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Discrete Löwner evolution (*)

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ABSTRACT. — We study a one parameter family of discrete Löwner evolutions driven by a random walk on the real line. We show that it converges to the stochastic Löwner evolution (SLE) under rescaling. We show that the discrete Löwner evolution satisfies Markovian-type and symmetry properties analogous to SLE, and establish a phase transition property for the discrete Löwner evolution when the parameter equals four.

RÉSUMÉ. — Nous étudions une famille d'évolution de temps discret des transformations conforme qui dépend d'une marche aléatoire sur les nombres réels. Nous démontrons que l'évolution stochastique de Löwner (SLE) est la limite d'échelle de cette famille. Nous démontrons que l'évolution en temps discret est Markovien et qu'elle a une propriété de symétrie comme SLE, et nous établissons une transition de phase pour l'évolution de temps discret quand le paramètre égal quatre.

1. Introduction

In this paper we study a discrete version of the stochastic Löwner evolution (SLE_κ) introduced by O. Schramm in [18]. Whereas SLE is driven by a one dimensional Brownian motion, our discrete Löwner evolution is driven by a random walk. SLE is a one parameter family of processes of growing random sets in a domain in the plane. We will only consider chordal SLE and our discrete version, where the random sets grow in the upper half-plane from 0 to ∞.

It has been shown that, in a sense that can be made precise ([8]), any random process of growing sets in the plane that satisfies a certain Markovian type property is given by SLE_κ for some $\kappa \in [0, \infty)$. Since SLE is

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amenable to computations this led to some spectacular calculations of various quantities long believed out of reach for mathematicians. For example, in a sequence of papers [9],[10],[11],[12], Lawler, Schramm, and Werner calculated all intersection exponents for Brownian motion in the plane. Many of these exponents had been predicted by physicists based on non-rigorous methods from conformal field theory. In particular, Lawler, Schramm, and Werner confirmed a conjecture of Mandelbrot, that the Brownian frontier has Hausdorff dimension 4/3. Furthermore, SLE has been shown to be the scaling limit of various discrete systems, e.g. loop-erased random walk and the outer boundar y of critical percolation clusters on the triangular lattice, and is conjectured to give the scaling limit of others, such as the self-avoiding random walk. To confirm the conjectures the existence of the scaling limit and the conformal invariance of the scaling limit need to be established, the latter usually being the main obstacle.

In this paper we study a discrete (in time) approximation of SLE. Instead of a continuous family of conformal maps $\{f_t\}_{t\in[0,\infty)}$ from the upper half-plane $\mathbb{H}$ into $\mathbb{H}$ so that $f_t(\mathbb{H}) \supseteq f_s(\mathbb{H})$ if $t \leqslant s$, we consider a sequence $\{f(m)\}_{m=0}^{\infty}$ of such maps. The "increments" $f_m^{-1} \circ f_{m+1}$ are all of the form

$$z \in \mathbb{H} \mapsto S(m) + \sqrt{(z - S(m))^2 - 4} \in \mathbb{H},$$

where $\{S(m)\}_{m=0}^{\infty}$ is a random walk on $\mathbb{R}$ with centered increments of variance κ. We show in Theorem 2.10 that the law of $\{f(m)\}_{m=0}^{\infty}$, properly rescaled, converges weakly to SLE_κ. The proof relies on Donsker's invariance principle and continuity properties of Löwner's differential equation, considered as a map from piecewise continuous curves ψ to 1-parameter families of conformal maps $f : \mathbb{H} \to \mathbb{H}$.

To establish continuity, we first choose a topology on the space of conformal maps $f : \mathbb{H} \to \mathbb{H}$. One natural choice is the topology of uniform convergence on compacts. In fact, in our context, this is equivalent to uniform convergence on $\{z \in \mathbb{H} : \Im(z) > a\}$ for every $a \in (0, \infty)$. However, the regularity of the Löwner equation allows us to choose a stronger topology that also takes some of the boundary behavior of f into account. We introduce this topology in the context of Cauchy transforms of probability measures in Lemmas 2.2, 2.3, and 2.4.

In Section 3 we study properties of the sequence $\{f(m)\}_{m=0}^{\infty}$. We show in Theorem 3.1 that if the increments $S(m+1) - S(m)$ have the appropriate properties, then $\{f(m)\}_{m=0}^{\infty}$ has the same Markovian-type and symmetry properties as SLE. We call $\{f(m)\}_{m=0}^{\infty}$ a discrete Löwner evolution with parameter κ (DLE_κ), if the increments $S(m+1) - S(m)$ are centered, independent and identically distributed random variables with variance κ.

Next, we study the dependency of the discrete Löwner evolution on κ. In the paragraphs following Proposition 3.3 we describe, graphically, DLE in the special case when the increments $S(m+1) - S(m)$ are Bernoulli random variables. The behavior of the omitted set, i.e. the image of $\mathbb{H}$ under $f(m)$, is rather easily understood in terms of the underlying random walk $\{S(m)\}$. In our view this connection is not as apparent in the continuous case and making it more explicit is one of our motivations for this paper. In Proposition 3.3 we note the transition from connected to disconnected complement of the image at $\kappa = 4$. In Theorem 3.4 we show that Markov chains (with uncountable state space) naturally associated to DLE have a transition from transient to recurrent at $\kappa = 4$. These Markov chains are the discrete analogues of Bessel processes naturally occurring in the study of SLE [20].

Finally, we collect in the appendix some facts about monotonic independence in noncommutative probability and its relation to the (deterministic) Löwner evolution. The impetus to build a discrete Löwner evolution from the maps $z \mapsto a + \sqrt{(z-a)^2 - 4}$ came from the preprint [16], which H. Bercovici had kindly brought to our attention. We would also like to thank an anonymous referee for bringing V. Beffara's thesis to our attention, where essentially the same discrete approximation is introduced, [2, Section 5.3]

2. A discrete approximation of SLE

Denote $\mathfrak{P}(\mathbb{R})$ the space $C([0,\infty);\mathbb{R})$ of continuous paths $\psi : [0,\infty) \to \mathbb{R}$ and endow $\mathfrak{P}(\mathbb{R})$ with the topology of uniform convergence on compact intervals. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent real-valued random variables on a probability space $(\Omega, \mathcal{F}, P)$, and assume that the X_n's have mean-value 0, variance $\kappa > 0$ and satisfy

$$\lim_{R\to\infty} \sup_{n\in\mathbb{Z}^+} \mathbb{E}^P[|X_n|^2, |X_n| \geqslant R] = 0.$$

Next, for $n \in \mathbb{Z}^+$, define $\omega \in \Omega \mapsto S_n(\cdot,\omega) \in \mathfrak{P}(\mathbb{R})$ so that $S_n(0,\omega) = 0$ and, for each $m \in \mathbb{Z}^+$, $S_n(\cdot,\omega)$ is linear on the interval $[\frac{m-1}{n}, \frac{m}{n}]$ with slope $n^{1/2}X_m(\omega)$. That is,

$$S_n(0,\omega) = 0, \quad S_n\left(\frac{m}{n},\omega\right) = n^{-1/2}\sum_{k=1}^{m} X_k,\ m \in \mathbb{Z}^+$$

and

$$S_n(t,\omega) = (m-nt)S_n\left(\frac{m-1}{n},\omega\right) + (1-(m-nt))S_n\left(\frac{m}{n},\omega\right)$$

for $t \in (\frac{m-1}{n}, \frac{m}{n})$. Finally, let

$$\mu_n \equiv (S_n)_* P$$

denote the distribution of $\omega \in \Omega \mapsto S_n(\cdot, \omega) \in \mathfrak{P}(\mathbb{R})$ under P. Then it is well known, see [19], that $\mu_n \Longrightarrow \mathcal{W}_\kappa$ as $n \to \infty$, where $\mathcal{W}_\kappa$ is the distribution of $\psi \in \mathfrak{P}(\mathbb{R}) \mapsto \sqrt{\kappa}\psi \in \mathfrak{P}(\mathbb{R})$ under Wiener's measure $\mathcal{W}$ on $\mathfrak{P}(\mathbb{R})$.

