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## Effective estimates for global relations on Euler-type series<sup>(\*)</sup>

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**ABSTRACT.** — In his work [Ch 1], [Ch 2], the second author has given a precise upper bound for the smallest prime for which a given non-trivial algebraic relation between values of Euler-type series ceases to be satisfied. This bound depends on the crucial, but ineffective, constant appearing in Shidlovsky's lemma. Using the relations of [BB], [Be 2] on the exponents of irregular differential equations and a method of the third author, we here make Shidlovsky's constant, hence the above bound, entirely effective. Furthermore, the upper bound is replaced by a sequence of intervals, and the non-vanishing by a lower bound, all again entirely effective.

**RÉSUMÉ.** — L'article comporte deux volets : d'une part montrer qu'il y a une infinité de nombres premiers pour lesquels une relation de dépendance algébrique non triviale entre valeurs de séries de type Euler cesse d'être satisfaite ; d'autre part, donner une version effective de cet énoncé. Le premier but est atteint au moyen d'un raffinement des techniques du second auteur, qui permet d'évaluer par défaut la répartition de ces nombres premiers. Pour remplir le second objectif, on établit une version entièrement effective du lemme de zéros de Shidlovsky : celle-ci repose sur une méthode du troisième auteur pour calculer les exposants d'une équation différentielle en une singularité irrégulière, jointe à la relation de Fuchs généralisée.

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## 1. Statement of the results

Let  $\mathbf{K}$  be an algebraic number field of degree  $\kappa$  over  $\mathbf{Q}$ . In 1981 E. Bombieri [Bo] introduced the notion of a global relation. Let  $P(y_1, \dots, y_m) \in \mathbf{K}[y_1, \dots, y_m]$ ,  $\xi \in \mathbf{K}$ ,  $f_i(z) \in \mathbf{K}[[z]]$ ,  $i = 1, \dots, m$ . The relation

$$P(f_1(\xi), \dots, f_m(\xi)) = 0$$

is called *global* if it holds in every completion  $\mathbf{K}_v$  of  $\mathbf{K}$  for which all  $f_i(\xi)$  converge.

Suppose that the considered power series are algebraically independent over  $\mathbf{K}(z)$  and converge at  $\xi$  in all non-archimedean completions  $\mathbf{K}_v$  of  $\mathbf{K}$  (with a possible exception of a finite number of them). Expecting that no non-trivial global relation occurs, one can try to characterize, in terms of numerical data attached to  $\mathbf{K}$ ,  $\xi$ , the  $f_i$ 's and the (non-zero) polynomial  $P$ , the set  $\mathcal{S} = \mathcal{S}(\xi, P(f_1, \dots, f_m))$  of all prime numbers  $p$  for which there exists a valuation  $v$  on  $\mathbf{K}$  extending the  $p$ -adic one such that when computed in  $\mathbf{K}_v$ :

$$P(f_1(\xi), \dots, f_m(\xi)) \neq 0.$$

In [A], Y. André gave a conditional criterion for this set  $\mathcal{S}$  to be *infinite* when the  $f_i$ 's are holonomic Gevrey series of arithmetic type of positive order (cf. [A], Thm. 3.2.1, assuming Conj. 3.1.3). His method is of a qualitative nature, and it seems difficult to quantify it, let alone make it effective. On the other hand, quantitative upper bounds for the least element of  $\mathcal{S}$  were obtained in [Ch 1], [Ch 2] in the (slightly less general) case of  $F$ -series. The aim of this paper is to sharpen the latter result on  $F$ -series in two ways: firstly, by showing, unconditionally, that  $\mathcal{S}$  is infinite, and secondly, by giving an entirely effective description an infinite number of disjoint intervals of  $\mathbf{R}$  which  $\mathcal{S}$  meets.

