ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

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Annales de la faculté des sciences de Toulouse 6e série, tome 13, nº 2 (2004), p. 149-177

<http://www.numdam.org/item?id=AFST_2004_6_13_2_149_0>

Annales de la Faculté des Sciences de Toulouse Vol. XIII, n° 2, 2004
pp. 149–177

Lévy Processes, Pseudo-differential Operators and Dirichlet Forms in the Heisenberg Group [(*)]

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ABSTRACT. — N. Jacob and his colleagues have recently made many interesting investigations of Markov processes in Euclidean space where the infinitesimal generator of the associated semigroup is a pseudo-differential operator in the Kohn-Nirenberg sense. We wish to extend this programme to the Heisenberg group where we can utilise the Weyl calculus to build pseudo-differential operators and we begin by considering Levy processes. We obtain the general form of symbol for infinitesimal generators. We then investigate a natural sub-class of group-valued processes whose components are a Lévy process in phase space and the associated Lévy area process on the real line. These are applied to clarify Gaveau's probabilistic proof for Mehler's formula.

In the second part of the paper, we describe some properties of the generator in its Schrödinger representation. In particular, when this operator is positive and symmetric, we show that it does not always give rise to a Dirichlet form and we obtain a Beurling-Deny type formula in which the jump measure may take negative values. When a bona fide Dirichlet form is induced, we give conditions under which there is an explicit description of the associated Hunt process which lives in extended Euclidean space.

RÉSUMÉ. — N. Jacob et ses collaborateurs ont récemment étudié des processus de Markov à valeurs dans des espaces préhilbertiens dont le générateur infinitésimal est un opérateur pseudo-différentiel au sens de Kohn Nirenberg. Nous souhaitons généraliser cette étude au groupe de Heisenberg quand on utilise le calcul de Weyl pour construire les opérateurs pseudo-différentiels. Nous commençons par le cas des processus de Lévy, et obtenons une forme générale pour le symbole de leur générateur infinitésimal. Ensuite nous considérons des processus dont les coordonnées

(*) Reçu le 29 août 2003, accepté le 14 janvier 2004
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dans l'espace de phase sont des processus de Lévy, et dont la partie réelle est l'aire de Lévy associée. Ceci permet de clarifier une preuve probabiliste de la formule de Mehler, dûe à Gaveau.

Dans une deuxième partie, nous décrivons quelques propriétés du générateur de la représentation de Schrödinger. Dans le cas où ces opérateurs sont symétriques, positifs, nous montrons qu'ils ne correspondent pas toujours à une forme de Dirichlet. Cependant dans le cas où ils correspondent à une forme de Dirichlet, on donne une description du processus de Hunt associé sous certaines conditions qui sont précisées.

1. Introduction

Let $X = (X(t), t \geqslant 0)$ be a time-homogeneous Markov process whose state space is a Lie group G, then we obtain a Markov semigroup $(T(t), t \geqslant 0)$ on the Banach space $C_b(G)$ of bounded, continuous functions on G, through the prescription

$$(T(t)f)(\sigma) = \mathbf{E}(f(X(t))|X(0) = \sigma),$$

for each $f \in C_b(G), \sigma \in G$, *i.e.* $(T(t), t \geqslant 0)$ is a strongly continuous one-parameter contraction semigroup which is positivity preserving and conservative, *i.e.* each $T(t)1 = 1$. In the case where G is $\mathbf{R}^n$, there has recently been much interesting work by N. Jacob and his collaborators in constructing and investigating such processes where the infinitesimal generator $\mathcal{A}$ of the semigroup is a pseudo-differential operator (in the Kohn-Nirenberg sense). More specifically we have, for sufficiently regular functions f in the domain of $\mathcal{A}$, and for each $x \in \mathbf{R}^n$,

$$(\mathcal{A}f)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{ix.\xi} q(x,\xi) \hat{f}(\xi) d\xi, \tag{1.1}$$

where $\hat{f}$ is the Fourier transform. In a classical paper by Courrège [8], it was shown that a generator of a Feller process has such a representation whenever the test functions are in its domain. The function q is the symbol of the operator $\mathcal{A}$ and a key feature of this approach is to extract information about the probabilistic behaviour of X from the properties of q. The prototype for such studies is the case where X is a Lévy process (*i.e.* a process with stationary and independent increments). In this case, q is a function of ξ alone, and is precisely the characteristic exponent of the process as determined by the celebrated Lévy-Khintchine formula. We refer readers to the monograph [19] and the recent survey article [18] for more details.

In this paper, we are motivated by the prospect of extending this analysis to more general Lie groups. Of course, a major obstacle here is the lack of a suitable Fourier transform, so we need to restrict our endeavours to a context where such objects exist. One possibility is to take the group to be semisimple with finite centre. Here we can employ the spherical transform based on Harish-Chandra's theory of spherical functions. For recent investigations of Lévy processes in this context – see [4]. We take a different approach in this article and consider the case where G is the Heisenberg group $\mathbf{H}^n$. This is interesting for a number of reasons. Its manifold structure is that of a $(2n+1)$-dimensional Euclidean space, and from a physical point of view it is useful to think of this as an extension of an even-dimensional phase space comprising n position and n momentum co-ordinates. In particular, the ordinary Fourier transform can now be utilised. As a group it is step 2 nilpotent and so noncommutativity enters quite mildly. Through the Schrödinger representation, this group plays a leading role in quantum mechanics, but it also figures prominently in a number of other branches of mathematics, particularly harmonic analysis (see [15]).

There is a rich theory of pseudo-differential operators in the Heisenberg group based on the so-called the Weyl functional calculus and by analogy with (1.1) we might aim to study Markov processes whose generators can be so represented as

$$(\mathcal{A}_\pi f)(x) = (2\pi)^{-n} \int_{\mathbf{R^{2n}}} \sigma(\frac{1}{2}(x+y), \xi) e^{i(x-y)\cdot\xi} f(y) dy d\xi. \tag{1.2}$$

We should emphasise that this calculus works through the Schrödinger representation π and so we deal not with the original semigroup generator $\mathcal{A}$, but its image $\mathcal{A}_\pi$ which can be thought of as its "quantisation". Indeed, operators of the form (1.2) were originally introduced by Hermann Weyl as a means of associating quantum mechanical observables to phase space functions q (see [30], Chapter 4, section D.14). For a more recent and extensive mathematical account of these operators see Chapter 2 of [9].

Our main focus in this paper is on the operators $\mathcal{A}_\pi$, and we seek to understand these from both an analytic and probabilistic viewpoint within the context of Lévy processes in $\mathbf{H}^n$. Indeed, these processes are more complicated than their counterparts in Euclidean space and it is clearly important to learn as much as possible about them before progressing to greater generality. The paper is organised as follows. In section 1, we briefly review the relationship between the Lévy-Khintchine formula and infinitesimal generators in Euclidean space. Section 2 summarises results we need about

Lévy processes in Lie groups, while section 3 is a primer on the Heisenberg group. Our main results can be found in sections 4 and 5. In section 4, we give the Weyl symbol for operators $\mathcal{A}_\pi$ associated to general Lévy processes in $\mathbf{H}^n$. We then introduce a class of such processes which are called phase-dominated as $\mathcal{A}_\pi$ only acts non-trivially on phase-space co-ordinates. We show that any such process can be written in the form $\rho = (A, X_Q, X_P)$ where $X = (X_Q, X_P)$ is a Lévy process in $\mathbf{R}^{2n}$ and A is the associated stochastic area process. In the case where X is a standard Brownian motion, we gain a new perspective on Gaveau's probabilistic proof of Mehler's formula for the symbol of the semigroup generated by $\mathcal{A}_\pi$ ([11]).

In section 5, we investigate analytic properties of the operator $\mathcal{A}_\pi$. In particular, we show that the smooth functions of compact support form a core for this operator. We find conditions under which $-\mathcal{A}_\pi$ is positive and symmetric and also, when it preserves the subspace of real-valued functions. At this stage, we would expect to obtain a symmetric Dirichlet form, and thus show that $\mathcal{A}_\pi$ also generates a sub-Markov semigroup in the L^2-space. In fact, this is not always the case. We find that the Beurling-Deny formula appears with a signed jump measure J which is not necessarily positive (although we can provide some classes of examples where it always is). This seems to be a new phenomenon which will, we hope, be more fully understood in the future.

In the case where we obtain a legitimate Dirichlet form, this induces a Hunt process on $\mathbf{R}^n \cup \{\Delta\}$, where Δ is the cemetery. An interesting feature is that both the diffusion and jump characteristics of the original Lévy process in G contribute to killing of the Dirichlet form (when it exists) and hence of the associated Hunt process. In proposition 8, we give conditions under which this induced process is nothing but translation through X_P, killed at a stopping time determined by X_Q.

For related work to this, readers might consult the monograph [22] which mainly studies limit theorems for Brownian motion in $\mathbf{H}^n$, including the central limit theorem and the law of the iterated logarithm. Stable processes in $\mathbf{H}^n$ are investigated in [12], see also [20]. Pap [23] has studied Lévy processes in general nilpotent Lie groups and has obtained a related result to that of our theorem 3 below. In [24], he has made an extensive study of Gaussian measures on $\mathbf{H}^1$. Hoh [14] has recently investigated Markov processes on manifolds whose generators are pseudo-differential operators in the Kohn-Nirenberg sense in local co-ordinates while Baldus [5, 6] has studied a very general framework for Markov processes in manifolds, employing a calculus of pseudo-differential operators due to Hörmander.

