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Decay of solutions of the elastic wave equation with a localized dissipation (*)

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ABSTRACT. — In this paper, we study the stabilization problem for the elastic wave equation in a bounded domain in two different situations. The boundary stabilization and the internal stabilization problem. Providing regular initial data we prove, without any assumption on the dynamics, that the energy decay is at last logarithmic. In order to prove that result we bound from below the spectrum of the infinitesimal generator of the associated semi-group.

RÉSUMÉ. — Dans ce papier on étudie le problème de stabilisation pour l'équation des ondes élastiques dans un domaine borné et dans deux situations différentes. On prouve, sans aucune condition sur la dynamique, que l'énergie decroît au moins comme l'inverse du logarithme du temps dès que les données sont suffisamment régulières. Pour montrer ce résultat nous donnons une estimation sur la distance entre le spectre du générateur infinitésimal du semi-groupe associé et l'axe réel.

1. Introduction and main results

We are mainly interested in the decaying mode of the energy (stabilization) of the solution of the initial boundary value problem in a connected and compact manifold M with compact boundary for the elastic wave equation as time tends to infinity. We will consider both the boundary stabilization problems with a boundary damping term supported in the boundary or the internal stabilization with a damping term supported in the interior of the domain. On one hand, it is well known that (see Lagnese [10]) the energy decreases to zero when the damping mechanism is effective in a non empty

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set $\omega \subset M$. On the other hand Bardos-Lebeau-Rauch [2], show that, when M is of class C^∞, the energy of the solution for general second order scalar initial-boundary value problems with regular coefficients satisfies the exponential decay if and only if the following "geometric control condition" is satisfied: there exists some $T > 0$ such that every ray of geometric optics intersects the set $\omega \times (0, T)$. The canonical example of open subset ω verifying the "control geometric condition" is when ω is a neighborhood of the boundary.

Taylor [16] gives a rigorous treatment of the singularity for the elastic wave equation with Neumann boundary condition, he prove there are three types of rays that carry singularities. The first types are classical rays reflecting at the boundary according to the laws of geometrical optics. The third type of rays lie on the boundary and singularities propagate along them with a slower propagation speed $C_R > 0$ (the Rayleigh speed).

In [8] Horn prove the uniform exponential decay of solution of elastic wave equation via linear velocity feedbacks acting through a portion of the boundary as traction forces. First these results are proven without the imposition of strong geometric assumptions on the controlled portion of the boundary, thus extending earlier work which required that the domain be "star shaped". Second, the feedback is only a function of velocity, as opposed to also containing the tangential derivative of the displacement but satisfies the Lions condition (see [14]).

In the present paper we show that even if the "geometrical control condition" is not fulfilled (i.e. without any assumption on the dynamics) then the energy decays with respect to time at least as fast as the inverse of the logarithm, providing the initial data belong to the domain of A^k (A stays for the infinitesimal generator of the evolution equation).

In order to prove that result we bound from below the spectrum of A. This bound is obtained by using a Carleman type estimate for the resolvent of A.

The originality of our method consists in the fact we can give a Carleman boundary estimate for the operator of the elasticity (with Dirichlet or Neumann conditions) without boundary tangential derivative of the displacement. To the best of knowledge, such sorts of estimates are not avaible in the literature. For the scalar elliptic operator Lebeau-Robbiano [13] obtained a similar estimate, and such an estimate played a crucial role in the proof of "*stabilization*" with Neumann dissipation see Horn [8] and Lasiecka and Triggiani [11]. For this end we use some idea from [3] where we study the problem of resonances in exterior domain and we have proved a Carleman estimate but with a tangential derivative terms. Here we rafined our study and we eliminated this tangential derivative terms. Moreover in the case of bounadry dissipation our boundary operator is different from [3] because here we have perturbed Neumann boundary operator by first order operator.

On the other hand, the results in [13]-[10] for the scalar wave equation need a interpolation estimate. This type of estimates seems difficult to show it for the elasticity system. In this paper we employ a direct method based on the Carleman estimate for the stationary associated operator.

1.1. Main results

Let (M,g) be a Riemannian, compact manifold with smooth and compact boundary ∂M. We set

$$\Delta_e = \mu\Delta + (\mu+\lambda)\nabla(\mathrm{div.}) \tag{1.1}$$

where μ, λ are real constants satisfy $\mu > 0$, $2\mu+\lambda > 0$, and Δ, ∇ and div are the Laplacian, gradiant and divergence operator associated to the metric g. We consider two classical examples of the elastic wave equation. The first damped by a boundary scalar velocity feedback $a_0 \in C^\infty(\partial M)$ and $a_0 \geqslant 0$

$$\begin{cases} \partial_t^2 u - \Delta_e u = 0 & \text{in} \quad M\times\mathbb{R} \\ u(x,0) = u_0, \quad \partial_t u(x,0) = u_1 & \text{in} \quad M \\ \sigma(u).\nu + a_0(x)\partial_t u = 0 & \text{on} \quad \partial M\times\mathbb{R} \end{cases} \tag{1.2}$$

Here $\sigma(u)$ is the stress tensor defined by

$$\sigma(u) = \lambda(\mathrm{div}\, u)Id + 2\mu\varepsilon(u) \tag{1.3}$$

where $\varepsilon(u) = 1/2(\nabla u + {}^t\nabla u)$ the strain tensor and $\nu(x)$ is the unit outer normal to M at $x \in \partial M$.

The second damped system by a internal matrix feedback $a \in C^\infty(M)$ satisfies for any $z \in \mathbb{C}$, $a(x)z\bar{z} \geqslant 0$ and the set $\left\{x \in M;\ a(x)z\bar{z} \geqslant \delta|z|^2\right\}$ is non empty

$$\begin{cases} \partial_t^2 u - \Delta_e u + a(x)\partial_t u = 0 & \text{in} \quad M\times\mathbb{R} \\ u(x,0) = u_0, \quad \partial_t u(x,0) = u_1 & \text{in} \quad M \\ u(x,t) = 0 & \text{on} \quad \partial M\times\mathbb{R} \end{cases} \tag{1.4}$$

The primary consideration of the paper is the decay rate of solutions of (1.2) and (1.4). For $(u_0, u_1) \in H = H^1 \oplus L^2$ we define the energy for the solution of (1.2) or (1.4) as

$$E(u,t) = \frac{1}{2}\int_M \Big(|\partial_t u|^2 + \lambda\, |div\, u|^2 + \mu \sum |\varepsilon_{ij}(u)|^2 \Big) dx \tag{1.5}$$

where dx is the Riemannian volume element in M.

1.1.1. Boundary stabilization

Let A_0 the linear operator defined by

$$A_0 = \begin{pmatrix} 0 & -iId \\ -i\Delta_e & 0 \end{pmatrix}. \tag{1.6}$$

Then the system (1.2) is transformed into

$$U_t = iA_0 U(t), \quad U(0) = U_0, \quad U(t) = {}^t(u(t,x), \partial_t u(t,x)) \tag{1.7}$$

The domain of A_0 is defined by

$$\mathcal{D}(A_0) = \Big\{ u = (u_0, u_1) \in H / A_0 u \in H,\ \ \sigma(u_0).\nu + a_0 u_1 = 0 \quad \text{in} \quad \partial M \Big\}. \tag{1.8}$$

Then, the immersion $\mathcal{D}(A_0) \hookrightarrow H$ is compact. With this definition, iA_0 is dissipative and is the infinitisimal generator of strongly continous semigroup e^{itA_0}, $t \geqslant 0$.

For any solution $u(t)$ of (1.2) the function $E(u,t)$ satisfies the following identity

$$E(u,t) - E(u,0) = -\int_0^t \int_{\partial M} a_0(x)\, |\partial_s u(x,s)|^2\, d\sigma ds \tag{1.9}$$

and therefore the energy is a decreasing function of time t.
Now we can able to state our first results

THEOREM 1.1.— *There exists C_1, $C_2 > 0$ such that if $Im z \leqslant C_1 e^{-C_2 |Re z|}$, $|z| > 1$ we have:*

$$\left\| (A_0 - z)^{-1} \right\|_{\mathcal{L}(H,H)} \leqslant C e^{C|Re z|}. \tag{1.10}$$

As immediate consequence (see theorem 1.6) of the previous theorem, we get the following result

THEOREM 1.2.— *For any $k > 0$ there exists $C > 0$ such that for any initial data $(u_0, u_1) \in \mathcal{D}(A_0^k)$ the solution $u(t,x)$ of (1.2) starting from (u_0, u_1) satisfy:*

$$E(u,t) \leqslant \frac{C}{\log(2+t)^{2k}} \left\|(u_0, u_1)\right\|^2_{\mathcal{D}(A_0^k)} . \tag{1.11}$$

1.1.2. Internal stabilization

Let us introduce the operator A defined as

$$A = \begin{pmatrix} 0 & -iId \\ -i\Delta_e & ia \end{pmatrix} \tag{1.12}$$

The domain of A is defined by $\mathcal{D}(A) = (H_0^1 \cap H^2) \oplus L^2$. Then we have $(A - z)$ is bijective from $\mathcal{D}(A)$ onto H for z in $\left\{z \in \mathbb{C}\ ; Imz \notin [0, \|a\|_{L^\infty}]\right\}$. The immersion $\mathcal{D}(A) \hookrightarrow H$ is compact, the spectrum of A is constituted a sequence z_j such that $Im(z_j) \in [0, \|a\|_{L^\infty}]$.

By integration by parts we have:

$$E(u,t) - E(u,0) = -\int_0^t \int_M a(x)\partial_s u(x,s)\overline{\partial_s u(x,s)}dxds \tag{1.13}$$

and therefore the energy is a decreasing function of time t.

Similarly we have the following Theorems

THEOREM 1.3.— *There exists C_1, $C_2 > 0$ such that if $Imz \leqslant C_1 e^{-C_2|Rez|}$, $|z| > 1$ we have*

$$\left\|(A-z)^{-1}\right\|_{\mathcal{L}(H,H)} \leqslant Ce^{C|Rez|}. \tag{1.14}$$

As immediate consequence of the last theorem (see theorem **??**), the followiong result holds

THEOREM 1.4.— *For any $k > 0$, there exists $C > 0$ such that for any initial data $(u_0, u_1) \in \mathcal{D}(A^k)$ the solution $u(t,x)$ of (1.4) starting from (u_0, u_1) satisfy*

$$E(u,t) \leqslant \frac{C}{\log(2+t)^{2k}} \left\|(u_0, u_1)\right\|^2_{\mathcal{D}(A^k)} . \tag{1.15}$$

Remark 1.5.— Theorems 1.1 and 1.2 hold if we change the Dirichlet condition by the Neumann Condition $\sigma(u) = 0$.

Theorems 1.1 and 1.3 prove in particulary that the resolvants $R(z) = (z - A)^{-1}$ and $R(z) = (z - A_0)^{-1}$ are analytic in the region

$$\Gamma = \left\{ z \in \mathbb{C}; 0 < Im(z) < C_1 e^{-C_2|Rez|};\ |z| > 1 \right\},$$

Theorems 1.2 and 1.4 follows from Theorem 1.1 and 1.3, this is proved in the general case by Burq (see Theorem 3 of [6])

THEOREM 1.6 ([6]).— *Let H be Hilbert space and iB a maximal dissipative unbounded operator with domain $\mathcal{D}(B)$. Assume that the resolvant $R(z) = (z - B)^{-1}$ is analytic in the region Γ and satisfies*

$$\|R(z)\|_{\mathcal{L}(H,H)} \leqslant Ce^{C|Rez|}, \quad \forall z \in \Gamma.$$

Then for any $k > 0$ there exist $C_k > 0$ such that the following estimate holds true

$$\left\| \langle iB \rangle^{-k} e^{itB} \right\|_{\mathcal{L}(H,H)} \leqslant \frac{C_k}{\log(2+t)^k}, \quad \text{where} \quad \langle iB \rangle = (I - iB) \tag{1.16}$$

1.2. Some remarks

(i) Theorems 1.1 and 1.3 are the analogous of Lebeau [10] and Lebeau-Robbiano [13] results in the case of the scalar wave equation, as well as our result [4] in the case of the transmission problem. Here our method is different to [13] and consist to use the Carleman estimate directly for the stationary operator without passing by the interpolation inequality.

