

On the other hand, by part 2) and Lemma 2.3, we see that

$$\lim_{\epsilon \to 0} \int_{\mathbb{B}_N} \frac{\epsilon |\zeta|^2 |\eta|^2 u(\Theta)}{(|\zeta \bullet \zeta|^2 + |\eta \bullet \eta|^2 + \epsilon)^3} d\overline{\Theta} \wedge d\Theta = C(N) \int_{\mathbb{M}_N} u \, dV$$

$$= C(N) \int_0^1 \rho^{2|N|-1} \psi(\rho) \int_{\mathcal{U} \cap \partial \mathbb{M}_N} u \, d\sigma d\rho = C(N) \int_{\mathcal{U} \cap \partial \mathbb{M}_N} u \, d\sigma,$$

Thus $d\mu = C(N) d\sigma$ and thereby completes the proof. $\qquad \square$

Next, set

$$K_s := \frac{s \wedge (\overline{\partial} s)^{|N|-1} \wedge \Phi \wedge \overline{\partial(\zeta \bullet \zeta)} \wedge \overline{\partial(\eta \bullet \eta)}}{|\zeta|^2 |\eta|^2 [s(\Theta, Z) \bullet (\Theta - Z)]^{|N|}}. \tag{2.5}$$

In view of (2.2), (2.3) and the equality which precedes Lemma 4 in [?], we see that K_s satisfies the identity

$$K_s^{\epsilon} = \frac{\epsilon |\zeta|^2 |\eta|^2}{(|\zeta \bullet \zeta|^2 + |\eta \bullet \eta|^2 + \epsilon)^3} K_s. \tag{2.6}$$

For every $1 \leqslant k \leqslant |N| + 2$, denote by $\omega_k(\overline{\Theta})$ the $(0, |N| + 1)$-form

$$(-1)^{k-1} d\overline{\Theta}_1 \wedge \ldots \wedge \widehat{d\overline{\Theta}_k} \wedge \ldots \wedge d\overline{\Theta}_{|N|+2}.$$

We can write K_s in the form

$$K_s = \sum_{k=1}^{|N|+2} h_k(\Theta, Z) \omega_k(\overline{\Theta}) \wedge d\Theta, \tag{2.7}$$

where h_k are the component functions of K_s with respect to the forms $\omega_1(\overline{\Theta}) \wedge d\Theta, \ldots, \omega_{|N|+2}(\overline{\Theta}) \wedge d\Theta$.

Let $\overline{\mathbb{M}}_N$ be the closure of $\mathbb{M}_N$ in $\overline{\mathbb{B}}_N$ and denote by $\mathcal{C}^k(\overline{\mathbb{M}}_N)$, $k \in \mathbb{N}$, the space of all $\mathcal{C}^k$ functions defined in a neighborhood of $\overline{\mathbb{M}}_N$ in $\overline{\mathbb{B}}_N$. If

$f := \sum\limits_{j=1}^{|N|+2} f_j d\overline{\Theta}_j$ is a $(0,1)$-form with coefficients in $\mathcal{C}(\overline{\mathbb{M}}_N)$, let $f|_{\mathbb{M}_N}$ denote the pull-back of f under the canonical injection of $\mathbb{M}_N$ in this neighborhood. Set

$$\|f\|_{\mathbb{M}_N,\infty} := \sup_{\Theta\in\mathbb{M}_N} \sum_{j=1}^{|N|+2} |f_j(\Theta)|. \tag{2.8}$$

Let $\overline{\partial}_{\mathbb{M}_N}$ be the $\overline{\partial}$-operator on $\mathbb{M}_N$. We end this section by the following

PROPOSITION 2.5.— *Consider a section s satisfying (2.1), a function $u \in \mathcal{C}^1(\overline{\mathbb{M}}_N)$ and a $(0,1)$-form $f := \sum\limits_{k=1}^{|N|+2} f_k d\overline{\Theta}_k$ with coefficients in $\mathcal{C}(\overline{\mathbb{M}}_N)$ that satisfy $\overline{\partial}_{\mathbb{M}_N} u = f|_{\mathbb{M}_N}$ on $\mathbb{M}_N$. Let h_k be the functions defined in (2.7). Then for $Z \in \mathbb{M}_N$,*

$$u(Z) = C\int_{\partial\mathbb{M}_N} u(\Theta)\left(\sum_{k=1}^{|N|+2}\Theta_k h_k(\Theta,Z)\right)d\sigma(\Theta) + C\int_{\mathbb{M}_N}\left(\sum_{k=1}^{|N|+2} f_k(\Theta)h_k(\Theta,Z)\right)dV(\Theta).$$

Proof.— For every $r \in]0,1[$ such that $Z \in \mathbb{M}_r$, consider a $\mathcal{C}^1$ extension of $u|_{\mathbb{M}_N}$ (which is also denoted by u) on $\overline{\mathcal{D}}_r$ that satisfies $\overline{\partial}u = f$ on $\mathbb{M}_r$. Suppose without loss of generality that $f = \overline{\partial}u$ on $\overline{\mathcal{D}}_r$. Parts 1) and 2) of Lemma 2.4, combined with (2.6) and (2.7), imply that $u(z) = CI_r^1 + CI_r^2$, where

$$I_r^1 := \int_{\mathbb{M}_r}\left(\sum_{k=1}^{|N|+2} f_k(\Theta)h_k(\Theta,Z)\right)dV(\Theta),$$

$$I_r^2 := \lim_{\epsilon\to 0}\int_{\partial\mathcal{D}_r}\frac{\epsilon|\zeta|^2|\eta|^2 u(\Theta)}{(|\zeta\bullet\zeta|^2+|\eta\bullet\eta|^2+\epsilon)^3}\left(\sum_{k=1}^{|N|+2} h_k(\Theta,Z)\omega_k(\overline{\Theta})\wedge d\Theta\right).$$

The proof is a consequence of the following two equalities

$$\lim_{r\to 0} I_r^1 = \int_{\mathbb{M}_N}\left(\sum_{k=1}^{|N|+2} f_k(\Theta)h_k(\Theta,Z)\right)dV(\Theta), \tag{2.9}$$

$$\lim_{r\to 0} I_r^2 = \int_{\partial\mathbb{M}_N} u(\Theta)\left(\sum_{k=1}^{|N|+2}\Theta_k h_k(\Theta,Z)\right)d\sigma(\Theta). \tag{2.10}$$

In order to prove these, fix a point $Z \in \mathbb{M}_N$. By (2.1), (2.5) and (2.7), there is a constant C such that

$$h_k(\Theta, Z) \leqslant \frac{C}{|\zeta|^2 |\eta|^2}, \qquad \text{for all } \Theta \in \mathbb{M}_N \setminus \mathbb{M}_r, \text{ with } 0 < r \ll 1. \quad (2.11)$$

We deduce easily from (2.11) and the hypothesis $\|f\|_{\mathbb{M}_N, \infty} < \infty$ that

$$\lim_{r \to 0} \int_{\mathbb{M}_N \setminus \mathbb{M}_r} \left| \sum_{k=1}^{|N|+2} f_k(\Theta) h_k(\Theta, Z) \right| dV(\Theta) \leqslant \lim_{r \to 0} \int_{\mathbb{M}_N \setminus \mathbb{M}_r} \frac{C dV(\Theta)}{|\zeta|^2 |\eta|^2} = 0,$$

where the equality follows from Corollary 2.2. This proves (2.9).

Next, we prove (2.10). Appealing to Corollary 2.2, Lemma 2.3, (2.11) and the fact that the function u is bounded, we see that

$$\lim_{r \to 0} \int_{\partial \mathbb{M}_N \setminus \partial \mathbb{M}_r} \left| \sum_{k=1}^{|N|+2} \Theta_k h_k(\Theta, Z) \right| |u(\Theta)| d\sigma(\Theta) \leqslant \lim_{r \to 0} \int_{\partial \mathbb{M}_N \setminus \partial \mathbb{M}_r} \frac{C d\sigma(\Theta)}{|\zeta|^2 |\eta|^2} = 0.$$

This, combined with part 3) of Lemma 2.4, implies that

$$\lim_{\epsilon \to 0} \int_{\partial \mathbb{B}_N \setminus \partial \mathcal{D}_r} \frac{\epsilon |\zeta|^2 |\eta|^2 u(\Theta)}{(|\zeta \bullet \zeta|^2 + |\eta \bullet \eta|^2 + \epsilon)^3} \left(\sum_{k=1}^{|N|+2} h_k(\Theta, Z) \omega_k(\overline{\Theta}) \wedge d\Theta \right)$$

$$= \int_{\partial \mathbb{M}_N \setminus \partial \mathbb{M}_r} u(\Theta) \left(\sum_{k=1}^{|N|+2} \Theta_k h_k(\Theta, Z) \right) d\sigma(\Theta) \longrightarrow 0, \quad \text{as } r \to 0. \quad (2.12)$$

from which it follows that (2.10) is a consequent of

$$\lim_{r \to 0} \lim_{\epsilon \to 0} \int_{\partial \mathcal{D}_r \setminus \partial \mathbb{B}_N} \frac{\epsilon |\zeta|^2 |\eta|^2 u(\Theta)}{(|\zeta \bullet \zeta|^2 + |\eta \bullet \eta|^2 + \epsilon)^3} \left(\sum_{k=1}^{|N|+2} h_k(\Theta, Z) \omega_k(\overline{\Theta}) \wedge d\Theta \right) = 0.$$

$$(2.13)$$

Next, we prove equality (2.13). We first make use of the following remark related to homogeneity properties of certain differential forms. Indeed, let $\alpha, \beta > 0$ and write the complex manifold $\mathbb{M}_N$ as a complete intersection of $\mathbb{B}_N$ and the two varieties given by the equations $\alpha^2 \zeta \bullet \zeta = 0$ and $\beta^2 \eta \bullet \eta = 0$. Applying Berndtsson's formulas to these two equations and observing that (2.13) corresponds to the particular case $\alpha = \beta = 1$, then we obtain

$$\lim_{\epsilon \to 0} \int_{\partial \mathcal{D}_r \setminus \partial \mathbb{B}_N} \frac{\epsilon \alpha^4 \beta^4 |\zeta|^2 |\eta|^2 u(\Theta)}{(\alpha^4 |\zeta \bullet \zeta|^2 + \beta^4 |\eta \bullet \eta|^2 + \epsilon)^3} \left(\sum_{k=1}^{|N|+2} h_k(\Theta, Z) \omega_k(\overline{\Theta}) \wedge d\Theta \right)$$

$$= \lim_{\epsilon \to 0} \int_{\partial \mathcal{D}_r \setminus \partial \mathbb{B}_N} \frac{\epsilon |\zeta|^2 |\eta|^2 u(\Theta)}{(|\zeta \bullet \zeta|^2 + |\eta \bullet \eta|^2 + \epsilon)^3} \left(\sum_{k=1}^{|N|+2} h_k(\Theta, Z) \omega_k(\overline{\Theta}) \wedge d\Theta \right), (2.14)$$

for all $0 < r < 1$.

We write $\partial \mathcal{D}_r \setminus \partial \mathbb{B}_N$ as a union of the two smooth manifolds

$$M_1^r \quad := \quad \{Z \in \mathbb{B}_N : |z| = r, |w| \geqslant r\};$$
$$M_2^r \quad := \quad \{Z \in \mathbb{B}_N : |z| \geqslant r, |w| = r\}.$$

Let $d\sigma_{rj}$ be the canonical volume form on the manifold M_j^r, $j = 1, 2$. Applying equality (3) in Proposition 16.4.4 of Rudin [?] yields that on M_j^r,

$$\omega_k(\overline{\Theta}) \wedge d\Theta = C(N, j, k) d\sigma_{rj}. \tag{2.15}$$

Choosing a function u and a section s appropriately and applying Lelong theory as in the proof of (2.4), it follows from (2.14) and (2.15) that on M_j^r, $0 < r < 1$, $j \in \{1, 2\}$, we have

$$\lim_{\epsilon \to 0} \frac{\epsilon \alpha^4 \beta^4 |\zeta|^2 |\eta|^2}{(\alpha^4 |\zeta \bullet \zeta|^2 + \beta^4 |\eta \bullet \eta|^2 + \epsilon)^3} d\sigma_{rj}(\Theta)$$

$$= \lim_{\epsilon \to 0} \frac{\epsilon |\zeta|^2 |\eta|^2}{(|\zeta \bullet \zeta|^2 + |\eta \bullet \eta|^2 + \epsilon)^3} d\sigma_{rj}(\Theta) = d\mu_{rj}(\Theta), \tag{2.16}$$

in the distribution sense, where $d\mu_{rj}$ is a $\mathcal{C}^\infty$ differential form of maximal degree on the manifold $M_j^r \cap \mathbb{H}_N$. In view of (2.11) and (2.16), equality (2.13) will follow from the following equalities

$$\lim_{r \to 0} \int_{M_j^r \cap \mathbb{H}_N} \frac{d\mu_{rj}(\Theta)}{|\zeta|^2 |\eta|^2} = 0, \qquad j = 1, 2. \tag{2.17}$$

We prove (2.17) for $j = 1$ which suffices to complete the proof. To do so, consider, for every $\alpha, \beta > 0$, the mapping $F_{\alpha, \beta}$ given by:

$$F_{\alpha, \beta}(x, z, w) := (x, \alpha z, \beta w), \qquad \text{for } Z \equiv (x, z, w) \in \mathbb{H}_N.$$

We remark immediately that we have the following property of homogeneity :

$$F_{r,s}^*(d\sigma_{r1})(\Theta) = C(N)r^{2n-1}s^{2m}d\sigma_n(\zeta) \wedge dV_l(\xi) \wedge dV_m(\eta),$$

for $0 < r, s \leqslant \frac{1}{2}$ and $\Theta \equiv (\xi, \zeta, \eta) \in \mathbb{C}^l \times \partial\mathbb{M}_n \times \mathbb{M}_m$. This, combined with equality (2.16), implies that

$$F_{\alpha,\beta}^*(d\mu_{r1}) = C(N,r)\alpha^{2n-1}\beta^{2m}d\mu_{\frac{r}{\alpha},1} \qquad \text{on } M_1^{\frac{r}{\alpha}}. \qquad (2.18)$$

Take $r_0 := \frac{1}{2}$. Since the differential form $d\mu_{r_0,1}$ is in $\mathcal{C}^\infty(M_1^{r_0})$, we see that

$$\int\limits_{|\zeta|=r_0,\frac{r_0}{2}<|\eta|<r_0} \frac{d\mu_{r_0,1}(\Theta)}{|\zeta|^2|\eta|^2} < \infty. \qquad (2.19)$$

Using (2.18) and (2.19), it is easy to show that

$$\int\limits_{M_1^r \cap \mathbb{H}_N} \frac{d\mu_{r1}(\Theta)}{|\zeta|^2|\eta|^2} \leqslant C\left(\frac{r}{r_0}\right)^{2n-3} \int\limits_{|\zeta|=r_0,\frac{r_0}{2}<|\eta|<r_0} \frac{d\mu_{r_0,1}(\Theta)}{|\zeta|^2|\eta|^2} \to 0, \quad \text{as } r \to 0.$$

This implies (2.17) and thus completes the proof. $\qquad \square$

3. Integral formulas on the manifold $\mathbb{M}_N$

In this section we establish integral formulas of Martinelli-Bochner type (Theorem 3.1) and those of Cauchy type (Theorems 3.3 and 3.6). These formulas will allow us to construct the $\bar{\partial}$-solving operators.

THEOREM 3.1.— *Suppose that $u \in \mathcal{C}^1(\overline{\mathbb{M}}_N)$ and $f := \sum\limits_{k=1}^{|N|+2} f_k d\overline{\Theta}_k$ is a $(0,1)$-form with coefficients in $\mathcal{C}(\overline{\mathbb{M}}_N)$ such that $\bar{\partial}_{\mathbb{M}_N} u = f|_{\mathbb{M}_N}$. Then for every $Z \in \mathbb{M}_N$,*

$$u(Z) = \left\{ \int\limits_{\partial\mathbb{M}_N} \frac{A(\Theta, Z)}{|Z - \Theta|^{2|N|}} u(\Theta) \frac{d\sigma(\Theta)}{|\zeta|^2|\eta|^2} \right.$$

$$\left. + \int\limits_{\mathbb{M}_N} \frac{1}{|Z - \Theta|^{2|N|}} \left(\sum\limits_{k=1}^{|N|+2} B_k(\Theta, Z) f_k(\Theta) \right) \frac{dV(\Theta)}{|\zeta|^2|\eta|^2} \right\},$$

where

$$A(\Theta, Z) := C(|\xi|^2 - \overline{x} \bullet \xi)(|\zeta|^2 + z \bullet \overline{\zeta})(|\eta|^2 + w \bullet \overline{\eta})$$

$$+ \, C\left(-|z \bullet \zeta|^2 + |z \bullet \overline{\zeta}|^2 - |\zeta|^2(|\zeta|^2 + z \bullet \overline{\zeta} - \overline{z} \bullet \zeta)\right)(|\eta|^2 + w \bullet \overline{\eta})$$

$$+ \, C\left(-|w \bullet \eta|^2 + |w \bullet \overline{\eta}|^2 - |\eta|^2(|\eta|^2 + w \bullet \overline{\eta} - \overline{w} \bullet \eta)\right)(|\zeta|^2 + z \bullet \overline{\zeta}),$$

and B_k are polynomials given by the following formulas:

(i) if $1 \leqslant k \leqslant l$, then

$$B_k(\Theta, Z) := C(\overline{\xi_k} - \overline{x_k})(|\zeta|^2 + z \bullet \overline{\zeta})(|\eta|^2 + w \bullet \overline{\eta});$$

(ii) if $l < k \leqslant l+n+1$ and $j = k - l$, then

$$B_k(\Theta, Z) := C\left((\overline{z}_j - \overline{\zeta}_j)(z \bullet \overline{\zeta} + |\zeta|^2) - (z_j + \zeta_j)\overline{z \bullet (\zeta - z)}\right)(|\eta|^2 + w \bullet \overline{\eta});$$

(iii) if $l+n+1 < k < l+n+m+2$ and $i = k-l-n-1$, then

$$B_k(\Theta, Z) := C\left((\overline{w}_i - \overline{\eta}_i)(w \bullet \overline{\eta} + |\eta|^2) - (w_i + \eta_i)\overline{w \bullet (\eta - w)}\right)(|\zeta|^2 + z \bullet \overline{\zeta}).$$

Proof.— Consider the Martinelli-Bochner section $s_b(Z, \Theta) := \overline{\Theta} - \overline{Z}$. In order to prove the theorem, we apply Proposition 2.5 to the section s_b. Using formulas (2.5), (2.7) and arguing as in the proof of Theorem 2.4 of [?], we compute explicitly the functions h_k associated to s_b and obtain the desired formula.   □

Remark 3.2.— If $u \in \mathcal{C}^1(\mathbb{M}_N)$ is bounded, then Proposition 2.5 and Theorem 3.1 hold for the dilated functions $u_r(Z) := u(rZ)$, $0 < r < 1$. This shows that Theorem 3.1 remains true if we only assume that $u \in \mathcal{C}^1(\mathbb{M}_N)$ is bounded and

$$\lim_{r \to 1^-} \int_{\partial \mathbb{M}_N} |u(\Theta) - u(r\Theta)| d\sigma(\Theta) = 0.$$

Following Charpentier [?] let

$$s_0(\Theta, Z) := \overline{\Theta}(1 - \Theta \bullet \overline{Z}) - \overline{Z}(1 - |\Theta|^2), \text{ and } D(\Theta, Z) := s_0(\Theta, Z) \bullet (\Theta - Z).$$

In what follows, $\mathrm{grad}_Z \, f$ denotes the gradient of a differentiable function f at a point Z.

THEOREM 3.3. — *There exist polynomials $R(\Theta, Z)$ and $P_k(\Theta, Z)$, $Q_k(\Theta, Z)$ for $1 \leqslant k \leqslant |N| + 2$, that satisfy the following properties:*

(i) $R(\Theta, Z) = \left(C|\xi|^2 + C|\zeta|^2 + C|\eta|^2\right)(|\zeta|^2 + z \bullet \overline{\zeta})(|\eta|^2 + w \bullet \overline{\eta})$.

(ii) For every $Z, \Theta \in \mathbb{B}_N$, and for every $1 \leqslant k \leqslant |N| + 2$,

$$
\begin{aligned}
P_k(\Theta, Z) &= O\left(|\Theta - Z|(|\zeta|^2 + |z||\zeta|)(|\eta|^2 + |w||\eta|)\right), \\
Q_k(\Theta, Z) &= O\left(|\Theta - Z|(|\zeta|^2 + |z||\zeta|)(|\eta|^2 + |w||\eta|)\right), \\
|\text{grad}_Z\, P_k(\Theta, Z)| &= O\left((|\zeta|^2 + |\zeta||z|)(|\eta|^2 + |\eta||w|)\right), \\
|\text{grad}_Z\, Q_k(\Theta, Z)| &= O\left((|\zeta|^2 + |\zeta||z|)(|\eta|^2 + |\eta||w|)\right).
\end{aligned}
$$

(iii) Given a function $u \in \mathcal{C}^1(\overline{\mathbb{M}}_N)$ and a $(0, 1)$-form $f := \sum\limits_{k=1}^{|N|+2} f_k\, d\overline{\Theta}_k \in \mathcal{C}(\overline{\mathbb{M}}_N)$ that satisfy $\overline{\partial}_{\mathbb{M}_N} u = f|_{\mathbb{M}_N}$, then for every $Z \in \mathbb{M}_N$,

$$
\begin{aligned}
u(Z) \ = \ &\int\limits_{\partial \mathbb{M}_N} \frac{R(\Theta, Z)}{(1 - Z \bullet \overline{\Theta})^{|N|}} u(\Theta) \frac{d\sigma(\Theta)}{|\zeta|^2|\eta|^2} \\
&+ \int\limits_{\mathbb{M}_N} \sum_{k=1}^{|N|+2} \frac{(1 - \Theta \bullet \overline{Z})^{|N|-2}}{D(\Theta, Z)^{|N|}}[(1 - \Theta \bullet \overline{Z})P_k(\Theta, Z) \\
&+ (1 - |\Theta|^2)Q_k(\Theta, Z)]f_k(\Theta) \frac{dV(\Theta)}{|\zeta|^2|\eta|^2}.
\end{aligned}
$$

Proof. — From the proof of Proposition 2.5 we may assume without loss of generality that there is a $\mathcal{C}^1$ extension of $u|_{\mathbb{M}_N}$, denoted again by u, such that $\overline{\partial} u = f$ on $\mathbb{B}_N$. Let K_0 be the kernel associated to the section s_0 by formula (2.5). By virtue of (2.6), when we integrate uK_0^ϵ over $\partial\mathbb{B}_N$, all terms which contain $\overline{\partial}|\Theta|^2$ vanish. In addition we have $1 - |\Theta|^2 = 0$ and $D(\Theta, Z) = \left|1 - Z \bullet \overline{\Theta}\right|^2$ so that

$$
\lim_{\epsilon \to 0} \int\limits_{\partial\mathbb{B}_N} uK_0^\epsilon = \lim_{\epsilon \to 0} \int\limits_{\partial\mathbb{B}_N} u(\Theta) \frac{\epsilon|\zeta|^2|\eta|^2}{(|\zeta \bullet \zeta|^2 + |\eta \bullet \eta|^2 + \epsilon)^3} \cdot \frac{1}{|\zeta|^2|\eta|^2(1 - Z \bullet \overline{\Theta})^{|N|}}
$$

$$
\left\{\left(\sum_{k=1}^{|N|+2} \overline{\Theta}_k d\Theta_k\right)\left(\sum_{k=1}^{|N|+2} d\overline{\Theta}_k \wedge d\Theta_k\right)^{|N|-1} \wedge \Psi \wedge \overline{\partial(\zeta \bullet \zeta)} \wedge \overline{\partial(\eta \bullet \eta)}\right\}.
$$

Rewriting the differential form in braces in the form $\sum\limits_{k=1}^{|N|+2} h_k(\Theta, Z)\omega_k(\overline{\Theta}) \wedge$ $d\Theta$ and applying part 3) of Lemma 2.4, we obtain

$$\lim_{\epsilon \to 0} \int\limits_{\partial \mathbb{B}_N} uK_0^\epsilon = \int\limits_{\partial \mathbb{M}_N} \frac{\Theta_k h_k(\Theta, Z)}{|\zeta|^2 |\eta|^2 (1 - Z \bullet \overline{\Theta})^{|N|}} d\sigma(\Theta).$$

A straightforward calculation of the functions $h_k(\Theta, Z)$ shows that

$$R(\Theta, Z) := \sum_{k=1}^{|N|+2} \Theta_k h_k(\Theta, Z) \tag{3.1}$$

satisfies assertion (i) of the theorem.