Acknowledgement. — This work was began while D.A. enjoyed the excellent hospitality of the Laboratoire de Statistique et de Probabilités at the Université Paul Sabatier, Toulouse. Both authors would like to thank Rene Schilling and Gyula Pap for helpful suggestions.

Notation. — We will use Einstein summation convention throughout and will sometimes be quite cavalier about writing matrix entries c^{ij} as c_{ij}. We emphasise that no metric tensor is involved in this purely notational raising and lowering of indices.

$|.|$ will always denote the Euclidean norm in $\mathbf{R}^d$.

We denote $D = (D_1, \ldots, D_n)$ where each $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index so each $\alpha_j \in \mathbf{N} \cup \{0\}$, we define

$$D^\alpha = \frac{1}{i^{|\alpha|}} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}},$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_n$. We call n the length of the multi-index, and note that multi-indices of the same length can be added component-wise.

We will sometimes find it notationally convenient to write first order partial derivatives $\frac{\partial}{\partial x_j}$ simply as ∂_j.

Throughout this paper all stochastic processes will be defined on a fixed probability space $(\Omega, \mathcal{F}, P)$.

All function spaces will consist of complex-valued functions unless otherwise indicated. In particular, $C_c^\infty(\mathbf{R}^n)$ comprises the smooth functions of compact support on $\mathbf{R}^n$ and $S(\mathbf{R}^n)$ is the Schwartz space of rapidly decreasing functions on $\mathbf{R}^n$ so that $f \in S(\mathbf{R}^n)$ if f is smooth and $\sup_{x \in \mathbf{R}^n} |x|^\beta |D^\alpha f(x)| < \infty$ for all $\beta \in \mathbf{N} \cup \{0\}$, and all multi-indices α.

2. Review of the Euclidean Case

Let $Y = (Y(t), t \geqslant 0)$ be a Lévy process in $\mathbf{R}^n$ then we have the Lévy-Khintchine formula

$$\mathbf{E}(e^{iu.Y(t)}) = e^{t\varphi(u)},$$

for all $u \in \mathbf{R}^n, t \geqslant 0$ where

$$\varphi(u) = im.u - \frac{1}{2}u.au + \int_{\mathbf{R}^n - \{0\}} \left(e^{iu.y} - 1 - i\frac{u.y}{1 + |y|^2} \right) \nu(dy). \tag{2.1}$$

Here $m \in \mathbf{R}^n$, a is a non-negative symmetric $n \times n$ matrix and ν is a Lévy measure on $\mathbf{R}^n - \{0\}$, *i.e.* $\int_{\mathbf{R}^n - \{0\}}(|y|^2 \wedge 1)\nu(dy) < \infty$ (see *e.g.* [27]).

Let $C_0(\mathbf{R}^n)$ denote the Banach space (when equipped with the supremum norm) of continuous functions on $\mathbf{R}^n$ which vanish at infinity. We obtain a Feller semigroup $(T(t), t \geqslant 0)$ on $C_0(\mathbf{R}^n)$ by the prescription

$$(T(t)f)(x) = \mathbf{E}(f(x + Y(t))),$$

for all $f \in C_0(\mathbf{R}^n), x \in \mathbf{R}^n, t \geqslant 0$. We denote the infinitesimal generator of the semigroup as $\mathcal{A}$. The following result is classical but we give a short proof for completeness.

PROPOSITION 2.1. —

1. *$\mathcal{A}$ is a pseudo-differential operator of the form*

$$\mathcal{A} = \varphi(D).$$

2. *For all $f \in S(\mathbf{R}^n), x \in \mathbf{R}^n$,*

$$\begin{aligned}\mathcal{A}(f)(x) &= i(m.D)f(x) - \frac{1}{2}a^{ij}D_iD_jf(x) + \\ &+ \int_{\mathbf{R}^n - \{0\}} \left(f(x+y) - f(x) - i\frac{(y.D)f(x)}{1+|y|^2} \right) \nu(dy).\end{aligned}$$

Proof. — If $f \in S(\mathbf{R}^n)$ we denote its Fourier transform by $\hat{f}$, so that for each $u \in \mathbf{R}^n, \hat{f}(u) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{-iu.x} f(x)dx$. We note that the mapping $f \to \hat{f}$ is a continuous linear bijection from $S(\mathbf{R}^n)$ to itself and Fourier inversion yields

$$f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{iu.x} \hat{f}(u)du.$$

By Fubini's theorem and the Lévy-Khintchine formula, we find that for all $x \in \mathbf{R}^n, t \geqslant 0$,

$$\begin{aligned}(T(t)f)(x) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{iu.x}\mathbf{E}(e^{i(u,Y(t))})\hat{f}(u)du \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{iu.x}e^{t\varphi(u)}\hat{f}(u)du.\end{aligned}$$

Hence by dominated convergence, $S(\mathbf{R}^n) \subset \mathrm{Dom}(\mathcal{A})$ and

$$\mathcal{A}f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{iu.x}\varphi(u)\hat{f}(u)du,$$

so that (1) is established. (2) then follows by standard use of the Fourier transform. □

Of course we can go much further in characterising the domain of $\mathcal{A}$. It is shown in Sato [27], pp. 208-11 that the twice continuously differentiable functions in $C_0(\mathbf{R}^n)$ are in the domain and that $C_c^\infty(\mathbf{R}^n)$ is a core. If we extend the semigroup to act in $L^2(\mathbf{R}^n)$, the entire domain forms a non-isotropic Sobolev space whose structure is determined by the symbol φ (see *e.g.* [19] p. 49-50).

3. Lévy Processes in Lie Groups

Let G be a Lie group with Lie algebra $\mathbf{g}$. A Lévy process in G is a G-valued stochastic process $\rho = (\rho(t), t \geqslant 0)$ which satisfies the following:

1. ρ has stationary and independent left increments, where the increment between s and t with $s \leqslant t$ is $\rho(s)^{-1}\rho(t)$.
2. $\rho(0) = e$ (a.s.)
3. ρ is stochastically continuous, *i.e.*

$$\lim_{s \to t} P(\rho(s)^{-1}\rho(t) \in A) = 0,$$

for all $A \in \mathcal{B}(G)$ with $e \notin \bar{A}$.

Let $C_0(G)$ be the Banach space (with respect to the supremum norm) of functions on G which vanish at infinity. Just as in the Euclidean case, we obtain a Feller semigroup $(T(t), t \geqslant 0)$ on $C_0(G)$ by the prescription

$$T(t)f(\tau) = \mathbf{E}(f(\tau\rho(t))),$$

for each $t \geqslant 0, \tau \in G, f \in C_0(G)$ and its infinitesimal generator will be denoted as $\mathcal{L}$.

We fix a basis $\{Z_1, \ldots, Z_n\}$ for $\mathbf{g}$ and define a dense subspace $C_2(G)$ of $C_0(G)$ as follows:

$$C_2(G) =$$
$$\{f \in C_0(G); Z_i^L(f) \in C_0(G) \text{ and } Z_i^L Z_j^L(f) \in C_0(G) \text{ for all } 1 \leqslant i, j \leqslant n\},$$

where Z^L denotes the left invariant vector field associated to $Z \in \mathbf{g}$ by differential left translation.

In [17], Hunt proved that there exist functions $y_i \in C_2(G), 1 \leqslant i \leqslant n$ so that each

$$y_i(e) = 0 \text{ and } Z_i^L y_j(e) = \delta_{ij},$$

and a map $h \in \mathrm{Dom}(\mathcal{L})$ which is such that:

1. $h > 0$ on $G - \{e\}$.
2. There exists a compact neighborhood of the identity V such that for all $\tau \in V$,
$$h(\tau) = \sum_{i=1}^{n} y_i(\tau)^2.$$

Any such function is called a *Hunt function* in G.

A positive measure ν defined on $\mathcal{B}(G - \{e\})$ is called a *Lévy measure* whenever

$$\int_{G-\{e\}} h(\sigma)\nu(d\sigma) < \infty,$$

for some Hunt function h.

We are now ready to state the main result of [17].

THEOREM 3.1 (HUNT'S THEOREM). — *Let ρ be a Lévy process in G with infinitesimal generator $\mathcal{L}$ then,*

1. *$C_2(G) \subseteq Dom(\mathcal{L})$.*
2. *For each $\tau \in G, f \in C_2(G)$*
$$\mathcal{L}(f)(\tau) = b^i Z_i^L f(\tau) + c^{ij} Z_i^L Z_j^L f(\tau) + \int_{G-\{e\}} (f(\tau\sigma) - f(\tau) - y^i(\sigma) Z_i^L f(\tau))\nu(d\sigma), \tag{3.1}$$
where $b = (b^1, \dots b^n) \in \mathbf{R}^n, c = (c^{ij})$ is a non-negative-definite, symmetric $n \times n$ real-valued matrix and ν is a Lévy measure on $G - \{e\}$.

Furthermore, any linear operator with a representation as in (3.1) is the restriction to $C_2(G)$ of the infinitesimal generator of a unique weakly continuous, convolution semigroup of probability measures in G.

Several obscure features of Hunt's paper were later clarified by Ramaswami in [26] and then incorporated into the seminal treatise of [13]. For a survey of this and related ideas see [2].