(ii) In particular Theorems 1.1 and 1.2 show that

$$Spect(A) \subset \left[C_1 e^{-C_2|Rez|}, \|a\|_\infty \right],$$

$$Spect(A_0) \backslash \{0\} \subset \left[C_1 e^{-C_2|Rez|}, \|a_0\|_\infty \right].$$

Any constants resolve the problem (1.2).

(iii) Theorems 1.1 and 1.3 are optimal if we do not assume condition on the dynamic. Moreover 1.1 and 1.2 hold if we change the Dirichlet condition by the Neumann Condition. We can even show that Theorem 1.1 is optimal under reasonable geometric framework, and the damped term $a(x)$ is positive on the whole domain M. Indeed, in this case the Railegh's rays on the boundary (with Neumann Condition) never hits the damped term (see Kawashita [9]).

(iv) In [5], we study the uniqueness problem for the elastic wave equation. We prove that we have the uniqueness property across any non characteristic surface. We also give two results which apply to the boundary controllability for the elastic wave equation.

(v) To prove theorems 1.1 and 1.3, we make use Carleman estimate to obtain information about the resolvent in a bounded domain, the cost is to use phases functions satisfying Hörmander's assumption and thus growing fast. D.Tataru [15] who was the first to consider the Carleman estimate and the uniform Lopatinskii condition for scalar operators with C^1 coefficients. Here our operator is a system with non diagonal boundary conditions and we can not applied [15].

This paper is organized as follow, in section 2 we gives a Carleman inequality adapted to our case, in section 3 we prove our results and the prove of Carleman estimate is in the section 4.

Finally I would like to thank Professors L.Robbiano and G.Lebeau for useful discussion during the preparation of this work.

2. Carleman estimate

The aim of this section is to explain Carleman estimate used to obtain information about the resolvent in bounded domain.

The point is to show estimations in a bounded domain $\Omega \subset M$ for the solutions of

$$\begin{cases} P_\tau(x,D)u = f & \text{in} \quad \Omega \\ \\ B_\tau(x,D)u = g & \text{on} \quad \partial\Omega \end{cases} \tag{2.1}$$

where $P_\tau(x,D)$ the differential operator with principal symbol

$$P_\tau(x,\xi) = \mu\,{}^t\xi.\xi + (\mu+\lambda)\xi\,{}^t\xi - \tau^2 Id \tag{2.2}$$

and the boundary operator

$$B_\tau(x,D)u = \begin{cases} u & \text{(Dirichlet Condition)} \\ \sigma(u).\nu + ia_0\tau u & \text{(Neumann Condition)} \end{cases} \tag{2.3}$$

Let $\varphi(x)$ be a real function in $C^\infty(\mathbb{R}^n)$, we define the operator

$$P(x,D,\tau) = e^{\tau\varphi}P_\tau(x,D)e^{-\tau\varphi} \tag{2.4}$$

with principal symbol given by

$$P(x,\xi,\tau) = P_\tau(x,\xi + i\tau\varphi'). \tag{2.5}$$

We introduce the scalar partial differential operator

$$a_\gamma(x,D) = \Delta + \frac{\tau^2}{\gamma}, \qquad \gamma \in \{\mu, 2\mu+\lambda\}, \tag{2.6}$$

and we define $a_\gamma(x,D,\tau)$ by $e^{\tau\varphi}a_\gamma(x,D)e^{-\tau\varphi}$ with principal symbol

$$a_\gamma(x,\xi,\tau) = a_\gamma(x,\xi+i\tau\varphi'). \tag{2.7}$$

We assume that φ satisfies Hörmander's assumption for the operator a_γ

$$\exists\, C\ \forall x \in \Omega,\ \forall \xi \in T^*\Omega\backslash\{0\};\quad a_\gamma(x,\xi;\tau) = 0 \Longrightarrow \Big\{Rea_\gamma; Ima_\gamma\Big\} > C, \tag{2.8}$$

where $\{,\}$ denote the Poisson brackets.

We have the following Carleman type estimate

PROPOSITION 2.1.— *Let φ satisfy (2.8). We assume that $\partial_\nu\varphi < -C_0$ (where $C_0 > 0$ large constant fixed in section 4) in $\Sigma_1 \subset \partial\Omega$, then there exist $C > 0$ and τ_0 such that for any $u \in C^\infty(\overline{\Omega})$ solution of (2.1), the following estimate holds*

$$\int_\Omega e^{2\tau\varphi}\ |f|^2\,dx + \tau\int_{\partial\Omega\backslash\Sigma_1} e^{2\tau\varphi}(\tau^2|u|^2 + |\sigma(u).\nu|^2)dx' \\ + \tau\int_{\Sigma_1} e^{2\tau\varphi}|g|^2\,dx' \geqslant C\tau\int_\Omega e^{2\tau\varphi}(\tau^2|u|^2 + |\nabla u|^2)dx \tag{2.9}$$

for large enough $\tau > \tau_0$. Here dx' is the Riemannian surface element in $\partial\Omega$.

Remark 2.2.— Unfortunately we can not assume that $\Sigma_1 = \partial\Omega$ in the previous proposition and eliminate the boundary terms $\partial\Omega\backslash\Sigma_1$. Indeed if we assume that $\partial_\nu\varphi < -C_0$ on $\partial\Omega$ then the function φ attain his global maximum in Ω and then the Hörmander assumption (2.8) are not satisfied (for more details see [3]).

In the next we treat the local extremum of the phase φ.

2.1. Construction and properties of the phases functions

The purpose of this subsection is to construct two phases φ_1, φ_2 which satisfy the Hörmander's assumption, excepting in a finite number of balls, such that on a ball where one of them do not satisfies these conditions the second does and is strictly greater.

PROPOSITION 2.3.— *(see [6] and [3]) Let Ω be a bounded smooth domain, and let Σ_1, Σ_2 two non empty parts of the boundary $\partial\Omega$ such that $\Sigma_1 \cup \Sigma_2 = \partial\Omega$. There exist two functions $\psi_{1,2} \in C^\infty(\bar{\Omega})$ satisfying $\partial_\nu \psi_{i|\Sigma_1} < 0$, $\partial_\nu \psi_{i|\Sigma_2} > 0$ having only no degenerate critical points, such that when $\nabla\psi_i = 0$ then $\nabla\psi_{i+1} \neq 0$ and $\psi_{i+1} > \psi_i$ (where $\psi_3 = \psi_1$).*

As immediate consequence of the last proposition, the following conclusion holds

COROLLARY 2.4.— *There exist a finite number of points $x_{ij} \in \Omega$; $i = 1,2$ $j = 1,...,N_i$ and $\varepsilon > 0$ such that $B(x_{ij}, 2\varepsilon) \subset \Omega$, $B(x_{1j}, 2\varepsilon) \cap B(x_{2j}, 2\varepsilon) = \emptyset$ and $\psi_{i+1} > \psi_i$ (where $\psi_3 = \psi_1$). Denote $\Omega_i = \Omega \cap (\cup_j B(x_{ij}, \varepsilon))^c$, and we take $\varphi_i = e^{\beta\psi_i}$ then for large β the phases φ_i satisfy Hörmander assumption in Ω_i.*

Remark 2.5.— By the previous construction the phases φ_1 and φ_2 attain his globals maximum in the portion Σ_2.

3. Proof of mains results

The main idea of our proofs is to use the boundary Carleman estimate (2.9) in Proposition 2.1 and we find an estimation of the norm of the resolvent $(A - z)^{-1}$ for z in the region

$$\Gamma = \left\{ z \in \mathbb{C}; 0 < Im(z) < C_1 e^{-C_2 |Rez|};\ |z| > 1 \right\},$$

with some constants $C_1, C_2 > 0$. Moreover we prove that

$$\left\| (A - z)^{-1} \right\|_{\mathcal{L}(H,H)} \leqslant C e^{C|Rez|}, \quad C > 0, \quad \forall z \in \Gamma. \tag{3.1}$$

3.1. Proof of Theorem 1.3

Let $f = (f_0, f_1) \in H$ and $u = (u_0, u_1) \in \mathcal{D}(A) = (H^2 \cap H_0^1) \oplus L^2$ such that $(A - z)u = f$ then we have

$$\begin{cases} -zu_0 - iu_1 = f_0 & \text{in } \ M \\ -i\Delta_e u_0 + ia(x)u_1 - zu_1 = f_1 & \text{in } \ M \\ u_0 = 0 & \text{on } \ \partial M \end{cases} \tag{3.2}$$

Then the solution (u_0, u_1) of (3.2) satisfies

$$\begin{cases} (\Delta_e + z^2 - iaz)u_0 = \Phi & \text{in} \quad M \\ u_1 = if_0 + izu_0 & \text{in} \quad M \\ u_0 = 0 & \text{on} \quad \partial M \end{cases} \tag{3.3}$$

where Φ given by

$$\Phi(x, z) = if_1(x) + iaf_0(x) - zf_0(x) \tag{3.4}$$

To prove Theorem 1.3 we need the following result, which is a consequence of proposition 2.1.

LEMMA 3.1.— *There exist a constant $C > 0$, such that for any $(u_0, u_1) \in \mathcal{D}(A)$ solution of (3.2), the following estimate holds*

$$\|u_0\|^2_{H^1_0(M)} \leqslant Ce^{C\tau}\left[\|\Phi(.,z)\|^2_{L^2} + |Imz|^2\|u_0\|^2_{L^2} + \int_M a(x)u_0\overline{u_0}dx\right] \tag{3.5}$$

for $\tau = |Rez|$ large enough, and $|Imz| \leqslant 1$.

Proof. — We need the following notations. First, denote $B_{4r} \subset M$ be a ball of radius $4r > 0$, such that $a(x) > 0$ in B_{4r}, and we set $\Omega_0 = M\backslash\overline{B_r}$. We next introduce the cutoff function $\chi \in C_0^\infty(\mathbb{R}^n)$ by setting

$$\chi(x) = \begin{cases} 1 & \text{in} \quad B^c_{3r} \\ 0 & \text{in} \quad B_{2r}. \end{cases} \tag{3.6}$$

Next, denote $v_0 = \chi u_0$. First of all, by (3.3), one sees that

$$(\Delta_e + z^2 - iaz)v_0 = \chi\Phi + [\Delta_e, \chi]u_0 := \widetilde{\Phi}. \tag{3.7}$$

Thus, for some constant $C > 0$ we have

$$\left\|\widetilde{\Phi}(.,z)\right\|^2_{L^2(M)} \leqslant C\left[\|\Phi(.,z)\|^2_{L^2} + \|[\Delta_e, \chi]\,u_0\|^2_{L^2}\right] \tag{3.8}$$

Moreover using (3.6) we get

$$v_0 = \sigma(v_0).\nu = 0 \quad \text{on} \quad \Sigma_2 := \partial B_r \tag{3.9}$$

Further, by the boundary condition in (3.3), we get

$$v_0 = 0, \quad \text{on} \quad \Sigma_1 := \partial M. \tag{3.10}$$

Let φ_i, $i = 1, 2$, satisfies the conclusion of Proposition 2.3 with

$$\Omega = \Omega_0 := M \backslash \overline{B_r}, \quad \Sigma_2 = \partial B_r, \quad \Sigma_1 = \partial M. \tag{3.11}$$

In order to eliminate the critical points of φ_i (and the failure of the Hörmander condition), let χ_i, $i = 1, 2$, two cutoff functions equal to 1 in $\left(\bigcup_j B(x_{ij}, 2\varepsilon) \right)^c$ and supported in $\left(\bigcup_j B(x_{ij}, \varepsilon) \right)^c$.