Denote $\mathbb{H}$ the upper half-plane $\{z \in \mathbb{C} : \Im(z) > 0\}$. For $z \in \mathbb{H}$, and $\psi \in \mathfrak{P}(\mathbb{R})$ consider the chordal Löwner equation

$$\frac{\partial}{\partial t} g(t, \psi; z) = \frac{2}{g(t, \psi; z) - \psi(t)}, \quad g(0, \psi; z) = z. \tag{2.1}$$

Then

$$\left| \frac{\partial}{\partial t} g(t, \psi; z) \right| \leqslant \frac{2}{\Im(g(t, \psi; z))}, \tag{2.2}$$

and

$$\frac{\partial}{\partial t} \Im(g(t, \psi; z)) = -\frac{2\Im(g(t, \psi; z))}{|g(t, \psi; z) - \psi(t)|^2} < 0. \tag{2.3}$$

The inequalities imply in particular that for each $z \in \mathbb{H}$ and $\psi \in \mathfrak{P}(\mathbb{R})$ the solution is well defined up to a time $\tau(\psi; z) \in (0, \infty]$, and that if $\tau(\psi; z) < \infty$, then $\lim_{t \nearrow \tau(\psi; z)} \Im(g(t, \psi; z)) = 0$. Let $\mathcal{K}(t, \psi)$ be the closure of $\{z \in \mathbb{H} : \tau(\psi; z) \leqslant t\}$.

Proposition 2.1 ([7]).— *For every $t \in (0, \infty)$ and $\psi \in \mathfrak{P}(\mathbb{R})$, $g(t, \psi; \cdot)$ is a conformal transformation of $\mathbb{H} \backslash \mathcal{K}(t, \psi)$ onto $\mathbb{H}$ satisfying*

$$g(t, \psi; z) = z + \frac{2t}{z} + O\left(\frac{1}{|z|^2}\right), \quad z \to \infty.$$

Let $\mathbf{M}_1(\mathbb{R})$ be the set of Borel probability measures on $\mathbb{R}$ and denote $\mathbf{M}_1^0(\mathbb{R})$ the subset of Borel probability measures with compact support. If $\mu \in \mathbf{M}_1^0(\mathbb{R})$ let $[A_\mu, B_\mu]$ denote the convex closure of $\mathrm{supp}(\mu)$. For $\mu, \nu \in \mathbf{M}_0^1(\mathbb{R})$ let

$$\rho(\mu, \nu) \doteq \rho_L(\mu, \nu) + \max\{|A_\mu - A_\nu|, |B_\mu - B_\nu|\},$$

where

$$\begin{aligned} \rho_L \equiv \inf\{\delta : \mu((-\infty, x - \delta]) - \delta &\leqslant \nu((-\infty, x]) \\ &\leqslant \mu((-\infty, x + \delta]) + \delta \text{ for all } x \in \mathbb{R}\} \end{aligned}$$

is the Lévy distance between μ and ν. Then ρ is a metric on $\mathbf{M}_1^0(\mathbb{R})$.

LEMMA 2.2.— *$(\mathbf{M}_1^0(\mathbb{R}), \rho)$ is a separable metric space.*

Proof.— It is easy to see that the set of all convex combinations $\sum_{k=1}^n \alpha_k \delta_{x_k}$, where $n \in \mathbb{Z}^+$, $\{\alpha_k\}_{k=1}^n \subset [0,1] \cap \mathbb{Q}$ with $\sum_{k=1}^n \alpha_k = 1$, and $\{x_k\}_{k=1}^n \subset \mathbb{Q}$, is a countable ρ-dense set in $\mathbf{M}_1^0(\mathbb{R})$. □

Given $\mu \in \mathbf{M}_1(\mathbb{R})$, denote G_μ its Cauchy transform

$$z \in \mathbb{C} \backslash \mathrm{supp}(\mu) \mapsto G_\mu(z) = \int_{\mathbb{R}} \frac{\mu(dx)}{z - x} \in \mathbb{C} \backslash \{0\}.$$

Note that G_μ is analytic, and that $G_\mu(\bar{z}) = \overline{G_\mu(z)}$. Furthermore, G_μ cannot be extended analytically beyond $\mathbb{C} \backslash \mathrm{supp}(\mu)$. Indeed, if G_μ extends analytically to $x \in \mathbb{R}$ then it must extend to a neighborhood $(x - \delta, x + \delta)$ for some $\delta > 0$. By continuity we then have $\lim_{y \searrow 0} \Im(G(a + \sqrt{-1}y)) = 0$, uniformly on compact subsets of $(x - \delta, x + \delta)$. But by Stieltjes' inversion formula [3, 2.20], for any $-\infty < x < x' < \infty$

$$\frac{1}{\pi} \lim_{y \searrow 0} \int_x^{x'} \Im(G_\mu(a + \sqrt{-1}y))\, da = \mu((x, x')) + \frac{1}{2}[\mu(\{x\}) + \mu(\{x'\})]. \quad (2.4)$$

Hence $\mu((x - \delta, x + \delta)) = 0$ and $x \notin \mathrm{supp}(\mu)$.

Since $G_\mu(z) \neq 0$ for all $z \in \mathbb{H}$ we may define the reciprocal Cauchy transform $f_\mu : \mathbb{H} \to \mathbb{H}$ by $f_\mu(z) = 1/G_\mu(z)$.

LEMMA 2.3.— *An analytic function $f : \mathbb{H} \to \mathbb{H}$ is the reciprocal Cauchy transform of some compactly supported probability measure μ on $\mathbb{R}$, if and only if*

$$\inf_{z \in \mathbb{H}} \frac{\Im(f(z))}{\Im(z)} = 1 \quad (2.5)$$

and $G = 1/f$ extends analytically to $\mathbb{C} \backslash [-N, N]$ for some $N \in \mathbb{N}$.