Write the kernel K_0 in the form (2.7) as $K_0 = \sum\limits_{k=1}^{|N|+2} h_k(\Theta, Z)\omega_k(\overline{\Theta}) \wedge d\Theta$. Then we have

$$I := \overline{\partial} u \wedge K_0 = \sum_{k=1}^{|N|+2} f_k(\Theta) h_k(\Theta, Z) d\Theta \wedge d\overline{\Theta}. \tag{3.2}$$

To finish the proof of the theorem, it suffices to prove the following lemma:

LEMMA 3.4. — *The functions h_k in the formula (3.2) can be rewritten in the form*

$$h_k(\Theta, Z) = \frac{(1 - \Theta \bullet \overline{Z})^{|N|-2}}{|\zeta|^2 |\eta|^2 D(\Theta, Z)^{|N|}} [(1 - \Theta \bullet \overline{Z}) P_k(\Theta, Z) + (1 - |\Theta|^2) Q_k(\Theta, Z)],$$
$$\tag{3.3}$$

where P_k and Q_k are some polynomials that satisfy assertion (ii) of the theorem.

End of the proof of Theorem 3.3 . — Suppose that the lemma above is proved. Applying Proposition 2.5 and using (3.1)–(3.3), the theorem follows. $\square$

Proof of Lemma 3.4 . — By virtue of (2.5) and (3.2), we can write $I = I_1 + I_2$, where

$$I_1 := \left(\sum_{k=1}^{|N|+2} f_k d\overline{\Theta}_k \right) \wedge \left\{ \frac{(1 - \Theta \bullet \overline{Z})^{|N|-1}}{|\zeta|^2 |\eta|^2 D(\Theta, Z)^{|N|}} \sum_{j=1}^{|N|+2} [\overline{\Theta}_j (1 - \Theta \bullet \overline{Z}) - \overline{Z}_j (1 - |\Theta|^2)] \right.$$

$$\left. d\Theta_j \wedge \Psi \wedge \overline{\partial(\zeta \bullet \zeta)} \wedge \overline{\partial(\eta \bullet \eta)} \wedge \left[\sum_{q=1}^{|N|+2} d\overline{\Theta}_q \wedge d\Theta_q \right]^{|N|-1} \right\},$$

and

$$I_2 := \left(\sum_{k=1}^{|N|+2} f_k d\overline{\Theta}_k \right)$$

$$\wedge \left\{ \frac{(1-\Theta \bullet \overline{Z})^{|N|-2}}{|\zeta|^2 |\eta|^2 D(\Theta, Z)^{|N|}} \sum_{j=1}^{|N|+2} \left[\overline{\Theta}_j (1-\Theta \bullet \overline{Z}) - \overline{Z}_j (1-|\Theta|^2) \right] d\Theta_j \right.$$

$$\wedge \overline{\partial} |\Theta|^2 \wedge \left[\sum_{k=1}^{n+1} \overline{Z}_k d\Theta_k \right] \wedge \Psi \wedge \overline{\partial(\zeta \bullet \zeta)} \wedge \overline{\partial(\eta \bullet \eta)}$$

$$\left. \wedge \left[\sum_{q=1}^{|N|+2} d\overline{\Theta}_q \wedge d\Theta_q \right]^{|N|-2} \right\}.$$

A straightforward computation shows that

$$I_1 := \frac{(1-\Theta \bullet \overline{Z})^{|N|-1}}{|\zeta|^2 |\eta|^2 D(\Theta, Z)^{|N|}} \left\{ C \sum_{k=1}^{l} f_k \left[-\overline{\xi}_k (1-\Theta \bullet \overline{Z}) + \overline{x}_k (1-|\Theta|^2) \right] \right.$$

$$(z \bullet \overline{\zeta} + |\zeta|^2)(w \bullet \overline{\eta} + |\eta|^2) + C \sum_{k=1}^{n+1} f_{k+l} \left\{ \left[-\overline{\zeta}_k (1-\Theta \bullet \overline{Z}) + \overline{z}_k (1-|\Theta|^2) \right] \right.$$

$$(z \bullet \overline{\zeta} + |\zeta|^2) - (1-|\zeta|^2)(z_k + \zeta_k)\overline{z \bullet \zeta} \right\} (w \bullet \overline{\eta} + |\eta|^2)$$

$$+ C \sum_{k=1}^{m+1} f_{k+l+n+1} \left\{ \left[-\overline{\eta}_k (1-\Theta \bullet \overline{Z}) + \overline{w}_k (1-|\Theta|^2) \right] (w \bullet \overline{\eta} + |\eta|^2) \right.$$

$$\left. - (1-|\eta|^2)(w_k + \eta_k)\overline{w \bullet \eta} \right\} (z \bullet \overline{\zeta} + |\zeta|^2) \right\} d\Theta \wedge d\overline{\Theta}. \tag{3.4}$$

Hence the functions h_k associated with I_1 are of the form (3.3).

To simplify notations we set

$$\omega_\xi := \sum_{k=1}^{l} d\xi_k \wedge d\overline{\xi_k}, \quad \omega_\zeta := \sum_{k=1}^{n+1} d\zeta_k \wedge d\overline{\zeta_k}, \quad \omega_\eta := \sum_{k=1}^{m+1} d\eta_k \wedge d\overline{\eta_k},$$

and we set for every form ω and every positive integer k,

$$\omega^k := \underbrace{\omega \wedge \ldots \wedge \omega}_{k}.$$

Then a simple calculation gives that

$$\Psi \wedge \overline{\partial(\zeta \bullet \zeta)} \wedge \overline{\partial(\eta \bullet \eta)} \wedge \left[\sum_{k=1}^{|N|+2} d\overline{\Theta}_k \wedge d\Theta_k\right]^{|N|-2} = \Psi \wedge \overline{\partial(\zeta \bullet \zeta)} \wedge \overline{\partial(\eta \bullet \eta)}$$

$$\wedge \left(C\omega_\xi^{l-2} \wedge \omega_\zeta^n \wedge \omega_\eta^m + C\omega_\xi^l \wedge \omega_\zeta^{n-2} \wedge \omega_\eta^m + C\omega_\xi^l \wedge \omega_\zeta^n \wedge \omega_\eta^{m-2} + C\omega_\xi^{l-1}\right.$$

$$\left.\wedge \omega_\zeta^{n-1} \wedge \omega_\eta^m + C\omega_\xi^l \wedge \omega_\zeta^{n-1} \wedge \omega_\eta^{m-1} + C\omega_\xi^{l-1} \wedge \omega_\zeta^n \wedge \omega_\eta^{m-1}\right)$$

$$\equiv \Psi \wedge \overline{\partial(\zeta \bullet \zeta)} \wedge \overline{\partial(\eta \bullet \eta)} \wedge \left(\sum_{k=1}^{6} J_k\right). \tag{3.5}$$

To conclude the proof of Lemma 3.4, it suffices to prove the following lemma :

LEMMA 3.5. — *For every $1 \leqslant k \leqslant 6$, the differential form*

$$I_{2k} \quad := \quad \left[\sum_{j=1}^{|N|+2} f_j d\overline{\Theta}_j\right] \wedge \sum_{j=1}^{|N|+2} \left[\overline{\Theta}_j(1 - \Theta \bullet \overline{Z}) - \overline{Z}_j(1 - |\Theta|^2)\right] d\Theta_j \wedge \overline{\partial}|\Theta|^2$$

$$\wedge \left[\sum_{j=1}^{|N|+2} \overline{Z}_j d\Theta_j\right] \wedge J_k \wedge \Psi \wedge \overline{\partial(\zeta \bullet \zeta)} \wedge \overline{\partial(\eta \bullet \eta)}$$

can be expressed as the product of the canonical volume form $d\Theta \wedge d\overline{\Theta}$ and a function of the form

$$\sum_{j=1}^{|N|+2} f_j \left((1 - \Theta \bullet \overline{Z})P_j(\Theta, Z) + (1 - |\Theta|^2)Q_j(\Theta, Z)\right),$$

where P_j, Q_j are some polynomials satisfying assertion (ii) of Theorem 3.3.

End of the proof of Lemma 3.4 . — Suppose that Lemma 3.5 is proved. We deduce from the definition of I_2, I_{2k} and (3.5) that

$$I_2 = \frac{(1 - \Theta \bullet \overline{Z})^{|N|-2}}{|\zeta|^2|\eta|^2 D(\Theta, Z)^{|N|}} \cdot \left(\sum_{k=1}^{6} I_{2k}\right).$$

Therefore Lemma 3.4 follows from Lemma 3.5. $\quad\square$

Proof of Lemma 3.5 . — We break the proof into 6 cases according to the integer k, $1 \leqslant k \leqslant 6$.

Case 1: $J_1 = \omega_\xi^{l-2} \wedge \omega_\zeta^n \wedge \omega_\eta^m$. In this case a direct computation shows that

$$
\begin{aligned}
I_{21} \;=\;& (z \bullet \overline{\zeta} + |\zeta|^2)(w \bullet \overline{\eta} + |\eta|^2) \left[\sum_{j=1}^{l} f_j d\overline{\xi}_j \right] \\
& \wedge \left[\sum_{j=1}^{l} [\overline{\xi}_j(1 - \Theta \bullet \overline{Z}) - \overline{x}_j(1 - |\Theta|^2)] \, d\xi_j \right] \wedge \overline{\partial}|\xi|^2 \\
& \wedge \left[\sum_{j=1}^{l} \overline{x}_j d\xi_j \right] \wedge \omega_\xi^{l-2} \\
=\;& (1 - \Theta \bullet \overline{Z})(z \bullet \overline{\zeta} + |\zeta|^2)(w \bullet \overline{\eta} + |\eta|^2) \left\{ \left[\sum_{j=1}^{l} f_j d\overline{\xi}_j \right] \right. \\
& \left. \wedge \left[\sum_{j=1}^{l} \overline{\xi}_j d\xi_j \right] \wedge \left[\sum_{j=1}^{l} \overline{x}_j d\xi_j \right] \wedge \overline{\partial}|\xi|^2 \wedge \omega_\xi^{l-2} \right\}.
\end{aligned}
$$

Since

$$
\left[\sum_{j=1}^{l} \overline{\xi}_j d\xi_j \right] \wedge \left[\sum_{j=1}^{l} \overline{x}_j d\xi_j \right] = \sum_{j,k=1, j<k}^{l} (\overline{\xi}_j \overline{x}_k - \overline{\xi}_k \overline{x}_j) \, d\xi_j \wedge d\xi_k,
$$

we see easily that I_{21} satisfies the conclusion of the lemma.

Case 2: $J_2 = \omega_\xi^l \wedge \omega_\zeta^{n-2} \wedge \omega_\eta^m$. In this case we can rewrite I_{22} in the form

$$
\begin{aligned}
(w \bullet \overline{\eta} + |\eta|^2) \left\{ \left[\sum_{j=1}^{n+1} f_{j+l} d\overline{\zeta}_j \right] \wedge \sum_{j=1}^{n+1} [\overline{\zeta}_j(1 - \Theta \bullet \overline{Z}) - \overline{z}_j(1 - |\Theta|^2)] \, d\zeta_j \wedge \overline{\partial}|\zeta|^2 \right. \\
\left. \wedge \left[\sum_{j=1}^{n+1} \overline{z}_j d\zeta_j \right] \wedge \left[\sum_{k,p=1}^{n+1} (z_k + \zeta_k)\overline{\zeta}_p d\zeta_k \wedge d\overline{\zeta}_p \right] \wedge \omega_\zeta^{n-2} \right\} \wedge \omega_\xi^l \wedge \omega_\eta^{m+1}.
\end{aligned}
$$

In view of the proof of Lemma 2.7 in [?], the differential form in braces can be expressed as the product of $d\zeta \wedge d\overline{\zeta}$ and a function of the form

$$
\sum_{j=1}^{n+1} f_{j+l} \left((1 - \Theta \bullet \overline{Z})S_j(\Theta, Z) + (1 - |\Theta|^2)T_j(\Theta, Z) \right),
$$

where S_j, T_j are some polynomials such that

$$
\begin{aligned}
S_j(\Theta, Z) \;=\;& O\left(|z - \zeta||\zeta|^2\right), \quad T_j(\Theta, Z) = O\left(|z - \zeta||\zeta|^2\right), \\
\mathrm{grad}_Z S_j(\Theta, Z) \;=\;& O\left(|\zeta|^2 + |\zeta||z|\right), \quad \mathrm{grad}_Z T_j(\Theta, Z) = O\left(|\zeta|^2 + |\zeta||z|\right).
\end{aligned}
$$

Combining what we have proved so far, we obtain that I_{22} satisfies the conclusion of the lemma.

Case 3: $J_3 = \omega_\xi^l \wedge \omega_\zeta^n \wedge \omega_\eta^{m-2}$. This case can be treated in the same way as the previous case.

Case 4: $J_4 = \omega_\xi^{l-1} \wedge \omega_\zeta^{n-1} \wedge \omega_\eta^m$. Then we have

$$
I_{24} = \left[\sum_{j=1}^{l+n+1} f_j d\overline{\Theta}_j \right]
$$
$$
\wedge \left\{ \sum_{t=1}^{l+n+1} \left[\overline{\Theta}_t (1 - \Theta \bullet \overline{Z}) - \overline{Z}_t (1 - |\Theta|^2) \right] d\Theta_t \wedge \overline{\partial}(|\xi|^2 + |\zeta|^2) \right.
$$
$$
\wedge \left[\sum_{s=1}^{l+n+1} \overline{Z}_s d\Theta_s \right] \wedge \left[\sum_{k,p=1}^{n+1} (z_k + \zeta_k) \overline{\zeta_p} d\zeta_k \wedge d\overline{\zeta_p} \right] \wedge \omega_\zeta^{n-1} \wedge \omega_\xi^{l-1} \right\}
$$
$$
\wedge (w \bullet \overline{\eta} + |\eta|^2) \omega_\eta^{m+1}.
$$

By splitting $\sum_{j=1}^{l+n+1} f_j d\overline{\Theta}_j$ into a sum of the two parts $\sum_{j=1}^{l} f_j d\overline{\xi_j}$ and $\sum_{k=1}^{n+1} f_{k+l} d\overline{\zeta_k}$, we also split I_{24} into two corresponding parts as $I_{24} = I_{241} + I_{242}$. A little calculation gives that

$$
I_{241} = \sum_{j=1}^{l} \sum_{k,p=1}^{n+1} f_j (z_k + \zeta_k) \overline{\zeta_p} d\overline{\xi_j} \wedge d\zeta_k \wedge d\overline{\zeta_p}
$$
$$
\wedge \left\{ \left[\sum_{t=1}^{l+n+1} \left[\overline{\Theta}_t (1 - \Theta \bullet \overline{Z}) - \overline{Z}_t (1 - |\Theta|^2) \right] d\Theta_t \right] \wedge \overline{\partial} |\zeta|^2 \right.
$$
$$
\wedge \left[\sum_{s=1}^{l+n+1} \overline{Z}_s d\Theta_s \right] \wedge \omega_\xi^{l-1} \wedge \omega_\zeta^{n-1} \wedge (w \bullet \overline{\eta} + |\eta|^2) \omega_\eta^{m+1}
$$
$$
= C(1 - \Theta \bullet \overline{Z})(w \bullet \overline{\eta} + |\eta|^2) \left\{ \sum_{j=1}^{l} f_j \left(\sum_{k,p=1}^{n+1} (z_k + \zeta_k) \overline{\zeta_p} \zeta_k \left(\overline{\xi_j} \overline{z_p} - \overline{x_j} \overline{\zeta_p} \right) \right) \right\}
$$
$$
\wedge \omega_\xi^l \wedge \omega_\zeta^{n+1} \wedge \omega_\eta^{m+1}.
$$

Similarly, since $\overline{\partial} |\xi|^2 = \sum_{s=1}^{l} \xi_s d\overline{\xi_s}$, we obtain

$$I_{242} = \sum_{s=1}^{l} \sum_{\substack{k,p=1 \\ k \neq p}}^{n+1} f_{k+l}(z_k + \zeta_k)\overline{\zeta_p}\xi_s d\zeta_k \wedge d\overline{\zeta_k} \wedge d\overline{\zeta_p} \wedge d\overline{\xi_s}$$

$$\wedge \left\{ \left[\sum_{t=1}^{l+n+1} \left[\overline{\Theta}_t(1 - \Theta \bullet \overline{Z}) - \overline{Z}_t(1 - |\Theta|^2)\right] d\Theta_t \right] \wedge \left[\sum_{t=1}^{l+n+1} \overline{Z}_t d\Theta_t \right] \right\}$$

$$\wedge \, \omega_\xi^{l-1} \wedge \omega_\zeta^{n-1} \wedge (w \bullet \overline{\eta} + |\eta|^2)\omega_\eta^{m+1}$$

$$= C(1 - \Theta \bullet \overline{Z})(w \bullet \overline{\eta} + |\eta|^2) \sum_{k=1}^{n+1} f_{k+l} \left\{ \sum_{s=1}^{l} \sum_{\substack{p=1 \\ p \neq k}}^{n+1} \left[(z_k + \zeta_k)\overline{\zeta_p}\xi_s \right. \right.$$

$$\left. \left. \left(\overline{x_s\zeta_p} - \overline{\xi_s z_p}\right) - (z_p + \zeta_p)\overline{\zeta_p}\xi_s \left(\overline{x_s\zeta_k} - \overline{\xi_s z_k}\right) \right] \right\} \wedge \omega_\xi^l \wedge \omega_\zeta^{n+1} \wedge \omega_\eta^{m+1}.$$

It can be checked that I_{241}, I_{242} and I_{24} satisfy the conclusion of the lemma.

Case 5: $J_5 = \omega_\xi^l \wedge \omega_\zeta^{n-1} \wedge \omega_\eta^{m-1}$. Observe that

$$I_{25} = \left[\sum_{j=l+1}^{|N|+2} f_j d\overline{\Theta}_j \right] \wedge \left\{ \sum_{t=l+1}^{|N|+2} \left[\overline{\Theta}_t(1 - \Theta \bullet \overline{Z}) - \overline{Z}_t(1 - |\Theta|^2)\right] d\Theta_t \right.$$

$$\wedge \overline{\partial}(|\zeta|^2 + |\eta|^2) \wedge \left[\sum_{s=l+1}^{|N|+2} \overline{Z}_s d\Theta_s \right] \wedge \left[\sum_{k,p=1}^{n+1} (z_k + \zeta_k)\overline{\zeta_p} d\zeta_k \wedge d\overline{\zeta_p} \right]$$

$$\left. \left[\sum_{r,s=1}^{m+1} (w_r + \eta_r)\overline{\eta_s} d\eta_r \wedge d\overline{\eta_s} \right] \wedge \omega_\zeta^{n-1} \wedge \omega_\eta^{m-1} \right\} \wedge \omega_\xi^l.$$

Rewriting $\sum_{j=l+1}^{|N|+2} f_j d\overline{\Theta}_j$ as the sum of two differential forms $\sum_{k=1}^{n+1} f_{k+l} d\overline{\zeta_k}$ and $\sum_{j=1}^{m+1} f_{j+l+n+1} d\overline{\eta_j}$, we thus divide I_{25} into two corresponding terms: $I_{25} = I_{251} + I_{252}$. A straightforward computation shows that

– 199 –

$$I_{251} = \sum_{k,p,q=1}^{n+1} \sum_{r,s=1}^{m+1} f_k(z_q + \zeta_q)\overline{\zeta_p}(\eta_r + w_r)\overline{\eta_s} d\overline{\zeta_k} \wedge d\zeta_q \wedge d\overline{\zeta_p} \wedge d\eta_r \wedge d\overline{\eta_s}$$

$$\wedge \left\{ \left[\sum_{t=l+1}^{|N|+2} [\overline{\Theta}_t(1 - \Theta \bullet \overline{Z}) - \overline{Z}_t(1 - |\Theta|^2)] \, d\Theta_t \right] \wedge \overline{\partial}|\eta|^2 \right.$$

$$\left. \wedge \left[\sum_{s=l+1}^{|N|+2} \overline{Z}_s d\Theta_s \right] \right\} \wedge \omega_\zeta^{n-1} \wedge \omega_\eta^{m-1} \wedge \omega_\xi^l$$

$$= (1 - \Theta \bullet \overline{Z})|\eta \bullet (w - \eta)|^2 \left\{ \sum_{k=1}^{n+1} f_k \left[C \sum_{p \neq k} (z_p + \zeta_p)\overline{\zeta_p \zeta_k} + C(z_k + \zeta_k)|\zeta_k|^2 \right] \right\}$$

$$\wedge \omega_\zeta^{n+1} \wedge \omega_\eta^{m+1} \wedge \omega_\xi^l. \tag{3.6}$$

We obtain in exactly the same way an explicit expression for I_{252}. Finally, we deduce from these expressions that I_{251}, I_{252} and I_{25} satisfy the conclusion of the lemma.

Case 6: $J_6 = \omega_\xi^{l-1} \wedge \omega_\zeta^n \wedge \omega_\eta^{m-1}$. This last case can be treated in the same way as Case 4. The proof of Lemma 3.5 is therefore complete. □

We end this section with the study of the particular case $N = (2,2)$. In this case we write for $Z, \Theta \in \mathbb{B}_N$:

$$Z \equiv (z, w) \equiv (z_1, z_2, z_3, w_1, w_2, w_3), \quad \text{and} \quad \Theta \equiv (\zeta, \eta) \equiv (\zeta_1, \zeta_2, \zeta_3, \eta_1, \eta_2, \eta_3).$$

To establish optimal Lipschitz estimates for the domain $\Omega_{(2,2)}$, we need a more precise formulation of the Cauchy type formula given in Theorem 3.3.

THEOREM 3.6. — *Let* $N := (2,2)$. *There are polynomials* $R(\Theta, Z)$ *and* $P_{jk}(\Theta, Z)$, $Q_{jk}(\Theta, Z)$, $1 \leqslant j \leqslant 2$, $1 \leqslant k \leqslant 4$, *that satisfy the following properties:*

(i) $R(\Theta, Z) = (C|\zeta|^2 + C|\eta|^2)(|\zeta|^2 + z \bullet \overline{\zeta})(|\eta|^2 + w \bullet \overline{\eta})$.

(ii) *For every* $Z, \Theta \in \mathbb{B}_N$, *and for every* $1 \leqslant j \leqslant 2$ *and* $1 \leqslant k \leqslant 4$,

$$P_{1k}(\Theta, Z) = O\left(|\Theta - Z|(|\zeta_3| + |\eta_3| + |\Theta - Z|)(|\zeta|^2 + |\zeta||z|)(|\eta|^2 + |\eta||w|)\right),$$

$$Q_{1k}(\Theta, Z) = O\left(|\Theta - Z|(|\zeta_3| + |\eta_3| + |\Theta - Z|)(|\zeta|^2 + |\zeta||z|)(|\eta|^2 + |\eta||w|)\right),$$

$$|\mathrm{grad}_Z\, P_{jk}(\Theta, Z)| \;=\; O\left((|\zeta|^2 + |\zeta||z|)(|\eta|^2 + |\eta||w|)\right),$$
$$|\mathrm{grad}_Z\, Q_{jk}(\Theta, Z)| \;=\; O\left((|\zeta|^2 + |\zeta||z|)(|\eta|^2 + |\eta||w|)\right).$$

(iii) Let $u \in \mathcal{C}^1(\overline{\mathbb{M}}_N)$ and $f := f_1 d\overline{\zeta}_1 + f_2 d\overline{\zeta}_2 + f_3 d\overline{\eta}_1 + f_4 d\overline{\eta}_2$ is a $(0,1)$-form with coefficients in $\mathcal{C}(\overline{\mathbb{M}}_N)$ that satisfy $\overline{\partial}_{\mathbb{M}_N} u = f|_{\mathbb{M}_N}$, then for every $Z \in \mathbb{M}_N$,

$$u(Z) = \int\limits_{\partial \mathbb{M}_N} \frac{R(\Theta, Z)}{\left(1 - Z \bullet \overline{\Theta}\right)^{|N|}} u(\Theta) \frac{d\sigma(\Theta)}{|\zeta|^2 |\eta|^2} +$$

$$\int\limits_{\mathbb{M}_N} \sum_{j=1}^{2} \sum_{k=1}^{4} \frac{(1 - \Theta \bullet \overline{Z})^{1+j}}{D(\Theta, Z)^4} [(1 - \Theta \bullet \overline{Z})P_{jk}(\Theta, Z) + (1 - |\Theta|^2)Q_{jk}(\Theta, Z)]$$

$$f_k(\Theta) \frac{dV(\Theta)}{|\zeta|^2 |\eta|^2}.$$

Proof.— We return to the arguments used in the proof of Theorem 3.3. By the hypothesis on f and (3.2), we have that

$$I := \overline{\partial} u \wedge K_0 = (f_1 H_1 + f_2 H_2 + f_3 H_3 + f_4 H_4) d\Theta \wedge d\overline{\Theta}, \qquad (3.7)$$

with $H_1 := h_1, H_2 := h_2, H_3 := h_4$ and $H_4 := h_5$. To complete the proof, it suffices to prove the following

LEMMA 3.7.— *The functions H_k in formula (3.7) can be expressed in the form*

$$H_k(\Theta, Z) = \sum_{j=1}^{2} \frac{(1 - \Theta \bullet \overline{Z})^{1+j}}{|\zeta|^2 |\eta|^2 D(\Theta, Z)^4} [(1 - \Theta \bullet \overline{Z})P_{jk}(\Theta, Z) + (1 - |\Theta|^2)Q_{jk}(\Theta, Z)],$$

$$(3.8)$$

where P_{jk} and Q_{jk} are some polynomials satisfying assertion (ii) of the theorem.