Now let h be a complex, separable Hilbert space and $U(h)$ be the group of all unitary operators in h. Let $\pi : G \to U(h)$ be a strongly continuous, unitary representation of G in h and let $C^\infty(\pi) = \{\psi \in h; g \to \pi(g)\psi \text{ is } C^\infty\}$ be the dense linear space of *smooth vectors* for π in h. We define a strongly continuous contraction semigroup $(\mathcal{T}_t^\pi, t \geqslant 0)$ of linear operators on h by

$$\mathcal{T}_t^\pi \psi = \mathbf{E}(\pi(\rho(t))\psi) = \int_G \pi(\sigma)\psi q_t(d\sigma),$$

for each $\psi \in h$. Here q_t is the law of ρ_t, and the integral is understood in the sense of Bochner.

Alternatively if we fix $\psi_1, \psi_2 \in h$ and define $f \in C_b(G)$ by $f(\sigma) = < \psi_1, \pi(\sigma)\psi_2 >$ where $\sigma \in G$, we have

$$(T(t)f)(e) = < \psi_1, \mathcal{T}_t^\pi \psi_2 > .$$

Let $\mathcal{L}^\pi$ denote the infinitesimal generator of this semigroup. It follows from the arguments of [3] (see also [28]), that $C^\infty(\pi) \subseteq \mathrm{Dom}(\mathcal{L}^\pi)$ and for all $\psi \in C^\infty(\pi)$ we have

$$\begin{aligned} \mathcal{L}^\pi \psi &= b^i d\pi(Z_i)\psi + c^{ij} d\pi(Z_i) d\pi(Z_j)\psi + \\ &\quad + \int_{G-\{e\}} (\pi(\sigma) - I - y^i(\sigma) d\pi(Z_i))\psi \nu(d\sigma). \end{aligned} \tag{3.2}$$

4. The Heisenberg Group

This section is based on Chapter 2 of [29] (see also the monograph [9]). The *Heisenberg group* $\mathbf{H}^n$ is a Lie group with underlying manifold $\mathbf{R}^{2n+1}$, equipped with the composition law

$$(a_1, q_1, p_1)(a_2, q_2, p_2) = (a_1 + a_2 + \frac{1}{2}(p_1 \cdot q_2 - q_1 \cdot p_2), q_1 + q_2, p_1 + p_2),$$

where each $a_i \in \mathbf{R}, q_i, p_i \in \mathbf{R}^n$ $(i = 1, 2)$.

A basis for the Lie algebra of left-invariant vector fields is $\{T, L_1, \ldots, L_n, M_1, \ldots, M_n\}$ where for $1 \leqslant j \leqslant n$,

$$T = \frac{\partial}{\partial t}, L_j = \frac{\partial}{\partial q_j} + \frac{1}{2} p_j \frac{\partial}{\partial t}, M_j = \frac{\partial}{\partial p_j} - \frac{1}{2} q_j \frac{\partial}{\partial t},$$

and we have the commutation relations

$$[L_j, L_k] = [M_j, M_k] = [M_j, T] = [L_j, T] = 0, [M_j, L_k] = \delta_{jk}T,$$

for $1 \leqslant j, k \leqslant n$, so that $\mathbf{H}^n$ is step-2 nilpotent.

By the Stone-von Neumann uniqueness theorem, all irreducible representations of $\mathbf{H}^n$ are either one-dimensional, or are unitarily equivalent to the *Schrödinger representations* in $L^2(\mathbf{R}^n)$ which are indexed by $\mathbf{R}$ and given, for each $\lambda \geqslant 0$ by

$$\pi_{\pm\lambda}(a, q, p) = e^{i(\pm\lambda a I \pm \lambda^{1/2} q.X + \lambda^{1/2} p.D)},$$

where $X = (X_1, \ldots, X_n)$ and each $X_i u(x) = x_i u(x)$ for $u \in S(\mathbf{R}^n)$. As the work in the remainder of this paper is not affected by the value of λ, we will from now on work only with π_1, which we will write simply as π. We note that the linear operator $p.D + q.X$ is essentially self-adjoint on $S(\mathbf{R}^n)$. A basis for the representation of the Lie algebra is

$$d\pi(T) = iI, d\pi(L_j) = iX_j, d\pi(M_j) = iD_j,$$

for $1 \leqslant j \leqslant n$, where I is the identity operator.

The Schrödinger representations allows us to define an interesting class of pseudo-differential operators using the *Weyl functional calculus.* Indeed let $\sigma \in S'(\mathbf{R}^{2n})$ then we may define

$$\sigma(X, D) = (2\pi)^{-n} \int_{\mathbf{R}^{2n}} \hat{\sigma}(q, p) e^{i(q.X + p.D)} dq dp,$$

and $\sigma(X, D)$ is a continuous linear operator from $S(\mathbf{R}^n)$ to $S'(\mathbf{R}^n)$.

Moreover, we have the following useful alternative form

$$(\sigma(X, D)f)(x) = (2\pi)^{-n} \int_{\mathbf{R}^{2n}} \sigma(\frac{1}{2}(x + y), \xi) e^{i(x-y).\xi} f(y) dy d\xi, \qquad (4.1)$$

for each $f \in S(\mathbf{R}^n), x \in \mathbf{R}^n$.

In particular we have the classical symbol class – if $\sigma \in C^\infty(\mathbf{R}^{2n})$ and for all multi-indices, α, β we can find $C_{\alpha,\beta} \geqslant 0, K \in \mathbf{R}$ and $\delta < \frac{1}{2}$ such that

$$|D_x^\alpha D_\xi^\beta \sigma(x, \xi)| \leqslant C_{\alpha,\beta}(1 + |\xi|^2)^{K + \delta(|\alpha| - |\beta|)},$$

then $\sigma(X, D)$ maps $S(\mathbf{R}^n)$ continuously into itself and extends to a continuous operator on $S'(\mathbf{R}^n)$.

Two straightforward examples which we will find useful below are:

1. If $\sigma(x,\xi) = e^{i(q.x+p.\xi)}$ then $\sigma(X,D) = \pi(0,q,p)$.
2. If $\sigma(x,\xi) = |x|^2 + |\xi|^2$ then $\sigma(X,D) = H_{osc} = -\Delta + |X|^2$.

The sub-Laplacian H_{osc} is of course the well-known harmonic oscillator Hamiltonian of elementary quantum mechanics.

5. Lévy Processes in the Heisenberg Group

Let ρ be a Lévy process in $\mathbf{H}^n$ and consider the unitary operator valued process $\pi(\rho)$ where π is the Schrödinger representation. We compute the form of the generator (3.2) on the domain $C^\infty(\pi) = S(\mathbf{R}^n)$. We will find it simplifies matters if we write the vector $b = (b_0, b^1, b^2)$, where $b_0 \in \mathbf{R}$ and $b^i \in \mathbf{R}^n, i = 1,2$. We also write the non-negative definite matrix $C = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{01} & C_1 & E \\ c_{02} & E^T & C_2 \end{pmatrix}$, where for $i = 1,2, c_{00} \geqslant 0, c_{0i} \in \mathbf{R}^n$ and C_i, E are $n \times n$ matrices, with each C_i symmetric.

PROPOSITION 5.1. —

$$\begin{aligned}\mathcal{L}^\pi &= i\left(b_0 + \sum_{j=1}^n E_{jj}\right) I + (ib^1_j - 2c^{01}_j)X^j + (ib^2_j - 2c^{02}_j)D^j \\ &- c_{00}I - c^1_{jk}X^jX^k - c^2_{jk}D^jD^k - 2E_{jk}X^jD^k \qquad (5.1)\\ &+ \int_{\mathbf{R}^{2n+1}-\{0\}} \left(e^{i(aI+q.X+p.D)} - I - \frac{i(aI+q.X+p.D)}{1+|a|^2+|q|^2+|p|^2}\right)\nu(da,dq,dp).\end{aligned}$$

Proof. — This follows directly from (3.2). The most interesting part is the quadratic term which we write as

$$c^{ij}d\pi(Z_i)d\pi(Z_j) = (iI\ iX\ iD)\begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{01} & C_1 & E \\ c_{02} & E^T & C_2 \end{pmatrix}\begin{pmatrix} iI \\ iX \\ iD \end{pmatrix}.$$

Using the Lie algebra commutation relations, we obtain for each $\psi \in S(\mathbf{R}^n)$

$$\begin{aligned}(E^{jk}X_jD_k + E^{kj}D_jX_k)\psi &= E^{jk}(X_jD_k + D_kX_j)\psi \\ &= 2E^{jk}X_jD_k\psi - i\sum_{j=1}^n E_{jj}\psi.\end{aligned}$$

□

The following is a fairly straightforward computation using the Weyl calculus:

PROPOSITION 5.2. — *$\mathcal{L}^{\pi}$ is a pseudo-differential operator with symbol*

$$\begin{aligned}\sigma_{\mathcal{L}^{\pi}}(x,\xi) &= i\left(b_0+\sum_{j=1}^{n}E_{jj}\right)+(ib_j^1-2c_j^{01})x^j+(ib_j^2-2c_j^{02})\xi^j \\ &- \quad c_{00}-c_{jk}^1x^jx^k-c_{jk}^2\xi^j\xi^k-2E_{jk}x^j\xi^k \\ &+ \quad \int_{\mathbf{R}^{2n+1}-\{0\}}\left(e^{i(a+q.x+p.\xi)}-1-\frac{i(a+q.x+p.\xi)}{1+|a|^2+|q|^2+|p|^2}\right)\nu(da,dq,dp),\end{aligned}$$

where $x,\xi\in\mathbf{R}^n$.