Proof.— By [14, Proposition 2.1], f is the reciprocal Cauchy transform of a probability measure μ on $\mathbb{R}$ if and only if (2.5) holds. If μ has compact support $K \subset \mathbb{R}$ and $f = f_\mu$, then $G = 1/f$ extends to $\mathbb{C} \backslash K$. Conversely, if f satisfies (2.5) and $G = 1/f$ extends analytically to $\mathbb{C} \backslash [-N, N]$, then by [14, Proposition 2.1], $f = f_\mu = 1/G_\mu$ for some $\mu \in \mathbf{M}_1(\mathbb{R})$, and then by Stieltjes' inversion formula $\mathrm{supp}(\mu) \subset [-N, N]$. □

Recall that for a domain $D \subseteq \mathbb{C}$ a function $f : D \to \mathbb{C}$ is univalent if it is analytic and 1-1. Let

$$\mathbf{M}^U = \{\mu \in \mathbf{M}_1^0(\mathbb{R}) : G_\mu : \mathbb{H} \to \mathbb{C} \backslash \overline{\mathbb{H}} \text{ is univalent}\}$$

If $f : \mathbb{H} \to \mathbb{H}$ is univalent, then we may extend f to $\mathbb{C}\backslash\mathbb{R}$ as a univalent function by the Schwarz reflection principle. We say f has a univalent extension to $\mathbb{C}\backslash[a,b]$, if f extends as a univalent function to $\mathbb{C}\backslash[a,b]$. Finally, define $A_f, B_f \in \mathbb{R}$ by

$$[A_f, B_f] = \bigcap\{[a,b] : f \text{ has a univalent extension to } \mathbb{C}\backslash[a,b]\},$$

whenever the right-hand side is nonempty.

LEMMA 2.4.— *If $\mu \in \mathbf{M}^U$, $A_\mu \neq B_\mu$ and $f = f_\mu$, then $[A_f, B_f] = [A_\mu, B_\mu]$. Furthermore, $\mathbf{M}^U$ is a closed subset of $(\mathbf{M}_0^1(\mathbb{R}), \rho)$. Finally, if $\{\mu_n\}_{n=1}^\infty \cup \{\mu\} \subset \mathbf{M}^U$ and $f = f_\mu$, $f_n = f_{\mu_n}$, $n \in \mathbb{Z}^+$, then $\rho(\mu_n, \mu) \to 0$, as $n \to \infty$, if and only if $f_n \to f$ uniformly on $\{z \in \mathbb{C} : \Im(z) > a\}$ for any $a \in (0,\infty)$, and* $\max\{|A_f - A_{f_n}|, |B_f - B_{f_n}|\}$ *converges to 0, as $n \to \infty$.*

Proof.— If $\mu \in \mathbf{M}^U$ and $G = G_\mu$, then it is easy to see that $[A_G, B_G] = [A_\mu, B_\mu]$. Indeed, since G_μ cannot be extended analytically beyond $\mathbb{C}\backslash\mathrm{supp}(\mu)$, it is clear that $[A_G, B_G] \supseteq [A_\mu, B_\mu]$. Furthermore, $G((B_\mu, \infty)) \subseteq \mathbb{R}^+$ and $G \restriction (B_\mu, \infty)$ is strictly decreasing. Similarly, $G((-\infty, A_\mu)) \subseteq \mathbb{R}^-$ and $G \restriction (-\infty, A_\mu)$ is strictly increasing. Hence $[A_G, B_G] = [A_\mu, B_\mu]$. Finally, since G does not assume the value zero outside of $[A_G, B_G]$, it follows that f extends as a univalent function to $\mathbb{C}\backslash[A_G, B_G]$. Now note that f, on a domain of univalence, can only assume the value zero once. Since $A_G \neq B_G$ we get $[A_f, B_f] = [A_G, B_G]$.

For the following statements, we begin by checking that weak convergence of a sequence $\{\mu_n\}_{n=1}^\infty \subset \mathbf{M}_1(\mathbb{R})$ to a probability measure μ is equivalent to the uniform convergence $G_{\mu_n}(z) \to G_\mu(z)$ on $\{z \in \mathbb{C} : \Im(z) > a\}$ for any $a > 0$. By [14, Theorem 2.5], $\mu_n \Longrightarrow \mu$ as $n \to \infty$, if and only if there exists $y > 0$ such that

$$\lim_{n\to\infty} G_{\mu_n}(x + \sqrt{-1}y) = G_\mu(x + \sqrt{-1}y), \quad x \in \mathbb{R}.$$

To complete this part assume now that $\mu_n \Longrightarrow \mu$ as $n \to \infty$. Then

$$\lim_{n\to\infty} G_{\mu_n}(z) = G_\mu(z), \quad z \in \mathbb{H}.$$

Now note that $|G_\nu(z)| \leqslant 1/\Im(z)$ for all $z \in \mathbb{H}$, and all $\nu \in \mathbf{M}_1(\mathbb{R})$. Hence the family $\{G_\nu, \nu \in \mathbf{M}_1(\mathbb{R})\}$ is locally bounded in $\mathbb{H}$ and it follows from Vitali's theorem, [5], that $G_{\mu_n}(z) \to G_\mu(z)$ uniformly on compacts. Since $\mu_n \Longrightarrow \mu$ as $n \to \infty$, for any $\epsilon > 0$ there exists $N > 0$ so that $\sup_n \mu_n(\mathbb{R}\backslash[-N,N]) \leqslant \epsilon$. Hence $|G_{\mu_n}(z)| \leqslant 1/N + \epsilon/a$ on $\{z \in \mathbb{C} : \Im(z) > a \text{ and } |z^2| > 2N\}$ and it follows that $G_{\mu_n}(z) \to G_\mu(z)$ uniformly on $\{z \in \mathbb{C} : \Im(z) > a\}$. Since uniform (on compacts) limits of univalent functions are either univalent or constant ([5]), and since G_μ cannot be constant as $G_\mu(z) \to 0$ as $z \to \infty$, it follows that $\mathbf{M}^U$ is ρ-closed in $\mathbf{M}_0^1(\mathbb{R})$.

Next, given a compact $A \subset \mathbb{H}$,

$$d \equiv \inf_{z\in A} |G_\mu(z)| > 0$$

and there is an $N \in \mathbb{Z}^+$ such that for all $n \geqslant N$, $\inf_{z\in A} |G_{\mu_n}(z)| \geqslant d/2$. Hence, for $n \geqslant N$,

$$\begin{aligned} \sup_{z\in A} |f_\mu(z) - f_{\mu_n}(z)| &= \sup_{z\in A} \frac{|G_{\mu_n}(z) - G_\mu(z)|}{|G_\mu(z) G_{\mu_n}(z)|} \\ &\leqslant \frac{2}{d^2} \sup_{z\in A} |G_{\mu_n}(z) - G_\mu(z)| \to 0, \end{aligned}$$

as $n \to \infty$. Since $\overline{\bigcup_{n=1}^\infty \operatorname{supp}(\mu_n)}$ is compact it follows in particular that the mean values $\{m_n\}_{n=1}^\infty$ of $\{\mu_n\}_{n=1}^\infty$ converge to the mean value m of μ and from Taylor's formula that there exists an $N > 0$ and a function $c_n(z)$ such that $\sup_{|z|>N} \sup_n |c_n(z)| < \infty$ and so that

$$G_{\mu_n}(z) = \frac{1}{z} + \frac{m_n}{z^2} + \frac{c_n(z)}{z^3}.$$

This implies that $f_n(z) = z - m_n + e_n(z)/z$, where $\sup_n |e_n(z)|$ is uniformly bounded for $|z| > N'$ for some $N' > 0$. Together with the uniform convergence on compacts this gives the uniform convergence of $\{f_n\}_{n=1}^\infty$ on $\{z \in \mathbb{C} : \Im(z) > a\}$ for any $a > 0$. □

Remark 2.5. — Based on the above proof it is easy to show that $\rho(\mu_n, \mu) \to 0$, as $n \to \infty$, if and only if for every $\epsilon > 0$ there exists an integer N so that

$$\{z \in \mathbb{C} : d(z, [A_f, B_f]) > \epsilon\} \subset \mathbb{C} \backslash [A_{f_n}, B_{f_n}]$$

and

$$|f(z) - f_n(z)| < \epsilon, \quad z \in \{z \in \mathbb{C} : d(z, [A_f, B_f]) > \epsilon\},$$

whenever $n \geqslant N$.