End of the proof of Theorem 3.6.— Suppose that the lemma is proved. Using the arguments that precede Lemma 3.4 in the proof of Theorem 3.3 and applying Proposition 2.5, the theorem follows. $\square$

Proof of Lemma 3.7.— Following the proof of Lemma 3.4, we write $I = I_1 + I_2$. By virtue of (3.4), the functions H_k associated to I_1 (similarly to those associated to I in (3.7)) are in the form (3.8) with $j = 2$.

Since $l = 0$ and $\omega_\xi = 0$, formula (3.5) becomes

$$\Psi \wedge \overline{\partial(\zeta \bullet \zeta)} \wedge \overline{\partial(\eta \bullet \eta)} \wedge \left[\sum_{k=1}^{6} d\overline{\Theta}_k \wedge d\Theta_k \right]^2$$

$$= \Psi \wedge \overline{\partial(\zeta \bullet \zeta)} \wedge \overline{\partial(\eta \bullet \eta)} \wedge \left(2\omega_\zeta \wedge \omega_\eta + \omega_\eta^2 + \omega_\zeta^2 \right)$$

$$\equiv \Psi \wedge \overline{\partial(\zeta \bullet \zeta)} \wedge \overline{\partial(\eta \bullet \eta)} \wedge (J_1 + J_2 + J_3). \tag{3.9}$$

Therefore, to conclude the proof of Lemma 3.7, it suffices to prove the following

LEMMA 3.8. — *For every* $1 \leqslant k \leqslant 3$, *the differential form*

$$I_{2k} := \left[f_1 d\overline{\zeta}_1 + f_2 d\overline{\zeta}_2 + f_3 d\overline{\eta}_1 + f_4 d\overline{\eta}_2 \right]$$

$$\wedge \sum_{r=1}^{6} \left[\overline{\Theta}_r (1 - \Theta \bullet \overline{Z}) - \overline{Z}_r (1 - |\Theta|^2) \right] d\Theta_r$$

$$\wedge \overline{\partial}|\Theta|^2 \wedge \left[\sum_{s=1}^{6} \overline{Z}_s d\Theta_s \right] \wedge J_k \wedge \Psi \wedge \overline{\partial(\zeta \bullet \zeta)} \wedge \overline{\partial(\eta \bullet \eta)}$$

can be expressed as the product of the canonical volume form $d\Theta \wedge d\overline{\Theta}$ *and a function of the form*

$$\sum_{t=1}^{4} f_t \left((1 - \Theta \bullet \overline{Z}) P_{1t}(\Theta, Z) + (1 - |\Theta|^2) Q_{1t}(\Theta, Z) \right),$$

where P_{1t}, Q_{1t} *are some polynomials satisfying assertion (ii) of the theorem.*

End of the proof of Lemma 3.7. — Suppose that Lemma 3.8 is proved. In view of (3.9) and the expression of I_2 given at the beginning of the proof of Lemma 3.4, we see that

$$I_2 = \frac{(1 - \Theta \bullet \overline{Z})^2}{|\zeta|^2 |\eta|^2 D(\Theta, Z)^4} \cdot \left(\sum_{k=1}^{3} I_{2k} \right).$$

Therefore, Lemma 3.7 follows from Lemma 3.8. □

Proof of Lemma 3.8. — We first remark that the case $k = 1$ corresponds to the case 5 in the proof of Lemma 3.5. Hence, by virtue of identity (3.6), I_{21} satisfies the conclusion of the lemma. Consider the case $k = 2$ which corresponds to case 2 in the proof of Lemma 3.5. Then we have

$$I_{22} = \left\{ [f_1 d\bar{\zeta}_1 + f_2 d\bar{\zeta}_2] \wedge \sum_{r=1}^{3} [\bar{\zeta}_r(1 - \Theta \bullet \bar{Z}) - \bar{z}_r(1 - |\Theta|^2)] \, d\zeta_r \wedge \bar{\partial}|\zeta|^2 \right.$$

$$\left. \wedge \left[\sum_{s=1}^{3} \bar{z}_s d\zeta_s \right] \wedge \left[\sum_{t,p=1}^{3} (z_t + \zeta_t)\bar{\zeta}_p d\zeta_t \wedge d\bar{\zeta}_p \right] \right\} \wedge (w \bullet \bar{\eta} + |\eta|^2)\omega_\eta^3.$$

A simple calculation gives that

$$I_{22} = \left\{ f_1(1 - \Theta \bullet \bar{Z})(C\bar{\zeta}_3\zeta_2 + C\zeta_3\bar{\zeta}_2) \left[\sum (-1)^{\epsilon(r,s,t)} \right. \right.$$

$$+ (\bar{\zeta}_r \bar{z}_s - \bar{\zeta}_s \bar{z}_r)(z_t + \zeta_t) \,_2 \, (1 - \Theta \bullet \bar{Z})(C\bar{\zeta}_3\zeta_1 + C\zeta_3\bar{\zeta}_1)$$

$$\left[\sum (-1)^{\epsilon(r,s,t)}(\bar{\zeta}_r \bar{z}_s - \bar{\zeta}_s \bar{z}_r)(z_t + \zeta_t) \right] \cdot (w \bullet \bar{\eta} + |\eta|^2)\omega_\zeta^3 \wedge \omega_\eta^3,$$

where the sum is taken over all permutations (r, s, t) of $\{1, 2, 3\}$ such that $r < s$ and where $\epsilon(r, s, t)$ is the sign of such permutations. It follows from this that I_{22} satisfies the conclusion of the lemma. Similarly, we have the same conclusion for I_{23}, which completes the proof. $\qquad \square$

4. Local coordinate systems on the complex manifolds $\mathbb{H}_n$ and $\mathbb{H}_m$

In the next theorem, we construct an open neighborhood $\mathcal{U}_n$ of $\mathbb{H}_n$ in $\mathbb{C}^{n+1}$, and for every $z \in \mathcal{U}_n$, a coordinate chart Φ^z defined on a coordinate patch $\mathcal{U}(z)$ of $\mathbb{H}_n$ that possess some interesting properties of homogeneity. The same construction will be applied to the complex manifold $\mathbb{H}_m$. These local coordinate systems will allow us in the next section to reduce certain types of integral estimates over $\mathbb{M}_N$ to simpler integral estimates over $\mathbb{C}^{|N|}$.

THEOREM 4.1.— *There are an open neighborhood $\mathcal{U}_n$ of $\mathbb{H}_n$ in $\mathbb{C}^{n+1}$ and constants $C_1, C_2, C_3 > 1$ that satisfy the following properties:*

1) If $z \in \mathbb{C}^{n+1} \setminus \mathcal{U}_n$ then $\mathrm{dist}(z, \mathbb{H}_n) > \frac{|z|}{C_1}$ with the understanding that $\mathrm{dist}(.,.)$ is the euclidean distance.

2) If $z \in \mathcal{U}_n$ and if the open set $\mathcal{U}(z) := \left\{ \zeta \in \mathbb{H}_n : |\zeta - z| < \frac{|z|}{C_1} \right\}$ is non-empty, then there exists a diffeomorphism Φ^z mapping $\mathcal{U}(z)$ into the open neighborhood $\widetilde{\mathcal{U}}(\widetilde{z}) := \left\{ \widetilde{\zeta} \in \mathbb{C}n : |\widetilde{\zeta} - \widetilde{z}| < \frac{|z|}{C_2} \right\}$ of a point $\widetilde{z} \in \mathbb{C}^n$ which is exactly $\Phi^z(z)$ in case $z \in \mathbb{H}_n$ such that

(i) $\zeta \bullet \overline{z} = \Phi^z(\zeta) \bullet \overline{\Phi^z(z)}$, for all $\zeta \in \mathcal{U}(z)$.

(ii) $|\Phi^z(z)| = |z|$ and $\frac{|\zeta|}{2} \leqslant |\Phi^z(\zeta)| \leqslant |\zeta|$, for all $\zeta \in \mathcal{U}(z)$.

(iii) For all $\zeta \in \mathcal{U}(z)$, we have $\frac{|\zeta|}{|\zeta_{n+1}|} \leqslant C_3 \frac{|\Phi^z(\zeta)|}{|\Phi^z_n(\zeta)|}$, where $\Phi^z := (\Phi^z_1, \ldots, \Phi^z_n)$.

(iv) For all compactly supported functions $f \in \mathcal{C}_0(\mathcal{U}(z))$ such that $f \geqslant 0$, we have

$$\int_{\mathcal{U}(z)} f dV_n \leqslant C_3 \int_{\widetilde{\mathcal{U}}(\widetilde{z})} (\Phi^z_* f)(\widetilde{\zeta}) dV_n(\widetilde{\zeta}),$$

where $dV_n(\widetilde{\zeta})$ denotes the Lebesgue measure on $\mathbb{C}^n$ and $\Phi^z_* f$ is the pushforward of f under the diffeomorphism Φ^z.

Remark 4.2. — We construct in the same way an open neighborhood $\mathcal{U}_m$ of $\mathbb{H}_m$ in $\mathbb{C}^{m+1}$, and for every $w \in \mathcal{U}_m$, a coordinate chart Φ^w defined on a coordinate patch $\mathcal{U}(w)$ of $\mathbb{H}_m$ that possess the same properties as $\mathcal{U}_n, \Phi^z$ and $\mathcal{U}(z)$.

To prove Theorem 4.1, we need the following

LEMMA 4.3. — *There exists a constant $C_0 > 0$ such that* $\max_{1 \leqslant j < k \leqslant n+1} |\mathrm{Im}(z_j \overline{z_k})| > C_0$, *for all* $z := (z_1, \ldots, z_{n+1}) \in \partial \mathbb{M}_n$. *Here* $\mathrm{Im}\lambda$ *denotes the imaginary part of* $\lambda \in \mathbb{C}$.

Proof. — Since the function $z \mapsto \max_{1 \leqslant j < k \leqslant n+1} |\mathrm{Im}(z_j \overline{z_k})|$ is continuous on the compact set $\partial \mathbb{M}_n$, it attains its minimum at a point z. Therefore it suffices to prove that there exist $1 \leqslant j < k \leqslant n+1$ such that $\mathrm{Im}(z_j \overline{z_k}) \neq 0$. Suppose the contrary. Since $|z| = 1$, there is an k such that $z_k \neq 0$. Hence for every $1 \leqslant j \leqslant n+1$, we have $z_j = \lambda_j z_k$ with $\lambda_j \in \mathbb{R}$, from which it follows that $0 = \left(\sum_{j=1}^{n+1} \lambda_j^2 \right) z_k^2$. Thus $z_k = 0$ and we obtain a contradiction. This completes the proof of the lemma. $\quad\square$

We now turn to the proof of Theorem 4.1.

Proof of Theorem 4.1. — The construction of the open neighborhood $\mathcal{U}_n$, the coordinate patches $\mathcal{U}(z)$ and the coordinate charts $\Phi^z : \mathcal{U}(z) \to \mathbb{C}^n$ for every $z \in \mathcal{U}_n$, will be done within two steps. First, by Lemma 4.3, we divide $\partial \mathbb{M}_n$ into $\frac{n(n+1)}{2}$ compact sets E_{jk}, $j < k$, where $E_{jk} := \{z \in \partial \mathbb{M}_n : |\mathrm{Im}(z_j \overline{z_k})| \geqslant C_0\}$.

Fix a sufficiently small number $\delta > 0$. The exact value of δ will be clear in the course of the proof. Let z be a point of $\mathbb{C}^{n+1}$.

Step 1: $\text{dist}(z, \partial\mathbb{M}_n) < \delta$.

According to the discussion above, suppose without loss of generality that there exist $j < k$ and $\hat z \in E_{jk}$ such that $|z - \hat z| < \delta$. Define the diffeomorphism Φ^z as follows : $\Phi^z := (\Phi_1^z, \ldots, \Phi_n^z)$, where

$$
\begin{cases}
\Phi_{k-1}^z(\zeta) := \dfrac{\zeta_j \bullet \overline{z}_j + \zeta_k \bullet \overline{z}_k}{\sqrt{|z_j|^2 + |z_k|^2}}; \\[2mm]
\Phi_l^z(\zeta) := \zeta_l, \quad \text{if } l < j; \\[2mm]
\Phi_l^z(\zeta) := \zeta_{l+1}, \quad \text{if } j \leqslant l < k-1 \text{ or } k \leqslant l \leqslant n.
\end{cases}
$$

We can choose the functions ζ_l, $l \neq j$, as the n-local coordinate functions of $\mathbb{H}_n$ at the point $\hat z$. Substituting ζ_j by $i\sqrt{\sum_{l \neq j} \zeta_l^2}$ in the expression of Φ^z, straightforward computations show that the real Jacobian of Φ^z at the point ζ corresponding to this local coordinate system is equal to $\dfrac{|\zeta_j \overline{z}_k - \zeta_k \overline{z}_j|^2}{|\zeta_j|^2(|z_j|^2 + |z_k|^2)}$. This quantity is uniformly bounded from above and from below by some positive constants as $\zeta \in \mathbb{H}_n$ and z are very near to $\hat z \in E_{jk}$. Therefore, when C_2 is sufficiently large, there exists a sufficiently small δ so that for every $\hat z \in E_{jk}$ and every z such that $|z - \hat z| < \delta$, Φ^z is a diffeomorphism from $\{\zeta \in \mathbb{H}_n : |\zeta - z| < 2\delta\}$ to $\left\{\widetilde\zeta \in \mathbb{C}^n : |\widetilde\zeta - \Phi^z(z)| < \frac{1}{2C_2}\right\}$.

Taking $C_1 > \frac{1}{2\delta}$ and observing that $|\Phi^z(z)| = |z| \approx 1$, it follows from the previous discussion that Φ^z is a diffeomorphism from $\mathcal{U}(z)$ onto an open neighborhood $\widetilde{\mathcal{U}}(\widetilde z)$ of the point $\widetilde z := \Phi^z(z) \in \mathbb{C}^n$.

To finish part 2) of the theorem, it remains to prove assertions (i)-(iv).

Assertions (i) and (ii) can be checked direcly. In particular, the estimate $|\Phi^z(\zeta)| \leqslant |\zeta|$ follows from the Cauchy-Schwarz inequality.

We prove now assertion (iii). Consider two cases according to k:
Case $k < n + 1$. In this case, in view of the definition of Φ^z, we have $\zeta_{n+1} = \Phi_n^z(\zeta)$. This, combined with (ii), implies assertion (iii).
Case $k = n + 1$. If $\zeta \in \mathcal{U}(z)$, then when C_1 is sufficiently large, we have $1 > |\zeta_{n+1}| \approx |\hat z_{n+1}| \geqslant C_0$. Hence assertion (iii) is almost obvious.

It now remains to prove assertion (iv). By Proposition 2.1 in [?], for $\zeta \in \mathcal{U}(z)$ we have the following identity:

$$
dV_n(\zeta) = C\frac{|\zeta|^2}{|\zeta_j|^2} d\zeta_1 \wedge d\overline{\zeta}_1 \wedge \ldots \wedge \widehat{d\zeta_j} \wedge \widehat{d\overline{\zeta}_j} \wedge \ldots \wedge d\zeta_{n+1} \wedge d\overline{\zeta}_{n+1}.
$$

Since $2 > |\zeta| > |\zeta_j| \approx |\hat z_j| \geqslant C_0$, it follows that $dV_n(\zeta) \approx (\Phi^z)^*\left(dV_n(\widetilde\zeta)\right)$ for $\widetilde\zeta = \Phi^z(\zeta)$. This implies assertion (iv).

Step 2: General case.

Set $\mathcal{U}_n := \{rz : r > 0 \text{ and } \mathrm{dist}(z, \partial\mathbb{M}_n) < \delta\}.$

If $z \in \mathcal{U}_n$, then according to the definition above, there exist $\hat{z} \in \partial\mathbb{M}_n$ and $r > 0$ such that $|rz - \hat{z}| < \delta$. Therefore, the construction given in Step 1 can then be applied to the point rz. Hence, we can define

$$\mathcal{U}(z) \quad := \quad \frac{1}{r} \cdot \mathcal{U}(rz);$$

$$\Phi^z(\zeta) \quad := \quad \frac{1}{r} \cdot \Phi^{rz}(r\zeta), \qquad \forall \zeta \in \mathcal{U}(z).$$

Using the homogeneous invariance of the complex manifold $\mathbb{H}_n$ with respect to the dilations, we conclude that for every $z \in \mathcal{U}_n$, the function Φ^z just defined satisfies part 2) of the theorem. To finish the proof of the theorem, it only remains to check part 1). Let $z \notin \mathcal{U}_n$. Then there exists a point $\hat{z} \in \mathbb{H}_n$ such that $|z - \hat{z}| = \mathrm{dist}(z, \mathbb{H}_n)$. Since $z \notin \mathcal{U}_n$, we deduce that $|z - \hat{z}| \geqslant \delta|\hat{z}|$. Hence

$$\left(\frac{1}{\delta} + 1\right) \mathrm{dist}(z, \mathbb{H}_n) > |z - \hat{z}| + |\hat{z}| \geqslant |z|.$$

Thus, if we choose $C_1 > \frac{1}{\delta} + 1$, then part 1) is satisfied. This completes the proof of the theorem. $\qquad \square$

5. Reduction of estimates from $\mathbb{M}_N$ to $\mathbb{B}_{|N|}$

This section proves the Theorem of reduction of estimates. We use the notations and the constants introduced in the previous section. In order to state this theorem, we need some more notations and definitions.

We denote by $\mathbb{B}_{|N|}$ the euclidean unit ball of $\mathbb{C}^{|N|}$. We often use the following notations for $\widetilde{\Theta}, \widetilde{Z} \in \mathbb{C}^{|N|}$:

$$\widetilde{\Theta} \equiv (\tilde{\xi}, \tilde{\zeta}, \tilde{\eta}) \in \mathbb{C}^l \times \mathbb{C}^n \times \mathbb{C}^m \quad \text{and} \quad \widetilde{Z} \equiv (\tilde{x}, \tilde{z}, \tilde{w}) \in \mathbb{C}^l \times \mathbb{C}^n \times \mathbb{C}^m.$$

Let $dV(\widetilde{\Theta})$ be the Lebesgue measure on $\mathbb{C}^{|N|}$. For every $i \in \{1, 2\}$, note $\mathbb{B}_{i,|N|}$ the euclidean ball of $\mathbb{C}^{|N|}$ centered at the origin with radius i. Thus $\mathbb{B}_{|N|} = \mathbb{B}_{1,|N|}$.

We shall define various notions of comparability.

DEFINITION 5.1.— *Consider two points $Z \equiv (x, z, w) \in \mathbb{B}_N$ and $\widetilde{Z} \equiv (\tilde{x}, \tilde{z}, \tilde{w}) \in \mathbb{C}^{|N|}$. Z is said to be comparable with $\widetilde{Z}$ if the following conditions are true:*

(1) $x = \widetilde{x}$.

(2) If $z \in \mathcal{U}_n$ and $\mathcal{U}(z) \neq \varnothing$, then $\widetilde{z} = \Phi^z(z)$, if not $|\widetilde{z}| = |z|$.

(3) If $w \in \mathcal{U}_m$ and $\mathcal{U}(w) \neq \varnothing$, then $\widetilde{w} = \Phi^w(w)$, if not $|\widetilde{w}| = |w|$.

Remark 5.2. — It should be noted that by this definition and Theorem 4.1 (ii), we have $|x| = |\widetilde{x}|, |z| = |\widetilde{z}|, |w| = |\widetilde{w}|$. Hence $\widetilde{Z} \in \mathbb{B}_{|N|}$.

DEFINITION 5.3. — *Let $i \in \{1, 2\}$ and fix two comparable points $Z \equiv (x, z, w) \in \mathbb{B}_N$ and $\widetilde{Z} \equiv (\widetilde{x}, \widetilde{z}, \widetilde{w}) \in \mathbb{C}^{|N|}$.*

We say that $\xi \in \mathbb{C}^l$ is i-comparable with $\widetilde{\xi} \in \mathbb{C}^l$ if $\xi = \widetilde{\xi}$.

We say that $\zeta \in \mathbb{H}_n$ is i-comparable with $\widetilde{\zeta} \in \mathbb{C}^n$ if the following conditions are true :

(1) If $|\zeta| > \sqrt{2}|z|$, then $|\widetilde{\zeta}| = |\zeta|$.

(2) If $|\zeta - z| < \frac{|z|}{C_1}$, then $\widetilde{\zeta} = \Phi^z(\zeta)$.

(3) If $|\zeta| \leqslant \sqrt{2}|z|$ and $|\zeta - z| \geqslant \frac{|z|}{C_1}$, then $|\widetilde{\zeta}| \leqslant \sqrt{2}|\widetilde{z}|$ and $|\widetilde{\zeta} - \widetilde{z}| > \frac{|\widetilde{z}|}{C_2}$; if moreover $i = 1$, then we have $|\widetilde{\zeta}| \leqslant |\zeta|$.

We can define in the same way the notion of i-comparability between $\eta \in \mathbb{H}_m$ and $\widetilde{\eta} \in \mathbb{C}^m$ upon substituting n by m and Φ^z by Φ^w.

Finally, two points $\Theta \equiv (\xi, \zeta, \eta) \in \overline{\mathbb{M}}_N$ and $\widetilde{\Theta} \equiv \left(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\eta} \right) \in \mathbb{C}^{|N|}$ are said to be i-comparable if ξ (resp. ζ and η) is i-comparable with $\widetilde{\xi}$ (resp. $\widetilde{\zeta}$ and $\widetilde{\eta}$).

Remark 5.4. — We deduce easily from this definition and Theorem 4.1 (ii) that if $\Theta \in \overline{\mathbb{M}}_N$ is i-comparable with $\widetilde{\Theta} \in \mathbb{C}^{|N|}$, then $\widetilde{\Theta} \in \overline{\mathbb{B}}_{i,|N|}$.

DEFINITION 5.5. — *Let $i \in \{1, 2\}$ and fix two comparable points $Z \in \mathbb{B}_N$ and $\widetilde{Z} \in \mathbb{B}_{|N|}$. Consider two non-negative measurable functions $K, \widetilde{K}$ defined respectively on $\overline{\mathbb{M}}_N$ and $\overline{\mathbb{B}}_{i,|N|}$.*

- *We write $K \lesssim C\widetilde{K}$ (respectively, $\widetilde{K} \lesssim CK$) at $(Z, \widetilde{Z})$ for a positive constant C if for all points $\Theta \in \overline{\mathbb{M}}_N$ i-comparable with $\widetilde{\Theta} \in \overline{\mathbb{B}}_{i,|N|}$,*

$$K(\Theta) \leqslant C\widetilde{K}(\widetilde{\Theta}) \qquad (\text{respectively, } \widetilde{K}(\widetilde{\Theta}) \leqslant CK(\Theta)).$$

- *We write $K \approx \widetilde{K}$ at $(Z, \widetilde{Z})$ if there exists $C > 0$ such that $K \lesssim C\widetilde{K} \lesssim C^2 K$.*

Now we are in a position to state the main theorem of this section.