Now let each of the vectors $c_{0i}=0$ for $i=1,2$ and define a Lévy process Y in $\mathbf{R}^{2n+1}$ with characteristics $(m,2C,\nu)$ where C and ν are as above and $m_0=b_0+\sum_{j=1}^{n}E_{jj}, m_j=b_j^1, m_{j+n}=b_j^2$ for $1\leqslant j\leqslant n$, then we can as in Proposition 2.1 associate a symbol φ_Y to the generator of Y via the Kohn-Nirenberg calculus and the Lévy-Khintchine formula.

Using propositions 2.1 and 5.1 we have the following correspondence between the processes ρ and Y at the level of their symbols.

THEOREM 5.3. —
$$\varphi_Y(1,u)=\sigma_{\mathcal{L}^{\pi}}(x,\xi),$$

where each $u=(x,\xi)\in\mathbf{R}^{2n}$.

It is interesting to contemplate theorem 5.3 from the following viewpoint. Suppose that we were only given the symbol as data from which we wish to construct a process, then we can build either the generator of a Lévy process Y in $\mathbf{R}^{2n+1}$ via the Kohn-Nirenberg calculus or of ρ on $\mathbf{H}^n$ by means of the Weyl calculus. If we know the diffusion matrix C a priori, we could argue that the non-commutativity of $\mathbf{H}^n$ can be detected in either context by the presence of its off-diagonal elements in the drift. We now introduce a class of Lévy processes in $\mathbf{H}^n$ for which this method of detecting non-commutativity no longer works.

We say that ρ is *phase-dominated* if $E=0$, each $c_{0i}=0 (i=1,2)$, $b_0=c_{00}=0$ and ν has support in $(\{0\}\times\mathbf{R}^{2n})-\{0\}$ (which we can identify with $\mathbf{R}^{2n}-\{0\}$).

We employ the phrase "phase-dominated" to indicate that the characteristics of ρ only operate on 'phase-space co-ordinates'.

We investigate phase-dominated processes more extensively. To this end we introduce a Lévy process $\Upsilon = (\Upsilon(t), t \geqslant 0)$ in $\mathbf{R}^{2n}$. It has the following Lévy-Itô decomposition (see *e.g.* [27], Chapter 4).

$$\Upsilon^j(t) = \alpha^j t + \tau_i^j B^i(t) + \int_0^{t+}\int_{|x|<1} x^j \tilde{N}(ds, dx) + \int_0^{t+}\int_{|x|\geqslant 1} x^j N(ds, dx),$$

for each $1 \leqslant j \leqslant 2n, t \geqslant 0$, where $\alpha = (\alpha^1, \ldots, \alpha^{2n}) \in \mathbf{R}^{2n}, (\tau_i^j)$ is a $2n \times r$ matrix, $B = (B^1, \ldots, B^r)$ is standard Brownian motion in $\mathbf{R}^r, N$ is a Poisson random measure on $\mathbf{R}^+ \times (\mathbf{R}^{2n} - \{0\})$ which is independent of B and has intensity measure η and $\tilde{N}$ is the associated compensator so $\tilde{N}(t, A) = N(t, A) - t\eta(A)$ for each $t \geqslant 0, A \in \mathcal{B}(\mathbf{R}^{2n} - \{0\})$. We will find it convenient to define

$$\Upsilon_Q(t) = (\Upsilon^1(t), \ldots, \Upsilon^n(t)), \quad \Upsilon_P(t) = (\Upsilon^{n+1}(t), \ldots, \Upsilon^{2n}(t)),$$

and we introduce the Lévy stochastic area of Υ_Q and Υ_P to be the process $(A(t), t \geqslant 0)$ comprising Itô stochastic integrals, which is defined by

$$\begin{aligned} A(t) &= \frac{1}{2}\sum_{j=1}^{n}\int_0^t (\Upsilon_P^j(s-)d\Upsilon_Q^j(s) - \Upsilon_Q^j(s-)d\Upsilon_P^j(s)) \\ &= \frac{1}{2}\sum_{j=1}^{n}\int_0^t (\Upsilon^{j+n}(s-)d\Upsilon^j(s) - \Upsilon^j(s-)d\Upsilon^{j+n}(s)). \end{aligned}$$

We say that the Lévy process Υ *diffuses phasewise* if $\tau = \begin{pmatrix} \tau_Q & 0 \\ 0 & \tau_P \end{pmatrix}$, where τ_Q and τ_P are $(n \times r_i)$ matrices for $i = 1, 2$ with $r_1 + r_2 = r$. We write $\gamma_Q = \tau_Q \tau_Q^T$ and $\gamma_P = \tau_P \tau_P^T$.

The next result is a special case of a theorem by Pap on the structure of Lévy processes in general nilpotent groups ([23], theorem 2 – see also Example 1 on page 154 therein). We include a proof for the reader's convenience.

THEOREM 5.4. — *A Lévy process $\rho = (\rho(t), t \geqslant 0)$ in $\mathbf{H}^n$ is phase-dominated if and only if there exists a Lévy process Υ in phase space $\mathbf{R}^{2n}$ which diffuses phasewise, for which*

$$\rho(t) = (A(t), \Upsilon_Q(t), \Upsilon_P(t)) \quad a.s. \tag{5.2}$$

for each $t \geqslant 0$. The infinitesimal generator of ρ is given by

$$\begin{aligned}(\mathcal{L}f)(\sigma) \quad = \quad & \alpha^j(L_jf)(\sigma) + \alpha^{j+n}(M_jf)(\sigma) + \frac{1}{2}\gamma_Q^{ij}(L_iL_jf)(\sigma) \\ & + \frac{1}{2}\gamma_P^{ij}(M_iM_jf)(\sigma) \\ & + \int_{\mathbf{R}^{2n}-\{0\}} (f(\sigma.(0,x_Q,x_P)) - f(\sigma) - [x^j(L_jf)(\sigma) \\ & - x^{j+n}(M_jf)(\sigma)]1_{|x|<1})\eta(dx),\end{aligned} \tag{5.3}$$

for each $f \in C^2(\mathbf{H}^n), \sigma \in \mathbf{H}^n$, where . is the composition law in $\mathbf{H}^n$, $x_Q = (x^1, \ldots, x^n)$ and $x_P = (x^{n+1}, \ldots, x^{2n})$.

Proof. — To ease the notation, we will take $n = 1$. There is also no great loss of generality in taking $\alpha^1 = \alpha^2 = 0, \tau_1^1 = \tau_2^2 = 1$ and assuming that η has support in $\{|x| \in \mathbf{R}^{2n}, |x| < 1, x \neq 0\}$. We then have the following

$$\Upsilon_Q(t) = B^1(t) + \int_0^{t+}\int_{|x|<1} x^1\tilde{N}(ds,dx),$$

$$\Upsilon_P(t) = B^2(t) + \int_0^{t+}\int_{|x|<1} x^2\tilde{N}(ds,dx).$$

Now assume that ρ is defined as in (5.2).

By Itô's formula for semimartingales with jumps (see *e.g.* [25], Chapter 2) we obtain for each $f \in C^2(\mathbf{R}^3)$,

$$\begin{aligned}f(\rho(t)) = f(0) &+ \int_0^t (\partial_a f)(\rho(s-))dA(s) + \int_0^t (\partial_q f)(\rho(s-))d\Upsilon_Q(s) \\ &+ \int_0^t (\partial_p f)(\rho(s-))d\Upsilon_P(s) + \int_0^t (\partial_a\partial_q f)(\rho(s-))d[[A,\Upsilon_Q]]_c(s) \\ &+ \int_0^t (\partial_a\partial_p f)(\rho(s-))d[[A,\Upsilon_P]]_c(s) + \int_0^t (\partial_q\partial_p f)(\rho(s-))d[[\Upsilon_Q,\Upsilon_P]]_c(s) \\ &+ \frac{1}{2}\int_0^t (\partial_a^2 f)(\rho(s-))d[[A,A]]_c(s) + \frac{1}{2}\int_0^t (\partial_q^2 f)(\rho(s-))d[[\Upsilon_Q,\Upsilon_Q]]_c(s) \\ &+ \frac{1}{2}\int_0^t (\partial_p^2 f)(\rho(s-))d[[\Upsilon_P,\Upsilon_P]]_c(s) + \sum_{0\leqslant s\leqslant t}[f(\rho(s-)+\Delta\rho(s)) - f(\rho(s-)) \\ &- (\partial_a f)(\rho(s-))\Delta A(s) - (\partial_q f)(\rho(s-))\Delta\Upsilon_Q(s) - (\partial_p f)(\rho(s-))\Delta\Upsilon_P(s)],\end{aligned}$$

where $[[.,.]]_c$ denotes the continuous part of quadratic variation (see *e.g.* [25] p.62). The following are then easily deduced from the quadratic variation of Brownian motion.

$$d[[\Upsilon_Q,\Upsilon_P]]_c(t)=0,\quad d[[\Upsilon_Q,\Upsilon_Q]]_c(t)=d[[\Upsilon_P,\Upsilon_P]]_c(t)=dt,$$
$$d[[\Upsilon_Q,A]]_c(t)=\frac{1}{2}\Upsilon_P(t)dt,\quad d[[\Upsilon_P,A]]_c(t)=-\frac{1}{2}\Upsilon_Q(t)dt,$$
$$d[[A,A]]_c(t)=\frac{1}{4}((\Upsilon_Q(t))^2+(\Upsilon_P(t))^2)dt.$$