Remark 2.6. — Note that $f = f_\mu$ extends as a univalent function to $\mathbb{C}$ if and only if μ is a point mass, i.e. $\mu = \delta_a$ for some $a \in \mathbb{R}$, and then $f(z) = z + a$, $z \in \mathbb{C}$.

Denote Σ the space of univalent functions $f : \mathbb{H} \to \mathbb{H}$ such that f is the reciprocal Cauchy transform of some $\mu \in \mathbf{M}^U$ and endow Σ with the metric ρ' induced from ρ, i.e. if $f_1, f_2 \in \Sigma$ and $f_1 = f_{\mu_1}$, $f_2 = f_{\mu_2}$, then $\rho'(f_1, f_2) = \rho(\mu_1, \mu_2)$. That ρ' is well-defined is a consequence of Stieltjes' inversion formula. Let $\mathfrak{P}(\Sigma)$ denote the space $C([0,\infty); \Sigma)$ of continuous paths $\Psi : [0,\infty) \to \Sigma$ with the topology of uniform convergence on compact intervals induced for example by the metric

$$D(\Phi, \Psi) \equiv \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\sup_{t\in[0,n]} \rho'(\Phi(t), \Psi(t))}{1 + \sup_{t\in[0,n]} \rho'(\Phi(t), \Psi(t))}.$$

Then $\mathfrak{P}(\Sigma)$ is a separable metric space. If $g(t, \psi; \cdot)$ are the values of the solutions of the Löwner equation (2.1) for fixed $(t, \psi) \in [0,\infty) \times \mathfrak{P}(\mathbb{R})$, and where z ranges over $\mathbb{H}\backslash\mathcal{K}_t$, then it follows from Proposition 2.1 and [1, Lemma 2] that $f \equiv g^{-1} \in \Sigma$. Finally, let L be the map defined by

$$\psi \in \mathfrak{P}(\mathbb{R}) \mapsto \{f(t, \psi; \cdot) : \mathbb{H} \to \mathbb{H}, t \in [0,\infty)\} \in \mathfrak{P}(\Sigma).$$

Then the probability measure $S_\kappa \equiv L_* \mathcal{W}_\kappa$ on $\mathfrak{P}(\Sigma)$ is the distribution of a stochastic Löwner evolution with parameter κ (SLE$_\kappa$).

PROPOSITION 2.7. — *The sequence $\{L \circ S_n\}_{n=1}^{\infty}$ converges in distribution to SLE$_\kappa$, i.e.*

$$L_*(S_n)_* P \Longrightarrow L_* \mathcal{W}_\kappa.$$

Proof. — Since $(S_n)_* P \Longrightarrow \mathcal{W}_\kappa$ as $n \to \infty$, it is enough to show that $L : \mathfrak{P}(\mathbb{R}) \to \mathfrak{P}(\Sigma)$ is continuous. For $t \in [0,\infty)$, $\psi \in \mathfrak{P}(\mathbb{R})$ and $z \in \mathbb{H}$, consider the initial value problem

$$\frac{\partial}{\partial s} h(s, \psi; z) = -\frac{2}{h(s, \psi; z) - \psi(t-s)}, \; 0 < s < t, \text{ and } h(0, \psi; z) = z. \quad (2.6)$$

Then $|(\partial/\partial s) h(s, \psi; z)| \leqslant 2/\Im(h(s, \psi; z))$ and

$$\frac{\partial}{\partial s} \Im(h(s, \psi; z)) = \frac{2\Im(h(s, \psi; z))}{|h(s, \psi; z) - \psi(t-s)|^2} > 0. \quad (2.7)$$

In particular, the initial value problem has a solution for $0 \leqslant s \leqslant t$. Given $\psi, \varphi \in \mathfrak{P}(\mathbb{R})$, let $u_1 = h(\cdot, \psi; z)$, $u_2 = h(\cdot, \varphi; z)$, and set

$$v(s, \psi; z) = -\frac{2}{z - \psi(t-s)}, \quad s \in [0,t], \psi \in \mathfrak{P}(\mathbb{R}), z \in \mathbb{H}.$$

Then

$$\begin{aligned} \dot{u}_2 - v(s, \psi; u_2) &= -\frac{2}{u_2 - \varphi(t-s)} + \frac{2}{u_2 - \psi(t-s)} \\ &= \frac{2(\psi(t-s) - \varphi(t-s))}{(u_2 - \varphi(t-s))(u_2 - \psi(t-s))}, \end{aligned}$$

and it follows from (2.7) that for $s \in [0,t]$

$$|\dot{u}_2 - v(s,\psi;u_2)| \leqslant \frac{2\sup_{s\in[0,t]}|\varphi(s)-\psi(s)|}{\Im(z)^2}. \tag{2.8}$$

Note also that

$$\left|\frac{\partial}{\partial z}v(s,\psi;z)\right| = \left|\frac{2}{(z-\psi(t-s))^2}\right| \leqslant \frac{2}{\Im(z)^2}. \tag{2.9}$$

Thus, by [4, 10.5.1.1], if $n \in \mathbb{Z}^+$, then

$$\sup_{s\in[0,t]}\sup_{z\in\mathbb{H}_{1/n}}|u_1-u_2| \leqslant \left(\exp\left[2n^2t\right]-1\right)\sup_{s\in[0,t]}|\varphi(s)-\psi(s)|.$$

Since the initial value problem (2.6) describes the reverse flow to the Löwner equation (2.1), we have $f(t,\psi;\cdot) \equiv h(t,\psi;\cdot)$. Thus, for $n \in \mathbb{Z}^+$,

$$\sup_{t\in[0,n]}\sup_{z\in\mathbb{H}_{1/n}}|f(t,\psi;z)-f(t,\varphi;z)| \leqslant \left(\exp\left[2n^3\right]-1\right)\sup_{t\in[0,n]}|\varphi(t)-\psi(t)|.$$

Consider now the initial value problem (2.6) with $z = x \in \mathbb{R}$ and let

$$A(t,\psi) = \mathbb{R}\backslash\{x\in\mathbb{R} : \min_{s\in[0,t]}|h(s,\psi;x)-\psi(t-s)| > 0\}.$$

By continuity, $A(t,\psi)$ is connected. In fact, $A(t,\psi) = [A_{f(t,\psi;\cdot)}, B_{f(t,\psi;\cdot)}]$. Indeed, it is clear that

$$A(t,\psi) \supseteq [A_{f(t,\psi;\cdot)}, B_{f(t,\psi;\cdot)}],$$

and also

$$f(t,\psi;A(t,\psi)\backslash[A_{f(t,\psi;\cdot)}, B_{f(t,\psi;\cdot)}]) \subset \mathbb{R}.$$

Since $\mathcal{K}(t,\psi) = \overline{\mathbb{H}}\backslash f(t,\psi;\overline{\mathbb{H}}\backslash A(t,\psi))$ and $\overline{\mathbb{H}\cap\mathcal{K}(t,\psi)} = \mathcal{K}(t,\psi)$ we have $f(t,\psi;A(t,\psi)\backslash[A_{f(t,\psi;\cdot)}, B_{f(t,\psi;\cdot)}]) = \emptyset$. It now follows from Lemma 2.8 that we can make the Hausdorff distance between $A(t,\psi)$ and $A(t,\varphi)$ as small as we like by choosing ψ close to φ. □