THEOREM 5.6. — *Let $i \in \{1, 2\}$ and fix two comparable points $Z \in \mathbb{B}_N$ and $\widetilde{Z} \in \mathbb{B}_{|N|}$. Let C be a positive constant. Consider non-negative measurable functions K, L defined on $\overline{\mathbb{M}}_N$ and $\widetilde{K}, \widetilde{L}$ defined on $\overline{\mathbb{B}}_{i, |N|}$ such that*

$$K \lesssim C\widetilde{K} \quad \text{and} \quad L \lesssim C\widetilde{L} \lesssim C^2 L \quad \text{at } (Z, \widetilde{Z}).$$

For every $\alpha := (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ such that $0 \leqslant \alpha_1 < 2n$, $0 \leqslant \alpha_2 < 2m$ and $0 \leqslant \alpha_3, \alpha_4 < 2$, we set

$$K_\alpha(\Theta) \quad := \quad K(\Theta)\left(1 + \frac{|z|}{|\zeta|}\right)^{\alpha_1}\left(1 + \frac{|w|}{|\eta|}\right)^{\alpha_2}\left|\frac{\zeta}{\zeta_{n+1}}\right|^{\alpha_3}\left|\frac{\eta}{\eta_{m+1}}\right|^{\alpha_4},$$

$$\widetilde{K}_{1,\alpha}(\widetilde{\Theta}) \quad := \quad \widetilde{K}(\widetilde{\Theta})\left(1 + \frac{|\widetilde{z}|}{|\widetilde{\zeta}|}\right)^{\alpha_1}\left(1 + \frac{|\widetilde{w}|}{|\widetilde{\eta}|}\right)^{\alpha_2}\left|\frac{\widetilde{\zeta}}{\widetilde{\zeta}_n}\right|^{\alpha_3}\left|\frac{\widetilde{\eta}}{\widetilde{\eta}_m}\right|^{\alpha_4},$$

$$\widetilde{K}_{2,\alpha}(\widetilde{\Theta}) \quad := \quad \widetilde{K}(\widetilde{\Theta})\left|\frac{\widetilde{\zeta}}{\widetilde{\zeta}_n}\right|^{\alpha_3}\left|\frac{\widetilde{\eta}}{\widetilde{\eta}_m}\right|^{\alpha_4}.$$

Then there exists a constant C_4 that depends only on N, α and C, C_1, C_2, C_3, (in particular this constant is independant of Z and $\widetilde{Z}$), such that
1)

$$\int_{\mathbb{M}_N} K_\alpha(\Theta)dV(\Theta) \leqslant C_4 \int_{\mathbb{B}_{i,|N|}} \widetilde{K}_{i,\alpha}(\widetilde{\Theta})dV(\widetilde{\Theta});$$

2) for $\delta > 0$,

$$\int_{\Theta \in \mathbb{M}_N, \ L(\Theta) \leqslant \delta} K_\alpha(\Theta)dV(\Theta) \leqslant C_4 \int_{\widetilde{\Theta} \in \mathbb{B}_{i,|N|}, \ \widetilde{L}(\widetilde{\Theta}) \leqslant C_4 \delta} \widetilde{K}_{i,\alpha}(\widetilde{\Theta})dV(\widetilde{\Theta});$$

3) for $0 < \delta_1 < \delta_2$,

$$\int_{\Theta \in \mathbb{M}_N, \ \delta_1 \leqslant L(\Theta) \leqslant \delta_2} K_\alpha(\Theta)dV(\Theta) \leqslant C_4 \int_{\widetilde{\Theta} \in \mathbb{B}_{i,|N|}, \ \frac{\delta_1}{C_4} \leqslant \widetilde{L}(\widetilde{\Theta}) \leqslant C_4 \delta_2} \widetilde{K}_{i,\alpha}(\widetilde{\Theta})dV(\widetilde{\Theta}).$$

Proof. — We shall only prove part 3). The two other assertions can be shown in exactly the same way. Firstly, we extend the domain of definition of the functions $K, L, \widetilde{K}, \widetilde{L}$ by setting

$$K(\Theta) \quad = \quad L(\Theta) := 0, \quad \text{if } \Theta \in \mathbb{H}_N \setminus \overline{\mathbb{M}}_N;$$
$$\widetilde{K}(\widetilde{\Theta}) \quad = \quad \widetilde{L}(\widetilde{\Theta}) := 0, \quad \text{if } \widetilde{\Theta} \in \mathbb{C}^{|N|} \setminus \overline{\mathbb{B}}_{i,|N|}.$$

By the hypothesis on L and $\widetilde{L}$, for every $\Theta \in \overline{\mathbb{M}}_N$ such that $\delta_1 \leqslant L(\Theta) \leqslant \delta_2$ and for every $\widetilde{\Theta} \in \overline{\mathbb{B}}_{i,|N|}$ i-comparable with Θ, we have

$$\frac{\delta_1}{C} \leqslant \widetilde{L}(\widetilde{\Theta}) \leqslant C\delta_2. \tag{5.1}$$

For every $\xi, \widetilde{\xi} \in \mathbb{B}_l$ and $\eta, \widetilde{\eta} \in \mathbb{H}_m$, consider the following integrals

$$R(\xi, \eta) \;\; := \int\limits_{\zeta \in \mathbb{H}_n,\; \delta_1 \leqslant L(\xi,\zeta,\eta) \leqslant \delta_2} K(\xi, \zeta, \eta) \left(1 + \frac{|z|}{|\zeta|}\right)^{\alpha_1} \left|\frac{\zeta}{\zeta_{n+1}}\right|^{\alpha_3} dV_n(\zeta),$$

$$\widetilde{R}_1(\widetilde{\xi}, \widetilde{\eta}) \;\; := \int\limits_{\widetilde{\zeta} \in \mathbb{C}^n,\; \frac{\delta_1}{C} \leqslant \widetilde{L}\left(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\eta}\right) \leqslant C\delta_2} \widetilde{K}\left(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\eta}\right) \left(1 + \frac{|\widetilde{z}|}{|\widetilde{\zeta}|}\right)^{\alpha_1} \left|\frac{\widetilde{\zeta}}{\widetilde{\zeta}_n}\right|^{\alpha_3} dV_n(\widetilde{\zeta}),$$

$$\widetilde{R}_2(\widetilde{\xi}, \widetilde{\eta}) \;\; := \int\limits_{\widetilde{\zeta} \in \mathbb{C}^n,\; \frac{\delta_1}{C} \leqslant \widetilde{L}\left(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\eta}\right) \leqslant C\delta_2} \widetilde{K}\left(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\eta}\right) \left|\frac{\widetilde{\zeta}}{\widetilde{\zeta}_n}\right|^{\alpha_3} dV_n(\widetilde{\zeta}),$$

where $dV_n(\widetilde{\zeta})$ denotes the Lebesgue measure on $\mathbb{C}^n$.

Next, consider the following integrals

$$S(\xi) \;\; := \int\limits_{\mathbb{H}_m} R(\xi, \eta) \left(1 + \frac{|w|}{|\eta|}\right)^{\alpha_2} \left|\frac{\eta}{\eta_{m+1}}\right|^{\alpha_4} dV_m(\eta),$$

$$\widetilde{S}_1(\widetilde{\xi}) \;\; := \int\limits_{\mathbb{C}^m} \widetilde{R}_1(\widetilde{\xi}, \widetilde{\zeta}) \left(1 + \frac{|\widetilde{w}|}{|\widetilde{\eta}|}\right)^{\alpha_2} \left|\frac{\widetilde{\eta}}{\widetilde{\eta}_m}\right|^{\alpha_4} dV_m(\widetilde{\eta}),$$

$$\widetilde{S}_2(\widetilde{\xi}) \;\; := \int\limits_{\mathbb{C}^m} \widetilde{R}_2(\widetilde{\xi}, \widetilde{\zeta}) \left|\frac{\widetilde{\eta}}{\widetilde{\eta}_m}\right|^{\alpha_4} dV_m(\widetilde{\eta}),$$

where $dV_m(\widetilde{\eta})$ is the Lebesgue measure on $\mathbb{C}^m$.

We outline the main ideas of the proof. Suppose that ξ (resp. η) is i-comparable with $\widetilde{\xi}$ (resp. $\widetilde{\eta}$). Using the hypothesis that $K \lesssim C\widetilde{K}$, we shall prove that

$$R(\xi, \eta) \leqslant C_4 \widetilde{R}_i(\widetilde{\xi}, \widetilde{\eta}), \quad i = 1, 2. \tag{5.2}$$

Next, we shall establish in the same way as in the proof of (5.2) the following estimate : (note that $\xi = \widetilde{\xi}$)

$$S(\xi) \leqslant C_4 \widetilde{S}_i(\widetilde{\xi}), \quad i = 1, 2. \tag{5.3}$$

Finally, an application of Fubini's theorem gives that

$$\int_{\Theta \in \mathbb{M}_N, \ \delta_1 \leqslant L(\Theta) \leqslant \delta_2} K_\alpha(\Theta) dV(\Theta) = \int_{\mathbb{B}_l} S(\xi) dV_l(\xi),$$

and

$$\int_{\widetilde{\Theta} \in \mathbb{B}_{i,|N|}, \frac{\delta_1}{C} \leqslant \widetilde{L}(\widetilde{\Theta}) \leqslant C\delta_2} \widetilde{K}_{i,\alpha}(\widetilde{\Theta}) dV(\widetilde{\Theta}) = \int_{\mathbb{B}_l} \widetilde{S}_i(\xi) dV_l(\xi).$$

Part 3) now follows by combining (5.3) with the latter two estimates. It now remains to prove inequality (5.2).

To do so, divide the domain of integration $\{\zeta \in \mathbb{H}_n : \frac{\delta_1}{C} \leqslant L(\xi, \zeta, \eta) \leqslant C\delta_2\}$ of $R(\xi, \eta)$ into the three subsets :

$$E_1 := \left\{ |\zeta - z| < \frac{|z|}{C_1} \right\}; \qquad E_2 := \left\{ |\zeta| > \sqrt{2}|z| \right\};$$

$$E_3 := \left\{ |\zeta| \leqslant \sqrt{2}|z| \text{ and } |\zeta - z| \geqslant \frac{|z|}{C_1} \right\}.$$

Also, divide the domain of integration $\left\{ \widetilde{\zeta} \in \mathbb{C}^n : \frac{\delta_1}{C} \leqslant \widetilde{L}\left(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\eta}\right) \leqslant C\delta_2 \right\}$ of $\widetilde{R}_i(\widetilde{\xi}, \widetilde{\eta})$ into three corresponding subsets :

$$\widetilde{E}_1 := \left\{ |\widetilde{\zeta} - \widetilde{z}| < \frac{|\widetilde{z}|}{C_2} \right\}; \qquad \widetilde{E}_2 := \left\{ |\widetilde{\zeta}| > \sqrt{2}|\widetilde{z}| \right\};$$

$$\widetilde{E}_3 := \left\{ |\widetilde{\zeta}| \leqslant \sqrt{2}|\widetilde{z}| \text{ and } |\widetilde{\zeta} - \widetilde{z}| \geqslant \frac{|\widetilde{z}|}{C_2} \right\}.$$

Estimate (5.2) will follow by combining three integral estimates of the form $\int_{E_j} \leqslant C_4 \int_{\widetilde{E}_j}$ with some appropriate integrands and $j = 1, 2, 3$. Therefore, we may assume without loss of generality that $E_j \neq \varnothing$, $j = 1, 2, 3$.

Combining Theorem 4.1, definition 5.3 and estimate (5.1), we see that (ξ, ζ, η) is i-comparable with $\left(\widetilde{\xi}, \Phi^z(\zeta), \widetilde{\eta}\right) \in \widetilde{E}_1$, for every $\zeta \in E_1$. Hence, the hypothesis $K \lesssim C\widetilde{K}$ implies that $K(\xi, \zeta, \eta) \leqslant C\widetilde{K}\left(\widetilde{\xi}, \Phi^z(\zeta), \widetilde{\eta}\right)$. Moreover, the fact that $\zeta \in E_1$ gives that $|\zeta| > \left(1 - \frac{1}{C_1}\right)|z|$. Therefore, applying Theorem 4.1 (iii)-(iv) gives that

$$\int_{E_1} K(\xi,\zeta,\eta)\left(1+\left|\frac{z}{\zeta}\right|\right)^{\alpha_1}\left|\frac{\zeta}{\zeta_{n+1}}\right|^{\alpha_3} dV_n(\zeta) \leqslant C_4 \int_{\widetilde{E}_1} \widetilde{K}\left(\widetilde{\xi},\Phi^z(\zeta),\widetilde{\eta}\right)\left|\frac{\widetilde{\zeta}}{\widetilde{\zeta}_n}\right|^{\alpha_3} dV_n(\widetilde{\zeta}).$$

(5.4)

Next, we prove the estimate of the form $\int_{E_2} \ \leqslant C_4 \int_{\widetilde{E}_2}\ $. Set $I := \{|\zeta| : \ \zeta \in E_2\}$.

We remark that $\frac{|z|}{|\zeta|} < \frac{1}{\sqrt{2}}$, for every $\zeta \in E_2$. Therefore, by integration in polar coordinates (Corollary 2.2), we obtain

$$\int_{E_2} K(\xi,\zeta,\eta)\left(1+\frac{|z|}{|\zeta|}\right)^{\alpha_1}\left|\frac{\zeta}{\zeta_{n+1}}\right|^{\alpha_3} dV_n(\zeta)$$

$$\lesssim \int_I \sup_{\zeta\in E_2,|\zeta|=r} K(\xi,\zeta,\eta)\, r^{2n-1} dr \cdot \int_{\partial \mathbb{M}}\left|\frac{\zeta}{\zeta_{n+1}}\right|^{\alpha_3} d\sigma_n(\zeta)$$

$$\lesssim \int_I \sup_{\zeta\in E_2,|\zeta|=r} K(\xi,\zeta,\eta)\, r^{2n-1} dr,$$

where the latter inequality holds by an application of Lemma 4.1 in [?] with $\alpha_3 < 2$.

On account of definition 5.3 and estimate (5.1), (ξ,ζ,η) is i-comparable with $\left(\widetilde{\xi},\widetilde{\zeta},\widetilde{\eta}\right) \in \widetilde{E}_2$, for every $\zeta \in E_2$ and $\widetilde{\zeta} \in \widetilde{E}_2$ such that $|\widetilde{\zeta}| = |\zeta|$. This, combined with the hypothesis $K \lesssim C\widetilde{K}$, implies that

$$\int_I \sup_{\zeta\in E_2,|\zeta|=r} K(\xi,\zeta,\eta)\, r^{2n-1} dr \lesssim \int_I \inf_{\widetilde{\zeta}\in\widetilde{E}_2,|\widetilde{\zeta}|=r} \widetilde{K}\left(\widetilde{\xi},\widetilde{\zeta},\widetilde{\eta}\right) r^{2n-1} dr.$$

The right side of the latter estimate is majorized by $C_4 \int_{\widetilde{E}_2} \widetilde{K}\left(\widetilde{\xi},\widetilde{\zeta},\widetilde{\eta}\right) dV_n(\widetilde{\zeta})$.

In summary, we have that

$$\int_{E_2} K(\xi,\zeta,\eta)\left(1+\frac{|z|}{|\zeta|}\right)^{\alpha_1}\left|\frac{\zeta}{\zeta_{n+1}}\right|^{\alpha_3} dV_n(\zeta) \leqslant C_4 \int_{\widetilde{E}_2} \widetilde{K}\left(\widetilde{\xi},\widetilde{\zeta},\widetilde{\eta}\right) dV_n(\widetilde{\zeta}).$$

(5.5)

It now remains to prove the estimate of the form $\int_{E_3} \ \leqslant C_4 \int_{\widetilde{E}_3}\ $. Consider two cases according to the value of i:

Case $i = 1$. We set $R := \sup_{\zeta \in E_3} |\zeta|$. In view of definition 5.3, Remark 5.2 and estimate (5.1), we see that (ξ, ζ, η) is 1-comparable with $\left(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\eta}\right) \in \widetilde{E}_3$ for every $\zeta \in E_3$ and $\widetilde{\zeta} \in \mathbb{C}^n$ such that $|\widetilde{\zeta} - \widetilde{z}| \geqslant \frac{\widetilde{z}}{C_2}$ and $|\widetilde{\zeta}| \leqslant |\zeta|$. Therefore, using integration in polar coordinates, we obtain

$$
\int_{E_3} K\left(\xi, \zeta, \eta\right) \left(1 + \frac{|z|}{|\zeta|}\right)^{\alpha_1} \left|\frac{\zeta}{\zeta_{n+1}}\right|^{\alpha_3} dV_n(\zeta)
$$

$$
\lesssim \int_0^R \sup_{\zeta \in E_3, |\zeta| = r} K\left(\xi, \zeta, \eta\right) r^{2n-1} \left(1 + \frac{|z|}{r}\right)^{\alpha_1} \cdot \int_{\partial \mathbb{M}} \left|\frac{\zeta}{\zeta_{n+1}}\right|^{\alpha_3} d\sigma_n(\zeta) dr
$$

$$
\lesssim \int_0^R \inf_{\widetilde{\zeta} \in \widetilde{E}_3, |\widetilde{\zeta}| = r} \widetilde{K}\left(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\eta}\right) r^{2n-1} \left(1 + \frac{|\widetilde{z}|}{r}\right)^{\alpha_1} \cdot \int_{\partial \mathbb{B}_n} \left|\frac{\widetilde{\zeta}}{\widetilde{\zeta}_n}\right|^{\alpha_3} d\sigma_n(\widetilde{\zeta}) dr
$$

$$
\leqslant C_4 \int_{\widetilde{E}_3} \widetilde{K}\left(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\eta}\right) \left(1 + \frac{|\widetilde{z}|}{|\widetilde{\zeta}|}\right)^{\alpha_1} \left|\frac{\widetilde{\zeta}}{\widetilde{\zeta}_n}\right|^{\alpha_3} dV_n(\widetilde{\zeta}), \tag{5.6}
$$

where on the third line, $d\sigma_n(\widetilde{\zeta})$ is the surface measure of the euclidean unit sphere $\partial \mathbb{B}_n$ of $\mathbb{C}^n$.

Case $i = 2$. We see easily that

$$
\widetilde{E}_3 = \left\{ \widetilde{\zeta} \in \mathbb{C}^n : \ |\widetilde{\zeta}| \leqslant \sqrt{2}|\widetilde{z}| \text{ and } |\widetilde{\zeta} - \widetilde{z}| \geqslant \frac{|\widetilde{z}|}{C_2} \right\}.
$$

Moreover, (ξ, ζ, η) is 2-comparable with $\left(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\eta}\right)$ for every $\zeta \in E_3$ and $\widetilde{\zeta} \in \widetilde{E}_3$. On the other hand, by Remark 5.2, we have $|z| = |\widetilde{z}|$. Thus,

$$
\int_{E_3} K\left(\xi, \zeta, \eta\right) \left(1 + \frac{|z|}{|\zeta|}\right)^{\alpha_1} \left|\frac{\zeta}{\zeta_{n+1}}\right|^{\alpha_3} dV_n(\zeta)
$$

$$
\lesssim \sup_{\zeta \in E_3} K\left(\xi, \zeta, \eta\right) \int_{|\zeta| \leqslant \sqrt{2}|z|} \left(1 + \frac{|z|}{|\zeta|}\right)^{\alpha_1} \left|\frac{\zeta}{\zeta_{n+1}}\right|^{\alpha_3} dV_n(\zeta) \lesssim |z|^{2n} \sup_{\zeta \in E_3} K\left(\xi, \zeta, \eta\right)
$$

$$
\lesssim |\widetilde{z}|^{2n} \inf_{\widetilde{\zeta} \in \widetilde{E}_3} \widetilde{K}\left(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\eta}\right) \leqslant C_4 \int_{\widetilde{E}_3} \widetilde{K}\left(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\eta}\right) dV_n(\widetilde{\zeta}). \tag{5.7}
$$

Now estimate (5.2) follows from (5.4)-(5.7). This completes the proof of part 3).   □

To conclude this section, we give without proof some properties of the relations " $\lesssim$ " and " $\approx$ ".

PROPOSITION 5.7. — *Let $Z, \widetilde{Z}$ and $K, L, \widetilde{K}, \widetilde{L}$ be as in the statement of Theorem 5.6. Suppose that $K \lesssim \widetilde{K}$ and $L \lesssim \widetilde{L}$ at $(Z, \widetilde{Z})$. Then $K + L \lesssim \widetilde{K} + \widetilde{L}$ and $K^\alpha L^\beta \lesssim \widetilde{K}^\alpha \widetilde{L}^\beta$, for every $\alpha, \beta \geqslant 0$.*

If in addition $\widetilde{K} \approx K$ and $\widetilde{L} \approx L$ then $K + L \approx \widetilde{K} + \widetilde{L}$ and $K^\alpha L^\beta \approx \widetilde{K}^\alpha \widetilde{L}^\beta$, for every $\alpha, \beta \in \mathbb{R}$.

6. Integral kernels

The pairs of integral kernels $K, \widetilde{K}$ satisfying the condition $K \approx \widetilde{K}$ that we shall use are studied here. Recall the function D introduced by Charpentier [?] :

$$D(\Theta, Z) := |1 - \Theta \bullet \overline{Z}|^2 - (1 - |\Theta|^2)(1 - |Z|^2), \text{ forall } \Theta, Z \in \mathbb{C}^k \text{ and } k \in \mathbb{N}.$$

THEOREM 6.1. — *Let $i \in \{1, 2\}$ and fix two comparable points $Z \in \mathbb{B}_N$ and $\widetilde{Z} \in \mathbb{B}_{|N|}$. Consider two functions $K, \widetilde{K}$ defined respectively on $\overline{\mathbb{M}}_N$ and $\overline{\mathbb{B}}_{i,|N|}$ that correspond to one of the following three cases :*

(1) $i = 2$ and $K(\Theta) := |\Theta - Z|$, $\quad \widetilde{K}(\widetilde{\Theta}) := |\widetilde{\Theta} - \widetilde{Z}|$;

(2) $i = 1$ and $K(\Theta) := |1 - \Theta \bullet \overline{Z}|$, $\quad \widetilde{K}(\widetilde{\Theta}) := |1 - \widetilde{\Theta} \bullet \overline{\widetilde{Z}}|$;

(3) $i = 1$ and $K(\Theta) := D(\Theta, Z)$, $\quad \widetilde{K}(\widetilde{\Theta}) := D(\widetilde{\Theta}, \widetilde{Z})$.

Then $K \approx \widetilde{K}$ at $(Z, \widetilde{Z})$.

Proof. — Using the definitions 5.1, 5.3 and 5.5, it can be easily checked that

$$|z - \zeta| \approx |\widetilde{z} - \widetilde{\zeta}| \quad \text{and} \quad |w - \eta| \approx |\widetilde{w} - \widetilde{\eta}| \quad \text{at } (Z, \widetilde{Z}).$$

Applying Proposition 5.7 to the latter two relations, assertion (1) follows.

To prove assertions (2) and (3), we need the following estimates of Bonami-Charpentier [?, p. 67] :

$$|1 - \Theta \bullet \overline{Z}| \approx (1 - |Z|^2) + (1 - |\Theta|^2) + \left| \text{Im } \Theta \bullet \overline{Z} \right| + |\Theta - Z|^2, \quad (6.1)$$

and

$$D(\Theta, Z) \approx (1 - |Z|^2)|\Theta - Z|^2 + \left(|\Theta|^2 - |Z|^2\right)^2 + \left|\text{Im } \Theta \bullet \overline{Z}\right|^2 + |\Theta - Z|^4, \quad (6.2)$$

for every $\Theta, Z \in \mathbb{B}_k$, where $\mathbb{B}_k$ is as usual the euclidean ball of $\mathbb{C}^k$.

Write $Z \equiv (x, z, w) \in \mathbb{B}_N$ and $\widetilde{Z} \equiv (\tilde{x}, \tilde{z}, \tilde{w}) \in \mathbb{B}_{|N|}$. Let $\Theta \equiv (\xi, \zeta, \eta) \in \overline{\mathbb{M}}_N$ be 1-comparable with $\widetilde{\Theta} \equiv \left(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\eta}\right) \in \mathbb{B}_{|N|}$. We break the proof into four cases.

Case 1: $\tilde{z} = \Phi^z(z), \widetilde{\zeta} = \Phi^z(\zeta)$ and $\tilde{w} = \Phi^w(w), \widetilde{\eta} = \Phi^w(\eta)$.