We can now simplify the expression for $f(\rho(t))$ by incorporating these results, the Lévy-Itô decompositions of Υ_Q and Υ_P and by introducing the Lie algebra basis $T=\partial_a, L=\partial_q+\frac{1}{2}pT$ and $M=\partial_p-\frac{1}{2}qT$. We then obtain

$$\begin{aligned}
f(\rho(t)) &= f(0)+\int_0^t (Lf)(\rho(s-))dB^1(s)+\int_0^t (Mf)(\rho(s-))dB^2(s)\\
&+ \frac{1}{2}\int_0^t (L^2f)(\rho(s-))ds+\frac{1}{2}\int_0^t (M^2f)(\rho(s-))ds+\int_0^{t+}\int_{|x|<1}\{f(A(s-)\\
&+ \frac{1}{2}(\Upsilon_P(s-)x^1-\Upsilon_Q(s-)x^2),\Upsilon_Q(s-)+x^1,\Upsilon_P(s-)+x^2)\\
&- f(\rho(s-))\}\tilde{N}(ds,dx)+\int_0^{t+}\int_{|x|<1}\{f(A(s-)\\
&+ \frac{1}{2}(\Upsilon_P(s-)x^1-\Upsilon_Q(s-)x^2),\Upsilon_Q(s-)+x^1,\Upsilon_P(s-)+x^2)\\
&- f(\rho(s-))-\frac{1}{2}(\Upsilon_P(s-)x^1-\Upsilon_Q(s-)x^2)(\partial_a f)(\rho(s-))\\
&- x^1(\partial_q f)(\rho(s-))-x^2(\partial_p f)(\rho(s-))\}ds\eta(dx)\\
&= f(0)+\int_0^t (Lf)(\rho(s-))dB^1(s)+\int_0^t (Mf)(\rho(s-))dB^2(s)\\
&+ \int_0^{t+}\int_{|x|<1}[f(\rho(s-).x)-f(\rho(s-))]\tilde{N}(ds,dx)+\int_0^t \mathcal{L}(f)(\rho(s-))ds,
\end{aligned}$$

where for each $\sigma\in\mathbf{H}^1$,

$$\begin{aligned}
\mathcal{L}(f)(\sigma) &= \frac{1}{2}(L^2f)(\sigma)+\frac{1}{2}(M^2f)(\sigma)\\
&\quad+\int_{|x|<1}(f(\sigma.x)-f(\sigma)-x^1(Lf)(\sigma)-x^2(Mf)(\sigma))\eta(dx).
\end{aligned}$$

Sufficiency now follows immediately from theorem 3.1 of [1]. For necessity, let ρ be an arbitrary phase-dominated Lévy process in $\mathbf{H}^n$, then it has an infinitesimal generator of the form (5.3), and hence we can define a Lévy process in $\mathbf{R}^{2n}$ with characteristics (α, τ, η) which diffuses phasewise. The required result then follows by reversing the steps in the argument given above and appealing to the uniqueness of solutions to stochastic differential equations of this type (see *e.g.* [1], [23]). □

Note. — For readers with a background in stochastic differential equations, we remark that since the exponential map from $\mathbf{R}^{2n+1}$ to $\mathbf{H}^n$ is surjective, the equation for $f \circ \rho$ which appears in the proof of theorem 5.4 is equivalent to

$$d\rho(t) = \sigma^i_j Z^j(\rho(t-)) \diamond d\Upsilon_i(t) \tag{5.4}$$

with initial condition $\rho(0) = e$ (a.s.), where $\diamond$ denotes the Marcus canonical form (see the appendix to [2] and references therein). Here $\sigma^i_j = \mathrm{diag}(0, 1, \ldots, 1)$, $Z^i (i = 0, 1, \ldots, 2n)$ are the usual Lie algebra generators and $\Upsilon = (\Upsilon_0, \Upsilon_Q, \Upsilon_P)$ is a Lévy process on $\mathbf{R}^{2n+1}$ (note the component Υ_0 is included simply for "accounting purposes" and plays no role in determining the process ρ).

Example 5.5 (Phase-dominated Brownian motion). —
Let $\rho(t) = (2A(t), \sqrt{2}B_Q(t), \sqrt{2}B_P(t))$, where $B = (B_0, B_Q, B_P)$ is a standard Brownian motion in $\mathbf{R}^{2n+1}$ and $2A(t) = \sum_{j=1}^n \int_0^t (B^j_P(s) dB^j_Q(s) - B^j_Q(s) dB^j_P(s)$, for each $t \geqslant 0$. In this case, the $\diamond$ in equation (5.4) is the Stratonovitch differential and $\Upsilon = B$. We note that $\mathcal{L}^\pi = -H_{osc}$ which has symbol $-(|x|^2 + |\xi|^2)$. This example also appears in [31] as a special case of a prescription for solutions of general SDEs driven by Brownian motion on nilpotent Lie groups. A related process is studied in [16].

Example 5.6 (Phase-dominated Poisson process). — In this example, we take $n = 1$. For each $t \geqslant 0$, let $\rho(t) = (A(t), N_Q(t), N_P(t))$, where N_Q and N_P are independent, one-dimensional Poisson processes. Let $(\tau_Q^{(m)}, m \in \mathbf{N})$ and $(\tau_P^{(m)}, m \in \mathbf{N})$ be the arrival times for N_Q and N_P, respectively. We compute

$$A(t) = \frac{1}{2} \sum_{m=0}^{\infty} N_P(t \wedge \tau_Q^{(m)} -) - N_Q(t \wedge \tau_P^{(m)} -).$$

Example 5.7 (Phase-dominated compound Poisson process). — We return to the case of general $\mathbf{H}^n$. Let $(X_Q(m), m \in \mathbf{N})$ and $(X_P(m), m \in \mathbf{N})$

be i.i.d. sequences of $\mathbf{R}^n$-valued random variables and let N_Q and N_P be as above. Assume further that they are independent of all the $X_Q(m)$'s and $X_P(m)$'s. Define the following compound Poisson processes in $\mathbf{R}^n$, where $t \geqslant 0$:-

$$\Theta_Q(t) = X_Q(1) + \cdots + X_Q(N_Q(t)), \quad \Theta_P(t) = X_P(1) + \cdots + X_P(N_P(t)).$$

Then our required process is $\rho(t) = (A(t), \Theta_Q(t), \Theta_P(t))$, where

$$A(t) = \frac{1}{2} \sum_{j=1}^{n} \sum_{m=0}^{\infty} (\Theta_P^j(t \wedge \tau_Q^{(m)}-) X_Q^j(m) - \Theta_Q^j(t \wedge \tau_P^{(m)}-) X_P^j(m)).$$

Even in the case of phase-dominated Lévy processes, we can detect the non-commutativity of the group by examining the passage from generator to semigroup. We begin by recalling the situation in $\mathbf{R}^n$. If Y is a Lévy process then its generator is a pseudo-differential operator with symbol φ and the semigroup $(T(t), t \geqslant 0)$ again consists of pseudo-differential operators with the symbol of each $T(t)$ being $e^{t\varphi}$, indeed this was established in the proof of Proposition 2.1.

The passage from the symbol of the generator to that of the semigroup is not so straightforward in the Heisenberg group case, as we will now demonstrate. In the following corollary a probabilistic representation of the symbol of the semigroup $\mathcal{T}_t^\pi$ is derived from theorem 5.4. Although for general phase-dominated Lévy processes it does not yield a closed form formula for the symbol, it is in fact, a generalization of the celebrated Mehler's formula, which we discuss below and which corresponds to phase-dominated Brownian motion.

COROLLARY 5.8.— *Let $\rho = (\rho(t), t \geqslant 0)$ be a phase-dominated Lévy process in $\mathbf{H}^n$, so that*

$$\rho(t) = (A(t), \Upsilon_Q(t), \Upsilon_P(t)) \quad a.s.$$

for each $t \geqslant 0$. The symbol of the associated semi-group is given by

$$\sigma_{\mathcal{T}_t^\pi}(x, \xi) = \int_\Omega e^{i(A(t)(\omega) + \Upsilon_Q(t)(\omega).x + \Upsilon_P(t)(\omega).\xi)} dP(\omega).$$

Proof. — Observe that for each $t \geqslant 0$, by theorem 5.4,

$$\pi(\rho(t)) = e^{i(A(t) + \Upsilon_Q(t)(t).X + \Upsilon_P(t).D)},$$

and so

$$\mathcal{T}_t^{\pi} = \int_{\Omega} e^{i(A(t)(\omega)+\Upsilon_Q(t)(\omega).X+\Upsilon_P(t)(\omega).D)} dP(\omega)$$

By a straightforward application of Fubini's theorem in (4.1) we see that

$$\sigma_{\mathcal{T}_t^{\pi}}(x,\xi) = \int_{\Omega} e^{i(A(t)(\omega)+\Upsilon_Q(t)(\omega).x+\Upsilon_P(t)(\omega).\xi)} dP(\omega).$$

□

Note 1. — In the phase-dominated Brownian-case, with $n = 1$, we see that

$$\sigma_{\mathcal{T}_t^{\pi}}(x,\xi) = \int_{\Omega} e^{i(2A(t)(\omega)+\sqrt{2}B^1(t)(\omega).x+\sqrt{2}B^2(t)(\omega).\xi)} dP(\omega),$$

and a classical formula for the Lévy area (see [32], equation (2.7), p.19) yields

$$\begin{aligned} &\mathbf{E}(\exp(i2A(t))|B^1(t) = a, B^2(t) = b) \\ &\qquad = \left(\frac{t}{\sinh(t)}\right) \exp\left(\frac{(a^2+b^2)(1 - t\coth(t))}{2t}\right). \end{aligned}$$

Hence, on carrying out a standard Gaussian integral, we find

$$\begin{aligned} \sigma_{\mathcal{T}_t^{\pi}}(x,\xi) &= \int_{\mathbf{R}^2} e^{i(\sqrt{2}ax+\sqrt{2}b\xi)} \mathbf{E}(\exp(i2A(t))|B^1(t) = a, B^2(t) = b) \\ &\quad \frac{e^{\frac{-(a^2+b^2)}{2t}}}{2\pi t} dadb \\ &= \frac{1}{\sinh(t)} \int_{\mathbf{R}^2} e^{i(\sqrt{2}ax+\sqrt{2}b\xi)} \exp\left(\frac{-(a^2+b^2)\coth(t)}{2}\right) \frac{dadb}{2\pi} \\ &= \frac{1}{\cosh(t)} \exp\left(-\tanh(t)(x^2+\xi^2)\right). \end{aligned}$$

This is precisely Mehler's formula in the case $n = 1$. The case for $n > 1$,

$$\sigma_{\mathcal{T}_t^{\pi}}(x,\xi) = (\cosh(t))^{-n} e^{-\tanh(t)(|x|^2+|\xi|^2)},$$

follows by a straightforward independence argument.