LEMMA 2.8.— *The Hausdorff distance between $A(t,\psi)$ and $A(t,\varphi)$ is less or equal $\delta > 0$ whenever*

$$\sup_{s\in[0,t]}|\psi(s)-\varphi(s)| < \frac{\delta}{3}\wedge\frac{2}{3}\cdot\frac{\delta}{\exp(9t/(2\delta^2))-1}. \tag{2.10}$$

Proof. — Let $\delta > 0$ be given. For $x, y \notin A(t,\psi)$ we have,

$$\begin{aligned}&\frac{\partial}{\partial s}[h(s,\psi;x) - \psi(t-s) - (h(s,\psi;y) - \psi(t-s))] \\ &\quad = 2\frac{h(s,\psi;x) - \psi(t-s) - (h(s,\psi;y) - \psi(t-s))}{(h(s,\psi;x) - \psi(t-s))(h(s,\psi;y) - \psi(t-s))}\end{aligned} \tag{2.11}$$

It follows that if for example $x > y > \max_{s\in[0,t]} \psi(t-s)$ and $d(y, A(t,\psi)) > 0$, then $\min_{s\in[0,t]} |h(s,\psi;x) - \psi(t-s)| \geqslant x - y$. In particular, $d(y, A(t,\psi)) \geqslant \delta$ implies $\min_{s\in[0,t]} |h(s,\psi;x) - \psi(t-s)| \geqslant \delta$. We will show that the latter together with (2.10) implies that $x \notin A(t,\varphi)$. Then

$$\sup_{x\in A(t,\varphi)} d(x, A(t,\psi)) \leqslant \delta.$$

By symmetry we then also have $\sup_{x\in A(t,\psi)} d(x, A(t,\varphi)) \leqslant \delta$.

If (2.10) holds, then $|\psi(0)-\varphi(0)| < \delta/3$. Since also $|h(0,\psi;x)-\psi(t)| \geqslant \delta$, there exists $t_0 \in (0,t]$ such that $\min_{s\in[0,t_0]} |h(s,\varphi;x) - \varphi(t-s)| > 0$. We claim that we may choose $t_0 = t$. For if not, then there exists $t_0 < t' \leqslant t$ such that $\lim_{s\nearrow t'} h(s,\varphi;x) = \varphi(t-t')$. Let $u = h(s,\psi;x)$ and set $v(s,\varphi;u) = -\frac{2}{u-\varphi(t-s)}$. Then

$$\left|\frac{\partial}{\partial u} v(s,\varphi;u)\right| \leqslant \frac{9}{2\delta^2}, \quad s \in [0,t]$$

since $|h(s,\psi;x) - \varphi(t-s)| \geqslant 2\delta/3$. Furthermore,

$$\dot{u} - v(s,\varphi;u) = \frac{2(\varphi(t-s) - \psi(t-s))}{(h(s,\psi;x) - \psi(t-s))(h(s,\psi;x) - \varphi(t-s))}$$

and so

$$|\dot{u} - v(s,\varphi;u)| \leqslant \frac{3}{\epsilon^2} \sup_{s\in[0,t]} |\varphi(s) - \psi(s)|.$$

Again by [4, 10.5.1.1]

$$\begin{aligned}&\sup_{s\in[0,t]} |h(s,\psi;x) - h(s,\varphi;x)| \\ &\quad \leqslant \frac{2}{3} \sup_{s\in[0,t]} |\varphi(s) - \psi(s)| \left(\exp\left(\frac{9t}{2\epsilon^2}\right) - 1\right) \\ &\quad \leqslant \frac{4}{9}\delta.\end{aligned} \tag{2.12}$$

Thus, if $t_0 \leqslant t$, then $|h(t_0,\psi;x) - \psi(t-t_0)| \leqslant \frac{7}{9}\delta$, a contradiction. Hence $x \notin A(t,\varphi)$. $\square$

Remark 2.9. — Since

$$\begin{aligned}\delta &\leqslant |h(s,\psi;x)-\psi(t-s)|\\ &\leqslant |h(s,\psi;x)-h(s,\varphi;x)|+|h(s,\varphi;x)-\varphi(t-s)|+|\varphi(t-s)-\psi(t-s)|\end{aligned}$$

the above proof together with (2.10) and (2.12) implies that

$$\min_{s\in[0,t]}|h(s,\varphi;x)-\varphi(t-s)|\geqslant\frac{2}{9}\delta.$$

For $t\in[0,\infty)$, $\psi\in\mathfrak{P}(\mathbb{R})$, $z\in\mathbb{H}$ and $n\in\mathbb{Z}^+$, consider the initial value problem $h_n(0,\psi;z)=z$ and

$$\frac{\partial}{\partial s}h_n(s,\psi;z)=-\frac{2}{h_n(s,\psi;z)-\psi\left(\frac{m}{n}\right)},$$

if $0<s<t$ and $t-s\in\left[\frac{m}{n},\frac{m+1}{n}\right)$ for some $m\in\mathbb{N}$. Then $|(\partial/\partial s)h_n(s,\psi;z)|\leqslant 2/\Im(h_n(s,\psi;z))$ and

$$\frac{\partial}{\partial s}\Im(h_n(s,\psi;z))>0. \tag{2.13}$$

In particular, the initial value problem has a solution for $0\leqslant s\leqslant t$. Proposition 2.1 extends to piecewise continuous ψ and thus $f_n(t,\psi;\cdot)\equiv h_n(t,\psi;\cdot)\in\Sigma$. Let L_n be the map defined by

$$\psi\in\mathfrak{P}(\mathbb{R})\mapsto\{f_n(t,\psi;\cdot):\mathbb{H}\to\mathbb{H},t\in[0,\infty)\}\in\mathfrak{P}(\Sigma).$$

We can consider the family of random variables $\{(L_n\circ S_n)(\frac{m}{n})\}_{m=0}^{\infty}$ as a random walk on Σ as follows. For $a\in\mathbb{R}$ let $r_n(a;\cdot)$ be the conformal map given by

$$z\in\mathbb{H}\mapsto r_n(a;z)=a+\sqrt{(z-a)^2-\frac{4}{n}}\in\mathbb{H}.$$

Then

$$r_n(a;\mathbb{H})=\mathbb{H}\backslash\{z\in\mathbb{H}:\Re(z)=a\text{ and }\Im(z)\in[0,\frac{2}{\sqrt{n}}]\}. \tag{2.14}$$

For $n\in\mathbb{Z}^+$, $\omega\in\Omega$, and $z\in\mathbb{H}$ set $D_n(0,\omega;z)=z$ and define inductively

$$D_n(m,\omega;z)=D_n\left(m-1,\omega;r_n\left(S_n\left(\frac{m-1}{n},\omega\right);z\right)\right),$$

if $m>0$. Then, for every $n\in\mathbb{Z}^+$ and $\omega\in\Omega$, $\{D_n(m,\omega;\cdot)\}_{m=0}^{\infty}$ is a family of conformal maps from $\mathbb{H}$ into $\mathbb{H}$ and,

$$D_n(m,\omega;\mathbb{H})\supsetneq D_n(m+1,\omega;\mathbb{H}),\qquad\text{for every }m\in\mathbb{N}. \tag{2.15}$$

In fact, $r_n(a;z)$ is the solution at time $t = 1/n$ of the initial value problem $(\partial/\partial s)h(s,a;z) = -2/(h(s,a;z) - a)$, $h(0,a;z) = z$. Thus $r_n(a;\cdot) \in \Sigma$ for every $a \in \mathbb{R}$, $D_n(m,\omega;\cdot) \in \Sigma$ for every $m \in \mathbb{N}$ and $\omega \in \Omega$, and finally

$$D_n(m,\omega;z) = f_n\left(\frac{m}{n}, S_n(\cdot,\omega); z\right).$$

By boundary correspondence, $D_n(m,\omega;\cdot)$ maps the real axis to a finite number of Jordan arcs. All prime ends are of the first kind and hence $D_n(m,\omega;\cdot)$ extends continuously to $\bar{\mathbb{H}}$, see [13, Theorem 2.21].