In this case by Theorem 4.1 (i)-(ii), we have that

$$\zeta \bullet \overline{z} = \widetilde{\zeta} \bullet \overline{\tilde{z}}, \ \eta \bullet \overline{w} = \widetilde{\eta} \bullet \overline{\tilde{w}}, \quad \text{and } |z| = |\tilde{z}|, \ |w| = |\tilde{w}|. \quad (6.3)$$

We deduce easily from the first two equalities of (6.3) that $|1 - \Theta \bullet \overline{Z}| = |1 - \widetilde{\Theta} \bullet \overline{\widetilde{Z}}|$, which proves assertion (2).

On the other hand, we have the following identity:

$$D(\Theta, Z) = (1 - |Z|^2)|\Theta - Z|^2 + \left|\overline{Z} \bullet (\Theta - Z)\right|^2.$$

By assertion (1), we have $|\Theta - Z|^2 \approx |\widetilde{\Theta} - \widetilde{Z}|^2$. This, combined with (6.3), implies that $D(\Theta, Z) \approx D\left(\widetilde{\Theta}, \widetilde{Z}\right)$, which completes the proof of assertion (3).

Case 2: $\tilde{z} = \Phi^z(z), \widetilde{\zeta} = \Phi^z(\zeta)$ and $|w - \eta| \geqslant \frac{|w|}{C_1}, |\tilde{w} - \widetilde{\eta}| \geqslant \frac{|\tilde{w}|}{C_2}$.

In this case it is easy to check that

$$\max\{|w|, |\eta|\} \lesssim |w - \eta|, \quad \text{and } |\text{Im } \eta \bullet \overline{w}| \lesssim |w - \eta|^2. \quad (6.4)$$

Now we set

$$Z' := (x, z), \ \widetilde{Z}' := (\tilde{x}, \tilde{z}), \ \Theta' := (\xi, \zeta), \widetilde{\Theta}' := (\widetilde{\xi}, \widetilde{\zeta}).$$

Combining (6.1),(6.3) and (6.4), we obtain

$$|1 - \Theta \bullet \overline{Z}| \approx |\eta - w|^2 + (1 - |Z'|^2) + (1 - |\Theta'|^2)$$

$$+ \left|\text{Im } \Theta' \bullet \overline{Z'}\right| + |\Theta' - Z'|^2 \approx |\eta - w|^2 + |1 - \Theta' \bullet \overline{Z'}|. \quad (6.5)$$

On the one hand, we have $|\eta - w|^2 \approx |\widetilde{\eta} - \widetilde{w}|^2$. On the other hand, proceeding as in the first case, we get $|1 - \Theta' \bullet \overline{Z'}| = |1 - \widetilde{\Theta}' \bullet \overline{\widetilde{Z}'}|$. This, combined with (6.5), shows that $|1 - \Theta \bullet \overline{Z}| \approx |1 - \widetilde{\Theta} \bullet \overline{\widetilde{Z}}|$, which completes the proof of assertion (2).

We now come to the proof of assertion (3). Applying (6.2),(6.3) and (6.4), we see easily that

$$D(\Theta, Z) \approx (1 - |Z|^2)|\Theta - Z|^2 + \left(|\Theta'|^2 - |Z'|^2\right)^2 + \left|\mathrm{Im}\ \Theta' \bullet \overline{Z'}\right|^2 + |\Theta' - Z'|^4 + |\eta - w|^4.$$

Since $(1 - |Z|^2)|\Theta - Z|^2 = \left(1 - |Z'|^2 - |w|^2\right)\left(|\Theta' - Z'|^2 + |\eta - w|^2\right)$, we obtain

$$D(\Theta, Z) \approx \left[(1 - |Z'|^2)|\Theta' - Z'|^2 + \left(|\Theta'|^2 - |Z'|^2\right)^2 + \left|\mathrm{Im}\ \Theta' \bullet \overline{Z'}\right|^2\right.$$

$$+ |\Theta' - Z'|^4 + |\eta - w|^2 \left(1 - |Z'|^2\right)$$

$$\approx D(\Theta', Z') + |\eta - w|^2 \left(1 - |Z'|^2\right). \tag{6.6}$$

On the one hand, we have $|\eta - w|^2 \approx |\widetilde{\eta} - \widetilde{w}|^2$ and $1 - |Z'|^2 = 1 - |\widetilde{Z}'|^2$. On the other hand, proceeding as in the first case, we can show that $D(\Theta', Z') \approx D\left(\widetilde{\Theta}', \widetilde{Z}'\right)$. This, combined with (6.6), shows that $D(\Theta, Z) \approx D\left(\widetilde{\Theta}, \widetilde{Z}\right)$ and the proof of assertion (3) is thereby completed.

Case 3: $\widetilde{w} = \Phi^w(w), \widetilde{\eta} = \Phi^w(\eta)$ and $|z - \zeta| \geqslant \frac{|z|}{C_1}, |\widetilde{z} - \widetilde{\zeta}| \geqslant \frac{|\widetilde{z}|}{C_2}$.

This case can be treated analogously as the previous case.

Case 4: $|z - \zeta| \geqslant \frac{|z|}{C_1}, |\widetilde{z} - \widetilde{\zeta}| \geqslant \frac{|\widetilde{z}|}{C_2}$ and $|w - \eta| \geqslant \frac{|w|}{C_1}, |\widetilde{w} - \widetilde{\eta}| \geqslant \frac{|\widetilde{w}|}{C_2}$.

We repeat the arguments used in the proof of the second case. More precisely, proceeding as in the proof of (6.5) and (6.6), one can show that

$$|1 - \Theta' \bullet \overline{Z'}| \approx |\zeta - z|^2 + |1 - \xi \bullet \overline{x}| D(\Theta', Z') \approx D(\xi, x) + |\zeta - z|^2 \left(1 - |x|^2\right). \tag{6.7}$$

On the other hand, it is clear that $|1 - \xi \bullet \overline{x}| = |1 - \widetilde{\xi} \bullet \overline{\widetilde{x}}|$ and $D(\xi, x) = D(\widetilde{\xi}, \widetilde{x})$.

These equalities, combined with (6.5), (6.6) and (6.7), imply that $|1 - \Theta \bullet \overline{Z}| \approx |1 - \widetilde{\Theta} \bullet \overline{\widetilde{Z}}|$ and $D(\Theta, Z) \approx D\left(\widetilde{\Theta}, \widetilde{Z}\right)$. The proof of the theorem is complete in this last case. $\square$

7. Integral estimates

In this section, we prove, with the help of Theorem 5.6, two important integral estimates that will be used repeatedly throughout the paper.

For every $\lambda > 0$ and $Z \in \mathbb{B}_N$, consider the function

$$K_{\lambda,Z}(\Theta) := \frac{1}{\left|1 - \Theta \bullet \overline{Z}\right|^{|N|+1+\lambda}}, \qquad \text{for all } \Theta \in \mathbb{B}_N.$$

The first result of this section is the following

THEOREM 7.1.— *For every* $\alpha := (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{R}^4$ *such that* $0 \leqslant \alpha_3, \alpha_4 < 2$ *and* $0 \leqslant \alpha_1 + \alpha_3 < 2n$, $0 \leqslant \alpha_2 + \alpha_4 < 2m$, *there exists a constant* C *independant of* $Z \in \mathbb{B}_N$ *such that*

$$\int_{\mathbb{M}_N} K_{\lambda,Z}(\Theta) \left(1 + \frac{|z|}{|\zeta|}\right)^{\alpha_1} \left(1 + \frac{|w|}{|\eta|}\right)^{\alpha_2} \left|\frac{\zeta}{\zeta_{n+1}}\right|^{\alpha_3} \left|\frac{\eta}{\eta_{m+1}}\right|^{\alpha_4} dV(\Theta)$$

$$\leqslant C \left(1 - |Z|^2\right)^{-\left(\lambda + \frac{\alpha_3}{2} + \frac{\alpha_4}{2}\right)}.$$

In order to state the second result of this section, we introduce some more notations. Let $\Theta := (\xi, \zeta, \eta)$, $Z := (x, z, w)$ and $Z' := (x', z', w')$ be three points of $\mathbb{M}_N$. Define

$$\Delta(\Theta, Z, Z') := \frac{1}{|\zeta|^2 |\eta|^2} \sum_{j=1}^{|N|+2} \left|\frac{B_j(\Theta, Z)}{|\Theta - Z|^{2|N|}} - \frac{B_j(\Theta, Z')}{|\Theta - Z'|^{2|N|}}\right|,$$

where $B_j(\Theta, Z)$ are the polynomials given in the statement of Theorem 3.1.

THEOREM 7.2.— *Let* q *be a real number such that* $1 \leqslant q < \frac{2|N|+4}{2|N|+3}$. *Then we have the estimate*

$$\int_{\mathbb{M}_N} \Delta(\Theta, Z, Z')^q \left(\frac{|\zeta||\eta|}{|\zeta_{n+1}||\eta_{m+1}|}\right)^{2(q-1)} dV(\Theta)$$

$$\leqslant \begin{cases} |Z - Z'|^{2|N|+4-(2|N|+3)q}, & \text{if } q > 1; \\ |Z - Z'| |\log|Z - Z'||, & \text{if } q = 1. \end{cases}$$

To prove these theorems, we need some preparatory lemmas.

LEMMA 7.3. — *Given $0 < R_1 \leqslant R_2$, $\alpha < 1$ and $0 \leqslant \beta, \gamma$, then*

$$\int_0^{R_1} \frac{dx}{x^\alpha (x + R_1)^\beta (x + R_2)^\gamma} \leqslant C(\alpha, \beta, \gamma) \int_0^{R_1} \frac{dx}{(x + R_1)^{\alpha + \beta} (x + R_2)^\gamma}.$$

LEMMA 7.4. — *Consider $\alpha := (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \in \mathbb{R}^5$ such that $0 \leqslant \alpha_4$, $\alpha_5 < 2$, and $\alpha_2 < 2n$, $\alpha_3 < 2m$. For $a := (a_1, a_2) \in \mathbb{C}^2$, we set*

$$I_{\alpha,a}(\widetilde{\Theta}) := \frac{1}{|\widetilde{\Theta}|^{\alpha_1} |\widetilde{\zeta}|^{\alpha_2} |\widetilde{\eta}|^{\alpha_3}} \left| \frac{\widetilde{\zeta}}{\widetilde{\zeta}_n - a_1} \right|^{\alpha_4} \left| \frac{\widetilde{\eta}}{\widetilde{\eta}_m - a_2} \right|^{\alpha_5}, \quad \widetilde{\Theta} \equiv (\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\eta}) \in \mathbb{C}^{|N|}.$$

Then

$$\int_{|\widetilde{\Theta}| < \delta} I_{\alpha,a}(\widetilde{\Theta}) dV(\widetilde{\Theta}) \leqslant C(N, \alpha) \delta^{2|N| - (\alpha_1 + \alpha_2 + \alpha_3)}, \quad \text{if } \alpha_1 + \alpha_2 + \alpha_3 < 2|N|;$$

$$\int_{\delta < |\widetilde{\Theta}|} I_{\alpha,a}(\widetilde{\Theta}) dV(\widetilde{\Theta}) \leqslant C(N, \alpha) \delta^{2|N| - (\alpha_1 + \alpha_2 + \alpha_3)}, \quad \text{if } \alpha_1 + \alpha_2 + \alpha_3 > 2|N|.$$

Proof. — Consider the case $a = (0, 0)$. We use integration in polar coordinates and then apply Lemma 7.3 four times to obtain

$$\int_{|\widetilde{\Theta}| < \delta} I_{\alpha,0}(\widetilde{\Theta}) dV(\widetilde{\Theta}) \lesssim \int_{0 < |\widetilde{\Theta}| < \delta} \frac{1}{|\widetilde{\Theta}|^{\alpha_1 + \alpha_2 + \alpha_3}} dV(\widetilde{\Theta}) \leqslant C(N, \alpha) \delta^{2|N| - (\alpha_1 + \alpha_2 + \alpha_3)},$$

which is the first estimate in the lemma. The second estimate can be proved in the same way. Now we consider the general case $a \in \mathbb{C}^2$. We write $\widetilde{\Theta} \equiv \left(\widetilde{\Theta}', \widetilde{\eta}_m \right)$ and observe that if $|\widetilde{\Theta}| < \delta$ and $|\widetilde{\eta}_m - a_2| < \frac{|a_2|}{2}$, then we have $\left| (\widetilde{\Theta}', t) \right| < 3\delta$, for every $t \in \mathbb{C}$ such that $|t - a_2| < \frac{|a_2|}{2}$. In addition, it is clear that

$$\int_{|t - a_2| < \frac{|a_2|}{2}} \frac{dt \wedge d\bar{t}}{|t - a_2|^{\alpha_5}} \leqslant C(\alpha_5) \int_{|t - a_2| < \frac{|a_2|}{2}} \frac{dt \wedge d\bar{t}}{|t|^{\alpha_5}}.$$

On the other hand, if $|\widetilde{\eta}_m - a_2| \geqslant \frac{|a_2|}{2}$, then $|\widetilde{\eta}_m| \leqslant 3|\widetilde{\eta}_m - a_2|$. It follows from these considerations that

$$\int_{|\widetilde{\Theta}| < \delta} I_{\alpha,a}(\widetilde{\Theta}) dV(\widetilde{\Theta}) \leqslant C(\alpha_5) \int_{|\widetilde{\Theta}| < 3\delta} \frac{1}{|\widetilde{\Theta}|^{\alpha_1} |\widetilde{\zeta}|^{\alpha_2} |\widetilde{\eta}|^{\alpha_3}} \left| \frac{\widetilde{\zeta}}{\widetilde{\zeta}_n - a_1} \right|^{\alpha_4} \left| \frac{\widetilde{\eta}}{\widetilde{\eta}_m} \right|^{\alpha_5}.$$

The same reasoning, applied to the variables $\widetilde{\zeta}$ and $\widetilde{\zeta}_n$, shows that

$$\int\limits_{|\widetilde{\Theta}|<\delta} I_{\alpha,a}(\widetilde{\Theta})dV(\widetilde{\Theta}) \leqslant C(\alpha_4,\alpha_5) \int\limits_{|\widetilde{\Theta}|<9\delta} I_{\alpha,0}(\widetilde{\Theta})dV(\widetilde{\Theta}).$$

The proof of the first estimate is now complete. The second estimate can be established in a similar way. $\qquad\square$

Proof of Theorem 7.1. — Let $\widetilde{Z} \equiv (\widetilde{x}, \widetilde{z}, \widetilde{w})$ be a point of $\mathbb{B}_{|N|}$ which is comparable with Z. Consider the function

$$\widetilde{K}_{\lambda,\widetilde{Z}}(\widetilde{\Theta}) := \frac{1}{\left|1 - \widetilde{\Theta} \bullet \overline{\widetilde{Z}}\right|^{|N|+1+\lambda}}, \qquad \text{for all } \widetilde{\Theta} \equiv \left(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\eta}\right) \in \mathbb{B}_{|N|}.$$

By Theorem 6.1(2) we have $K_{\lambda,Z} \approx \widetilde{K}_{\lambda,\widetilde{Z}}$. By Remark 5.2 we obtain $|Z| = |\widetilde{Z}|$. Therefore, in view of Theorem 5.6 we see that Theorem 7.1 will follow from the estimate

$$\int\limits_{\mathbb{B}_{|N|}} \widetilde{K}_{\lambda,\widetilde{Z}}(\widetilde{\Theta}) \left(1 + \frac{|\widetilde{z}|}{|\widetilde{\zeta}|}\right)^{\alpha_1} \left(1 + \frac{|\widetilde{w}|}{|\widetilde{\eta}|}\right)^{\alpha_2} \left|\frac{\widetilde{\zeta}}{\widetilde{\zeta}_n}\right|^{\alpha_3} \left|\frac{\widetilde{\eta}}{\widetilde{\eta}_m}\right|^{\alpha_4} dV(\widetilde{\Theta})$$

$$\leqslant C \left(1 - |\widetilde{Z}|^2\right)^{-\left(\lambda + \frac{\alpha_3}{2} + \frac{\alpha_4}{2}\right)}. \tag{7.1}$$

Now we go back to the proof of Lemma I.5 in the work of Bonami-Charpentier [?, p. 68-69]. We may assume without loss of generality that $|\widetilde{w}_1| \geqslant \frac{1}{2\sqrt{|N|}}$ and set

$$\widetilde{w}' := (\widetilde{w}_2, \ldots, \widetilde{w}_m) \quad \text{and} \quad A := 1 - |\widetilde{Z}|^2.$$

As in [?, p. 68], observe that

$$u := 1 - |\widetilde{\Theta}|^2, \ v := \operatorname{Im}(\widetilde{\Theta} \bullet \overline{\widetilde{Z}}), \ \widetilde{\xi}, \widetilde{\zeta} \quad \text{and} \quad \widetilde{\eta}' := (\widetilde{\eta}_2, \ldots, \widetilde{\eta}_m)$$

form a system of coordinates whose jacobian is bounded from above and from below by positive constants uniformly in $\widetilde{Z} \in \mathbb{B}_{|N|}$ and that satisfies $|\widetilde{Z} - \widetilde{\Theta}| \leqslant \frac{1}{4\sqrt{|N|}}$. Using the following estimate (see [?, p. 68]) :

$$|1 - \widetilde{\Theta} \bullet \overline{\widetilde{Z}}| \approx A + |u| + |v| + |\widetilde{x} - \widetilde{\xi}|^2 + |\widetilde{z} - \widetilde{\zeta}|^2 + |\widetilde{w}' - \widetilde{\eta}'|^2,$$

we see that in order to prove (7.1), it suffices to establish

$$A^{\lambda+\frac{\alpha_3}{2}+\frac{\alpha_4}{2}} \int\limits_{\mathbb{C}^{|N|}} \frac{\left(1+\frac{|\widetilde{z}|}{|\zeta|}\right)^{\alpha_1} \frac{1}{|\widetilde{\zeta_n}|^{\alpha_3}} \frac{1}{|\widetilde{\eta_m}|^{\alpha_4}} \cdot du\,dv\,dV(\widetilde{\xi})dV(\widetilde{\zeta})dV(\widetilde{\eta}')}{\left(A+|u|+|v|+|\widetilde{x}-\widetilde{\xi}|^2+|\widetilde{z}-\widetilde{\zeta}|^2+|\widetilde{w}'-\widetilde{\eta}'|^2\right)^{|N|+1+\lambda}}$$

$$\leqslant C(N,\alpha,\lambda) \int\limits_0^\infty \frac{du}{(1+u)^{1+\frac{\lambda}{5}}} \int\limits_{-\infty}^\infty \frac{dv}{(1+|v|)^{1+\frac{\lambda}{5}}} \int\limits_{\mathbb{C}^l} \frac{dV(\widetilde{\xi})}{(1+|\widetilde{x}-\widetilde{\xi}|^2)^{l+\frac{\lambda}{5}}}$$

$$\int\limits_{\mathbb{C}^n} \frac{\left(1+\frac{|\widetilde{z}|}{|\zeta|}\right)^{\alpha_1} \frac{1}{|\widetilde{\zeta_n}|^{\alpha_3}}}{(1+|\widetilde{z}-\widetilde{\zeta}|^2)^{n+\frac{\lambda}{5}}} dV(\widetilde{\zeta}) \int\limits_{\mathbb{C}^{m-1}} \frac{\frac{1}{|\widetilde{\eta_m}|^{\alpha_4}}}{(1+|\widetilde{w}'-\widetilde{\eta}'|^2)^{m-1+\frac{\lambda}{5}}} dV(\widetilde{\eta}').$$

To finish the proof we only need show that the last two integrals in the last line are finite. But this will follow from the following

LEMMA 7.5. — *For every real numbers α, β, γ such that $0 < \gamma$, $0 \leqslant \beta < 2$ and $0 \leqslant \alpha + \beta < 2n$, there is a constant $C := C(n, \alpha, \beta, \gamma)$ such that*

$$I := \int\limits_{\mathbb{C}^n} \frac{\left(1+\frac{|z|}{|\zeta|}\right)^{\alpha} \frac{1}{|\zeta_n|^{\beta}}}{(1+|z-\zeta|^2)^{n+\gamma}} dV(\zeta) < C, \qquad \text{for all } z \in \mathbb{C}^n,$$

where dV is the Lebesgue measure of $\mathbb{C}^n$.

Proof. — Dividing the domain of integration $\mathbb{C}^n$ of I into the three subsets $\left\{|\zeta| < \frac{|z|}{2}\right\}$, $\{|\zeta| > 2|z|\}$ and $\left\{\frac{|z|}{2} \leqslant |\zeta| \leqslant 2|z|\right\}$, we thus divide I into three corresponding terms I_1, I_2 and I_3. We now estimate each of these terms. On the one hand, we have

$$I_1 \lesssim \int\limits_{\left\{|\zeta|<\frac{|z|}{2}\right\}} \frac{\frac{|z|^{\alpha}}{|\zeta|^{\alpha}} \frac{1}{|\zeta_n|^{\beta}}}{(1+|z|^2)^{n+\gamma}} dV(\zeta) \lesssim \frac{|z|^{\alpha}}{(1+|z|^2)^{n+\gamma}} \cdot |z|^{2n-\alpha-\beta} < C,$$

where the second inequality holds by applying a variant of Lemma 7.4. On the other hand,

$$I_2 \lesssim \int\limits_{\{|\zeta|>2|z|\}} \frac{\frac{1}{|\zeta_n|^{\beta}}}{(1+|\zeta|^2)^{n+\gamma}} dV(\zeta) \lesssim \int\limits_{|\zeta|<1} \frac{dV(\zeta)}{|\zeta_n|^{\beta}} + \int\limits_{|\zeta|\geqslant 1} \frac{dV(\zeta)}{|\zeta|^{2n+2\gamma}|\zeta_n|^{\beta}} < C,$$

where the last inequality follows from applying twice a variant of Lemma 7.4. Finally,

$$I_3 \lesssim \int\limits_{\left\{ \frac{|z|}{2} \leqslant |\zeta| \leqslant 2|z| \right\}} \frac{\frac{1}{|\zeta_n|^\beta}}{(1 + |z - \zeta|^2)^{n+\gamma}} dV(\zeta)$$

$$\lesssim \int\limits_{|z-\zeta| < 1} \frac{dV(\zeta)}{|\zeta_n|^\beta} + \int\limits_{|z-\zeta| \geqslant 1} \frac{dV(\zeta)}{|z - \zeta|^{2n+2\gamma} |\zeta_n|^\beta} < C,$$

where the last inequality holds by applying twice a variant of Lemma 7.4 and an obvious change of variable. The lemma is now proved. □

In order to prove Theorem 7.2, we need the following

LEMMA 7.6. — *There is a constant $C = C(n) > 1$ such that for all points $z, z' \in \mathbb{H}_n$, there is a smooth curve $\gamma = \gamma_{z,z'} : [0, 1] \to \mathbb{H}_n$ satisfying*

$$\gamma(0) = z, \quad \gamma(1) = z', \quad |\gamma(t)| \leqslant \max\{|z|, |z'|\}, \quad |\gamma'(t)| \leqslant C|z - z'|.$$

Proof. — Suppose without loss of generality that $|z| \geqslant |z'|$. We set $\hat{z} := |z'|\frac{z}{|z|}$. Then a little geometric argument shows that $|z' - \hat{z}| \leqslant |z - z'|$ and $|z - \hat{z}| \leqslant |z - z'|$. Since the group $SO(n + 1, \mathbb{R})$ acts transitively on $\partial \mathbb{M}_n$, there exists a curve $\gamma_1 : [0, 1] \to \mathbb{H}_n$ satisfying

$$\gamma_1(0) = \hat{z}, \quad \gamma_1(1) = z', \quad |\gamma_1(t)| = |z'|, \quad |\gamma_1'(t)| \leqslant C(n)|z' - \hat{z}|.$$

Define

$$\gamma_2(t) := \begin{cases} (1 - 2t)z + 2t\hat{z}, & \text{if } 0 \leqslant t \leqslant \frac{1}{2}; \\ \gamma_1(2t - 1), & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

It is easy to see that for every $t \neq \frac{1}{2}$, the curve $\gamma_2(t)$ satisfies all the properties stated in the lemma. To conclude the proof, it suffices to approximate in $\mathbb{H}_n$ the curve γ_2 by a smooth curve γ. □