The probabilistic approach which we have just given is essentially that of Gaveau in [11]. See [29] pp 68-76 for three different proofs of this result.

Note 2. — In related work to this, Pap [24] has recently found a probabilistic interpretation of the formula for the fundamental solution of $\frac{\partial u}{\partial t} = \mathcal{L}^\pi u$, in the case where ρ is an arbitrary symmetric Brownian motion in $\mathbf{H}^1$. This result can be interpreted as a generalisation of Mehler's formula.

6. Properties of the Operator $\mathcal{L}^\pi$

In this section we will make frequent use of the projection-valued measure $P_{a,q,p}$ in $L^2(\mathbf{R}^n)$ associated to the spectral decomposition of the self-adjoint operator $\overline{aI + q.X + p.D}$, where $(a,q,p) \in \mathbf{R}^{2n+1}$.

We know that $\mathcal{L}^\pi$ is a densely defined, closed linear operator in $L^2(\mathbf{R}^n)$. Our first task is to show that $C_c^\infty(\mathbf{R}^n)$ is a core for $\mathcal{L}^\pi$. The key is the following technical result.

PROPOSITION 6.1. — *There exists $C > 0$ such that for all $f \in C_c^\infty(\mathbf{R}^n)$,*

$$||\mathcal{L}^\pi f|| \leqslant \tag{6.1}$$

$$C\left[||f||+\sum_{j=1}^{n}(||X_j f||+||D_j f||)+\sum_{j,k=1}^{n}(||X^j X^k f||+||X^j D^k f||+||D^j D^k f||)\right].$$

Proof. — Using the defining property of the Lévy measure ν, we can rewrite $\mathcal{L}^\pi$ (as given by (5.1)) as

$$\mathcal{L}^\pi = \mathcal{L}_1^\pi + \mathcal{L}_2^\pi + \mathcal{L}_3^\pi,$$

where for some $\alpha, \beta_j, \gamma_j \in \mathbf{C}, 1 \leqslant j \leqslant n$,

$$\mathcal{L}_1^\pi = \alpha I + \beta_j X^j + \gamma_j D^j - c_{00} I - c_{jk}^1 X^j X^k - c_{jk}^2 D^j D^k - 2E_{jk} X^j D^k,$$

$$\mathcal{L}_2^\pi = \int_B (e^{i(aI+q.X+p.D)} - I - i(aI + q.X + p.D))\nu(da, dq, dp),$$

$$\mathcal{L}_3^\pi = \int_{B^c} (e^{i(aI+q.X+p.D)} - I)\nu(da, dq, dp),$$

and where $B = \{y \in \mathbf{R}^{2n+1}, ||y|| \leqslant 1\}$. Now for each $f \in C_c^\infty(\mathbf{R}^n)$, we find that

$$\begin{aligned} ||\mathcal{L}_3^\pi f|| &\leqslant \int_{B^c} ||(e^{i(aI+q.X+p.D)} - I)f||\nu(da, dq, dp) \\ &\leqslant 2\nu(B^c)||f||. \qquad \ldots(i) \end{aligned}$$

By the spectral theorem and Taylor's theorem, we see that for each $(a,q,p) \in \mathbf{R}^{2n+1}$,

$$\begin{aligned}
||[e^{i(aI+q.X+p.D)} - I \; - i(aI + q.X + p.D)]f||^2 \\
&= \int_{\mathbf{R}^{2n+1}} |e^{i\lambda} - 1 - i\lambda|^2 ||P_{a,q,p}(d\lambda)f||^2 \\
&\leqslant \frac{1}{4}\int_{\mathbf{R}^{2n+1}} |\lambda|^4 ||P_{a,q,p}(d\lambda)f||^2 \\
&= \frac{1}{4}||(aI + q.X + p.D)^2 f||^2.
\end{aligned}$$

We thus obtain

$$\begin{aligned}
||\mathcal{L}_2^\pi f|| &\leqslant \frac{1}{2}\int_B ||(aI + q.X + p.D)^2 f||\nu(da,dq,dp) \\
&\leqslant C_1 \int_B (|a|^2 + |q|^2 + |p|^2) \Bigg[||f|| + \sum_{j=1}^n (||X_j f|| + ||D_j f||) + \\
&\quad + \sum_{j,k=1}^n (||X^j X^k f|| + ||X^j D^k f|| + ||D^j D^k f||) \Bigg] \nu(da,dq,dp) \\
&\leqslant C_2 \Bigg[||f|| + \sum_{j=1}^n (||X_j f|| + ||D_j f||) + \\
&\quad + \sum_{j,k=1}^n (||X^j X^k f|| + ||X^j D^k f|| + ||D^j D^k f||) \Bigg], \ldots (ii)
\end{aligned}$$

where $C_1, C_2 > 0$.

The required result now follows on combining (i), (ii) and the expression for $||\mathcal{L}_1^\pi f||$. □

THEOREM 6.2. — *$C_c^\infty(\mathbf{R}^n)$ is a core for $\mathcal{L}^\pi$.*

Proof. — Let $f \in \mathrm{Dom}(\mathcal{L}^\pi)$, then we can find $(f_n, n \in \mathbf{N})$ in $C_c^\infty(\mathbf{R}^n)$ such that $f = \lim_{n\to\infty} f_n$. By Proposition 6.1, we deduce that $\lim_{m,n\to\infty} |||\mathcal{L}^\pi(f_n - f_m)|| = 0$, hence the sequence $(\mathcal{L}^\pi f_n, n \in \mathbf{N})$ is Cauchy and so convergent to some $g \in L^2(\mathbf{R}^n)$. But $\mathcal{L}^\pi$ is closed, hence $g = \mathcal{L}^\pi f$ and the result is established. □

In the following we will find it convenient to denote the restriction of $\mathcal{L}^\pi$ to $C_c^\infty(\mathbf{R}^n)$ by $\mathcal{L}_0^\pi$.

By (5.1) we see that $\mathcal{L}_0^\pi$ is symmetric if the following conditions hold:

- $b_0 = b_j^i = 0 (i = 1, 2, j = 1, \ldots n), E = 0$.
- ν is a symmetric measure *i.e.* $\nu(A) = \nu(-A)$ for all $A \in \mathcal{B}(\mathbf{R}^{2n+1})$.

If we make the further assumption that each $c_{0i} = 0 (i = 1, 2)$, then $-\mathcal{L}_0^\pi$ is also a positive operator in the sense that $< \psi, (-\mathcal{L}_0^\pi)\psi > \geqslant 0$ for all $\psi \in C_c^\infty(\mathbf{R}^n)$ as the following result shows:-

PROPOSITION 6.3. — *$-\mathcal{L}_0^\pi$ is a positive symmetric operator where*

$$\begin{aligned} -\mathcal{L}_0^\pi &= c_{00}I + c_{jk}^1 X^j X^k + c_{jk}^2 D^j D^k \\ &\quad + \int_{\mathbf{R}^{2n+1}-\{0\}} (I - \cos(aI + q.X + p.D))\nu(da, dq, dp). \end{aligned} \tag{6.2}$$

Proof. — Since C is a non-negative definite symmetric matrix, it is easily verified that the first line of (6.2) is a positive symmetric operator. We write

$$M_\pi = \int_{\mathbf{R}^{2n+1}-\{0\}} (1 - \cos(aI + q.X + p.D))\nu(da, dq, dp).$$

By the spectral theorem, for each $\psi \in S(\mathbf{R}^n)$, we have

$$< \psi, M_\pi \psi >= \int_{\mathbf{R}^{2n+1}-\{0\}} \int_{\mathbf{R}} (1 - \cos(\lambda)) ||P_{a,q,p}(d\lambda)\psi||^2 \nu(da, dq, dp) \geqslant 0,$$

and the result follows. □

Now since $-\mathcal{L}_0^\pi$ is positive symmetric, we can define a positive quadratic form $\mathcal{E}_\pi$ with domain $C_c^\infty(\mathbf{R}^n)$ by the prescription

$$\mathcal{E}_\pi(f) = - < f, \mathcal{L}_0^\pi f >,$$

for each $f \in C_c^\infty(\mathbf{R}^n)$. Then $\mathcal{E}_\pi$ is closable with closure $\overline{\mathcal{E}_\pi}$, and there exists a positive self-adjoint operator A^π, which is an extension of $-\mathcal{L}_0^\pi$, such that $\mathrm{Dom}(A^\pi) \subseteq \mathrm{Dom}(\overline{\mathcal{E}_\pi})$ and $\overline{\mathcal{E}_\pi}(f) =< f, A^\pi f >$, for all $f \in \mathrm{Dom}(A^\pi)$. A^π is called the *Friedrichs extension* of $-\mathcal{L}_0^\pi$ (see *e.g.* [7], p.4).