THEOREM 2.10. — *The sequence $\{L_n \circ S_n\}_{n=1}^{\infty}$ converges in distribution to SLE_κ, i.e.*

$$(L_n)_*(S_n)_*P \Longrightarrow L_*\mathcal{W}_\kappa.$$

Proof. — With the notation from above we have

$$\begin{aligned}\dot{h}_n - v(s,\psi;h_n) &= -\frac{2}{h_n - \psi\left(\frac{m}{n}\right)} + \frac{2}{h_n - \psi(t-s)} \\ &= \frac{2\left(\psi(t-s) - \psi\left(\frac{m}{n}\right)\right)}{\left(h_n - \psi\left(\frac{m}{n}\right)\right)\left(h_n - \psi(t-s)\right)},\end{aligned}$$

and it follows from (2.13) that for $s \in [0,t]$

$$|\dot{h}_n - v(s,\psi;h_n)| \leqslant \frac{2\rho(n,t;\psi)}{\Im(z)^2}, \tag{2.16}$$

where $\rho(n,t;\psi) \equiv \sup\{|\psi(r) - \psi(s)| : 0 \leqslant s < r \leqslant t \text{ with } r - s \leqslant \frac{1}{n}\}$ is the modulus of continuity of ψ. Thus, from [4, 10.5.1.1], if $N \in \mathbb{Z}^+$,

$$\sup_{s\in[0,t]} \sup_{z\in\mathbb{H}_{1/N}} |h_n(s,\psi;z) - h(s,\psi;z)| \leqslant \left(\exp[2N^2t] - 1\right)\rho(n,t;\psi).$$

Similarly, the proof of Lemma 2.8 extends to show that $A(t,\psi_n) \to A(t,\psi)$ in the Hausdorff distance, as $n \to \infty$, where ψ_n is defined by $\psi_n(t-s) = \psi(m/n)$ if $t - s \in [m/n,(m+1)/n)$, and where we now define $A(t,\psi)$ as the convex closure of $\mathbb{R}\backslash\{x \in \mathbb{R} : \min_{s\in[0,t]} |h(s,\psi;x) - \psi(t-s)| > 0\}$. It follows that, for each $\psi \in \mathfrak{P}(\mathbb{R})$, $D(f_n(\cdot,\psi;\cdot), f(\cdot,\psi;\cdot)) \to 0$ as $n \to \infty$. In particular, $D(L_n \circ S_n, L \circ S_n) \to 0$ in probability as $n \to \infty$ and by the principle of accompanying laws, [19, 3.1.14], and Proposition 2.7 we get

$$(L_n)_*(S_n)_*P \Longrightarrow L_*\mathcal{W}_\kappa.$$

□

3. Properties of discrete Löwner evolution

For all $n \in \mathbb{Z}^+$,

$$r_n(a;z) = \frac{1}{\sqrt{n}} r_1\left(\sqrt{n}a; \sqrt{n}z\right), \quad a \in \mathbb{R}, z \in \mathbb{H},$$

and

$$D_n\left(0, \omega; \frac{z}{\sqrt{n}}\right) = \frac{1}{\sqrt{n}} D_1(0, \omega; z), \quad \omega \in \Omega, z \in \mathbb{H}.$$

Using

$$\sqrt{n} S_n\left(\frac{m}{n}, \omega\right) = \sum_{k=1}^{m} X_k(\omega) = S_1(m, \omega), \quad \omega \in \Omega, m \in \mathbb{N},$$

it follows by induction that

$$D_n\left(m, \omega; \frac{z}{\sqrt{n}}\right) = \frac{1}{\sqrt{n}} D_1(m, \omega; z), \quad \omega \in \Omega, z \in \mathbb{H}, m \in \mathbb{N}.$$

Thus to study the families $\{D_n(m)\}_{m=0}^{\infty}$ we may as well restrict to $n = 1$.

Writing D, S, r for D_1, S_1, and r_1, respectively, $\{D(m)\}_{m=0}^{\infty}$ is defined by

$$D(0, \omega; z) = z, \quad \omega \in \Omega, z \in \mathbb{H}, \tag{3.1}$$

and

$$D(m, \omega; z) = D(m-1, \omega; r[S(m-1, \omega); z]), \quad \omega \in \Omega, z \in \mathbb{H}, \tag{3.2}$$

if $m \in \mathbb{Z}^+$.

THEOREM 3.1.— *For $(m, n) \in \mathbb{N}^2$ such that $m \leqslant n$, and $\omega \in \Omega$, define conformal maps $H(m, n, \omega; \cdot) : \mathbb{H} \to \mathbb{H}$ by*

$$H(m, n, \omega; \cdot) = D(m, \omega; \cdot)^{-1} \circ D(n, \omega; \cdot)$$

and set

$$\tilde{H}(m, n, \omega; z) = H(m, n, \omega; z + S(m, \omega)) - S(m, \omega).$$

Then the family $\{\tilde{H}(m, n)\}_{n=m}^{\infty}$ is independent of the family $\{D(k)\}_{k=0}^{m+1}$. Furthermore, if the random variables $\{X_k\}_{k=0}^{\infty}$ are identically distributed, then the distribution of the random variable $\omega \in \Omega \mapsto \tilde{H}(m, n, \omega; \cdot) \in \Sigma$ under P is the same as the distribution of $\omega \in \Omega \mapsto D(n-m, \omega; \cdot) \in \Sigma$ under P. Finally, if X_k is symmetric for each $k \in \mathbb{N}$, then, for each $m \in \mathbb{N}$, $\omega \in \Omega \mapsto D(m, \omega; \cdot) \in \Sigma$ and $\omega \in \Omega \mapsto \chi \circ D(m, \omega; \cdot) \circ \chi \in \Sigma$ have the same distribution under P, where the map χ is given by

$$x + \sqrt{-1}y \in \mathbb{H} \mapsto -x + \sqrt{-1}y \in \mathbb{H}.$$

Proof. — For the first statement, note that the case $n = m$ is trivial and consider the case $n > m$. From (3.2) and induction on n it follows that

$$D(n; z) = D\left(m; r\left[S(m); r\left[S(m+1); \ldots; r[S(n-1); z] \cdots\right]\right]\right). \qquad (3.3)$$

In particular, for any $m \in \mathbb{Z}^+$, $D(m)$ is $\sigma(X_1, \ldots, X_{m-1})$-measurable. Furthermore, we now get

$$\tilde{H}(m, n; z) = r\left[S(m); r\left[S(m+1); \ldots; r[S(n-1); z + S(m)] \cdots\right]\right] - S(m).$$