Proof of Theorem 7.2. — We only give the proof for the case $q > 1$. For every points $Z, Z' \in \mathbb{H}_N$, consider the smooth curve

$$\gamma = \gamma_{Z,Z'} := \left(\gamma_{x,x'}, \gamma_{z,z'}, \gamma_{w,w'} \right) : [0, 1] \to \mathbb{H}_N = \mathbb{B}_l \times \mathbb{H}_n \times \mathbb{H}_m,$$

where $\gamma_{z,z'}, \gamma_{w,w'}$ are given by Lemma 7.6 and $\gamma_{x,x'}(t) := (1 - t)x + tx'$. Then it follows from Lemma 7.6 that there is a constant $C := C(N)$ such that

$$\gamma(0) = Z, \ \gamma(1) = Z', \ |\gamma'(t)| \leqslant C|Z - Z'|,$$

$$|\zeta| \leqslant \max\{|z|, |z'|\}, \ |\eta| \leqslant \max\{|w|, |w'|\},$$

where $\Theta \equiv (\xi, \zeta, \eta) = \gamma(t)$. Set

$$E := \left\{ \Theta \in \mathbb{H}_N : |\Theta - Z| \geqslant 2C|Z' - Z| \right\}.$$

On the one hand, for $\Theta \notin E$, using the definition of Δ and Theorem 3.1, we check easily that

$$I_1 \ := \ \int_{\mathbb{M}_N \setminus E} \Delta(\Theta, Z, Z')^q \left(\frac{|\zeta||\eta|}{|\zeta_{n+1}||\eta_{m+1}|} \right)^{2(q-1)} dV(\Theta)$$

$$\lesssim \ \int_{|\Theta - Z| \leqslant 2C|Z' - Z|} \frac{\left(1 + \frac{|z|}{|\zeta|}\right)^{2q} \left(1 + \frac{|w|}{|\eta|}\right)^{2q}}{|\Theta - Z|^{(2|N|-1)q}} \left(\frac{|\zeta||\eta|}{|\zeta_{n+1}||\eta_{m+1}|} \right)^{2(q-1)} dV(\Theta)$$

$$+ \ \int_{|\Theta - Z'| \leqslant 3C|Z' - Z|} \frac{\left(1 + \frac{|z'|}{|\zeta|}\right)^{2q} \left(1 + \frac{|w'|}{|\eta|}\right)^{2q}}{|\Theta - Z'|^{(2|N|-1)q}} \left(\frac{|\zeta||\eta|}{|\zeta_{n+1}||\eta_{m+1}|} \right)^{2(q-1)} dV(\Theta)$$

$$= I_{11} + I_{12}.$$

To estimate I_{11} and I_{12}, it suffices to apply part 2) of Theorem 5.6 with $i = 2$ and Theorem 6.6 (1). This can be reduced to majorizing I_{11} and I_{12} by $\int_{|\widetilde{\Theta}| < C|Z - Z'|} I_{\alpha, a}(\widetilde{\Theta}) dV(\widetilde{\Theta})$, where

$$\alpha := ((2|N| - 1)q, 2(q - 1), 2(q - 1), 2(q - 1), 2(q - 1)).$$

An application of Lemma 7.4 shows that the latter integral is bounded from above by $C|Z - Z'|^{2|N|+4-(2|N|+3)q}$. Hence

$$I_1 \lesssim |Z - Z'|^{2|N|+4-(2|N|+3)q}. \tag{7.2}$$

On the other hand, if $\Theta \in E$, then for every $0 \leqslant t \leqslant 1$ and $\gamma := \gamma_{Z,Z'}$, we have that $|\gamma(t) - \Theta| \approx |\Theta - Z|$. Therefore, using the explicit formula of $B_j(\Theta, Z)$ and taking into account the properties of the curve γ stated at the beginning of the proof, the Mean Value Theorem, applied to the functions of variable $Z : \frac{B_j(\Theta, Z)}{|\Theta - Z|^{2|N|}}$, shows that

$$\Delta\left(\Theta, Z, Z'\right) \lesssim |Z - Z'| \frac{\left(1 + \frac{|z|}{|\zeta|} + \frac{|w|}{|\eta|}\right)^3}{|\Theta - Z|^{2|N|}}$$

$$+ |Z - Z'| \frac{\left(1 + \frac{|z'|}{|\zeta|} + \frac{|w'|}{|\eta|}\right)^3}{|\Theta - Z'|^{2|N|}}. \tag{7.3}$$

Proceeding exactly as in estimating I_{11} and I_{12}, we get

$$I_{21} = \left|Z - Z'\right|^q \int_E \frac{\left(1 + \frac{|z|}{|\zeta|} + \frac{|w|}{|\eta|}\right)^{3q}}{|\Theta - Z|^{2|N|q}} \left(\frac{|\zeta||\eta|}{|\zeta_{n+1}||\eta_{m+1}|}\right)^{2(q-1)} dV(\Theta)$$

$$\lesssim |Z - Z'|^{2|N|+4-(2|N|+3)q}. \tag{7.4}$$

Also,

$$I_{22} = \left|Z - Z'\right|^q \int_E \frac{\left(1 + \frac{|z'|}{|\zeta|} + \frac{|w'|}{|\eta|}\right)^{3q}}{|\Theta - Z'|^{2|N|q}} \left(\frac{|\zeta||\eta|}{|\zeta_{n+1}||\eta_{m+1}|}\right)^{2(q-1)} dV(\Theta)$$

$$\lesssim |Z - Z'|^{2|N|+4-(2|N|+3)q}. \tag{7.5}$$

Therefore, it follows from (7.3), (7.4) and (7.5) that

$$\int_E \Delta(\Theta, Z, Z')^q \left(\frac{|\zeta||\eta|}{|\zeta_{n+1}||\eta_{m+1}|}\right)^{2(q-1)} dV(\Theta)$$

$$\lesssim I_{21} + I_{22} \lesssim |Z - Z'|^{2|N|+4-(2|N|+3)q}.$$

This, combined with estimate (7.2), completes the proof of the theorem. $\square$

8. Lipschitz estimates on the complex manifold $\mathbb{M}_N$

Let u be a function in $\mathcal{C}^1(\mathbb{M}_N)$. For every $Z \in \mathbb{M}_N$, define

$$\left(\text{grad}_{\mathbb{M}_N} u\right)(Z) := \sup |(f \circ \gamma)'(0)|,$$

where the supremum is taken over all smooth curves $\gamma : [0, 1] \longrightarrow \mathbb{M}_N$ such that $\gamma(0) = Z$ and $|\gamma'(t)| \leqslant 1$.

We begin this section with the following Hardy-Littlewood type lemma.

LEMMA 8.1.— *For every $0 < \alpha \leqslant 1$, there exists a constant $C = C(N, \alpha)$ with the following property: Suppose $u \in \mathcal{C}^1(\mathbb{M}_N)$ and K is some finite constant such that*

$$\left(\text{grad}_{\mathbb{M}_N} u\right)(Z) \leqslant K \left(1 - |Z|\right)^{\alpha - 1} \qquad \text{for all } Z \in \mathbb{M}_N.$$

Then $|u(Z) - u(Z^{'})| \leqslant CK|Z - Z^{'}|^\alpha$ for all $Z, Z^{'} \in \mathbb{M}_N$.

Proof.— First we make the following remark :

Write $Z = (x, z, w)$, $Z^{'} = (x^{'}, z^{'}, w^{'})$, $X := (x, z)$, $Y := w$ and $X^{'} := (x^{'}, z^{'})$, $Y^{'} := w^{'}$. Suppose without loss of generality that $|Z^{'}| \leqslant |Z|$. Then $|X^{'}|^2 + |Y^{'}|^2 \leqslant |X|^2 + |Y|^2 < 1$.

- *If $|X^{'}| \leqslant |X|$, by noticing that $|(X^{'}, Y)| \leqslant |Z|$, then we write*

$$|u(Z) - u(Z^{'})| = |u(X, Y) - u(X^{'}, Y^{'})|$$

$$\leqslant |u(X, Y) - u(X^{'}, Y)| + |u(X^{'}, Y) - u(X^{'}, Y^{'})|.$$

- *If $|Y^{'}| \leqslant |Y|$, by noticing that $|(X, Y^{'})| \leqslant |Z|$, then we write*

$$|u(Z) - u(Z^{'})| = |u(X, Y) - u(X^{'}, Y^{'})|$$

$$\leqslant |u(X, Y) - u(X, Y^{'})| + |u(X, Y^{'}) - u(X^{'}, Y^{'})|.$$

Let $Z, Z^{'}$ be two points of $\mathbb{M}_N$ such that $0 < |Z^{'}| \leqslant |Z| < 1$ and set $\delta := |Z - Z^{'}|$.

First assume that $\delta < 1 - |Z|$. Applying the previous remark three times, we only need prove the lemma in one of the following three cases:

1) $x = x^{'}, z = z^{'}$;   2) $x = x^{'}, w = w^{'}$;   3) $z = z^{'}, w = w^{'}$.

Suppose for example we are in the first case $x = x^{'}, z = z^{'}$. In this case, take the curve $\gamma = \gamma_{Z,Z'}$. According to the hypothesis of the lemma and the properties of the curve γ given in the proof of Theorem 7.2, we have

$$\left(\text{grad}_{\mathbb{M}_N} u\right)(\Theta) \leqslant K\delta^{\alpha - 1}, \qquad \text{for all } \Theta \in \gamma([0, 1]).$$

Therefore,

$$|u(Z) - u(Z^{'})| \leqslant CK\delta^{\alpha - 1}|Z - Z^{'}| = CK|Z - Z^{'}|^\alpha.$$

The remaining cases $1 - |Z| < \delta \leqslant 1 - |Z'|$ and $1 - |Z'| < \delta$ can be checked using the same argument as in Lemma 6.4.8 of [?]. $\qquad \square$

In order to state the main result of this section, we consider, for $1 \leqslant p < \infty$, the space

$$
\mathrm{L}^p(\mathbb{M}_N) := \left\{ f : \left(\int_{\mathbb{M}_N} |f(\Theta)|^p \frac{|\zeta_{n+1}|^2 |\eta_{m+1}|^2}{|\zeta|^2 |\eta|^2} dV(\Theta) \right)^{\frac{1}{p}} \equiv \|f\|_{\mathbb{M}_{N,p}} < \infty \right\}.
$$

If $f := \sum_{j=1}^{|N|+2} f_j d\overline{\Theta}_j$ is a $(0,1)$-form defined in a neighborhood of $\overline{\mathbb{M}}_N$ in $\overline{\mathbb{B}}_N$, we set

$$
\|f\|_{\mathbb{M}_{N,p}} := \sum_{j=1}^{|N|+2} \|f_j\|_{\mathbb{M}_{N,p}}. \tag{8.1}
$$

Recall that the norm $\| \; \|_{\mathbb{M}_{N,\infty}}$ was defined by formula (2.8).

Next, for every $0 < \alpha < \beta \leqslant 1$ and for $X = \mathbb{M}_N$ or $\partial \mathbb{M}_N$, we define

$$
\Gamma_{\alpha,\beta}(X) := \left\{ f \; : \|f\|_{\Lambda_\alpha(X)} + \sup_{\substack{\gamma \in \mathcal{C}^2_1(\mathbb{B}_N) \\ \gamma \subset X}} \|f \circ \gamma\|_{\Lambda_\beta([0,1])} \right.
$$

$$
\left. \equiv \|f\|_{\Gamma_{\alpha,\beta}(X)} < \infty \right\}.
$$

We can say informally that $\Gamma_{\alpha,\beta}(X)$ is the trace of the non-isotropic Lipchitz space $\Gamma_{\alpha,\beta}(\mathbb{B}_N)$ (see Definition 1.1 in Krantz [?]) on the manifold X.

THEOREM 8.2. — *Suppose that $u \in \mathcal{C}^1(\overline{\mathbb{M}}_N)$ and consider a $(0,1)$-form*

$$
f := \begin{cases} \sum_{k=1}^{|N|+2} f_k d\overline{\Theta}_k, & \text{if } N \neq (0,2,2); \\ f_1 d\overline{\zeta}_1 + f_2 d\overline{\zeta}_2 + f_3 d\overline{\eta}_1 + f_4 d\overline{\eta}_2, & \text{if } N = (0,2,2); \end{cases}
$$

with coefficients in $\mathcal{C}(\overline{\mathbb{M}}_N)$ such that $\overline{\partial}_{\mathbb{M}_N} u = f|_{\mathbb{M}_N}$ on $\mathbb{M}_N$. Define $T_1 f$ on $\partial \mathbb{M}_N$ as follows:

• *for $N \neq (0,2,2)$,*

$$
(T_1 f)(Z) := \int_{\mathbb{M}_N} \sum_{k=1}^{|N|+2} \left[\frac{(1 - \Theta \bullet \overline{Z}) P_k(\Theta, Z) + (1 - |\Theta|^2) Q_k(\Theta, Z)}{(1 - \overline{\Theta} \bullet Z)^{|N|} (1 - \Theta \bullet \overline{Z})^2} \right]
$$

$$
f_k(\Theta) \frac{dV(\Theta)}{|\zeta|^2 |\eta|^2};
$$

- *for $N = (0, 2, 2)$,*

$$(T_1 f)(Z) := \int\limits_{\mathbb{M}_N} \sum_{j=1}^{2} \sum_{k=1}^{4} \frac{(1 - \Theta \bullet \overline{Z})^{1+j}}{|1 - \Theta \bullet \overline{Z}|^8} [(1 - \Theta \bullet \overline{Z}) P_{jk}(\Theta, Z)$$

$$+ (1 - |\Theta|^2) Q_{jk}(\Theta, Z)] f_k(\Theta) \frac{dV(\Theta)}{|\zeta|^2 |\eta|^2},$$

where P_k, Q_k and P_{jk}, Q_{jk} are the polynomials given by Theorems 3.3 and 3.6.

Then the definition of $T_1 f$ can be extended to $\mathbb{M}_N$ by setting

$$(T_1 f)(Z) := J_1(Z) + J_2(Z), \tag{8.2}$$

where

$$J_1(z) := \int\limits_{\partial \mathbb{M}_N} \frac{A(\Theta, Z)}{|Z - \Theta|^{2|N|}} (T_1 f)(\Theta) \frac{d\sigma(\Theta)}{|\zeta|^2 |\eta|^2},$$

$$J_2(z) := \int\limits_{\mathbb{M}_N} \frac{1}{|Z - \Theta|^{2|N|}} \left(\sum_{k=1}^{|N|+2} B_k(\Theta, Z) f_k(\Theta) \right) \frac{dV(\Theta)}{|\zeta|^2 |\eta|^2}$$

and the operator $T_1 f$ satisfies
(i) $\overline{\partial}_{\mathbb{M}_N}(T_1 f) = f|_{\mathbb{M}_N}$.

    Moreover, for every $p > 0$, we set (as in the statement of Theorem 1.1) :

$$\alpha = \alpha(N, p) := \begin{cases} \frac{1}{2} - \frac{|N|+3}{p}, & \text{if } N \neq (0, 2, 2) \text{ and } p > 2(|N| + 3); \\ \frac{1}{2} - \frac{6}{p}, & \text{if } N = (0, 2, 2) \text{ and } p > 12. \end{cases}$$

Then there exists a constant $C := C(N,p)$ such that

(ii)

$$\begin{cases} T_1 f|_{\partial \mathbb{M}_N} \in \Gamma_{\alpha, 2\alpha}(\partial \mathbb{M}_N) \text{ and } \|T_1 f|_{\partial \mathbb{M}_N}\|_{\Gamma_{\alpha, 2\alpha}(\partial \mathbb{M}_N)} \leqslant C \|f\|_{\mathbb{M}_{N,p}}, & \text{if } p < \infty; \\ T_1 f|_{\partial \mathbb{M}_N} \in \Gamma_{\frac{1}{2}, \overline{1}}(\partial \mathbb{M}_N) \text{ and } \|T_1 f|_{\partial \mathbb{M}_N}\|_{\Gamma_{\frac{1}{2}, \overline{1}}(\partial \mathbb{M}_N)} \leqslant C \|f\|_{\mathbb{M}_{N,\infty}}, & \text{if } p = \infty; \end{cases}$$

(iii) $T_1 f \in \Lambda_\alpha(\mathbb{M}_N)$ and $\|T_1 f\|_{\Lambda_\alpha(\mathbb{M}_N)} \leqslant C \|f\|_{\mathbb{M}_{N,p}}$.

    Proof. — We only give the proof in the case $N \neq (0, 2, 2)$ and $p < \infty$. The first remaining case $N = (0, 2, 2)$ can be proved in exactly the same way by applying Theorem 3.6 instead of Theorem 3.3. The second remaining case $p = \infty$ follows essentially along the same lines as in our previous work [?] basing on the work of Greiner-Stein [?].

We first introduce two new integral operators T_2 and T_3 :

$$(T_2 f)(Z) \quad := \quad \int\limits_{\mathbb{M}_N} \sum_{k=1}^{|N|+2} \frac{(1 - \Theta \bullet \overline{Z})^{|N|-2}}{D(\Theta, Z)^{|N|}} [(1 - \Theta \bullet \overline{Z}) P_k(\Theta, Z)$$

$$+ (1 - |\Theta|^2) Q_k(\Theta, Z)] f_k(\Theta) \frac{dV(\Theta)}{|\zeta|^2 |\eta|^2},$$

$$(T_3 f)(Z)$$

$$:= \int\limits_{\mathbb{M}_N} \sum_{k=1}^{|N|+2} \left[\frac{(1 - \Theta \bullet \overline{Z}) P_k(\Theta, Z) + (1 - |\Theta|^2) Q_k(\Theta, Z)}{(1 - \overline{\Theta} \bullet Z)^{|N|}(1 - \Theta \bullet \overline{Z})^2} \right] f_k(\Theta) \frac{dV(\Theta)}{|\zeta|^2 |\eta|^2},$$

for all $Z \in \mathbb{B}_N$.

Applying Theorem 3.3 to the function u gives that

$$(T_2 f)(Z) = u(Z) - \int\limits_{\partial \mathbb{M}_N} \frac{R(\Theta, Z)}{(1 - Z \bullet \overline{\Theta})^{|N|}} u(\Theta) \frac{d\sigma(\Theta)}{|\zeta|^2 |\eta|^2}, \quad \text{for all } Z \in \mathbb{M}_N. \quad (8.3)$$

Moreover, we note that

$$(T_1 f)(Z) = (T_2 f)(Z) = (T_3 f)(Z), \qquad \text{for all } Z \in \partial \mathbb{M}_N. \qquad (8.4)$$

Arguing as in the proof of Lemma 3.5 in [?] and using Theorems 5.6, 6.1 and 7.1, one can show that

$$\lim_{r \to 1^-} \int\limits_{\partial \mathbb{M}_N} |(T_2 f)(\Theta) - (T_2 f)(r\Theta)| d\sigma(\Theta) = 0.$$

Therefore, in view of Remark 3.2, we can apply Theorem 3.1 to the function $T_2 f$. Next observe that (8.2) is just the Martinelli-Bochner formula. Hence by virtue of (8.3), the hypothesis and the fact that $R(\Theta, Z)$ is holomorphic in the variable Z, we obtain

$$T_1 f = T_2 f|_{\mathbb{M}_N} \quad \text{and} \quad \overline{\partial}_{\mathbb{M}_N}(T_1 f) = \overline{\partial}_{\mathbb{M}_N} u = f|_{\mathbb{M}_N}. \qquad (8.5)$$

This completes the proof of assertion (i). In view of (8.4), assertion (ii) will follow from the following lemma.

LEMMA 8.3. —

$$\begin{cases} \|T_3 f|_{\mathbb{M}_N}\|_{\Gamma_{\alpha, 2\alpha}(\mathbb{M}_N)} \leqslant C \|f\|_{\mathbb{M}_{N,p}} & \text{if } p < \infty; \\ \|T_3 f|_{\mathbb{M}_N}\|_{\Gamma_{\frac{1}{2}, \bar{1}}(\mathbb{M}_N)} \leqslant C \|f\|_{\mathbb{M}_{N,\infty}} & \text{if } p = \infty. \end{cases}$$

Proof. — Using the properties of the polynomials $P_k(\Theta, Z)$ and $Q_k(\Theta, Z)$ in Theorem 3.3(ii), we see that

$$|(\text{grad } T_3 f)(Z)| \lesssim \sum_{k=1}^{|N|+2} \int_{\mathbb{M}_N} \frac{|\Theta - Z|}{\left|1 - \Theta \bullet \overline{Z}\right|^{|N|+2}} \left(1 + \frac{|z|}{|\zeta|}\right) \left(1 + \frac{|w|}{|\eta|}\right) f_k(\Theta) dV(\Theta)$$

$$+ \sum_{k=1}^{|N|+2} \int_{\mathbb{M}_N} \frac{1}{\left|1 - \Theta \bullet \overline{Z}\right|^{|N|+1}} \left(1 + \frac{|z|}{|\zeta|}\right) \left(1 + \frac{|w|}{|\eta|}\right) f_k(\Theta) dV(\Theta).$$

Since $|\Theta - Z| \leqslant 2\sqrt{|1 - \Theta \bullet \overline{Z}|}$, this implies by Hölder's inequality that $|(\text{grad } T_3 f)(Z)|$ is bounded from above by

$$C \|f\|_{\mathbb{M}_{N,p}} \left(\int_{\mathbb{M}_N} \frac{1}{\left|1 - \Theta \bullet \overline{Z}\right|^{(|N|+\frac{3}{2})q}} \left(1 + \frac{|z|}{|\zeta|}\right)^q \left(1 + \frac{|w|}{|\eta|}\right)^q \right.$$

$$\left. \left|\frac{\zeta}{\zeta_{n+1}}\right|^{\frac{2q}{p}} \left|\frac{\eta}{\eta_{m+1}}\right|^{\frac{2q}{p}} dV(\Theta) \right)^{\frac{1}{q}},$$

where q verifies $\frac{1}{p} + \frac{1}{q} = 1$. Now applying Theorem 7.1 yields

$$|(\text{grad } T_3 f)(Z)| \leqslant C \|f\|_{\mathbb{M}_{N,p}} (1 - |Z|)^{-\frac{1}{2} - \frac{|N|+3}{p}}. \tag{8.6}$$

so that by the classical Hardy-Littlewood lemma for the euclidean ball $\mathbb{B}_N$ we see that

$$|(T_3 f)(Z) - (T_3 f)(Z')| \leqslant C \|f\|_{\mathbb{M}_{N,p}} |Z - Z'|^{\frac{1}{2} - \frac{|N|+3}{p}}, \quad \text{for all } Z, Z' \in \mathbb{B}_N.$$

Therefore, choosing $Z' = 0$, we obtain

$$\|T_3 f\|_{L^\infty(\mathbb{B}_N)} \leqslant |(T_3 f)(0)| + C \|f\|_{\mathbb{M}_{N,p}} \leqslant C \|f\|_{\mathbb{M}_{N,p}}. \tag{8.7}$$

For every $u \in \mathcal{C}^1(\mathbb{B}_N)$, set

$$\left(\text{grad}^t u\right)(Z) := \sup_{\gamma \in \mathcal{C}_1^2(\mathbb{B}_N): \, \gamma(0)=Z} |(u \circ \gamma)'(0)|.$$

By the proof Lemma 4.8 in [?], we see that $\left(\text{grad}^t |1 - \overline{\Theta} \bullet Z|\right)(Z) \leqslant C |\Theta - Z|$. Therefore, a straightforward calculation shows that

$$\left(\operatorname{grad}^t T_3 f\right)(Z) \lesssim \sum_{k=1}^{|N|+2} \int\limits_{\mathbb{M}_N} \frac{|\Theta - Z|^2}{|1 - \Theta \bullet \overline{Z}|^{|N|+2}} \left(1 + \frac{|z|}{|\zeta|}\right)\left(1 + \frac{|w|}{|\eta|}\right) f_k(\Theta) dV(\Theta)$$

$$+ \sum_{k=1}^{|N|+2} \int\limits_{\mathbb{M}_N} \frac{1}{|1 - \Theta \bullet \overline{Z}|^{|N|+1}} \left(1 + \frac{|z|}{|\zeta|}\right)\left(1 + \frac{|w|}{|\eta|}\right) f_k(\Theta) dV(\Theta).$$

Hence, arguing as in the proof of (8.6), we see that

$$\left(\operatorname{grad}^t T_3 f\right)(Z) \leqslant C\|f\|_{\mathbb{M}_{N,p}} \left(1 - |Z|\right)^{-\frac{|N|+3}{p}}. \tag{8.8}$$

Combining (8.6), (8.7) and (8.8), the lemma follows from Lemma 4.7 in [?]. □

To prove assertion (iii), we need the following

LEMMA 8.4. —

$$|J_1(Z) - J_1(Z')| \leqslant C\|f\|_{\mathbb{M}_{N,p}} |Z - Z'|^{\frac{1}{2} - \frac{|N|+3}{p}}, \qquad \text{for all } Z, Z' \in \mathbb{M}_N.$$