LEMMA 6.4. — *$A^\pi = -\mathcal{L}^\pi$ and $(\mathcal{T}_t^\pi, t \geqslant 0)$ is a self-adjoint semigroup in $L^2(\mathbf{R}^n)$.*

Proof. — Since A^π is positive and self-adjoint, $-A^\pi$ is the generator of a self-adjoint, strongly continuous, contraction semigroup $(\mathcal{S}_t^\pi, t \geqslant 0)$ in $L^2(\mathbf{R}^n)$. $\mathcal{S}_t^\pi f$ and $\mathcal{T}_t^\pi f$ are both solutions of the initial value problem $f'(t) = \mathcal{L}^\pi f(t), f(0) = f$, where $f \in C_c^\infty(\mathbf{R}^n)$, hence by uniqueness of such solutions we conclude that $\mathcal{S}_t^\pi f = \mathcal{T}_t^\pi f$, for each $t \geqslant 0, f \in C_c^\infty(\mathbf{R}^n)$. A standard density argument then yields that $\mathcal{S}_t^\pi f = \mathcal{T}_t^\pi f$, for all $f \in L^2(\mathbf{R}^n)$, and the result follows. □

In order to construct Dirichlet forms, we need to know when the operator $\mathcal{L}^\pi$ preserves real-valued functions.

PROPOSITION 6.5. — *$-\mathcal{L}_0^\pi$ maps real-valued functions to real-valued functions if and only if*

$$\int_{\mathbf{R}^{2n+1}-\{0\}} \sin(a + q.x)\sin(p.\xi)\nu(da, dq, dp) = 0, \forall \xi \in \mathbf{R}^n. \tag{6.3}$$

Proof. — Because of (6.2) the only part of $-\mathcal{L}_0^\pi$ that can map real-valued functions to complex-valued functions is M_π. Hence without loss of generality we state a necessary and sufficient condition for M_π.

We will employ the Weyl functional calculus which was introduced in section 4. Since equation (4.1) is valid for all $f \in C_c^\infty(\mathbf{R}^n)$, an operator of the form $\sigma(X, D)$ maps real-valued functions to real-valued functions if and only if

$$\int_{\mathbf{R^{2n}}} \sigma(\frac{1}{2}(x+y), \xi)e^{i(x-y).\xi}d\xi \in \mathbf{R},$$

for all $x, y \in \mathbf{R}^n$. Hence for all $x, p \in \mathbf{R}^n$

$$\int_{\mathbf{R^{2n}}} \sigma(x, \xi)e^{ip.\xi}d\xi \in \mathbf{R},$$

and this is equivalent to the symmetry condition

$$\sigma(x, \xi) = \sigma(x, -\xi) \quad \text{for all } x,\ \xi \in \mathbf{R^n}. \tag{6.4}$$

Now we return to the operator M_π. By Proposition 5.2, we have for all $x, \xi \in \mathbf{R^n}$,

$$\sigma_{M_\pi}(x, \xi) = \int_{\mathbf{R}^{2n+1}-\{0\}} (1 - \cos(a + q.x + p.\xi))\nu(da, dq, dp).$$

Equation (6.4) then yields

$$\int_{\mathbf{R}^{2n+1}-\{0\}} (\cos(a+q.x+p.\xi) - \cos(a+q.x-p.\xi))\nu(da,dq,dp),$$

for all $x, \xi \in \mathbf{R^n}$, which is equivalent to equation (6.3), as required. □

Note that if ν is symmetric with respect the variable p that is

$$\nu(A) = \nu(s(A))$$

for all $A \in \mathcal{B}(\mathbf{R^n})$, where $s(a,q,p) = (a,q,-p)$, then (6.3) is satisfied. A less trivial probabilistic example where (6.3) is true is when Υ_P and Υ_Q are independent. This is easily verified from the observation that, by the Lévy-Khintchine formula, we have

$$\nu(da,dq,dp) = \delta_0(da) \otimes \nu_Q(dq) \otimes \delta_0(dp) + \delta_0(da) \otimes \delta_0(dq) \otimes \nu_P(dp), \quad (6.5)$$

where ν_P is the Lévy measure of Y_P, ν_Q is the Lévy measure of Y_Q, and δ_0 denotes the Dirac mass at the point 0.

For the remainder of this paper, we will assume that the condition (6.3) is satisfied.

In this case, $-\mathcal{L}_0^\pi$ maps real-valued functions to real-valued functions and Theorem 6.2 allows us to extend this property to the generator $\mathcal{L}^\pi$ and to the semigroup $(\mathcal{T}_t^\pi, t \geqslant 0)$. From now on, we will restrict the action of the positive symmetric operator $-\mathcal{L}_0^\pi$ and its closure $-\mathcal{L}^\pi$ to the real Hilbert space $L^2(\mathbf{R}^n, \mathbf{R})$, where the domain of the former operator comprises the real-valued smooth functions of compact support, $C_c^\infty(\mathbf{R}^n, \mathbf{R})$. We may similarly restrict the domain of the form $\mathcal{E}$, and of its closure $\overline{\mathcal{E}_\pi}$, to comprise real-valued functions.

PROPOSITION 6.6.— *For each $f \in C_c^\infty(\mathbf{R}^n, \mathbf{R})$,*

$$\begin{aligned}
\mathcal{E}_\pi(f,f) &= \sum_{j,k=1}^n c_{jk}^2 \int_{\mathbf{R}^n} \partial_j f(x) \partial_k f(x) dx \\
&+ \int_{\mathbf{R}^n} \Bigg[c_{00} + c_{jk}^1 x^j x^k + \int_{\mathbf{R}^{2n+1}-\{0\}} \sin^2(\frac{1}{2}(a+q.x+\frac{1}{2}q.p)) \\
&+ \sin^2(\frac{1}{2}(a+q.x-\frac{1}{2}q.p))\nu(da,dq,dp) \Bigg] f^2(x) dx \\
&+ \frac{1}{2} \int_{\mathbf{R}^n} \int_{\mathbf{R}^{2n+1}-\{0\}} \cos(a+q.x+\frac{1}{2}p.q)(f(x)-f(x+p))^2 \nu(da,dq,dp) dx.
\end{aligned} \quad (6.6)$$

Proof. — Using the notation M_π from the proof of Proposition 6.3 and applying the symmetry of ν, we obtain for each $f \in C_c^\infty(\mathbf{R}^n), x \in \mathbf{R}^n$,

$$(M_\pi f)(x) = \int_{\mathbf{R}^{2n+1}-\{0\}} [f(x) - e^{i(a+q.x+\frac{1}{2}p.q)} f(x+p)]\nu(da, dq, dp).$$

(6.6) follows by a straightforward computation using Fubini's theorem, a change of variable and the symmetry of ν. □

The structure of (6.6) can be written more succinctly as follows. For each $f \in C_c^\infty(\mathbf{R}^n, \mathbf{R})$,

$$\begin{aligned} \mathcal{E}_\pi(f,f) &= \mathcal{E}_c(f,f) + \int_{\mathbf{R}^n} f(x)^2 k(dx) \\ &\quad + \int_{\mathbf{R}^n \times (\mathbf{R}^{2n+1}-\{0\})} (f(x) - f(x+p))^2 J(da, dq, dp, dx), \end{aligned} \tag{6.7}$$

where

$$\mathcal{E}_c(f,f) = \sum_{j,k=1}^{n} c_{jk}^2 \int_{\mathbf{R}^n} \partial_j f(x) \partial_k f(x) dx,$$

$$\begin{aligned} k(dx) &= \Big(c_{00} + c_{jk}^1 x^j x^k \\ &\quad + \int_{\mathbf{R}^{2n+1}-\{0\}} \sin^2(\frac{1}{2}(a + q.x + \frac{1}{2}q.p)) \\ &\quad + \sin^2(\frac{1}{2}(a + q.x - \frac{1}{2}q.p))\nu(da, dq, dp) \Big) \, dx, \end{aligned}$$

and $J(da, dq, dp, dx) = \frac{1}{2}\cos(a + q.x + \frac{1}{2}p.q)\nu(da, dq, dp)dx$.

(6.7) is closely related to the Beurling-Deny formula for regular Dirichlet forms (cf. [10], p.108) and it is tempting to give the formula a probabilistic interpretation wherein $\mathcal{E}_c$ is the local part, associated to a diffusion process, the term controlled by k represents killing, while that determined by J represents jumps. In general, however J is not a positive measure, take for instance $\nu(da, dq, dp) = \delta_{(\pi,0,1)}(da, dq, dp) + \delta_{(-\pi,0,-1)}(da, dq, dp)$ where δ is Dirac mass.

The interpretation of $\mathcal{E}_\pi$ in general, is an interesting problem. Here we will concentrate on the case where we obtain a legitimate Dirichlet form and so, from now on, we will make the assumption that J is a positive measure on $\{(a,q,p,x) \in \mathbf{R}^{2n+1}, p \neq 0\}$. This holds, for example, when $\mathrm{supp}(\nu) = \{(0,0,p), p \in \mathbf{R}^n\}$ or, when ν is the Lévy measure corresponding to Υ_P and Υ_Q independent, as can easily be deduced from (6.5).