By a simple calculus we get for $\tau^2 = Re(z^2)$

$$\begin{cases} (-\Delta_e - \tau^2)(\chi_i v_0) = \Psi_i & \text{in} \quad M \\ \Psi_i = [-\Delta_e, \chi_i] v_0 - \chi_i \widetilde{\Phi} + iIm(z^2) \chi_i v_0 - iaz\chi_i v_0 \end{cases} \tag{3.12}$$

Taking into account the boundary conditions (3.9)-(3.10) and applying Proposition 2.1 with $B = I$ (Dirichlet condition), we arrive at

$$\int_{\Omega_0} e^{2\tau\varphi_i} \left| \Psi_i(x, z) \right|^2 dx \geqslant C\tau \int_{\Omega_0} (\tau^2 \left| \chi_i v_0 \right|^2 + \left| \nabla (\chi_i v_0) \right|^2) e^{2\tau\varphi_i} dx. \tag{3.13}$$

However, by (3.12) we get

$$\begin{aligned} \int_{\Omega_0} e^{2\tau\varphi_i} \Big(\left| [-\Delta_e, \chi_i] v_0 \right|^2 + \left| \widetilde{\Phi}(x, z) \right|^2 + \left| Imz \right|^2 \left| Rez \right|^2 \left| u_0 \right|^2 + \left| z \right|^2 a(x) u_0 \overline{u_0} \Big) dx \\ \geqslant C\tau \int_{\Omega_0} e^{2\tau\varphi_i} (\tau^2 \left| \chi_i v_0 \right|^2 + \left| \nabla (\chi_i v_0) \right|^2) \end{aligned} \tag{3.14}$$

We addition the last two estimates for $i = 1, 2$ and using the properties of the phases $\varphi_i > \varphi_{i+1}$ in $\left(\bigcup_j B(x_{i+1,j}, 2\varepsilon) \right)$ then we can absorb the term $[-\Delta_e, \chi_i] v_0$ at the left hand side of (3.14) into the right hand side for large $\tau > 0$. More precisely, for large enough τ, the following estimate holds

$$\begin{aligned} \int_{\Omega_0} (e^{2\tau\varphi_1} + e^{2\tau\varphi_2}) \Big(\left| \widetilde{\Phi}(x, z) \right|^2 + \left| Imz \right|^2 \left| Rez \right|^2 \left| u_0 \right|^2 + \left| z \right|^2 a(x) u_0 \overline{u_0} \Big) dx \\ \geqslant C\tau \int_{\Omega_0} (\tau^2 \left| v_0 \right|^2 + \left| \nabla v_0 \right|^2)(e^{2\tau\varphi_1} + e^{2\tau\varphi_2}) \end{aligned} \tag{3.15}$$

Consequently, by (3.8), and using $M = \Omega_0 \cup B_{2r}$, we see that for $|Imz| \leqslant 1$, $\tau = |Rez|$, it holds that

$$\begin{aligned} \int_M \left| u_0 \right|^2 + \left| \nabla u_0 \right|^2 dx \leqslant Ce^{C\tau} \Big(\int_M \left| \Phi(x, z) \right|^2 dx + \int_{B_{2r}} \left| u_0 \right|^2 + \left| \nabla u_0 \right|^2 dx \\ + \left| Imz \right|^2 \left| Rez \right|^2 \int_M \left| u_0 \right|^2 dx + \int_M a(x) u_0 \overline{u_0} dx + \int_M \left| [\Delta_e, \chi] u_0 \right|^2 dx \Big) \end{aligned}$$

$$\leqslant Ce^{C\tau}\Big(\|\Phi(.,z)\|_{L^2}^2 + |Imz|^2 \|u_0\|_{L^2}^2 + \int_M a(x)u_0.\overline{u}_0 dx \\ + \int_{\Omega_0} |[\Delta_e,\chi]u_0|^2 dx + \int_{B_{2r}} |\nabla u|^2 dx\Big) \tag{3.16}$$

where we have used $|Rez| = \tau \leqslant e^{C\tau}$.

To accomplish the proof of the lemma we estimate the last two terms in the R.H.S of (3.16). We set $\widetilde{\chi}$ a cutoff function equal to 1 in a neighborhood of B_{3r} and supported in B_{4r} then we have

$$(-I+\Delta_e)\chi u_0 = \widetilde{\chi}(z^2-1+iaz)u_0 + \widetilde{\chi}\Phi + [\Delta_e,\widetilde{\chi}]u_0 \tag{3.17}$$

and hence we get, by elliptic estimates (see for example [19])

$$\begin{aligned} \|u_0\|_{H^1(B_{3r})}^2 &\leqslant C\Big(\|(-1+\Delta_e)\widetilde{\chi}u_0\|_{H^{-1}}^2 + \|u_0\|_{L^2(B_{4r})}^2 \Big) \\ &\leqslant C\Big(\|\Phi(.,z)\|_{L^2}^2 + (1+|z|^2)^2 \|u_0\|_{L^2(B_{4r})}^2 + \int_M a(x)u_0\overline{u_0}\Big)dx \\ &\leqslant C\Big(\|\Phi(.,z)\|_{L^2}^2 + (1+|z|^2)^2 \int_M a(x)u_0\overline{u_0}\Big). \end{aligned} \tag{3.18}$$

Using (3.6) we obtain that $supp(\chi) \subset B_{3r}$ and we deduce from (3.18)

$$\int_{\Omega_0} |[\Delta_e,\chi]u_0|^2 + \int_{B_{2r}} |\nabla u|^2 \leqslant C \|u_0\|_{H^1(B_{3r})}^2 \\ \leqslant C\Big(\|\Phi(.,z)\|_{L^2}^2 + (1+|z|^2)^2 \int_M a(x)u_0\overline{u_0}\Big). \tag{3.19}$$

Collecting (3.19) and (3.16) we obtain

$$\|u_0\|_{H_0^1(M)}^2 \leqslant Ce^{C\tau}\left[\|\Phi(.,z)\|_{L^2}^2 + |Imz|^2 \|u_0\|_{L^2}^2 + \int_M a(x)u_0\overline{u_0}dx \right] \tag{3.20}$$

This complete the proof of lemma 3.1. $\square$

We now turn to the proof of Theorem 1.3. In the next we estimate the last term of inequality (3.5). In all the estimates that follows, we shall indicate by C a universal positive constant, possibly different from line to line, even within the same inequality, depending on n, M and $\|a\|_\infty$, but always independent of τ.

Integrating by parts we obtain

$$\begin{aligned}\int_M \Phi\overline{u}_0 dx &= \int_M (\Delta_e + z^2 - iaz)u_0\overline{u}_0 dx \\ &= \int_M z^2 |u_0|^2 dx + \int_M E(u_0, \overline{u}_0)dx - iz\int_M a(x)u_0\overline{u}_0 dx \quad (3.21)\end{aligned}$$

Keeping only the imaginary part of (3.21), we arrive at the inequality

$$\tau \int_M a(x)u_0\overline{u}_0 dx \leqslant C\left[|Imz|\,|Rez|\,\|u_0\|_{L^2}^2 + \|\Phi(.,z)\|_{L^2}\|u_0\|_{L^2}\right] \quad (3.22)$$

Inserting (3.22) into (3.5), we get for any $\varepsilon > 0$

$$\begin{aligned}\|u_0\|_{H_0^1}^2 &\leqslant Ce^{C\tau}\left[\|\Phi(.,z)\|_{L^2}^2 + \|\Phi(.,z)\|_{L^2}\|u_0\|_{L^2} + |Imz|\,\|u_0\|_{L^2}^2\right] \\ &\leqslant Ce^{C\tau}\left[\|\Phi(.,z)\|_{L^2}^2 + \varepsilon^{-1}\|\Phi(.,z)\|_{L^2}^2 + \varepsilon\|u_0\|_{L^2}^2 + |Imz|\,\|u_0\|_{L^2}^2\right] \quad (3.23)\end{aligned}$$

Taking into account (3.4), then for $|Imz| \leqslant \frac{1}{2C}e^{-C\tau}$ and $\varepsilon = e^{-2C\tau}$, the following inequality holds

$$\|u_0\|_{H_0^1}^2 \leqslant Ce^{C\tau}\Big(\|f_0\|_{H_0^1}^2 + \|f_1\|_{L^2}^2\Big). \quad (3.24)$$

On the ohter hand by (3.3) we can obtain easily

$$\|u_1\|_{L^2}^2 \leqslant Ce^{C\tau}\Big(\|f_0\|_{H_0^1}^2 + \|f_1\|_{L^2}^2\Big). \quad (3.25)$$

Hence (3.25) and (3.24), yield the final inequality

$$\|u\|_H^2 \leqslant C\,\|(A-z)u\|_H^2, \quad \forall z \in \Gamma. \quad (3.26)$$

Which completes the proof.

3.2. Proof of Theorem 1.1

Let $f = (f_0, f_1) \in H$, and $u = (u_0, u_1) \in \mathcal{D}(A_0)$ such that $(A_0 - z)u = f$, which implies further

$$\begin{cases} -zu_0 - iu_1 = f_0 \\ -i\Delta_e u_0 - zu_1 = f_1 \quad \text{in} \quad M \\ \sigma(u_0).\nu + a_0 u_1 = 0 \quad \text{on} \quad \partial M \end{cases} \quad (3.27)$$

then the solution (u_0, u_1) of (3.27) satisfies

$$\left\{\begin{array}{lll}(\Delta_e + z^2)u_0 = \Phi & \text{in} & M \\ \sigma(u_0).\nu + a_0 izu_0 = \Phi_0 & \text{on} & \partial M \\ u_1 = if_0 + izu_0. & & \end{array}\right. \tag{3.28}$$

where Φ and Φ_0 given by

$$\Phi(x, z) = if_1 - zf_0, \quad \Phi_0 = -ia_0(f_0)_{|\partial M} \tag{3.29}$$

To prove Theorem 1.1 we need the following lemma, which, also, is a consequence of Proposition 2.1.

LEMMA 3.2. — *There exist a constant $C > 0$ such that for any $(u_0, u_1) \in \mathcal{D}(A_0)$ solution of (3.28) the following estimate holds*

$$\|u_0\|^2_{H^1(M)} \leqslant Ce^{C\tau}\Big(\|\Phi(.,z)\|^2_{L^2(M)} + \|\Phi_0(.,z)\|^2_{L^2(\partial M)} + |Imz|^2 \|u_0\|^2 + \int_{\partial M} a_0 |u_0|^2\Big) \tag{3.30}$$

for $\tau = |Rez|$ large enough, and $|Imz| \leqslant 1$.

Proof. — Here we are choosing the following partition of $\partial M = \Sigma_1 \cup \Sigma_2$, where

$$\Sigma_2 \subset \Big\{x \in \partial M; a_0(x) > \delta\Big\}, \quad \Sigma_1 = \partial M \backslash \Sigma_2. \tag{3.31}$$

Let φ_i, $i = 1, 2$, satisfies the conclusion of Proposition 2.3 with Σ_1, Σ_2 defined by (3.31). Finally let χ_i, $i = 1, 2$, two cutoff functions equal to 1 in $\Big(\bigcup_j B(x_{ij}, 2\varepsilon)\Big)^c$ and supported in $\Big(\bigcup_j B(x_{ij}, \varepsilon)\Big)^c$ (in order to eliminate the critical points of the phase function φ_i).