Proof. — Observe that the polynomial $A(\Theta, Z)$ satisfies

$$\frac{1}{|\zeta|^2|\eta|^2} \left|\operatorname{grad}_Z \frac{A(\Theta, Z)}{|\Theta - Z|^{2|N|}}\right|$$

$$\lesssim \frac{\left(1 + \frac{|z|}{|\zeta|}\right)\left(1 + \frac{|w|}{|\eta|}\right)}{|\Theta - Z|^{2|N|}} + \frac{\frac{1}{|\zeta|}\left(1 + \frac{|w|}{|\eta|}\right)}{|\Theta - Z|^{2|N|-1}} + \frac{\frac{1}{|\eta|}\left(1 + \frac{|z|}{|\zeta|}\right)}{|\Theta - Z|^{2|N|-1}}.$$

In addition, if we set $u \equiv 1$ in Theorem 3.1, then we see that

$$\int\limits_{\partial \mathbb{M}_N} \frac{A(\Theta, Z)}{|Z - \Theta|^{2|N|}} \frac{d\sigma(\Theta)}{|\zeta|^2|\eta|^2} = 1.$$

Setting $Z := rZ'$, $Z' \in \partial \mathbb{M}_N$, this implies that

$$\left(\operatorname{grad}_{\mathbb{M}_N} J_1\right)(Z) \lesssim \int\limits_{\partial \mathbb{M}_N} \left|\operatorname{grad}_Z \frac{A(\Theta, Z)}{|\Theta - Z|^{2|N|}}\right| \left|(T_1 f)(\Theta) - (T_1 f)(Z')\right| \frac{d\sigma(\Theta)}{|\zeta|^2|\eta|^2}.$$

Combining (8.4) and Lemma 8.3, we obtain

$$\left|(T_1 f)(\Theta) - (T_1 f)(Z')\right| \leqslant C\|f\|_{\mathbb{M}_{N,p}}|\Theta - Z'|^{\frac{1}{2} - \frac{|N|+3}{p}}.$$

Hence,
$$\left(\mathrm{grad}_{\mathbb{M}_N} J_1\right)(Z) \leqslant C\|f\|_{\mathbb{M}_{N,p}}$$

$$\left(\int_{\partial\mathbb{M}_N} \frac{\left(1 + \frac{|z'|}{|\zeta|}\right)\left(1 + \frac{|w'|}{|\eta|}\right)}{|\Theta - rZ'|^{2|N|}} |\Theta - Z'|^{\frac{1}{2} - \frac{|N|+3}{p}} d\sigma(\Theta) \right.$$

$$+ \int_{\partial\mathbb{M}_N} \frac{\frac{1}{|\zeta|}\left(1 + \frac{|w'|}{|\eta|}\right)}{|\Theta - rZ'|^{2|N|-1}} |\Theta - Z'|^{\frac{1}{2} - \frac{|N|+3}{p}} d\sigma(\Theta)$$

$$\left. + \int_{\partial\mathbb{M}_N} \frac{\frac{1}{|\eta|}\left(1 + \frac{|z'|}{|\zeta|}\right)}{|\Theta - rZ'|^{2|N|-1}} |\Theta - Z'|^{\frac{1}{2} - \frac{|N|+3}{p}} d\sigma(\Theta) \right).$$

We shall establish in Proposition 9.4 below that the latter three integrals are dominated by $C\left(1 - |Z|\right)^{-\frac{1}{2} - \frac{|N|+3}{p}}$. Taking for granted Proposition 9.4, it follows that

$$\left(\mathrm{grad}_{\mathbb{M}_N} J_1\right)(Z) \leqslant C\|f\|_{\mathbb{M}_{N,p}}\left(1 - |Z|\right)^{-\frac{1}{2} - \frac{|N|+3}{p}}.$$

Finally, applying Lemma 8.1 to this gives the desired conclusion. $\qquad\square$

We now complete the proof of Theorem 8.2. By Hölder's inequality and Theorem 7.2, we have

$$\left|J_2(Z) - J_2(Z')\right| \leqslant C\|f\|_{\mathbb{M}_{N,p}}|Z - Z'|^{1 - \frac{2|N|+4}{p}}, \qquad \text{for all } Z, Z' \in \mathbb{M}_N.$$

This, combined with Lemma 8.4 gives that

$$\left|(T_1 f)(Z) - (T_1 f)(Z')\right| \leqslant C\|f\|_{\mathbb{M}_{N,p}}|Z - Z'|^{\frac{1}{2} - \frac{|N|+3}{p}}, \qquad \text{for all } Z, Z' \in \mathbb{M}_N.$$

Arguing as in the proof of (8.7), one can show that

$$\|T_1 f\|_{L^\infty(\mathbb{M}_N)} \leqslant C\|f\|_{\mathbb{M}_{N,p}}.$$

This proves assertion (iii). $\qquad\square$

9. A Stokes type theorem on the manifold $\mathbb{M}_N$ and applications

The main result of this section is the following Stokes type theorem:

THEOREM 9.1. — *Consider for every function* $v \in \mathcal{C}^1\left(\overline{\mathbb{M}}_N\right)$ *and every real numbers* $\lambda < 2n - 1$ *and* $\mu < 2m - 1$, *the function* u *given by* $u(\Theta) := \frac{v(\Theta)}{|\zeta|^\lambda |\eta|^\mu}$, *for* $\Theta \in \mathbb{M}_N$. *Then there is a constant* $C := C(N)$ *such that*

$$\left| \int_{\partial \mathbb{M}_N} u \, d\sigma \right|$$

$$\leqslant C \int_{\mathbb{M}_N} \Big(|\xi| |(\mathrm{grad}_\xi u)(\Theta)| + |\zeta| |(\mathrm{grad}_\zeta u)(\Theta)| + |\eta| |(\mathrm{grad}_\eta u)(\Theta)|$$

$$+ |u(\Theta)| \Big) dV(\Theta).$$

REMARK 9.2. — We do not know whether it is possible to establish a theorem of reduction of estimates from $\partial \mathbb{M}_N$ to $\partial \mathbb{B}_{|N|}$, similar to Theorem 5.6. To overcome this, we use Theorem 9.1 to estimate difficult integrals taken over $\partial \mathbb{M}_N$ by simpler ones taken over $\mathbb{M}_N$ and then apply Theorem 5.6. We have already encountered this type of integral estimates in the proof of Lemma 8.4.

Proof. — Set $d\xi := d\xi_1 \wedge \ldots \wedge d\xi_l$ and

$$\alpha_n(\zeta) \quad := \quad \frac{1}{n+1} \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{\zeta_j} d\zeta_1 \wedge \ldots \wedge \widehat{d\zeta_j} \wedge \ldots \wedge d\zeta_{n+1},$$

$$\alpha_m(\eta) \quad := \quad \frac{1}{m+1} \sum_{k=1}^{m+1} \frac{(-1)^{k-1}}{\eta_k} d\eta_1 \wedge \ldots \wedge \widehat{d\eta_k} \wedge \ldots \wedge d\eta_{m+1}.$$

By Proposition 2.1 in [?] and Proposition 2.1 above, we see that

$$dV_l(\xi) = C d\xi \wedge d\bar{\xi},$$

$$dV_n(\zeta) = C|\zeta|^2 \alpha_n(\zeta) \wedge \alpha_n(\bar{\zeta})\big|_{\mathbb{H}_n}, dV_m(\eta) = C|\eta|^2 \alpha_m(\eta) \wedge \alpha_m(\bar{\eta})\big|_{\mathbb{H}_m};$$

$$dV(\Theta) = dV_l(\xi) \wedge dV_n(\zeta) \wedge dV_m(\eta). \tag{9.1}$$

Next, put

$$\omega_j(\xi) \quad := \quad d\xi_1 \wedge \ldots \wedge \widehat{d\xi_j} \wedge \ldots \wedge d\xi_l, \qquad (1 \leqslant j \leqslant l);$$

$$\omega_k(\zeta) \quad := \quad d\zeta_1 \wedge \ldots \wedge \widehat{d\zeta_k} \wedge \ldots \wedge d\zeta_{n+1}, \qquad (1 \leqslant k \leqslant n+1);$$

$$\omega_p(\eta) \quad := \quad d\eta_1 \wedge \ldots \wedge \widehat{d\eta_p} \wedge \ldots \wedge d\eta_{m+1}, \qquad (1 \leqslant p \leqslant m+1).$$

Finally, we define $\omega_{jk}(\zeta), \widetilde{\omega}_{jk}(\zeta)$ $(1 \leqslant j, k \leqslant n + 1)$ and $\omega_{pq}(\eta), \widetilde{\omega}_{pq}(\eta)$ $(1 \leqslant p, q \leqslant m + 1)$ in just the same way as $\omega_{pj}(z), \widetilde{\omega}_{jk}(z)$ in [?, p. 507–508].

Consider the mapping $g :]0, +\infty[\times \mathbb{C}^{|N|+2} \longrightarrow \mathbb{C}^{|N|+2}$ given by

$$g(t\Theta) := t\Theta.$$

Using (9.1) and proceeding as in the proof of Lemma 2.1 in [?], we see that

$$(g^* dV)(t, \Theta) = t^{2|N|-1} dt \wedge [I_\xi \wedge dV_n(\zeta) \wedge dV_m(\eta) + I_\zeta \wedge dV_l(\xi) \wedge dV_m(\eta)$$

$$+ I_\eta \wedge dV_l(\xi) \wedge dV_n(\zeta)] + t^{2|N|} dV(\Theta), \tag{9.2}$$

where

$$I_\xi \quad := \quad C \sum_{p=1}^{l} (-1)^{p-1} \left(\overline{\xi}_p d\xi \wedge \omega_p(\overline{\xi}) + \xi_p \omega_p(\xi) \wedge d\overline{\xi} \right),$$

$$I_\zeta \quad := \quad C|\zeta|^2 \sum_{j,k=1}^{n+1} \frac{(-1)^{j+k}}{\zeta_j \overline{\zeta}_k} \widetilde{\omega}_{jk}(\zeta), \text{ and } I_\eta := C|\eta|^2 \sum_{p,q=1}^{m+1} \frac{(-1)^{p+q}}{\eta_p \overline{\eta}_q} \widetilde{\omega}_{pq}(\eta).$$

Now set

$$\omega(\Theta) := I_\xi \wedge dV_n(\zeta) \wedge dV_m(\eta) + I_\zeta \wedge dV_l(\xi) \wedge dV_m(\eta) + I_\eta \wedge dV_l(\xi) \wedge dV_n(\zeta). \tag{9.3}$$

Since g is a diffeomorphism from $]0, +\infty[\times \partial\mathbb{M}_N \longrightarrow \mathbb{H}_N$, it follows from (9.2) and (9.3) that

$$\int_{\mathbb{H}_N} u(\Theta) dV(\Theta) = \int_0^\infty t^{2|N|-1} \int_{\partial\mathbb{M}_N} u(t\Theta)\omega(\Theta), \qquad \text{for all } u \in \mathcal{C}_0(\mathbb{H}_N),$$

so that by Lemma 2.3, we obtain $d\sigma = C\omega|_{\partial\mathbb{M}_N}$. Therefore, since by the hypothesis $\lambda < 2n - 1$, $\mu < 2m - 1$, $|u(\Theta)| \lesssim \frac{1}{|\zeta|^\lambda |\eta|^\mu}$ for all $\Theta \in \mathbb{M}_N$, the homogeneity properties of the differential form $\omega(\Theta)$ and the same arguments as in the proof of (2.12), (2.13) and (2.17) of Proposition 2.5 imply that

$$\int_{\partial\mathbb{M}_N} u d\sigma = \lim_{r \to 0} \int_{\partial\mathbb{M}_r} u\omega. \tag{9.4}$$

Stokes theorem gives that

$$\int_{\partial\mathbb{M}_r} u\omega = \int_{\mathbb{M}_r} du \wedge \omega + \int_{\mathbb{M}_r} u d\omega. \tag{9.5}$$

We shall estimate $\left| \int_{\mathbb{M}_r} du \wedge \omega \right|$. Let Z be a point of $\mathbb{M}_r$. Choose j and k with $1 \leqslant j \leqslant n+1$, $1 \leqslant k \leqslant m+1$ so that in a sufficiently small neighborhood $\mathcal{U} := \mathcal{U}(Z)$ in $\mathbb{M}_r$, we have

$$|\zeta_j| \geqslant \frac{1}{2} \max_{p \neq j} |\zeta_p| \qquad \text{and} \qquad |\eta_k| \geqslant \frac{1}{2} \max_{q \neq k} |\eta_q|. \tag{9.6}$$

By (9.3) and (9.5), we obtain

$$\left| \int_{\mathcal{U}} du \wedge \omega \right| \leqslant \left| \int_{\mathcal{U}} du \wedge I_\xi \wedge dV_n(\zeta) \wedge dV_m(\eta) \right| + \left| \int_{\mathcal{U}} du \wedge I_\zeta \wedge dV_l(\xi) \wedge dV_m(\eta) \right|$$

$$+ \left| \int_{\mathcal{U}} du \wedge I_\eta \wedge dV_l(\xi) \wedge dV_n(\zeta) \right|. \tag{9.7}$$

We shall estimate for example $\left| \int_{\mathcal{U}} du \wedge I_\zeta \wedge dV_l(\xi) \wedge dV_m(\eta) \right|$. It should be noted that the following identity is implicit in the proof of Lemma 2.1 of [?] :

$$\frac{1}{(n+1)^2} I_\zeta \bigg|_{\mathbb{H}_n} = \frac{(-1)^{j+k} |\zeta|^2}{\zeta_j \bar{\zeta}_k} \widetilde{\omega}_{jk}(\zeta) \bigg|_{\mathbb{H}_n}, \qquad \text{for } 1 \leqslant j, k \leqslant n+1.$$

Therefore, $I_\zeta|_{\mathbb{H}_n}$ is equal to

$$C \frac{|\zeta|^2}{|\zeta_j|^2} \widetilde{\omega}_{jj}(\zeta) \bigg|_{\mathbb{H}_n} = |\zeta|^2 \sum_{p=1}^{j-1} \frac{(-1)^{p-1} \zeta_p}{\zeta_j} \omega_{pj}(\zeta) \wedge \alpha_n(\bar{\zeta})$$

$$+ |\zeta|^2 \sum_{p=j+1}^{n+1} \frac{(-1)^p \zeta_p}{\zeta_j} \omega_{jp}(\zeta) \wedge \alpha_n(\bar{\zeta}) + |\zeta|^2 \sum_{q=1}^{j-1} \frac{(-1)^{n+q-1} \bar{\zeta}_q}{\bar{\zeta}_j} \omega_{qj}(\bar{\zeta}) \wedge \alpha_n(\zeta)$$

$$+ |\zeta|^2 \sum_{q=j+1}^{n+1} \frac{(-1)^{n+q} \bar{\zeta}_q}{\bar{\zeta}_j} \omega_{jq}(\bar{\zeta}) \wedge \alpha_n(\zeta).$$

Combining the identity $\alpha_n(\zeta) = (n+1) \frac{(-1)^{p-1}}{\zeta_p} \omega_p(\zeta)$, $1 \leqslant p \leqslant n+1$, (see formula (2.6) in [?]) and formula (9.1), a straightforward calculation gives that

$$\int\limits_{\mathcal{U}} du \wedge I_\zeta \wedge dV_l(\xi) \wedge dV_m(\eta)$$

$$= \int\limits_{\mathcal{U}} d_\zeta u \wedge I_\zeta \wedge dV_l(\xi) \wedge dV_m(\eta)$$

$$= C \int\limits_{\mathcal{U}} \sum_{p=1, p\neq j}^{n+1} \left(\frac{\partial u}{\partial \zeta_j}(-1)^p \zeta_p + \frac{\partial u}{\partial \zeta_p}(-1)^{p-1}\frac{\zeta_p^2}{\zeta_j} + \frac{\partial u}{\partial \overline{\zeta}_j}(-1)^{p+n}\overline{\zeta}_p \right.$$

$$\left. + \frac{\partial u}{\partial \overline{\zeta}_p}(-1)^{n+p-1}\frac{\overline{\zeta}_p^2}{\overline{\zeta}_j} \right) dV(\Theta).$$

By virtue of (9.6), we majorize easily the latter integral and obtain

$$\left| \int\limits_{\mathcal{U}} du \wedge I_\zeta \wedge dV_l(\xi) \wedge dV_m(\eta) \right| \leqslant C(N) \int\limits_{\mathcal{U}} |\zeta||(\mathrm{grad}_\zeta u)(\Theta)|dV(\Theta).$$

Hence, in view of (9.7), it follows that

$$\left| \int\limits_{\mathcal{U}} du \wedge \omega \right| \leqslant C(N) \int\limits_{\mathcal{U}} \left(|\xi||(\mathrm{grad}_\xi u)(\Theta)| + |\zeta||(\mathrm{grad}_\zeta u)(\Theta)| \right.$$

$$\left. + |\eta||(\mathrm{grad}_\eta u)(\Theta)|dV(\Theta). \right.$$

On the other hand, we can prove in just the same way that

$$\left| \int\limits_{\mathcal{U}} ud\omega \right| \leqslant C(N) \int\limits_{\mathcal{U}} |u|dV.$$

These two estimates, combined with (9.4) and (9.5), complete the proof. $\square$

We now present two applications of Theorem 9.1.

PROPOSITION 9.3. — *Let* $\lambda, \alpha_1, \alpha_2$ *be real numbers such that* $0 < \lambda < 1$, *and* $0 \leqslant \alpha_1 < 2n$, $0 \leqslant \alpha_2 < 2m$. *Then there exists a constant* $C :=$ $C(N, \lambda, \alpha_1, \alpha_2)$ *such that for every* $Z \in \mathbb{B}_N$,

$$\int\limits_{\partial\mathbb{M}_N} \frac{1}{|1 - Z \bullet \overline{\Theta}|^{|N|+1-\lambda}} \left(1 + \frac{|z|}{|\zeta|} \right)^{\alpha_1} \left(1 + \frac{|w|}{|\eta|} \right)^{\alpha_2} d\sigma(\Theta) \leqslant C \left(1 - |Z|^2 \right)^{\lambda-1}.$$

Proof.— Applying Theorem 9.1 gives that

$$\int\limits_{\partial\mathbb{M}_N} \frac{1}{|1 - Z \bullet \overline{\Theta}|^{|N|+1-\lambda}} \left(1 + \frac{|z|}{|\zeta|}\right)^{\alpha_1} \left(1 + \frac{|w|}{|\eta|}\right)^{\alpha_2} d\sigma(\Theta)$$

$$\leqslant C \int\limits_{\mathbb{M}_N} \frac{1}{|1 - Z \bullet \overline{\Theta}|^{|N|+2-\lambda}} \left(1 + \frac{|z|}{|\zeta|}\right)^{\alpha_1} \left(1 + \frac{|w|}{|\eta|}\right)^{\alpha_2} dV(\Theta)$$

$$\leqslant C \left(1 - |Z|^2\right)^{\lambda-1},$$

where the latter inequality follows from Theorem 7.1. $\square$

The following proposition completes the missing point in the proof of Lemma 8.4 on page 51.

PROPOSITION 9.4.— *Suppose that $0 < \lambda < 1$. Then there is a constant $C := C(N, \lambda)$ such that for every $0 < r < 1$ and $Z \in \partial\mathbb{M}_N$,*

$$I_1 := \int\limits_{\partial\mathbb{M}_N} \frac{\left(1 + \frac{|z|}{|\zeta|}\right)\left(1 + \frac{|w|}{|\eta|}\right)}{|\Theta - rZ|^{2|N|}} |\Theta - Z|^{\lambda} d\sigma(\Theta) \leqslant C \left(1 - r\right)^{\lambda-1},$$

$$I_2 := \int\limits_{\partial\mathbb{M}_N} \frac{\frac{1}{|\zeta|}\left(1 + \frac{|w|}{|\eta|}\right)}{|\Theta - rZ|^{2|N|-1}} |\Theta - Z|^{\lambda} d\sigma(\Theta) \qquad \leqslant C \left(1 - r\right)^{\lambda-1}.$$

Proof.— We only give the proof of the estimate for I_2. Starting from the elementary estimate $|\Theta - rZ| \approx (1 - r) + |\Theta - Z|$ for all $\Theta \in \partial\mathbb{M}_N$, we see that

$$I_2 \lesssim \int\limits_{\partial\mathbb{M}_N} \frac{\frac{1}{|\zeta|}\left(1 + \frac{|w|}{|\eta|}\right)}{[(1 - r) + |\Theta - Z|]^{2|N|-1}} |\Theta - Z|^{\lambda} d\sigma(\Theta)$$

$$\lesssim \int\limits_{\mathbb{M}_N} \frac{\frac{1}{|\zeta|}\left(1 + \frac{|w|}{|\eta|}\right)}{[(1 - r) + |\Theta - Z|]^{2|N|-1}} |\Theta - Z|^{\lambda-1} dV(\Theta)$$

$$\lesssim \int\limits_{\mathbb{M}_N} \frac{\left(1 + \frac{|z|}{|\zeta|}\right)\left(1 + \frac{|w|}{|\eta|}\right)}{[(1 - r) + |\Theta - Z|]^{2|N|-1} |\zeta - z|} |\Theta - Z|^{\lambda-1} dV(\Theta),$$

where the last two estimates follow respectively from Theorem 9.1 and the very elementary inequality $\frac{1}{|\zeta|} \leqslant \left(1 + \frac{|z|}{|\zeta|}\right) \frac{1}{|\zeta - z|}$. Hence, by part 3) of Theorem 5.6, the latter integral is dominated by $C\widetilde{I}_2$, where

$$\widetilde{I}_2 := \int\limits_{\mathbb{B}_{2,|N|}} \frac{|\widetilde{\Theta} - \widetilde{Z}|^{\lambda-1}}{\left[(1-r) + |\widetilde{\Theta} - \widetilde{Z}|\right]^{2|N|-1} |\widetilde{\zeta} - \widetilde{z}|} dV(\widetilde{\Theta}).$$

Dividing the domain of integration of $\widetilde{I}_2$ into two regions

$$\widetilde{E}_1 := \left\{\Theta \in \mathbb{B}_{2,|N|} : |\widetilde{\Theta} - \widetilde{Z}| < 1 - r\right\}$$

and

$$\widetilde{E}_2 := \left\{\Theta \in \mathbb{B}_{2,|N|} : |\widetilde{\Theta} - \widetilde{Z}| \geqslant 1 - r\right\},$$

we thus break $\widetilde{I}_2$ into two corresponding terms $\widetilde{I}_{21}$ and $\widetilde{I}_{22}$. We then apply Lemma 7.4 to estimate each of these terms and obtain

$$\widetilde{I}_{21} \lesssim \frac{1}{(1-r)^{2|N|-1}} \int\limits_{\widetilde{E}_1} \frac{|\widetilde{\Theta} - \widetilde{Z}|^{\lambda-1}}{|\widetilde{\zeta} - \widetilde{z}|} dV(\widetilde{\Theta}) \lesssim (1-r)^{\lambda-1},$$

$$\widetilde{I}_{22} \lesssim \int\limits_{\widetilde{E}_2} \frac{|\widetilde{\Theta} - \widetilde{Z}|^{\lambda-1}}{|\widetilde{\Theta} - \widetilde{Z}|^{2|N|-1} |\widetilde{\zeta} - \widetilde{z}|} dV(\widetilde{\Theta}) \lesssim (1-r)^{\lambda-1}.$$

In summary, we have

$$I_2 \lesssim \widetilde{I}_2 = \widetilde{I}_{21} + \widetilde{I}_{22} \leqslant C(1-r)^{\lambda-1},$$

which completes the proof of the proposition. $\qquad\square$

10. Proof of the main results

In this section we prove Theorems 1.1 and 1.2. For this purpose, we first establish some preparatory results.