Applying repeatedly the identity $r(a - b; z - b) + b = r(a; z)$ gives

$$\begin{aligned} &\tilde{H}\ (m, n; z) \qquad (3.4) \\ &\quad = r[S(m) - S(m); r[S(m+1) - S(m); \ldots; r[S(n-1) - S(m); z] \cdots]]. \end{aligned}$$

This implies the first statement because the random variables $\{X_k\}_{k=0}^{\infty}$ are mutually independent. The expression (3.4) also implies the second statement, under the assumption that the X_k's are identically distributed. Regarding the third statement, note that $r(a; \cdot) \circ \chi = \chi \circ r(-a; \cdot)$ and that $\chi^{-1} = \chi$. Thus it follows from (3.3) that

$$\chi \circ D(m, \omega; \cdot) \circ \chi = \tilde{D}(m, \omega; \cdot),$$

where $\tilde{D}(m)$ is defined as $D(m)$ in (3.2), with $-S(m)$ in place of $S(m)$, $m \in \mathbb{N}$. The symmetry of the X_k's implies the symmetry of the $S(m)$'s and so the distributions of $\omega \in \Omega \mapsto D(m, \omega) \in \Sigma$ and $\omega \in \Omega \mapsto \tilde{D}(m, \omega) \in \Sigma$ under P are equal. $\square$

Remark 3.2. — The above theorem gives the discrete version of corresponding results for SLE, see [7]. The first statement shows that $\{D(m)\}_{m=0}^{\infty}$ has, up to a shift, independent increments relative to composition of maps, the second statement shows that the shifted increments are stationary (assuming the X_k's are identically distributed) and the third statement is a kind of reflection symmetry of $D(m)$ (for symmetric X_k's) . Note that if we look at the images of the maps $D(m, \omega; \cdot)$, then we find that $\omega \mapsto D(m, \omega; \mathbb{H})$ and $\omega \mapsto \chi(D(m, \omega; \mathbb{H})$ have the same distribution under P on a suitably defined space of domains in the upper half-plane, because $\chi(\mathbb{H}) = \mathbb{H}$. Equivalently, the distributions of the "hulls" $\mathbb{H} \backslash D(m; \mathbb{H})$ is invariant under χ. This is the reflection symmetry statement in [7]. In fact, the weak convergence of the increments and the continuity of the map L imply that the continuous results for SLE can be deduced directly from Theorem 3.1.

Assume now that $\{X_n\}_{n=0}^{\infty}$ is a sequence of independent and identically distributed random variables of mean-value 0 and variance $\kappa \geqslant 0$ on a probability space $(\Omega, \mathcal{F}, P)$. In that case we call the family $\{D(m)\}_{m=0}^{\infty}$ a *discrete Löwner evolution with parameter* κ (DLE_κ).

When the X_n's are Bernoulli random variables, i.e.

$$P(X_n = \sqrt{\kappa}) = P(X_n = -\sqrt{\kappa}) = \frac{1}{2}, \quad n \in \mathbb{N},$$

then the corresponding discrete Löwner evolution has a trivial "phase transition" at $\kappa = 4$.

PROPOSITION 3.3. — *Let $\{D(m)\}_{m=0}^{\infty}$ be a discrete Löwner evolution with parameter κ driven by a sequence of Bernoulli random variables as above. If $\kappa \leqslant 4$, then $\overline{\mathbb{H}\backslash D(m,\omega;\mathbb{H})}$ is connected in $\overline{\mathbb{H}}$ for all $\omega \in \Omega$ and $m \in \mathbb{N}$. If $\kappa > 4$, then $\overline{\mathbb{H}\backslash D(m,\omega;\mathbb{H})}$ is not connected in $\overline{\mathbb{H}}$, for all $\omega \in \Omega$ and $m \geqslant 2$.*

Proof. — This follows immediately by considering the composition of maps $r(0;\cdot) \circ r(\sqrt{\kappa};\cdot)$. The closure of the complement of the image of $\mathbb{H}$ under this map is connected in $\overline{\mathbb{H}}$ if and only if $\kappa \leqslant 4$. $\square$

Graphically, for $\kappa \leqslant 4$ the omitted set, i.e. $\overline{\mathbb{H}\backslash D(m,\omega;\mathbb{H})}$, is a single tree made up of m curvy branches. The tree grows one branch at each step. Orient the branches in the direction of the root and label the end-point closest to the root "bottom" and the other end-point "top". If $0 < \kappa < 4$, then the $(m+1)$st branch "branches off" the mth branch somewhere between the mth branch's top and bottom. We call the segment of the mth branch between the branch-point to the $(m+1)$st branch and the top of of the mth branch the overshoot. For $\kappa = 4$ the branch-point is at the bottom and the tree looks like a bushel, all branches emanating from the point $z = 0$, while for $\kappa = 0$ the branch point is at the top and the tree degenerates to a vertical line segment in the closed upper half-plane beginning at $z = 0$. As κ decreases from 4 to 0 the branch-point increases from bottom to top. Using the orientation to wards the root, the $(m+1)$st branch branches off to the right of the mth branch if $X_m > 0$, and to the left if $X_m < 0$.

If $\kappa > 4$, $\overline{\mathbb{H}\backslash D(m,\omega;\mathbb{H})}$ consists of m branches forming at least $\min(m,2)$ trees and at most m trees. The latter will be the case for instance if the driving random walk makes all of its first $m-1$ steps in one direction, while the former picture emerges if the walk changes direction at every step. Typically, for large m neither will be the case and it would be interesting for example to calculate the expected number of trees, or the distribution of the distance of the roots of neighboring trees. For example, by first letting the random walk alternate directions for a long time and then stepping only in one direction for a long time, it is easy to see that roots may be spaced arbitrarily far apart.

If X_n is centered and of variance κ but not necessarily a Bernoulli random variable then the above picture should still be approximately right. Of course, even for $\kappa \leqslant 4$ we may now get several trees. But their number or spacing should be small as $m \to \infty$ compared to the case when $\kappa > 4$.

The phase transition for SLE at $\kappa = 4$ is the fact that $\mathcal{K}(t)$ is a simple curve for $\kappa \leqslant 4$, P–a.s., and that it is not a simple curve for $\kappa > 4$, P–a.s, [17]. Thus, in the scaling limit, the overshoots disappear, creating a simple curve if $\kappa \leqslant 4$. For $\kappa \geqslant 4$, the disjoint trees become connected in the scaling limit (if they are too small, some might also disappear).

We now study a question related to this phase transition following ideas in [7].