By a simple calculus we get for $\tau^2 = Re(z^2)$

$$\left\{\begin{array}{lll}(-\Delta_e - \tau^2)(\chi_i u_0) = \Psi_i & \text{in} & M \\ \Psi_i = [-\Delta_e, \chi_i]u_0 - \chi_i \Phi + iIm(z^2)\chi_i u_0 & & \\ \sigma(u_0).\nu + ia_0\tau u_0 = \Phi_0 + Im(z)a_0 u_0 := \Psi_0 & \text{on} & \partial M\end{array}\right. \tag{3.32}$$

Taking into account the boundary conditions in (3.32) and applying Proposition 2.1 with the boundary condition $B_\tau u_0 = \sigma(u_0).\nu + ia_0\tau u_0$ on Σ_1, we arrive at

$$\begin{aligned}\int_M e^{2\tau\varphi_i}\left|\Psi_i(x,z)\right|^2 dx + \int_{\Sigma_2} e^{2\tau\varphi_i}(\tau^2\left|u_0\right|^2 + \left|\sigma(u_0).\nu\right|^2)dx' \\ + \int_{\Sigma_1} e^{2\tau\varphi_i}\left|\Psi_0\right|^2 dx' \\ \geqslant C\tau\int_M(\tau^2\left|\chi_i v_0\right|^2 + \left|\nabla(\chi_i v_0)\right|^2)e^{2\tau\varphi_i}dx. \end{aligned} \tag{3.33}$$

However, by (3.32) and (3.31) we get

$$\begin{aligned}\int_M e^{2\tau\varphi_i}\Big(\left|[-\Delta_e,\chi_i]u_0\right|^2 \quad + \quad & \left|\Phi\right|^2 + \left|Imz\right|^2\left|Rez\right|^2\left|u_0\right|^2\Big)dx \\ & + \int_{\partial M} e^{2\tau\varphi_i}a_0\left|u_0\right|^2 dx' + \int_{\partial M} e^{2\tau\varphi_i}\left|\Phi_0\right|^2 dx' \\ \geqslant \quad & C\tau\int_M(\tau^2\left|\chi u_0\right|^2 + \left|\nabla(\chi u_0)\right|^2)e^{\tau\varphi_i}dx. \end{aligned} \tag{3.34}$$

We addition the last two estimates for $i=1,2$ and using the properties of the phases $\varphi_i > \varphi_{i+1}$ in $\left(\bigcup_j B(x_{i+1,j},2\varepsilon)\right)$ then we can absorb the term $[-\Delta_e,\chi_i]v_0$ at the left hand side of (3.34) into the right hand side for large $\tau>0$. More precisely we obtain for large enough τ the following inequality

$$\begin{aligned}\int_M(e^{2\tau\varphi_1}+e^{2\tau\varphi_2})\Big(\left|\Phi\right|^2 + \left|Imz\right|^2\left|Rez\right|^2\left|u_0\right|^2\Big)dx \\ + \int_{\partial M}(e^{2\tau\varphi_1}+e^{2\tau\varphi_2})\Big(a_0\left|u_0\right|^2 + \left|\Phi_0\right|^2\Big)dx' \\ \geqslant C\tau\int_M(\tau^2\left|u_0\right|^2 + \left|\nabla u_0\right|^2)(e^{\tau\varphi_1}+e^{2\tau\varphi_2})dx. \end{aligned} \tag{3.35}$$

Consequently we have

$$\begin{aligned}\int_M\left|u_0\right|^2 + \left|\nabla u_0\right|^2 dx \quad \leqslant \quad & Ce^{C\tau}\Big(\left\|\Phi(.,z)\right\|^2_{L^2(M)} + \left\|\Phi_0\right\|^2_{L^2(\partial M)} \\ & + \left|Imz\right|^2\left\|u_0\right\|^2 + \int_{\partial M} a_0\left|u_0\right|^2\Big) \end{aligned} \tag{3.36}$$

This complete the proof of lemma 3.2. $\square$

We now turn to the proof of Theorem 1.1. For this end we estimate the last term of inequality (3.30). In all the estimates that follows, we shall indicate by C a universal positive constant, possibly different from line to line, even within the same inequality, depending on n, M and $\left\|a_0\right\|_\infty$, but always independent of τ.

Integrating by part we get

$$\begin{aligned}\int_M \Phi\overline{u}_0 dx &= \int_M (\Delta_e + z^2)u_0\overline{u}_0 dx \\ &= z^2\int_M |u_0|^2 dx + \int_M E(u_0,\overline{u}_0)dx + \int_{\partial M}\sigma(u_0).\nu\overline{u}_0 dx' \\ &= z^2\|u_0\|^2_{L^2} + \int_M E(u_0,\overline{u}_0)dx \\ &\quad -iz\int_{\partial M} a_0(x)|u_0|^2 dx' + \int_{\partial M}\Phi_0.\overline{u}_0 dx' \end{aligned} \tag{3.37}$$

Keeping only the imaginary part of (3.37), we arrive at the inequality

$$\begin{aligned}\tau\int_{\partial M} a_0(x)|u_0|^2 d\sigma \leqslant C\Big(&\|\Phi(.,z)\|_{L^2}\|u_0\|_{L^2} + |Imz|\,|Rez|\,\|u_0\|^2_{L^2} \\ &+ \|\Phi_0\|_{H^1(M)}\|u_0\|_{H^1(M)}\Big)\end{aligned} \tag{3.38}$$

where terms on the boundary have been bounded using the trace theorem.

Combining (3.38) with (3.30) we obtain

$$\begin{aligned}\|u_0\|^2_{H^1(M)} \leqslant\; & Ce^{C\tau}\Big(\|\Phi(.,z)\|^2_{L^2(M)} + \|\Phi_0\|^2_{H^1(M)} + \|\Phi(.,z)\|_{L^2}\|u_0\|_{L^2} \\ &+ \|\Phi_0\|_{H^1}\|u_0\|_{H^1} + |Imz|\,\|u_0\|^2_{L^2}\Big).\end{aligned} \tag{3.39}$$

Now assume that $|Imz| \leqslant \frac{1}{2C}e^{-C\tau}$ then we have by the previous estimate and (3.29)

$$\|u_0\|^2_{H^1} \leqslant Ce^{C\tau}\Big(\|f_0\|^2_{H^1} + \|f_1\|^2_{L^2}\Big). \tag{3.40}$$

Using (3.28) we get

$$\|u_1\|^2_{L^2} \leqslant \|f_1\|^2_{L^2} + |z|^2\|u_0\|^2_{L^2} \leqslant Ce^{C\tau}\|f\|^2_H \tag{3.41}$$

and hence

$$\|u\|^2 \leqslant Ce^{C|Rez|}\|(A_0 - z)\|^2 \tag{3.42}$$

then $(A_0 - z)$ is injective then bijective in $\mathcal{D}(A_0)$ and we get

$$\left\|(A_0 - z)^{-1}\right\|_{\mathcal{L}(H,H)} \leqslant Ce^{-C|Rez|} \tag{3.43}$$

for any $z \in \Big\{z \in \mathbb{C},\quad |Imz| < \frac{1}{C}e^{-C|Rez|};\quad |Rez| > 1\Big\}$. This complete the proof.

4. Proof of Carleman estimate

The aim of this section is to prove the estimate of Carleman's type near the boundary for the solutions of the following boundary value problem

$$\begin{cases} P_\tau(x,D)u = f & \text{in} \quad \Omega \\ B_\tau(x,D)u = g & \text{on} \quad \partial\Omega \end{cases} \tag{4.1}$$

where $P_\tau(x,D)$ a partial differential operator with principal symbol given by

$$P_\tau(x,\xi) = \mu|\xi|^2 + (\mu+\lambda)\xi^t\xi - \tau^2 Id.$$

We prove Propsition 2.1 only in the case of Neumann boundary condition $B(x,D)u = \sigma(u).\nu + ia_0\tau u$. The case of the Dirichlet condition can be proved in the some way and is much simpler (see [3]).

We define the sobolev spaces with a parameter τ, H^s_τ by

$$u(x,\tau) \in H^s_\tau \quad \Longleftrightarrow \quad <\xi,\tau>^s \hat{u}(\xi,\tau) \in L^2.$$

$\hat{u}$ denoted the partial Fourier transform with respect to x.

We introduce the following norms in the Sobolev spaces $H^k(\Omega)$ and $H^k(\partial\Omega)$

$$\|u\|^2_{k,\tau} = \sum_{j=0}^{k} \tau^{2(k-j)} \|u\|^2_{H^j(\Omega)} \qquad |u|^2_{k,\tau} = \sum_{j=0}^{k} \tau^{2(k-j)} |u|^2_{H^j(\partial\Omega)}$$

and we set

$$|u|^2_{1,0,\tau} = |u|^2_{1,\tau} + |\sigma(u).\nu|^2.$$

For a differential operator

$$P(x,D,\tau) = \sum_{|\alpha|+k\leqslant m} a_{\alpha,k}(x)\tau^k D^\alpha,$$

we note the associated symbol by

$$p(x,\xi,\tau) = \sum_{|\alpha|+k\leqslant m} a_{\alpha,k}(x)\xi^\alpha\tau^k,$$

we define the class of symbols of order m by

$$S^m_\tau = \left\{ a(x,\xi,\tau) \in C^\infty,\ |D^\alpha_x D^\beta_\xi a(x,\xi,\tau)| \leqslant C_{\alpha,\beta} <\xi,\tau>^{m-|\beta|} \right\},$$

and the class of tangential symbols of order m by

$$\mathcal{TS}_\tau^m = \Big\{a(x,\xi',\tau) \in C^\infty,\ |D_x^\alpha D_{\xi'}^\beta a(x,\xi',\tau)| \leqslant C_{\alpha,\beta} < \xi',\tau >^{m-|\beta|}\Big\}.$$

We shall frequently use the symbol $< \xi', \tau >= \sqrt{|\xi'|^2 + \tau^2} = \Lambda$.

Finally let $a(x,\xi',\tau) \in \mathcal{TS}_\tau^2$ such that

$$a(x,\xi',\tau) + a(x,\xi',\tau)^* \geqslant C < \xi',\tau >^2,$$

then we have the following Gårding estimate

$$Re(A(x,D',\tau)u,u) \geqslant C\tau^2 \|u\|^2. \tag{4.2}$$

4.1. Reduction of the problems

4.1.1. Reduction of the Laplacian

Let Ω be a bounded smooth domain of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^∞. In a neighborhood of a given $x_0 \in \partial\Omega$ we denote by $x = (x', x_n)$ the system of normal geodesic coordinates where $x' \in \partial\Omega$ and $x_n \in \mathbb{R}$ are characterized by

$$|x_n| = dist(x,\partial\Omega);\ \Omega = \big\{x_n > 0\big\};\ \mathrm{dist}(x',x) = \mathrm{dist}(x,\partial\Omega), \\ \nu = (0,...,0,-1) \tag{4.3}$$

In this system of coordinates there exist $\ell(x,\xi)$ such that

$$\ell(x,\xi) = \ell_0(x)\xi_n + \ell_1(x,\xi') \tag{4.4}$$

and the principal symbol of Laplace operator takes the form

$$\widetilde{a}(x,\xi) = {}^t\ell(x,\xi)\ell(x,\xi) = \xi_n^2 + r(x,\xi') \tag{4.5}$$

where $r(x,\xi')$ is a quadratic form, such that there exists $C > 0$

$$r(x,\xi') \geqslant C|\xi'|^2, \quad \text{for any} \quad x \in K, \quad \xi' \in T^*(\partial\Omega) \tag{4.6}$$

where K is a fixed compact in $\overline{\Omega}$. Finally we have the vectors field ℓ_0, ℓ_1 satisfying

$${}^t\ell_1\ell_1 = r(x,\xi'), \quad {}^t\ell_0\ell_1 = 0 \tag{4.7}$$

Let $\varphi(x)$ be a $C^\infty(\mathbb{R}^n)$ function with values in $\mathbb{R}$, defined in a neighborhood of K. We define the operator

$$a(x,D,\tau) = e^{\tau\varphi}\widetilde{a}(x,D)e^{-\tau\varphi} := op(a) \tag{4.8}$$

Denote by

$$a(x,\xi,\tau) = a(x,\xi + i\tau\varphi'), \quad \varphi' = \nabla\varphi, \quad \partial_\nu\varphi = \nabla\varphi.\nu = -\varphi'_{x_n} \tag{4.9}$$

the principal symbol of the operator, and we set

$$op(\widetilde{q}_2) = \frac{1}{2}\Big(op(a) + op(a)^*\Big); \qquad op(\widetilde{q}_1) = \frac{1}{2i}\Big(op(a) - op(a)^*\Big),$$

its real and imaginary part. Then we have

$$\begin{cases} op(a) = op(\widetilde{q}_2) + iop(\widetilde{q}_1) \\ \widetilde{q}_2 = \xi_n^2 + q_2(x,\xi',\tau); \qquad \widetilde{q}_1 = 2\tau\xi_n\varphi'_{x_n} + 2q_1(x,\xi',\tau). \end{cases} \tag{4.10}$$

Where $q_1 \in \mathcal{TS}_\tau^1$ and $q_2 \in \mathcal{TS}_\tau^2$ are tangential symbols given by

$$\begin{cases} q_2(x,\xi,\tau) = r(x,\xi') - (\tau\varphi'_{x_n})^2 - \tau^2 r(x,\varphi'_{x'}) \\ q_1(x,\xi,\tau) = \widetilde{r}(x,\xi',\tau\varphi'_{x'}) \end{cases} \tag{4.11}$$

and $\widetilde{r}(x,\xi',\eta')$ the bilinear form attached to the quadratic form $r(x,\xi')$.