Consider the holomorphic mapping $F_N : \overline{\mathbb{M}}_N \longrightarrow \overline{\Omega}_N \setminus \{0\}$ which maps every $Z \equiv (x, z, w) \equiv (x_1, \ldots, x_l, z_1, \ldots, z_{n+1}, w_1, \ldots, w_{m+1})$, element of $\overline{\mathbb{M}}_N$ to

$$F_N(Z) := \widetilde{Z} := \left(\frac{x_1}{\sqrt{2}}, \ldots, \frac{x_l}{\sqrt{2}}, z_1, \ldots, z_n, w_1, \ldots, w_m\right).$$

Recall that $dV(\widetilde{\Theta})$ is the canonical volume form of $\mathbb{C}^{|N|}$. It follows from formula (5.2) in [?] and formula (9.1) that

$$\frac{|\zeta_{n+1}|^2 |\eta_{m+1}|^2}{|\zeta|^2 |\eta|^2} dV(\Theta) = CF_N^* \left(dV(\widetilde{\Theta}) \right), \quad \text{for } \Theta \in \mathbb{M}_N \text{ and } \widetilde{\Theta} = F_N(\Theta).$$

(10.1)

PROPOSITION 10.1. — *Consider a $\bar{\partial}$-closed $(0,1)$-form f of class $\mathcal{C}^1$ defined in a neighborhood of $\overline{\Omega}_N$. Then the solution $T_1(F_N^* f)$ given by Theorem 8.2 satisfies*

$$(T_1(F_N^* f))(Z) = (T_1(F_N^* f))(Z'),$$

for all $Z, Z' \in \mathbb{M}_N$ such that $F_N(Z) = F_N(Z')$.

Proof. — Suppose that $f \in \mathcal{C}^1_{0,1}(r\Omega_N)$ for some $r > 1$. Since $r\Omega_N$ is pseudoconvex, there exists a function $u \in \mathcal{C}^1(\overline{\Omega}_N)$ such that $\bar{\partial} u = f$ in Ω_N. Therefore, it follows from (8.3) and (8.5) that for every $Z \in \mathbb{M}_N$,

$$(T_1(F_N^* f))(Z) = (u \circ F_N)(Z) - \int\limits_{\partial \mathbb{M}_N} \frac{R(\Theta, Z)}{(1 - Z \bullet \overline{\Theta})^{|N|}} (u \circ F_N)(\Theta) \frac{d\sigma(\Theta)}{|\zeta|^2 |\eta|^2}$$

Using this and the explicit formula of $R(\Theta, Z)$, we see that the proof follows. $\square$

THEOREM 10.2. — *For every $0 < \lambda \leqslant \frac{1}{2}$, there is a constant $C := C(N, \lambda)$ such that $\|u\|_{\Gamma_{\lambda, 2\lambda}(\mathbb{M}_N)} \leqslant C\|u\|_{\Lambda_\lambda(\partial \mathbb{M}_N)}$*

for all functions u in $\mathcal{C}(\overline{\mathbb{M}}_N)$ which are holomorphic in $\mathbb{M}_N$.

Proof. — Consider the holomorphic function $U \in H(\mathbb{B}_N)$ defined by

$$U(Z) := \int\limits_{\partial \mathbb{M}_N} \frac{R(\Theta, Z)}{\left(1 - Z \bullet \overline{\Theta}\right)^{|N|}} u(\Theta) \frac{d\sigma(\Theta)}{|\zeta|^2 |\eta|^2}, \quad \text{for all } Z \in \mathbb{B}_N. \quad (10.2)$$

Applying Theorem 3.3 to the function u yields

$$U(Z) = u(Z), \quad \text{for all } Z \in \mathbb{M}_N.$$

This shows that the theorem will follow from the estimate

$$\|U\|_{\Gamma_{\lambda, 2\lambda}(\mathbb{B}_N)} \leqslant C\|u\|_{\Lambda_\lambda(\partial \mathbb{M}_N)}. \quad (10.3)$$

To prove this, observe by (10.2) and formula (2) in [?, Section 6.4.4] that the radial derivative $(\mathcal{R}U)$ of U is given by
$(\mathcal{R}U)(Z) =$

$$\int_{\partial \mathbb{M}_N} \left[\frac{\left(C|\xi|^2 + C|\zeta|^2 + C|\eta|^2\right)\left((|\eta|^2 + w \bullet \overline{\eta})z \bullet \overline{\zeta} + (|\zeta|^2 + z \bullet \overline{\zeta})w \bullet \overline{\eta}\right)}{\left(1 - Z \bullet \overline{\Theta}\right)^{|N|}} \right.$$

$$\left. + \frac{N\left(C|\xi|^2 + C|\zeta|^2 + C|\eta|^2\right)(|\zeta|^2 + z \bullet \overline{\zeta})(|\eta|^2 + w \bullet \overline{\eta})Z \bullet \overline{\Theta}}{\left(1 - Z \bullet \overline{\Theta}\right)^{|N|+1}} \right] u(\Theta) \frac{d\sigma(\Theta)}{|\zeta|^2|\eta|^2} \, .$$

Using this and arguing as in the proof of Theorem 6.4.9 of [?], it can be shown that

$$|(\mathcal{R}U)(Z)| \leqslant C\|u\|_{\Lambda_\lambda(\partial \mathbb{M}_N)} \int_{\partial \mathbb{M}_N} \frac{1}{\left|1 - Z \bullet \overline{\Theta}\right|^{|N|+1-\lambda}} \left(1 + \frac{|z|}{|\zeta|}\right)\left(1 + \frac{|w|}{|\eta|}\right) d\sigma(\Theta).$$

Therefore, by Proposition 9.3,

$$|(\mathcal{R}U)(Z)| \leqslant C\|u\|_{\Lambda_\lambda(\partial \mathbb{M}_N)} \left(1 - |Z|^2\right)^{\lambda-1}, \qquad \text{for all } Z \in \mathbb{B}_N$$

so that by Theorem 6.4.10 of [?], inequality (10.3) follows and the proof is now complete. $\quad\square$

THEOREM 10.3. — *For every $(0,1)$-form f and real numbers p, α satisfying the hypothesis of Theorem 8.2, we have*

$$\begin{cases} T_1 f \in \Gamma_{\alpha,2\alpha}(\mathbb{M}_N) \text{ and } \|T_1 f\|_{\Gamma_{\alpha,2\alpha}(\mathbb{M}_N)} \leqslant C\|f\|_{\mathbb{M}_{N,p}} & \text{if } p < \infty, \\ T_1 f \in \Gamma_{\frac{1}{2},\tilde{1}}(\mathbb{M}_N) \text{ and } \|T_1 f\|_{\Gamma_{\frac{1}{2},\tilde{1}}(\mathbb{M}_N)} \leqslant C\|f\|_{\mathbb{M}_{N,\infty}} & \text{if } p = \infty, \end{cases}$$

where the constant $C := C(N,p)$.

Proof. — Let $\frac{1}{2} < r < 1$ and set

$$r\mathbb{M}_N := \{Z \in \mathbb{M}_N : |Z| < r\} \quad \text{and} \quad r\partial\mathbb{M}_N := \{rZ, \ Z \in \partial\mathbb{M}_N\} \, .$$

Now we define the norm $\|f\|_{r\mathbb{M}_{N,p}}$ in the same way as $\|f\|_{\mathbb{M}_{N,p}}$ given in (8.1) by substituting the domain of integration $\mathbb{M}_N$ by $r\mathbb{M}_N$. It is obvious that $\|f\|_{r\mathbb{M}_{N,p}} \leqslant \|f\|_{\mathbb{M}_{N,p}}$.

Applying Theorem 8.2 to the complex manifold $r\mathbb{M}_N$ gives an integral operator T_r that satisfies the following properties:

$$\overline{\partial}_{\mathbb{H}_N}(T_r f) = f \quad \text{on } r\mathbb{M}_N, \tag{10.4}$$

and

$$(T_r f)|_{r\partial\mathbb{M}_N} \in \Gamma_{\alpha,2\alpha}(r\partial\mathbb{M}_N) \text{ and } \|(T_r f)|_{r\partial\mathbb{M}_N}\|_{\Gamma_{\alpha,2\alpha}(r\partial\mathbb{M}_N)} \leqslant C\|f\|_{\mathbb{M}_{N,p}}. \tag{10.5}$$

Setting
$$u(Z) := (T_1 f)(Z) - (T_r f)(Z), \qquad \text{for all } Z \in \mathbb{M}_N, \tag{10.6}$$

Theorem 8.2 and (10.4) imply that u is holomorphic on $r\partial\mathbb{M}_N$. On the other hand, by Theorem 8.2(ii) and (10.5), (10.6), we get

$$\|u|_{r\partial\mathbb{M}_N}\|_{\Lambda_\alpha(r\partial\mathbb{M}_N)} \leqslant \|(T_r f)|_{r\partial\mathbb{M}_N}\|_{\Lambda_\alpha(r\partial\mathbb{M}_N)} + \|(T_1 f)|_{r\partial\mathbb{M}_N}\|_{\Lambda_\alpha(r\partial\mathbb{M}_N)}$$
$$\leqslant C\|f\|_{\mathbb{M}_{N,p}}.$$

Applying Theorem 10.2 to this estimate yields

$$\|u|_{r\partial\mathbb{M}_N}\|_{\Gamma_{\alpha,2\alpha}(r\partial\mathbb{M}_N)} \leqslant C\|f\|_{\mathbb{M}_{N,p}},$$

so that by (10.5), we get $T_1 f = u + T_r f \in \Gamma_{\alpha,2\alpha}(r\partial\mathbb{M}_N)$ and

$$\|(T_1 f)|_{r\partial\mathbb{M}_N}\|_{\Gamma_{\alpha,2\alpha}(r\partial\mathbb{M}_N)} \leqslant C\|f\|_{\mathbb{M}_{N,p}}.$$

Since all admissible curves $\gamma \in \mathcal{C}_1^2(\mathbb{B}_N)$ such that $\gamma \subset \mathbb{M}_N$ lie on some manifold $r\partial\mathbb{M}_N$, the proof of the theorem is now complete. $\quad\square$

Proof of Theorem 1.1. — Consider first the case where f is a $\overline{\partial}$-closed $(0,1)$-form of class $\mathcal{C}^1$ defined in a neighborhood of $\overline{\Omega}_N$. The general case will be treated later.

In view of (8.1) and (10.1), it can be checked that

$$\|f\|_{L^p(\Omega_N)} = C(N,p)\|F_N^* f\|_{\mathbb{M}_{N,p}}. \tag{10.7}$$

By Proposition 10.1, we can define the $\overline{\partial}$-solving operator T on Ω_N as

$$(Tf)(\widetilde{Z}) := (T_1(F_N^* f))(Z), \tag{10.8}$$

for every $\widetilde{Z} \in \Omega_N$ and $Z \in \mathbb{M}_N$ such that $F_N(Z) = \widetilde{Z}$.

Combining Proposition 10.1, Theorem 10.3 and equalities (10.7) and (10.8), we see that the operator T satisfies

$$\overline{\partial}(Tf) \;=\; f \quad \text{on } \Omega_N; \tag{10.9}$$
$$\|F_N^*(Tf)\|_{\Gamma_{\alpha,2\alpha}(\mathbb{M}_N)} \;\leqslant\; C\|f\|_{L^p(\Omega_N)}. \tag{10.10}$$

Let $\widetilde{Z} \equiv (\widetilde{x}, \widetilde{z}, \widetilde{w})$ and $\widetilde{Z'} \equiv (\widetilde{x'}, \widetilde{z'}, \widetilde{w'})$ be two elements of Ω_N. We shall show that there exists a constant $C := C(N,p)$ such that

$$\left|(Tf)(\widetilde{Z}) - (Tf)(\widetilde{Z'})\right| \leqslant C\|f\|_{L^p(\Omega_N)}|\widetilde{Z} - \widetilde{Z'}|^\alpha. \tag{10.11}$$

Using the remark made at the beginning of the proof of Lemma 8.1, we only need prove (10.11) in one of the following three cases:

1) $\widetilde{x} = \widetilde{x}'$, $\widetilde{z} = \widetilde{z}'$; 2) $\widetilde{x} = \widetilde{x}'$, $\widetilde{w} = \widetilde{w}'$; 3) $\widetilde{z} = \widetilde{z}'$, $\widetilde{w} = \widetilde{w}'$.

Consider for example the case $\widetilde{x} = \widetilde{x}'$, $\widetilde{w} = \widetilde{w}'$. In this case, estimate (10.11) becomes

$$\left| (Tf)(\widetilde{x}, \widetilde{z}, \widetilde{w}) - (Tf)(\widetilde{x}, \widetilde{z}', \widetilde{w}) \right| \leqslant C\|f\|_{L^p(\Omega_N)} |\widetilde{z} - \widetilde{z}'|^\alpha,$$

which can be proved by using (10.10) and arguing as in the proof of case 2 in Section 5 of [?].

It remains to treat the general case. If merely $f \in L^p(\Omega_N)$, we can regularize f by convolution with a $\mathcal{C}_0^\infty$ function of sufficiently small support. Then the same limiting argument as in [?, p. 361-362] shows that the conclusion of the theorem holds also for such f. This completes the proof of the theorem. □

Proof of Theorem 1.2. — First suppose that $p < \infty$. We break the proof into four cases. In the course of the proof, we shall see that the general case can be reduced to one of these four cases. In the sequel, we write for every $Z \in \mathbb{C}^{|N|}$, $Z \equiv (x, z, w) \in \mathbb{C}^l \times \mathbb{C}^n \times \mathbb{C}^m$.

Case 1: $n > 2$ and $m \geqslant 2$.

For every real number λ_0 such that $\frac{1}{4} < \lambda_0 < \frac{1}{2}$, consider two real numbers $\lambda, \mu > 0$ related by $2\mu^2 = \frac{1}{2}\left(\frac{1}{2} - \lambda\right) = \frac{1}{2} - \lambda_0$. Let $c \in \mathbb{C}$ such that $|c| \leqslant 1$, and consider the following elements of Ω_N :

$$Z_{\lambda,c} \; := \; \left(\underbrace{0, \ldots, 0}_{l}, \lambda, i\lambda, \mu c, \underbrace{0, \ldots, 0}_{n-3}, \frac{1}{2}, \frac{i}{2}, \underbrace{0, \ldots, 0}_{m-2} \right) ;$$

$$Z_{\lambda_0,c} \; := \; \left(\underbrace{0, \ldots, 0}_{l}, \lambda_0, i\lambda_0, \mu c, \underbrace{0, \ldots, 0}_{n-3}, \frac{1}{2}, \frac{i}{2}, \underbrace{0, \ldots, 0}_{m-2} \right) .$$

Now we put $f := \bar{\partial}u_0$, where the function u_0 is given by

$$u_0(Z) := \frac{|z_3|^2}{\left(1 - \frac{z_1}{2} + \frac{iz_2}{2} - \frac{w_1}{2} + \frac{iw_2}{2}\right)^{\frac{1}{2} + \frac{|N|+3}{p}}}, \qquad \text{for all } Z \in \Omega_N.$$

Then we have

$$f(Z) := \frac{z_3 d\bar{z}_3}{\left(1 - \frac{z_1}{2} + \frac{iz_2}{2} - \frac{w_1}{2} + \frac{iw_2}{2}\right)^{\frac{1}{2} + \frac{|N|+3}{p}}}.$$

Suppose that u is a solution of the equation $\overline{\partial} u = f$ on Ω_N. Since $u - u_0$ is holomorphic on Ω_N and $u_0(Z_{\lambda,0}) = u_0(Z_{\lambda_0,0}) = 0$, by Cauchy formula we have

$$\frac{1}{2\pi} \int_0^{2\pi} u(Z_{\lambda,e^{i\theta}}) d\theta - u(Z_{\lambda,0}) = \frac{|\mu|^2}{\left(\frac{1}{2} - \lambda\right)^{\frac{1}{2} + \frac{|N|+3}{p}}},$$

$$\frac{1}{2\pi} \int_0^{2\pi} u(Z_{\lambda_0,e^{i\theta}}) d\theta - u(Z_{\lambda_0,0}) = \frac{|\mu|^2}{\left(\frac{1}{2} - \lambda_0\right)^{\frac{1}{2} + \frac{|N|+3}{p}}}.$$

If $u \in \Lambda_{\alpha+\epsilon}(\Omega_N)$ for some $\epsilon > 0$, then the difference between the two left hand sides is $O(|\lambda - \lambda_0|^{\alpha+\epsilon})$. On the other hand, the difference between the two right sides is greater than $C|\lambda - \lambda_0|^\alpha$. Letting λ_0 tend to $\frac{1}{2}$, we reach a contradiction. Hence $u \notin \Lambda_{\alpha+\epsilon}(\Omega_N)$.

It now remains to check that $f \in L^{p-\epsilon}(\Omega_N)$ for all $\epsilon > 0$. Applying (10.7) and using the local coordinates Φ^z and Φ^w of Theorem 4.1 with $z := (\frac{1}{2}, \frac{i}{2}, \underbrace{0, \ldots, 0}_{n-1}) \in \mathbb{H}_n$ and $w := (\frac{1}{2}, \frac{i}{2}, \underbrace{0, \ldots, 0}_{m-1}) \in \mathbb{H}_m$, it follows that for every $\epsilon \geqslant 0$,

$$\|f\|_{L^{p-\epsilon}(\Omega_N)}^{p-\epsilon} = C \|F_N^* f\|_{\mathbb{M}_{N,p-\epsilon}}^{p-\epsilon} \approx \int_{\mathcal{U} \cap \mathbb{B}_{|N|}} \frac{|z_2|^{p-\epsilon} |z_3|^2 |z_4|^2}{|1 - z_1|^{(p-\epsilon)\left(\frac{1}{2} + \frac{|N|+3}{p}\right)}} dV(Z),$$

where $\mathcal{U}$ is a sufficiently small neighborhood of the point $(1, 0, \ldots, 0) \in \mathbb{C}^{|N|}$ and $dV(Z)$ is the Lebesgue measure of $\mathbb{C}^{|N|}$. We now explain briefly how the estimate $\approx$ in the latter line could be obtained. Indeed, using the local coordinates Φ^z and Φ^w, the function $|\zeta_{n+1}|$ (resp. $|\eta_{m+1}|$) appearing in the $\|\cdot\|_{\mathbb{M}_{N,p}}$ norm in (8.1) becomes the function $|z_3|$ (resp. $|z_4|$) defined in $\mathbb{C}^{|N|}$.

By integration in polar coordinates, it is easy to reduce the estimate of the latter integral to that of the following one

$$\int_{z \in \mathbb{C}: |1-z| < 1} \frac{dz \wedge d\bar{z}}{|1 - z|^{2 - \frac{\epsilon(|N|+3)}{p}}}.$$

From this integral, we see that $f \in L^{p-\epsilon}(\Omega_N)$ for all $\epsilon > 0$. This completes the proof in the first case. Furthermore, we remark that the method presented here can be applied to all domains Ω_N where $N := (n_1, \ldots, n_m)$ satisfies the condition $n_m > 2$.

Case 2: $l \geqslant 1$ and $n, m \geqslant 2$.

Choose λ_0, λ and μ as in case 1. Let $c \in \mathbb{C}$ such that $|c| \leqslant 1$ and consider the following points of Ω_N:

$$Z_{\lambda,c} \quad := \quad \left(\mu c, \underbrace{0, \ldots, 0}_{l-1}, \lambda, i\lambda, \underbrace{0, \ldots, 0}_{n-2}, \frac{1}{2}, \frac{i}{2}, \underbrace{0, \ldots, 0}_{m-2} \right) ;$$

$$Z_{\lambda_0,c} \quad := \quad \left(\mu c, \underbrace{0, \ldots, 0}_{l-1}, \lambda_0, i\lambda_0, \underbrace{0, \ldots, 0}_{n-2}, \frac{1}{2}, \frac{i}{2}, \underbrace{0, \ldots, 0}_{m-2} \right) .$$

We set $f := \overline{\partial} u_0$, where the function u_0 is given by

$$u_0(Z) := \frac{|x_1|^2}{\left(1 - \frac{z_1}{2} + \frac{iz_2}{2} - \frac{w_1}{2} + \frac{iw_2}{2} \right)^{\frac{1}{2} + \frac{|N|+3}{p}}}, \qquad \text{for all } Z \in \Omega_N.$$

The rest of the proof follows along the same lines as that of case 1. Finally, we remark that the method used in this second case works also for all domains Ω_N where $N := (n_1, \ldots, n_m)$ satisfies the condition $n_1 = 1$ and $n_m > 1$.

Case 3: $l = 0$ and $n = m = 2$.

For every λ_0 such that $\frac{1}{2\sqrt{2}} < \lambda_0 < \frac{1}{\sqrt{2}}$, let λ and μ be two positive real numbers satisfying $\mu^2 = \frac{1}{2} \left(\frac{1}{\sqrt{2}} - \lambda \right) = \frac{1}{\sqrt{2}} - \lambda_0$. Let $c \in \mathbb{C}$ such that $|c| \leqslant 1$ and consider the following elements of Ω_N

$$Z_{\lambda,c} := (\mu c, 0, \lambda, i\lambda), \qquad \text{and} \qquad Z_{\lambda_0,c} := (\mu c, 0, \lambda_0, i\lambda_0).$$

We set $f := \overline{\partial} u_0$, where the function u_0 is defined by

$$u_0(Z) := \frac{|z_1|^2}{\left(1 - \frac{w_1}{\sqrt{2}} + \frac{iw_2}{\sqrt{2}} \right)^{\frac{1}{2} + \frac{6}{p}}}, \qquad \text{for all } Z \equiv (z_1, z_2, w_1, w_2) \in \Omega_N.$$

Proceeding as in the proof of case 1, it can be checked that if a function u satisfies $\overline{\partial} u = f$ then $u \notin \Lambda_{\alpha+\epsilon}(\Omega_N)$, $\forall \epsilon > 0$. It now remains to establish that $f \in L^{p-\epsilon}(\Omega_N)$ for all $\epsilon > 0$.

We first apply (10.7), then use the local coordinates Φ^w in Theorem 4.1 with $w := \left(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 0 \right) \in \mathbb{H}_2$, and conclude that for every $\epsilon \geqslant 0$,

$$\|f\|_{L^{p-\epsilon}(\Omega_N)}^{p-\epsilon} = C\|F_N^* f\|_{\mathbb{M}_{N,p-\epsilon}}^{p-\epsilon} \approx \int\limits_{\mathcal{U}\cap\mathbb{B}_4} \frac{|z_2|^{p-\epsilon}|z_3|^2}{|1-z_1|^{(p-\epsilon)\left(\frac{1}{2}+\frac{6}{p}\right)}} dV_4(Z).$$

Here $\mathcal{U}$ is a sufficiently small neighborhood of the point $(1,0,0,0)$ in $\mathbb{C}^4$ and $\mathbb{B}_4$ (resp. $dV_4(Z)$) is the euclidean unit ball (resp. the Lebesgue measure) of $\mathbb{C}^4$.

By integration in polar coordinates, the estimate of the latter integral is reduced to that of the integral

$$\int\limits_{z\in\mathbb{C}:|1-z|<1} \frac{dz \wedge d\bar{z}}{|1-z|^{2-\frac{6\epsilon}{p}}}.$$

From this integral we conclude that $f \in L^{p-\epsilon}(\Omega_N)$. The proof of the theorem is complete in this third case. It should be noted that this method is applicable to all domains Ω_N where $N := (n_1,\ldots,n_m)$ satisfies the condition $n_1 = \cdots = n_m = 2$.

Case 4: $l = m = 0$ and $n = 2$.

In this case $\alpha(N,p) = \frac{1}{2} - \frac{3}{p}$. Let z be a strongly convex point of the boundary $\partial\Omega_N$. It then follows from the work of Krantz in [?, Section 6] that there exists a $(0,1)$-form $f \in \mathcal{C}^\infty(\mathcal{U})$ that satisfies the conclusion of the theorem if Ω_N is replaced by $\mathcal{U}$. Here $\mathcal{U}$ is an open strongly convex neighborhood of z in $\Omega_N \cup \{z\}$. In view of [?], we see easily that the form f can be extended to a form of class $\mathcal{C}^\infty(\Omega_N)$ satisfying the conclusion of the theorem. The proof is thus complete in this last case.

This argument also shows that the Lipschitz $\left(\frac{1}{2} + \epsilon\right)$-estimates $(\epsilon > 0)$ do not hold for the case $p = \infty$. This completes the proof of Theorem 1.2. $\square$

Finally, we conclude this paper by some remarks and open problems.

1. It seems to be of some interest to establish the (L^p, L^q) type optimal regularity for the $\bar{\partial}$-equation on Ω_N.

2. We conjecture that the Lipschitz $\frac{\tilde{1}}{2}$-regularity corresponding to the case $p = \infty$ in Theorem 1.1 is optimal. More precisely, this regularity can not be improved to Lipschitz $\frac{1}{2}$.

3. Does there exist a natural way to define the Nevanlinna class on the non-smooth domains Ω_N and find a related Blaschke type condition that characterizes the zeroes of the functions of this class?

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