Let $g_n(m) = (D_n(m))^{-1}$, $n \in \mathbb{Z}^+$, $m \in \mathbb{N}$. Then

$$g_n(m, \omega; \cdot) = \left(r_n \left[S_n \left(\frac{m-1}{n}, \omega \right); \cdot \right] \right)^{-1} \circ g_n(m-1, \omega; \cdot),$$

that is

$$g_n(m, \omega; \cdot) = S_n \left(\frac{m-1}{n}, \omega \right) + \sqrt{\left[g_n(m-1, \omega; \cdot) - S_n \left(\frac{m-1}{n}, \omega \right) \right]^2 + \frac{4}{n}}.$$

In particular, if we set

$$Y_n(m, \omega; \cdot) = \frac{g_n(m, \omega; \cdot) - S_n \left(\frac{m-1}{n}, \omega \right)}{\sqrt{\kappa}},$$

then

$$Y_n(m, \omega; \cdot) = \sqrt{\left(Y_n(m-1, \omega; \cdot) - \frac{X_m}{\sqrt{n\kappa}} \right)^2 + \frac{4}{n\kappa}}.$$

Note that $Y_1(m, \omega; \cdot) = \sqrt{n} Y_n(m, \omega; \cdot)$ and $X'_m \equiv X_m / \sqrt{\kappa}$ is centered with variance 1. For $z = x \in \mathbb{R} \backslash \{0\}$ set $Y_0 = x$ and $Y_m = Y_1(m, \omega; x)$. Then $\{Y_m\}_{m=0}^{\infty}$ is a Markov chain satisfying the evolution equation

$$Y_m = \sqrt{(Y_{m-1} - X'_m)^2 + \frac{4}{\kappa}}, \quad m \in \mathbb{Z}^+,$$

or equivalently

$$Y_m^2 - Y_{m-1}^2 = -2 Y_{m-1} X'_m + (X'_m)^2 + \frac{4}{\kappa}, \quad m \in \mathbb{Z}^+.$$

THEOREM 3.4.— *Suppose that the moment generating function of X_1' has a positive radius of convergence. Then the Markov chain $\{Y_m\}_{m=0}^{\infty}$ is transient if $\kappa < 4$, and it is recurrent if $\kappa \geqslant 4$.*

Proof. — It is easy to see that $\limsup_{m\to\infty} Y_m = +\infty$, P-a.s. Using Taylor series together with the assumption that the moment generating function of X_1' has a positive radius of convergence, we see that

$$\begin{aligned}
&\lim_{y\to\infty} 2y\,\mathbb{E}^P[Y_m - Y_{m-1}|Y_{m-1} = y] \\
&\quad = \lim_{y\to\infty} 2y^2\,\mathbb{E}^P\left[\sqrt{\left(1-\frac{X_m'}{y}\right)^2 + \frac{4}{y^2\kappa}} - 1\right] \\
&\quad = \lim_{y\to\infty}\left(2y^2\,\mathbb{E}^P\left[-\frac{X_m'}{y} + \frac{(X_m')^2}{2y^2} + \frac{2}{y^2\kappa} - \frac{(X_m')^2}{2y^2}\right] + O(1/y)\right) = \frac{4}{\kappa}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\mathbb{E}^P[(Y_m - Y_{m-1})^2|Y_{m-1} = y] \\
&\quad = \mathbb{E}^P[-2Y_{m-1}X_m' + (X_m')^2 + 4/\kappa - 2Y_{m-1}(Y_m - Y_{m-1})|Y_{m-1} = y] \\
&\quad = 1 + \frac{4}{\kappa} - 2y\,\mathbb{E}^P[Y_m - Y_{m-1}|Y_{m-1} = y] \to 1,
\end{aligned}$$

as $y \to \infty$. Thus, by [6, Theorem 3.2], the result follows. □

The continuous analogoue of the Markov chain $\{Y_m\}_{m=0}^{\infty}$ is a Bessel process of dimension $1 + 4/\kappa$. It is well known that Bessel processes are recurrent for dimension $d \leqslant 2$ and transient for $d > 2$. The above result thus is exactly as expected.

A. Monotonic Independence and Löwner Map

The following definition is taken from [15]. Let $(\mathcal{A}, \phi)$ be a C^*-probability space consisting of a unital C^*-algebra $\mathcal{A}$ and a state ϕ over $\mathcal{A}$. The elements X of $\mathcal{A}$ are called random variables and $\phi(X)$ their expectation.

DEFINITION A.1.— *A family $\{X_i\}_{i\in I} \subset \mathcal{A}$ of random variables on $(\mathcal{A}, \phi)$ with totally ordered index set I is said to be monotonically independent with respect to a state ϕ if the following two conditions are satisfied.*

(*a*) *Whenever $i < j$, $k < j$, and $p \in \mathbb{N}$, then*

$$X_i X_j^p X_k = \phi(X_j^p) X_i X_k.$$

(*b*) *Whenever* $i_m > \cdots > i_1 > i$, $j_n > \cdots > j_1 > i$, *and* $p, p_k, q_l \in \mathbb{N}$, *then*

$$\begin{aligned}\phi(X_{i_m}^{p_m} \cdots X_{i_1}^{p_1} X_i^p X_{j_1}^{q_1} \cdots X_{j_n}^{q_n}) \\ = \phi(X_{i_m}^{p_m}) \cdots \phi(X_{i_1}^{p_1}) \phi(X_i^p) \phi(X_{j_1}^{q_1}) \cdots \phi(X_{j_n}^{q_n}).\end{aligned}$$

THEOREM A.2. — *Let* $X_1, X_2, \ldots, X_n \in \mathcal{A}$ *be monotonically independent self-adjoint random variables on* $(\mathcal{A}, \phi)$, *in the natural order of* $\{1, 2, \ldots, n\}$. *If* $f_{X_k} : \mathbb{H} \to \mathbb{H}$ *denotes the reciprocal Cauchy transform of the distribution of* X_k, *for* $1 \leqslant k \leqslant n$, *then*

$$f_{X_1+X_2+\cdots+X_n} = f_{X_1} \circ f_{X_2} \circ \cdots \circ f_{X_n}.$$

Define for a pair of probability measures μ, ν on $\mathbb{R}$ the monotonic convolution λ of μ and ν, denoted by $\lambda = \mu \rhd \nu$, as the unique probability measure λ satisfying $f_\lambda(z) = f_\mu(f_\nu(z))$.

COROLLAY A.3. — *For* $\psi \in \mathfrak{P}(\mathbb{R})$ *let* $f(t, \psi) = g^{-1}(t, \psi)$ *be the solution to the Löwner equation (2.1). For* $0 \leqslant s \leqslant t$ *set* $f_{s,t} = g_s \circ f_t$. *Then* $f_{s,t} = f_{\mu_{s,t}}$ *for a unique probability measure* $\mu_{s,t}$, *and, for* $r \leqslant s \leqslant t$,

$$\mu_{r,s} \rhd \mu_{s,t} = \mu_{r,t}.$$

Similarly, for $\omega \in \Omega$ *and* $0 \leqslant m \leqslant n$, $H(m, n, \omega)$ *is the reciprocal Cauchy transform of a unique probability measure* $\mu_{m,n}$, *and if* $l \leqslant m \leqslant n$, *then*

$$\mu_{l,m} \rhd \mu_{m,n} = \mu_{l,n}.$$

Thus $\{f(t, \psi) : t \in [0, \infty)\}$ and $\{D(m, \omega) : m \in \mathbb{N}\}$ correspond to monotonically independent increment processes in some noncommutative probability space $(\mathcal{A}, \phi)$. In fact, the "building blocks" for our discrete Löwner evolution, the functions $r_n(a; z) = a + \sqrt{(z-a)^2 - 4/n}$, are the reciprocal Cauchy transforms of some well known distributions: the arcsine distribution supported in $(-2/\sqrt{n}, 2/\sqrt{n})$ if $a = 0$, and a deformation of the arcsine distribution if $a \neq 0$. Note that the arcsine distribution plays for monotonic convolution the role the Gaussian distribution plays for "classical convolution." For example, the monotonic central limit theorem establishes convergence to an arcsine distribution.

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