4.1.2. Reduction of the elasticity system

In the system of normal geodesic coordinates the principal symbol of elasticity operator can be written as

$$L(x,\xi) = \mu\,{}^t\ell(x,\xi)\ell(x,\xi)Id + (\lambda+\mu)\ell(x,\xi)^t\ell(x,\xi) \tag{4.12}$$

where $\ell(x,\xi)$ defined by (4.4) and $\ell(x,\xi)^t\ell(x,\xi)$ the orthogonal projection onto the space spanned by $\ell(x,\xi)$. The principal symbol of $P(x,D,\tau) = (e^{\tau\varphi}L(x,D)e^{-\tau\varphi} - \tau^2 Id)$ is given by

$$P(x,\xi,\tau) = \sum_{j=0}^{2} P_{2-j}(x,\xi',\tau)\xi_n^j \tag{4.13}$$

where $P_j(x,\xi',\tau) \in \mathcal{TS}_\tau^j$ a tangential symbols defined by

$$\begin{cases} P_0 = \mu Id + (\mu+\lambda)\ell_0^t\ell_0 \\ P_1 = 2i\tau\varphi'_{x_n}P_0 + (\mu+\lambda)(\ell_0^t\ell_1(x,\xi'+i\tau\varphi'_{x'}) + \ell_1(x,\xi'+i\tau\varphi'_{x'})^t\ell_0) \\ P_2 = \mu\ell_1^t\ell_1 + i\tau\varphi'_{x_n}P_1(x,\xi') - (\tau\varphi'_{x_n})^2P_0 + (\mu+\lambda)\ell_1^t\ell_1 - \tau^2 Id. \end{cases}$$

For a fixed $(x,\xi') \in T^*(\partial\Omega)$ let $\alpha(x,\xi',\tau) \in \mathbb{C}$ such that

$$a(x,\xi,\tau) = (\xi_n + i\tau\varphi'_{x_n} + i\alpha)(\xi_n + i\tau\varphi'_{x_n} - i\alpha) \tag{4.14}$$

then we have also by (4.7)

$$r(x,\xi' + i\tau\varphi'_{x'}) = -(i\alpha)^2 = {}^t\ell(x,\xi' + i\tau\varphi'_{x'})\ell(x,\xi' + i\tau\varphi'_{x'}). \tag{4.15}$$

the determinant of $P(x,\xi,\tau)$ is given by (see [3])

$$det(P(x,\xi,\tau)) = \mu^{n-1}(2\mu+\lambda)a_\mu(x,\xi,\tau)^{n-1}a_{2\mu+\lambda}(x,\xi,\tau).$$

where $a_\gamma(x,\xi,\tau) = a(x,\xi,\tau) - \frac{\tau^2}{\gamma}$.

4.2. Study of the eigenvalues

The proof of Carleman estimate rely on a cutting argument based on the nature of the roots with respect ξ_n of $a_\gamma(x,\xi',\xi_n,\tau)$.

Let use now introduce the following micro-local regions:

(i) $$\mathcal{E}^+ = \left\{(x,\xi',\tau) \in K \times S^{n-1};\, q_2 - \frac{\tau^2}{\mu} + \frac{q_1^2}{(\tau\varphi'_{x_n})^2} > 0\right\},$$

(ii) $$\mathcal{Z}_\gamma = \left\{(x,\xi',\tau) \in K \times S^{n-1};\, q_2 - \frac{\tau^2}{\gamma} + \frac{q_1^2}{(\tau\varphi'_{x_n})^2} = 0\right\},$$

(iii) $$\mathcal{E}^- = \left\{(x,\xi',\tau) \in K \times S^{n-1};\, q_2 - \frac{\tau^2}{2\mu+\lambda} + \frac{q_1^2}{(\tau\varphi'_{x_n})^2} < 0\right\},$$

(iv)
$$\mathcal{M} = \left\{(x,\xi',\tau) \in K\times S^{n-1};\, q_2 - \frac{\tau^2}{\mu} + \frac{q_1^2}{(\tau\varphi'_{x_n})^2} < 0 < q_2 - \frac{\tau^2}{2\mu+\lambda} + \frac{q_1^2}{(\tau\varphi'_{x_n})^2}\right\}.$$

And for fixed (x,ξ',τ) let $\alpha_\gamma(x,\xi',\tau) \in \mathbb{C}$ such that

$$\begin{aligned} a_\gamma(x,\xi,\tau) &= a(x,\xi,\tau) - \frac{\tau^2}{\gamma} \\ &= (\xi_n + i\tau\varphi'_{x_n} + i\alpha_\gamma)(\xi_n + i\tau\varphi'_{x_n} - i\alpha_\gamma). \end{aligned} \tag{4.16}$$

Taking into account (4.16) and (4.14) we get

$$(i\alpha_\gamma)^2 = (i\alpha)^2 + \frac{\tau^2}{\gamma}. \tag{4.17}$$

then we get by (4.15) and (4.11)

$$\alpha_\gamma^2 = (\tau\varphi'_{x_n})^2 + q_2 - \frac{\tau^2}{\gamma} + 2i\tau q_1 \tag{4.18}$$

For a fixed (x, ξ', τ) we decompose $a_\gamma(x, \xi, \tau)$ as polynomial in ξ_n. Then the following Lemma holds

LEMMA 4.1.— *We have the following:*

1. *For any $(x, \xi', \tau) \in \mathcal{E}^+$ the roots $z_1^\pm, z_2^\pm$ of a_μ and $a_{2\mu+\lambda}$ as a polynomial with respect to ξ_n satisfy $\pm Imz_k^\pm > 0$.*
2. *For any $(x, \xi', \tau) \in \mathcal{Z}_\gamma$, one of the roots of a_γ is real and the other root lies in the upper half-plane if $\varphi'_{x_n} < 0$ (resp. in the lower half-plane if $\varphi'_{x_n} > 0$).*
3. *For any $(x, \xi', \tau) \in \mathcal{E}^-$ the roots of a_γ for $\gamma \in \{\mu, 2\mu + \lambda\}$ are in the upper half-plane if $\varphi'_{x_n} < 0$ (resp. in the lower half-plane if $\varphi'_{x_n} > 0$).*
4. *For any $(x, \xi', \tau) \in \mathcal{M}$ the roots of $a_{2\mu+\lambda}$ satisfy $\pm Imz_2^\pm > 0$ and the roots of a_μ satisfy 3).*

Proof. — The imaginary parts of the roots of a_γ are

$$-\tau\varphi'_{x_n} \pm Re(\alpha_\gamma)$$

but the lines $Re(z) = \pm\tau\varphi'_{x_n}$ can be transformed by $z \to z' = z^2$ in the parabola defined by $Rez' = (\tau\varphi'_{x_n})^2 - \frac{|Imz'|^2}{4(\tau\varphi'_{x_n})^2}$. Then the Lemma is proved if we change z' by α_γ^2 and using (4.18) $\square$

4.3. Proof of the Carleman estimate in the region $\mathcal{E}^+$

The purpose of this section is to proof the Carleman estimate in the region $\mathcal{E}^+$. In this region we can not need any assumption for φ'_{x_n}, indeed by lemma 4.1 the signes of the imaginary parts of the roots are independent of the signe of φ'_{x_n}. Here we us the methods of projectors of Calderon to show the following proposition. Let $\chi(x, \xi', \tau)$ be a homogeneous cutoff function of degree 0 in the region $\mathcal{E}^+$, then we have the following result

PROPOSITION 4.2.— *There exist $C > 0$ such that for any large enough τ the following inequality holds*

$$\|P(x, D, \tau)u\|^2 + \tau\,|B(x, D, \tau)u|_{0,\tau}^2 + \|u\|_{1,\tau}^2 + |u|_{1,\tau}^2 \geqslant C\tau\Big(\|op(\chi)u\|_{1,\tau}^2 + \tau\,|op(\chi)u|_{1,\tau}^2\Big) \tag{4.19}$$

whenever $u \in C_0^\infty(K)$.

4.3.1. Preliminaries

We consider the boundary operator $B(x,D)u = \sigma(u).\nu + a_0 i\tau u$, and the conjugate operator $e^{\tau\varphi}B(x,D)e^{-\tau\varphi}$ with principal symbol given by (see [18] and [1])

$$B(x,\xi,\tau) = B_0(x)\xi_n + B_1(x,\xi',\tau) \tag{4.20}$$

where $B_0 \in \mathcal{TS}^0_\tau$ and $B_1 \in \mathcal{TS}^1_\tau$ two tangential symbols given by

$$\begin{cases} B_0 = \mu Id + (\mu+\lambda)\ell_0^t\ell_0 \\ \\ B_1 = i\tau\varphi'_{x_n}B_0 + \mu\ell_1^t\ell_0 + \lambda\ell_0^t\ell_1 - a_0\tau Id. \end{cases} \tag{4.21}$$

Let $u \in C_0^\infty(K)$, denote

$$\widetilde{u} = op(\chi)u \qquad \widetilde{f} = op(P)\widetilde{u} \tag{4.22}$$

where $op(P)$ is the differential operator with principal symbol

$$P(x,\xi,\tau) = P_0(x)\xi_n^2 + P_1(x,\xi',\tau)\xi_n + P_2(x,\xi',\tau)$$

It is easy to see that

$$\begin{cases} op(P)\widetilde{u} = \widetilde{f} & \text{in} \quad x_n > 0 \\ \\ op(B)\widetilde{u} = \widetilde{g} & \text{on} \quad x_n = 0 \end{cases} \tag{4.23}$$

where

$$\widetilde{f} = op(\chi)f + [op(P);op(\chi)]u \quad \text{and} \quad \widetilde{g} = op(\chi)g + [op(B);op(\chi)]u \tag{4.24}$$

It is helpful to replace the system (4.23) by an equivalent first order system. Put

$$v = {}^t\Big(\langle D',\tau\rangle\,\tilde{u}, D_n\tilde{u}\Big) \tag{4.25}$$

then the system (4.23) is reduced to the following system

$$\begin{cases} D_n v - op(\mathcal{A})v = F & \text{in} \quad x_n > 0 \\ \\ op(\mathcal{B})v = \widetilde{g} & \text{on} \quad x_n = 0 \end{cases} \tag{4.26}$$

Where the principal symbol of $op(\mathcal{A})$ is given by

$$\mathcal{A} = \begin{pmatrix} 0 & -\Lambda \\ \\ -\Lambda^{-1}P_0^{-1}P_2 & -P_0^{-1}P_1 \end{pmatrix}$$

and $\mathcal{B} \in \mathcal{TS}^0_\tau$ is the tangential symbol

$$\mathcal{B} = (\Lambda^{-1}B_1, B_0) \tag{4.27}$$

with B_0 and B_1 are defined by (4.21) and $F = {}^t(0, P_0^{-1}\tilde{f})$.