THEOREM 6.7. — *If J is a positive measure on $\{(a,q,p,x) \in \mathbf{R}^{2n+1}, p \neq 0\}$ then $\overline{\mathcal{E}_\pi}$ is a symmetric Dirichlet form.*

Proof. — By standard use of mollifiers, see *e.g.* [10], p.7-8, for each $\varepsilon > 0$, there exists a family of infinitely differential functions $(\phi_\varepsilon(x), x \in \mathbf{R})$ such that $\phi_\varepsilon(f) \in C_c^\infty(\mathbf{R}^n, \mathbf{R})$ for all $f \in C_c^\infty(\mathbf{R}^n, \mathbf{R})$. Moreover we have $\phi_\varepsilon(x) = x$, for all $x \in [0,1]$, $-\varepsilon \leqslant \phi_\varepsilon(x) \leqslant 1+\varepsilon$, for all $x \in \mathbf{R}$ and $0 \leqslant \phi_\varepsilon(y) - \phi_\varepsilon(x) \leqslant y - x$, whenever $x, y \in \mathbf{R}$ with $x < y$. Now it follows immediately from the representation (6.6), that for each $f \in C_c^\infty(\mathbf{R}^n, \mathbf{R}), \varepsilon > \mathbf{0}$,

$$\mathcal{E}_\pi(\phi_\varepsilon(f), \phi_\varepsilon(f)) \leqslant \mathcal{E}_\pi(f, f),$$

and so $\mathcal{E}_\pi$ is Markovian. Hence $\overline{\mathcal{E}_\pi}$ is a symmetric Dirichlet form by Theorem 3.1.1 of [10], p.98-9. □

By general properties of Dirichlet forms (see *e.g.* [10], [7]), $\mathcal{L}^\pi$ generates a symmetric sub-Markov semigroup in $L^2(\mathbf{R}^n, \mathbf{R})$, so that whenever $0 \leqslant f \leqslant 1$ (a.e.), we have $0 \leqslant \mathcal{T}_t^\pi f \leqslant 1$ (a.e.).

By Theorem 6.2, $\overline{\mathcal{E}}$ is regular and by the construction of Chapter 7 in [10], we can assert the existence of a symmetric (with respect to Lebesgue measure) Hunt process $X = (X(t), t \geqslant 0)$ with state space $\mathbf{R}^n \cup \{\Delta\}$ (where Δ is the cemetery point), which is unique up to exceptional sets and whose transition semigroup is a quasi-continuous version of $(\mathcal{T}_t^\pi, t \geqslant 0)$. It is an interesting problem to relate this equivalence class of processes more directly to ρ. From (6.7), we see that the local part $\overline{\mathcal{E}}_c$ is simply given by a Brownian motion in $\mathbf{R}^n$ with covariance matrix (c_{jk}^2), while both the diffusion and jump characteristics of ρ contribute to killing of X. In the case where $\mathrm{supp}(\nu) = \{(0,0,p), p \in \mathbf{R}^n\}$ and $c_{00} = c_{jk}^1 = 0$, for all $1 \leqslant j, k \leqslant n$, then X is a symmetric Lévy process in $\mathbf{R}^n$. The next proposition describes the Hunt process in a more interesting setting. We will take $n = 1$, for simplicity.

PROPOSITION 6.8. — *Let $\rho = (A(t), \Upsilon_Q(t), \Upsilon_P(t), t \geqslant 0)$ be a phase-dominated Lévy process for which Υ_Q and Υ_P are symmetric and independent. For each $x \in \mathbf{R}$ consider the sub-Feller process $(Y_t^x, t \geqslant 0)$ defined* *by*

$$\begin{aligned} Y_t^x &= x + \Upsilon_P(t) \quad \textit{if} \quad T^x > t, \\ &= \Delta \quad \textit{if} \quad T^x \leqslant t, \end{aligned} \tag{6.8}$$

where Δ is a cemetery point, and

$$T^x = \inf\{s \geqslant 0,\ \tau < \Phi_Q(\Upsilon_P, x, s)\},$$

where τ is a random variable with an exponential distribution with parameter 1 *which is independent of ρ.*

We then have

$$(\mathcal{T}_t^\pi f)(x) = \mathbf{E}(f(Y_t^x)),$$

for every $f \in C_c^\infty(\mathbf{R}, \mathbf{R}), t \geqslant 0, x \in \mathbf{R}$. The functional Φ_Q is defined by the relation

$$\mathbf{E}\left(\exp(i \int_0^t (g(s-) + x) d\Upsilon_Q(s))\right) = \exp(-\Phi_Q(g, x, t)),$$

for every function $g : \mathbf{R}^+ \to \mathbf{R}$ which is right continuous with left limits.

Proof. — For every $f \in C_c^\infty(\mathbf{R}, \mathbf{R}), t \geqslant 0, x \in \mathbf{R}$, the Schrödinger representation yields

$$(\mathcal{T}_t^\pi f)(x) = \mathbf{E}\left(\exp(i(A(t) + \Upsilon_Q(t).x + 1/2\Upsilon_Q(t).\Upsilon_P(t)))f(x + \Upsilon_P(t))\right),$$

see *e.g.* [29], p. 49 equation (2.23).

Since

$$\Upsilon_Q(t).\Upsilon_P(t)) = \int_0^t \Upsilon_P(s-)d\Upsilon_Q(s) + \Upsilon_Q(s-)d\Upsilon_P(s),$$

we may also write

$$\begin{aligned} (\mathcal{T}_t^\pi f)(x) &= \mathbf{E}\left(\exp(i(\int_0^t \Upsilon_P(s-)d\Upsilon_Q(s) + \Upsilon_Q(t).x))f(x + \Upsilon_P(t))\right) \\ &= \mathbf{E}\left(\exp(i(\int_0^t (\Upsilon_P(s-) + x)d\Upsilon_Q(s)))f(x + \Upsilon_P(t))\right). \end{aligned}$$

Then the conditional expectation,

$$\mathbf{E}\left(\exp(i \int_0^t (\Upsilon_P(s-) + x)d\Upsilon_Q(s)) \mid \Upsilon_P\right) = \exp(-\Phi_Q(\Upsilon_P, x, t)),$$

because Υ_P and Υ_Q are independent.

Now the Lévy-Itô decomposition for the symmetric Lévy process Υ_Q takes the form

$$\Upsilon_Q(t) = \theta_Q B(t) + \int_{|u|<1} u\tilde{N}(t,du) + \int_{|u|\geqslant 1} uN(t,du),$$

where $\theta_Q \geqslant 0, B = (B(t), t \geqslant 0)$ is a standard Brownian motion and N is a Poisson random measure on $\mathbf{R}^+ \times (\mathbf{R} - \{0\})$ which is independent of B and has a compensated measure $\tilde{N}$ and a symmetric Lévy measure η_Q.

Let $g : \mathbf{R}^+ \to \mathbf{R}$ be right continuous with left limits. We apply Itô's formula to the semimartingale $\exp\left(i\int_0^t (g(s-)+x)d\Upsilon_Q(s)\right)$ and then take expectations, to obtain for each $t \geqslant 0, x \in \mathbf{R}$,

$$\begin{aligned}
\Phi_Q(g,x,t) &= \frac{\theta_Q^2}{2}\int_0^t (g(s-)+x)^2 ds \\
&- \int_0^t \int_{|u|<1} [\exp(iu(g(s-)+x)) - 1 - iu(g(s-)+x)]\eta_Q(du)ds \\
&- \int_0^t \int_{|u|\geqslant 1} [\exp(iu(g(s-)+x)) - 1]\eta_Q(du)ds.
\end{aligned}$$

Observe that $\Phi_Q(f,x,t)$ is a non-negative functional since η_Q is symmetric, indeed if we interpret all integrals with respect to η_Q as principal values, we may write

$$\Phi_Q(g,x,t) = \frac{\theta_Q^2}{2}\int_0^t (g(s-)+x)^2 ds + \int_0^t \int_{\mathbf{R}-\{0\}} (1-\cos(ug(s-)))\eta_Q(du)ds.$$

We now conclude that

$$(\mathcal{T}_t^\pi f)(x) = \mathbf{E}\left(\exp(-\Phi_Q(\Upsilon_P, x, t)) f(x + \Upsilon_P(t))\right),$$

which completes the proof of the proposition. $\square$

We remark that much of the analysis described in this section can also be carried out for non-symmetric Dirichlet forms, provided the sector condition described in [21] holds.

Note.— When $-\mathcal{L}^\pi$ is positive and symmetric, the symbol of $\mathcal{L}^\pi$ takes the form

$$
\begin{aligned}
\sigma_{\mathcal{L}^\pi}(x,\xi) &= -c_{00} - c^1_{jk}x^j x^k - c^2_{jk}\xi^j\xi^k \\
&\quad + \int_{\mathbf{R}^{2n+1}-\{0\}} (\cos(a+q.x+p.\xi)-1)\nu(da,dq,dp),
\end{aligned}
$$

and following a calculation in [12], p.109, we can write the associated closed form in the alternative form

$$
\overline{\mathcal{E}_\pi}(f,f) = -(2\pi)^{-n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\sigma_{\mathcal{L}^\pi}(x,\xi)\Gamma(f)(x,\xi)dxd\xi,
$$

where $\Gamma(f)(x,\xi) = \int_{\mathbf{R}^n} e^{-iy.\xi}f(x+\frac{1}{2}y)f(x-\frac{1}{2}y)dy$, for each $f \in C_c^\infty(\mathbf{R}^n)$.

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