For (x_0, ξ_0', τ_0) fixed in supp(χ) then the eigenvalues of the matrix $\mathcal{A}$ are $z_1^\pm, z_2^\pm$ with multiplicity respectively $(n-1)$ and 1 defined by

$$z_1^\pm = -i\tau\varphi'_{x_n} \pm i\alpha_\mu, \quad z_2^\pm = -i\tau\varphi'_{x_n} \pm i\alpha_{2\mu+\lambda}, \quad \pm Im(z_k^\pm) > 0. \tag{4.28}$$

Denote by $S = (s_1^+, ..., s_n^+, s_1^-, ..., s_n^-)$ the matrix of the eigenvectors of the matrix $\mathcal{A}(x_0, \xi_0', \tau_0)$ corresponding to eigenvalues with positive or non positive imaginary parts. Then we can extended S as a smooth positively homogeneous function of degree zero in a small conic neighborhood of (x_0, ξ_0', τ_0) . Let $S(x, D_{x'}, \tau)$ the pseudo-differential operator with principal symbol $S(x, \xi', \tau)$. Then by the argument in Taylor [17] (see also Yamamoto [18]) there exist a pseudo-differential matrix operator $K(x, D_{x'}, \tau)$ of order -1 such that the boundary value problem (4.26) is reduced to the following

$$\begin{cases} D_n w - op(\mathcal{H})w = \psi & \text{in} \quad x_n > 0 \\ \\ op(\widetilde{\mathcal{B}})w = \widetilde{g} & \text{on} \quad x_n = 0 \end{cases} \tag{4.29}$$

where

$$w = (I+K)^{-1}S^{-1}v, \quad \psi = (I+K)^{-1}S^{-1}F, \quad \widetilde{\mathcal{B}} = \mathcal{B}S(I+K)^{-1} \tag{4.30}$$

and $op(\mathcal{H})$ is a tangential operator of order 1 with principal symbol $\mathcal{H} = diag(\mathcal{H}^+, \mathcal{H}^-) \in \mathcal{TS}^1_\tau$ which satisfies

$$\pm Im(\mathcal{H}^\pm) \geqslant C\Lambda I_n. \tag{4.31}$$

We can now state the Lemma which will be the key ingredient in the proof of the Proposition 4.2

LEMMA 4.3.— *There exist $C_1, C_2 > 0$ and a tangential symbols $\mathcal{R}(x, \xi', \tau) \in \mathcal{TS}^0_\tau$, $e(x, \xi', \tau) \in \mathcal{TS}^1_\tau$ such that*

$$Im(\mathcal{RH}) = e(x, \xi', \tau)$$

and we have the following properties

(i) $e(x, \xi', \tau) \geqslant C_1\Lambda I_{2n}$ *in* $supp(\chi)$

(ii) $-\mathcal{R}(x, \xi', \tau) + C_2\widetilde{\mathcal{B}}^*\widetilde{\mathcal{B}} \geqslant C_2 I_{2n}$ *on* $\{x_n = 0\} \cap supp(\chi)$

Proof. — Let $\widetilde{\mathcal{B}}$ the principal symbol of $op(\widetilde{\mathcal{B}})$ where $\widetilde{\mathcal{B}} = (\widetilde{\mathcal{B}}^+, \widetilde{\mathcal{B}}^-)$, and $\widetilde{\mathcal{B}}^+ = \widetilde{\mathcal{B}}S^+$ the projection of $\widetilde{\mathcal{B}}$ on the subspace generated by the vectors of S^+. First we prove that $\widetilde{\mathcal{B}}^+$ is an isomorphism.

Let $X = (X_1, X_2) \in \mathbb{C}^n \oplus \mathbb{C}^n$ be an eigenvector of $\mathcal{A}$ associated to z_0. Then X satisfy

$$\Lambda X_2 = z_0 X_1 \quad P(z_0).X_1 = 0. \tag{4.32}$$

a-Calculus of eigenvector associated to z_1^+:

Denote by $\left\{\omega_j^+\right\}_{j=1,\dots,n-2}$ a basis of $\left\{\ell_0, \Lambda^{-1}\ell_1\right\}^\perp$. Then we have

$$P(z_1^+)\omega_j^+ = 0 \qquad \text{for} \quad j \in \left\{1, \dots, n-2\right\}$$

where

$$P(z_1^+) = (\mu + \lambda)(\ell_1 + i\alpha_\mu \ell_0)^t(\ell_1 + i\alpha_\mu \ell_0)$$

Now we set the following vector in $\mathbb{C}^n$

$$\omega_{n-1}^+ = \Lambda^{-2}((i\alpha)^2\ell_0 + i\alpha_\mu \ell_1) \tag{4.33}$$

then we have by a simple calculation

$$P(z_1^+)\omega_{n-1}^+ = 0$$

b-Calculus of eigenvector associated to z_2^+:

We get

$$P(z_2^+) = -\frac{\mu+\lambda}{2\mu+\lambda}\tau^2 Id + (\mu+\lambda)(\ell_1 + i\alpha_{2\mu+\lambda}\ell_0)^t(\ell_1 + i\alpha_{2\mu+\lambda}\ell_0). \tag{4.34}$$

Let now

$$\omega_n^+ = \Lambda^{-1}((i\alpha_{2\mu+\lambda})\ell_0 + \ell_1) \tag{4.35}$$

then we have $P(z_2^+)\omega_n^+ = 0$, with reference to (4.32) we denote

$$\begin{cases} s_j^+ = (\omega_j^+; \Lambda^{-1}z_1^+\omega_j^+), \quad j \in \left\{1, \dots, n-1\right\} \\ s_n^+ = (\omega_n^+; \Lambda^{-1}z_2^+\omega_n^+) \end{cases} \tag{4.36}$$

and by $(b_1^+, \dots, b_n^+, b_1^-, \dots, b_n^-)$ the principal symbol of $\widetilde{\mathcal{B}}$. By elementary calculus we get (we refer to Appendix at the end of this paper for the sketch of the proof) $\widetilde{\mathcal{B}}^{+} = \widetilde{\mathcal{B}}S^{+} = (b_1^+, ..., b_n^+)$ where

$$\begin{cases} b_j^+ = \Lambda^{-1}(\mu(i\alpha_\mu) - a_0\tau)\omega_j^+ \\ b_{n-1}^+ = \Lambda^{-3}[2\mu(i\alpha_\mu) - a_0\tau)](i\alpha)^2\ell_0 + \Lambda^{-3}[2\mu(i\alpha)^2 - a_0\tau(i\alpha_\mu) + \tau^2]\ell_1 \\ b_n^+ = \Lambda^{-2}[2\mu(i\alpha)^2 - a_0\tau(i\alpha_{2\mu+\lambda}) + \tau^2]\ell_0 + \Lambda^{-2}[2\mu(i\alpha_{2\mu+\lambda}) - a_0\tau)]\ell_1. \end{cases} \tag{4.37}$$

Then we get (see also Appendix)

$$det(\widetilde{\mathcal{B}}^{+}) = (\mu(i\alpha_\mu) - a_0\tau)^{n-2}R_0\Big(\frac{\sqrt{\mu}}{\tau}\alpha(x,\xi',\tau)\Big) \tag{4.38}$$

where R_0 is a function given by

$$\begin{aligned} R_0(s) \;&=\; (1-2s^2)^2 + (\frac{a_0^2}{\mu} - 4s^2)\left(\sqrt{s^2-1}\sqrt{s^2 - \frac{\mu}{2\mu+\lambda}}\right) \\ &\quad -\frac{a_0^2}{\mu}s^2 + \frac{ia_0}{\sqrt{\mu}}(1-2s^2)\left(\sqrt{s^2-1} + \sqrt{s^2 - \frac{\mu}{2\mu+\lambda}}\right). \end{aligned} \tag{4.39}$$

We define the characteristic manifold by

$$\mathcal{Q} = \left\{(x',\xi',\tau) \in T^*(\partial\Omega), \quad R_0\Big(\frac{\sqrt{\mu}}{\tau}\alpha(x,\xi',\tau)\Big) = 0\right\}.$$

Therefore $\widetilde{\mathcal{B}}^{+}$ is elliptic outside $\mathcal{Q}$.

Let z_0 a root of R_0 such that $\frac{\sqrt{\mu}}{\tau}\alpha(x,\xi',\tau) = z_0$ then we have

$$\begin{aligned} \alpha^2(x,\xi',\tau) \;&=\; (\tau\varphi'_{x_n})^2 + q_2 + 2i\tau q_1 \\ &=\; \frac{\tau^2}{\mu}z_0^2 \end{aligned} \tag{4.40}$$

then we have in the region $\mathcal{E}^+$.

$$\frac{\tau^2}{\mu}Rez_0^2 - (\tau\varphi'_{x_n})^2 - \frac{\tau^2}{\mu} + \frac{(Imz_0^2)^2}{4(\varphi'_{x_n}}\tau)^2 > 0. \tag{4.41}$$

which is again a contradiction if we choose $\varphi'_{x_n} > C_0$ for large $C_0 > 0$. Then $\mathcal{E}^+ \cap \mathcal{Q} = \emptyset$, and $\mathcal{B}^+$ is an isomorphism in the elliptic region $\mathcal{E}^+$.

Now let us search that the symetrizer $\mathcal{R}$ in the form $diag(Id, -\rho Id)$, $\rho > 0$, then we have

$$\begin{aligned} Im(\mathcal{R}\mathcal{H}) &= diag\Big(Im(\mathcal{H}^+), -\rho Im(\mathcal{H}^-)\Big) \\ &:= e(x,\xi',\tau) \end{aligned} \tag{4.42}$$

and by (4.31) we obtain

$$e(x,\xi',\tau) \geqslant C\Lambda I_{2n}.$$

which prove the first part of lemma 4.3.

Let now $w = (w^+, w^-) \in \mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$, then we have $\widetilde{\mathcal{B}}w = \widetilde{\mathcal{B}}^+ w^+ + \widetilde{\mathcal{B}}^- w^-$, since $\widetilde{\mathcal{B}}^+$ is an isomorphism, then there exist $C > 0$, such that for any $w \in \mathbb{C}^{2n}$ we get

$$\left|w^+\right|^2 \leqslant C\left|\widetilde{\mathcal{B}}^+ w^+\right|^2 \leqslant C\left[\left|\widetilde{\mathcal{B}}^- w^-\right|^2 + \left|\widetilde{\mathcal{B}}w\right|^2\right] \leqslant C\left[\left|w^-\right|^2 + \left|\widetilde{\mathcal{B}}w\right|^2\right] \tag{4.43}$$

we deduce

$$\begin{aligned} -(\mathcal{R}w, w) &= -\left|w^+\right|^2 + \rho\left|w^-\right|^2 \\ &\geqslant \left|w^+\right|^2 + (\rho - 2C)\left|w^-\right|^2 - 2C\left|\widetilde{\mathcal{B}}w\right|^2. \end{aligned} \tag{4.44}$$

Then we have the desirable estimation, for large ρ. □

4.3.2. Proof of Proposition 4.2

We need the following notations. First, denote by $\langle .,. \rangle$ the product scalar in $\mathbb{R}^{n-1}$ and $|.|$ the associated norm. Let G be a function defined by

$$G(x_n) = \frac{d}{dx_n}\Big[\langle op(\mathcal{R})w, w\rangle\Big]. \tag{4.45}$$

Using $D_n w - op(\mathcal{H})w = \psi$ then we obtain

$$G(x_n) = -2Im\,\langle op(\mathcal{R})op(\mathcal{H})w, w\rangle - 2Im\,\langle op(\mathcal{R})w, \psi\rangle + \langle op(\mathcal{R}'_{x_n})w, w\rangle \tag{4.46}$$

The integration in the normal direction gives

$$\begin{aligned} \langle op(\mathcal{R})w, w\rangle_{x_n=0} &= 2\int_0^\infty Im\,\langle op(\mathcal{R})op(\mathcal{H})w, w\rangle\, dx_n \\ &+ 2Im\int_0^\infty \langle op(\mathcal{R})w, \psi\rangle\, dx_n - \int_0^\infty \langle op(\mathcal{R}'_{x_n})w, w\rangle\, dx_n \end{aligned} \tag{4.47}$$

Taking into account Lemma 4.3, and $\tau \leqslant \Lambda$, we obtain by Gårding inequality

$$\int_0^\infty Im\,\langle op(\mathcal{R})op(\mathcal{H})w, w\rangle\, dx_n \geqslant C\tau \int_0^\infty \|w(x_n)\|^2_{L^2(\mathbb{R}^{n-1})}\, dx_n = C\tau\,\|w\|^2 \tag{4.48}$$

Moreover by lemma 4.3 (ii) we obtain

$$\langle op(\mathcal{R})w, w\rangle_{x_n=0} \leqslant C\left|\widetilde{\mathcal{B}}w\right|^2 - C'\,|w|^2 . \tag{4.49}$$

Furthermore, for any $\varepsilon > 0$, we have

$$\int_0^\infty |\langle op(\mathcal{R})w, \psi\rangle|\, dx_n \leqslant \varepsilon\tau\,\|w\|^2 + \frac{C_\varepsilon}{\tau}\,\|\psi\|^2 \tag{4.50}$$

Collecting (4.50), (4.49) and (4.48), (4.47) yields, for large enough τ and ε small

$$C\tau(\|w\|^2 + |w|^2) \leqslant \|\psi\|^2 + \tau\left|\widetilde{\mathcal{B}}w\right|^2 . \tag{4.51}$$

We now turn to the proof of proposition 4.2.

Using that (4.25)-(4.30) and the ellipticity of the tangential operator $(I+K)^{-1}S^{-1}$ we obtain

$$\begin{aligned}\|\widetilde{u}\|^2_{1,\tau} + |\widetilde{u}|^2_{1,\tau} = \|v\|^2 + |v|^2 &\leqslant C\left[\left\|(I+K)^{-1}S^{-1}v\right\|^2 + \left|(I+K)^{-1}S^{-1}v\right|^2\right]\\ &\leqslant C\left[\|w\|^2 + |w|^2\right]\end{aligned} \tag{4.52}$$

Collecting (4.52)-(4.51), (4.30) yields

$$\begin{aligned}\|\widetilde{u}\|^2_{1,\tau} + |\widetilde{u}|^2_{1,\tau} &\leqslant \frac{C}{\tau}\left[\|\psi\|^2 + |\mathcal{B}v|^2\right]\\ &\leqslant \frac{C}{\tau}\Big(\|P(x,D,\tau)u\|^2 + \tau\,|B(x,D,\tau)u|^2_{0,\tau}\\ &\qquad + \|u\|^2_{1,\tau} + |u|^2_{1,\tau}\Big)\end{aligned} \tag{4.53}$$

This complete the proof of Proposition 4.2.

Remark 4.4.— In the elliptic region $\mathcal{E}^+$, τ can be degenerate to zero. But for $\tau = 0$ the problem $L(x,D)u = \mu\Delta u + (\mu+\lambda)\nabla(\mathrm{div} u)$ with Neumann boundary condition $N(u) = \sigma(u).\nu$ is a coercive elliptic boundary problem, and we can prove the proposition 4.2 in this case (see [3]).

4.4. Carleman estimate outside $\mathcal{E}^+$

To prove a precise Carleman estimates outside $\mathcal{E}^+$ we invoque the following estimate proved in [3] where we have controlled the norm $H^1_\tau(\Omega)$ by the all traces terms on the all boundary.

PROPOSITION 4.5. — *Let φ satisfy (2.8), with respect $P(x, D, \tau)$. Then there exists $C > 0$, and τ_0 such that for any $u \in C^\infty(\overline{\Omega})$ we have*

$$\int_\Omega |P(x, D, \tau)u|^2 \, dx + \tau \int_{\partial\Omega} (\tau^2 |u|^2 + |\nabla u|^2) dx' \geqslant C\tau \int_\Omega (\tau^2 |u|^2 + |\nabla u|^2) dx \tag{4.54}$$

for large $\tau > \tau_0$.

Moreover outside $\mathcal{E}^+$ we estimate the tangential derivative by the trace of displacement. This is possible in this region because τ and $< \xi', \tau >$ are equivalent. More precisely we have the following Lemma.

LEMMA 4.6. — *Let $\chi(x, \xi', \tau)$ be a cutoff function homogeneous of degree zero in the regions $\mathcal{Z}_\gamma \cup \mathcal{E}^- \cup \mathcal{M}$. Then there exist a constant $C > 0$ such that*

$$|op(\chi)u|_{1,\tau} \leqslant C\tau \, |op(\chi)u|_{0,\tau} \tag{4.55}$$

for any $u \in C^\infty(K)$.

Proof. — It is enough to prove that: there exist C such that $< \xi', \tau > \leqslant C\tau$ for any $(x, \xi', \tau) \in \mathcal{Z}_\gamma \cup \mathcal{E}^- \cup \mathcal{M}$. We argue by contradiction. Otherwise there exist a sequence $(x_k, \xi'_k, \tau_k) \in \mathcal{Z}_\gamma \cup \mathcal{E}^- \cup \mathcal{M}$ and

$$\lim_{k\to\infty} \frac{\tau_k}{< \xi'_k, \tau_k >} = 0 \tag{4.56}$$

using that the definition of $\mathcal{Z}_\gamma \cup \mathcal{E}^- \cup \mathcal{M}$, if $(x, \xi', \tau) \in \mathcal{Z}_\gamma \cup \mathcal{E}^- \cup \mathcal{M}$ then there exist $\gamma \in \{\mu, 2\mu + \lambda\}$ such that

$$q_2(x, \xi', \tau) - \frac{\tau^2}{\gamma < \xi', \tau >^2} \leqslant 0 \tag{4.57}$$

In particular we obtain

$$q_2\Big(x_k, \frac{\xi'_k}{< \xi'_k, \tau_k >}, \frac{\tau_k}{< \xi'_k, \tau_k >}\Big) - \frac{\tau_k^2}{\gamma < \xi'_k, \tau_k >^2} \leqslant 0 \tag{4.58}$$

and by (4.11) we get

$$r(x_k, \frac{\xi'_k}{<\xi'_k, \tau_k>}) \leqslant \frac{\tau_k^2}{<\xi'_k, \tau_k>^2} \left[\frac{1}{\gamma} + (\varphi'_{x_n})^2 + r(x_k, \varphi'_{x'}(x_k)) \right]$$
$$\leqslant \mathcal{O}\Big(\frac{\tau_k^2}{<\xi'_k, \tau_k>^2} \Big) \tag{4.59}$$

Moreover (4.6) implies

$$\frac{|\xi'_k|^2}{<\xi'_k, \tau_k>^2} = 1 - \frac{\tau_k^2}{<\xi'_k, \tau_k>^2} \leqslant \mathcal{O}\Big(\frac{\tau_k^2}{<\xi'_k, \tau_k>^2} \Big) \tag{4.60}$$

Then (4.60) contradict (4.56). This ends the proof of lemma 4.6. □

4.4.1. Carleman estimate in the region $\mathcal{E}^-$

The purpose of this section is to prove the Carleman estimate in the region $\mathcal{E}^-$. In this region we prove a better estimates (without boundary traces) under the condition $\varphi'_{x_n} > 0$, this is connected by in this region and under $\varphi'_{x_n} > 0$ there are no roots with respect ξ_n in the upper half-plane $Im\,\xi_n > 0$ for a_γ. Let $\chi_1(x, \xi', \tau)$ be a homogeneous cutoff function of degree zero in the region $\mathcal{E}^-$. Denote $\widetilde{u} = op(\chi_1)u$, then we have the following Lemma.

LEMMA 4.7.— *There exist $C > 0$ such that for any large enough τ we have*

$$\|P(x, D, \tau)u\|^2 + \tau(\tau^2 |\widetilde{u}|^2_{0,\tau} + |\sigma(\widetilde{u}).\nu|^2) + \|u\|^2_{1,\tau} + |u|^2_{1,\tau} \geqslant C\tau \, \|\widetilde{u}\|^2_{1,\tau} \tag{4.61}$$

whenever $u \in C_0^\infty(K)$. Furthermore if we assume that $\varphi'_{x_n} > 0$ on $supp\chi_1 \cap \{x_n = 0\}$, then there exist $C > 0$ and τ_0 such that for any $\tau \geqslant \tau_0$ we have

$$\|P(x, D, \tau)u\|^2 + \|u\|^2_{1,\tau} + |u|^2_{1,\tau} \geqslant C\tau \Big(\|op(\chi_1)u\|^2_{1,\tau} + |op(\chi_1)u|^2_{1,\tau} \Big), \tag{4.62}$$

whenever $u \in C_0^\infty(K)$.

Proof. — For the first part of the lemma it is enough to apply proposition 4.5 to $op(\chi_1)u$ and using the lemma 4.6.
Now we will prove the second part. First we take

$$\widetilde{u} = op(\chi_1)u, \qquad \widetilde{f} = op(P)\widetilde{u}, \tag{4.63}$$

by an argument similar to the one of section 4.3.1 we have

$$\begin{cases} D_n w - op(\mathcal{H}^-)w = \psi & \text{in} \quad x_n > 0 \\ op(\widetilde{\mathcal{B}})w = \widetilde{g} & \text{on} \quad x_n = 0 \end{cases} \tag{4.64}$$

$$w = (I+K)^{-1}S^{-1}v, \quad \psi = (I+K)^{-1}S^{-1}F, \quad \widetilde{\mathcal{B}} = \mathcal{B}S(I+K)^{-1} \tag{4.65}$$

and $op(\mathcal{H}^-)$ is an $2n \times 2n$ square matrix, whose components are tangential pseudo-differential operators of order 1 with principal symbol $\mathcal{H}^- \in \mathcal{TS}^1_\tau$, such that $Im(\mathcal{H}^-) \geqslant C < \xi', \tau > I_{2n}$, let

$$G(x_n) = -\frac{d}{dx_n} \langle w, w \rangle \tag{4.66}$$

using (4.64) then we obtain

$$G(x_n) = 2Im \left\langle op(\mathcal{H}^-)w, w \right\rangle + 2Im \left\langle w, \psi \right\rangle. \tag{4.67}$$

The integration in x_n gives

$$-|w|^2 = 2\int_0^\infty -Im \left\langle op(\mathcal{H}^-)w, w \right\rangle dx_n - 2Im \int_0^\infty \left\langle w, \psi \right\rangle dx_n. \tag{4.68}$$

By Gårding inequality we get

$$\int_0^\infty -Im \left\langle op(\mathcal{H}^-)w, w \right\rangle dx_n \geqslant C\tau \int_0^\infty \|w(x_n)\|^2 \, dx_n \tag{4.69}$$

Furthermore, for any ε

$$\int_0^\infty | \left\langle w, \psi \right\rangle | dx_n \leqslant \varepsilon C\tau \|w\|^2 + \frac{C_\varepsilon}{\tau} \|\psi\|^2 . \tag{4.70}$$

Combining (4.70) (4.69) with (4.68) we obtain

$$C\tau \|w\|^2 + C' |w|^2 \leqslant \frac{C}{\tau} \|\psi\|^2 . \tag{4.71}$$

We deduce, by (4.65) and (4.63), the following

$$C\tau\Big(\|\widetilde{u}\|^2_{1,\tau} + |\widetilde{u}|^2_{1,\tau} \Big) \leqslant \|P(x, D, \tau)u\|^2 + l.o.t(u)$$

This complete the proof of (4.62). □

4.4.2. Carleman estimate in a non elliptic regions

The purpose of this section is to get the Carleman estimate in the non elliptic region. Our goals here are firstly estimated the traces of the solution in the part of boundary where φ'_{x_n} has the good sign, second we eliminate the tangential derivative (independently to the sign of φ'_{x_n}). Precisely we take a cutoff function $\chi_0(x, \xi', \tau)$ homogeneous of degree zero in a neighborhood of the regions $\mathcal{Z}_\gamma \cup \mathcal{M}$. Let $\widetilde{u} = op(\chi_0)u$, our purpose here is to prove the following Lemma.

LEMMA 4.8. — *There exist $C > 0$ such that for any large enough τ we have*

$$\|P(x,D,\tau)u\|^2+\tau(\tau^2\,|\widetilde{u}|^2_{0,\tau}+|\sigma(\widetilde{u}).\nu|^2)+\|u\|^2_{1,\tau}+|\widetilde{u}|^2_{1,\tau}\geqslant C\tau\,\|\widetilde{u}\|^2_{1,\tau} \quad (4.72)$$

whenever $u \in C_0^\infty(K)$.

Furthermore if we assume that $\varphi'_{x_n} > C_0$ on $\{x_n = 0\} \cap supp\chi_0$ then we have

$$\begin{aligned}\|P(x,D,\tau)u\|^2 \quad &+ \quad \|u\|^2_{1,\tau}+|u|^2_{1,\tau}+\tau\,|B(x,D,\tau)u|^2_{0,\tau}\\ &\geqslant \quad C\Big(\tau\,\|op(\chi_0)u\|^2_{1,\tau}+\tau\,|op(\chi_0)u|^2_{1,\tau}\Big) \qquad (4.73)\end{aligned}$$

whenever $u \in C_0^\infty(K)$.

To show the first part of the previous Lemma it is enough to apply proposition 4.5 to $op(\chi_0)u$ and we use the lemma 4.6 in order to eliminate the tangential derivatives.

To show the second part of lemma 4.8 we need the following estimate.

LEMMA 4.9. — *Assume that $\varphi'_{x_n} > C_0$ on $\{x_n = 0\} \cap supp\chi_0$ then the following estimate holds*

$$\|P(x,D,\tau)u\|^2+\tau\,|B(x,D,\tau)u|^2_{0,\tau}+\|u\|^2_{1,\tau}+|u|^2_{1,\tau}\geqslant C\tau\,|op(\chi_0)u|^2_{1,0,\tau} \quad (4.74)$$

whenever $u \in C_0^\infty(K)$.

To proof lemma 4.9 we need the following notations. First we take

$$\widetilde{u}=op(\chi_0)u, \qquad \widetilde{f}=op(P)\widetilde{u} \quad (4.75)$$

by an argument similar to the one of section 4.3.1, we have

$$\begin{cases} D_n w - op(\mathcal{H})w = \psi & \text{in} \quad x_n > 0 \\ op(\widetilde{\mathcal{B}})w = \widetilde{g} & \text{on} \quad x_n = 0 \end{cases} \quad (4.76)$$

where

$$\begin{aligned} w=(I+K)^{-1}S^{-1}v, &\quad v={}^t\Big(\langle D',\tau\rangle\,\widetilde{u},D_n\widetilde{u}\Big),\\ \psi=(I+K)^{-1}S^{-1}F, &\quad \widetilde{\mathcal{B}}=\mathcal{B}S(I+K)^{-1} \end{aligned} \quad (4.77)$$

and $op(\mathcal{H})$ is an $2n \times 2n$ square matrix, whose components are tangential pseudo-differential operators of order 1 with principal symbol $\mathcal{H} = diag(\mathcal{H}^+, \mathcal{H}^-)$, such that $\mathcal{H}^-$ satisfy $-Im(\mathcal{H}^-) \geqslant C < \xi', \tau >$.

For the proof of the lemma 4.9 we use the following result which can be proved in the same way as Lemma 4.4 (see [3] For more details).

LEMMA 4.10. — *Let $\mathcal{R} = diag(0, -\rho Id_n)$ then there exist $C > 0$ and $e(x, \xi', \tau) \in \mathcal{TS}^1_\tau$ such that*

i) $Im(\mathcal{RH}) = diag(0, e(x, \xi', \tau))$

ii) $e(x, \xi', \tau) \geqslant C < \xi', \tau > I_n$ *in* $supp\chi_0$

iii) $-\mathcal{R} + \widetilde{\mathcal{B}}^* \widetilde{\mathcal{B}} \geqslant C.I_{2n}$ *on* $\{x_n = 0\} \cap supp\chi_0$

4.4.3. Proof of Lemma 4.9

Denote the function

$$G(x_n) = \frac{d}{dx_n} \langle op(\mathcal{R})w, w\rangle_{L^2(\mathbb{R}^{n-1})}. \tag{4.78}$$

Taking into account $D_n w - op(\mathcal{H})w = \psi$ then we obtain

$$G(x_n) = -2Im \langle op(\mathcal{R})op(\mathcal{H})w, w\rangle + 2Im \langle op(\mathcal{R})w, \psi\rangle + \langle op(\mathcal{R}'_{x_n})w, w\rangle \tag{4.79}$$

The integration in the normal direction gives

$$\begin{aligned}\langle op(\mathcal{R})w, w\rangle_{x_n=0} &= 2\int_0^\infty Im \langle op(\mathcal{R})op(\mathcal{H})w, w\rangle \, dx_n \\ &\quad -2Im \int_0^\infty \langle op(\mathcal{R})w, \psi\rangle - \int_0^\infty \langle op(\mathcal{R}'_{x_n})w, w\rangle \, dx_n \end{aligned} \tag{4.80}$$

Taking into account Lemma 4.10, $\tau \leqslant \Lambda$, and Gårding inequality we have for large τ

$$\int_0^\infty Im \langle op(\mathcal{R})op(\mathcal{H})w, w\rangle \, dx_n \geqslant C\tau \left\|w^-\right\|^2 \tag{4.81}$$

and further, for any ε

$$\int_0^\infty |\langle op(\mathcal{R})w, \psi\rangle| \, dx_n \leqslant \varepsilon C\tau \left\|w^-\right\|^2 + \frac{C_\varepsilon}{\tau} \|\psi\|^2. \tag{4.82}$$

Apply now Lemma 4.10 we obtain

$$-\langle op(\mathcal{R})w, w\rangle + C\left|\widetilde{\mathcal{B}}w\right|^2 \geqslant C\,|w|^2. \tag{4.83}$$

Collecting (4.83) (4.82) (4.81) with (4.80) we get

$$C\tau \left\|w^-\right\|^2 + C' \left|w\right|_0^2 \leqslant \frac{C}{\tau} \left\|\psi\right\|^2 + \left|\widetilde{\mathcal{B}}w\right|^2. \tag{4.84}$$

This implies, by (4.77) the following inequality

$$\tau \left|\widetilde{u}\right|_{1,0,\tau}^2 \leqslant C\Big(\left\|\psi\right\|^2 + \tau \left|\widetilde{\mathcal{B}}w\right|^2 \Big) \leqslant C\Big(\left\|\psi\right\|^2 + \tau \left|\mathcal{B}v\right|^2 \Big) \tag{4.85}$$

Collecting (4.85) and (4.77) and using that (4.75), we obtain (4.74). Finally Combining (4.74) with (4.72) we get (4.73). This complete the proof of Lemma 4.9.

4.4.4. End of the proof of Proposition 2.1

Let φ satisfy the hypotheses of Proposition 2.1

$$-\partial_\nu \varphi = -\nabla\varphi.\nu = \varphi'_{x_n} > C_0 \quad on \quad \Sigma_1 \tag{4.86}$$

Via a partition of unity $(\theta_j)_j$, near the boundary and by Proposition 4.3, Lemma 4.7 and Lemma 4.8 the following estimate holds

$$\begin{aligned} C\tau(\left\|\theta_j u\right\|_{1,\tau}^2 + \left|\theta_j u\right|_{1,\tau}^2) &\leqslant \left\|P(x,D,\tau)u\right\|^2 + \tau \int_{\Sigma_1 \cap Supp(\theta_j)} \left|B(x,D,\tau)u\right|^2 dx' \\ &+ \tau \int_{\Sigma_2 \cap Supp(\theta_j)} (\tau^2 \left|u\right|^2 + \left|\sigma(u)\right|^2) dx' + l.o.t(u) \end{aligned} \tag{4.87}$$

This complete the proof of Proposition 2.1.

5. Appendix

This appendix is devoted to a proof of (4.37) and (4.38). For this purpose we need the following

$${}^t\ell_0.\omega_j^+ = {}^t\ell_1.\omega_j^+ = 0,\ j \in \{1,...,n-2\},\ {}^t\ell_1\ell_1 = -(i\alpha)^2,\ (i\alpha_\gamma)^2 = (i\alpha)^2 + \frac{\tau^2}{\gamma} \tag{5.1}$$

and

$$\begin{cases} {}^t\ell_0\omega_{n-1}^+ = \Lambda^{-2}(i\alpha)^2, & {}^t\ell_1\omega_{n-1}^+ = -\Lambda^{-2}(i\alpha)^2(i\alpha_\mu) \\ {}^t\ell_0\omega_n^+ = \Lambda^{-1}(i\alpha_{2\mu+\lambda}), & {}^t\ell_1\omega_n^+ = -\Lambda^{-1}(i\alpha)^2. \end{cases} \tag{5.2}$$

Moreover, for $j \in \{1, ..., n-1\}$, we have

$$\begin{aligned} b_j^+ = \widetilde{\mathcal{B}} s_j^+ &= \Lambda^{-1}(B_1 \omega_j^+ + z_1^+ B_0 \omega_j^+), \quad j \in \{1, ..., n-1\} \\ &= \Lambda^{-1}\Big((i\alpha_\mu) B_0 \omega_j^+ + (\mu \ell_1{}^t \ell_0 + \lambda \ell_0{}^t \ell_1)\omega_j^+ - a_0 \tau \omega_j^+\Big) \quad (5.3) \end{aligned}$$

and

$$\begin{aligned} b_n^+ = \widetilde{\mathcal{B}} s_n^+ &= \Lambda^{-1}(B_1 \omega_j^+ + z_2^+ B_0 \omega_j^+) \\ &= \Lambda^{-1}\Big((i\alpha_{2\mu+\lambda}) B_0 \omega_n^+ + (\mu \ell_1{}^t \ell_0 + \lambda \ell_0{}^t \ell_1)\omega_n^+ - a_0 \tau \omega_n^+\Big) \quad (5.4) \end{aligned}$$

By means of (5.1) and (5.2) we obtain (4.37).

Moreover by (4.37) we have

$$\begin{aligned} \Lambda^5 R_0 &= [2\mu(i\alpha_\mu) - a_0\tau)](i\alpha)^2[2\mu(i\alpha_{2\mu+\lambda}) - a_0\tau)] \\ &\quad - [2\mu(i\alpha)^2 - a_0\tau(i\alpha_\mu) + \tau^2][2\mu(i\alpha)^2 - a_0\tau(i\alpha_{2\mu+\lambda}) + \tau^2] \quad (5.5) \end{aligned}$$

furether, by (4.17) we have

$$\begin{aligned} (i\alpha_\mu)^2(i\alpha_{2\mu+\lambda})^2 &= (i\alpha)^4 + \Big(\frac{3\mu+\lambda}{\mu(2\mu+\lambda)}\Big)(i\alpha)^2 + \frac{\tau^4}{\mu(2\mu+\lambda)} \\ &= \frac{\tau^4}{\mu^2}\left[\Big(\Big(\frac{\sqrt{\mu}}{\tau}\alpha\Big)^2 - 1\Big)\Big(\Big(\frac{\sqrt{\mu}}{\tau}\alpha\Big)^2 - \frac{\mu}{2\mu+\lambda}\Big)\right] \quad (5.6) \end{aligned}$$

using that (5.6) in (5.5) we obtain (4.39).

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