Recall that for given positive real numbers  $c_1, c_2, c_3$  and a natural integer  $d$ , the class  $F(\mathbf{K}, c_1, c_2, c_3, d)$  consists of power series  $f(z) = \sum_{n=0}^{\infty} a_n n! z^n$  for which:

1. for all  $n = 0, 1, \dots, a_n \in \mathbf{K}$ , and  $\exp(c_1 n)$  is an upper bound for the size of  $a_n$  (i.e. the maximum of the moduli of the algebraic conjugates of  $a_n$ ; for more on sizes and heights, see Remark 1.5 below);

2. there exists a sequence  $d_n$  of positive integers  $d_n = d_{0,n} d^n$  such that for all indices  $n \geq k \geq 0$ ,  $d_n a_k$  lies in the ring of integers  $\mathbf{Z}_{\mathbf{K}}$  of  $\mathbf{K}$ , while  $d_{0,n}$

is divisible only by prime numbers  $p \leq c_2 n$  satisfying  $ord_p d_{0,n} \leq c_3 (\log_p n + \frac{n}{p^2})$ . In particular,  $f$  is a Gevrey series of arithmetic type, of order 1 <sup>(1)</sup>. A typical example is given by Euler's series  $f(z) = \sum_{n=0}^{\infty} n! z^n$  (cf. [Re]), which satisfies the differential equation  $z^2 f' + (z-1)f + 1 = 0$ .

We consider such  $F$ -series  $f_2, \dots, f_m$  and assume that

3. the vector  ${}^t(f_1(z) \equiv 1, f_2(z), \dots, f_m(z))$  is a solution of a differential system  $\mathbf{D}$  with coefficients  $A_{i,j} \in \mathbf{K}(z)$ :

$$Y' = A(z)Y : y'_i = \sum_{j=1}^m A_{i,j}(z)y_j, \quad i = 1, \dots, m. \quad (\mathbf{D})$$

We denote by  $q = \text{deg}(\mathbf{D})$  and  $H(\mathbf{D})$  the degree and height of the differential system  $\mathbf{D}$ . On letting  $T(z) \in \mathbf{K}[z]$  be the monic polynomial of minimal degree such that  $T(z)A_{i,j}(z) \in \mathbf{K}[z]$  for all  $i, j$ , this means that the collection of polynomials  $T, TA_{i,j}$  has degree  $q$  and height  $H(\mathbf{D})$  in the sense of Remark 1.5 below. We further denote by  $n_0 := n_0(\mathbf{D})$  the well-known *Shidlovsky's constant* appearing in the theory of  $E, G$  and  $F$ -functions (cf. [Sh], Chapter 3, Formula (83)). This constant is the main source of ineffectivity in this subject; it will be dealt with in Theorem 1.2.

In what follows, we fix a non-zero ordinary point  $\xi \in \mathbf{K}$  of the system  $\mathbf{D}$ :

$$\xi = \frac{a}{b}, a \in \mathbf{Z}_{\mathbf{K}}, b \in \mathbf{N}, \xi T(\xi) \neq 0$$

and set  $h(\xi) = \ln(1 + \text{size}(\xi)) + \ln b$ , which is essentially the logarithm of the height  $H(\xi)$  of  $\xi$ . We also write  $\text{Disc}(\mathbf{K})$  for the discriminant of the number field  $\mathbf{K}$ , and associate to the numerical data above the numbers  $c_4 = c_1 + \ln d + \frac{5}{4}c_2c_3 + 2c_3 + 5$ ,  $c_5 = m^2c_4 + 2$ , and

$$N_0 = \max(n_0(\mathbf{D}), (\ln 2 + \frac{1}{\kappa} \text{Disc}(\mathbf{K}))^2, \exp(4(2(m+3) + c_4(m-1))^2), \exp(h(\xi)^2 + \text{deg}(\mathbf{D})(h(\xi) + 2) + 2 \ln H(\mathbf{D}) + 1)).$$

Let further

$$H_0 = \exp(N_0 \ln N_0 (1 - \frac{m+3 + (m-1)c_5}{(\ln N_0)^{\frac{1}{2}}}). \quad (\mathbf{A})$$

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<sup>(1)</sup> Conversely, the denominators of Gevrey series of such type, if holonomic, satisfy the first condition  $p \leq c_2 n$ , but we do not know whether the second one always holds.

Notice that this real number  $H_0$  is effectively computable in terms of the data  $c_1, c_2, c_3, d, m, \kappa = [\mathbf{K} : \mathbf{Q}], \text{Disc}(\mathbf{K}), H(\xi), \text{deg}(\mathbf{D}), H(\mathbf{D})$ , and the constant  $n_0(\mathbf{D})$ .

Now, consider the positive functions in the real variable  $x > 3$  defined by

$$l(x) := e^{(\ln x)^{\frac{1}{2}}}$$

$$u(x) = m(x + 1) - [x(\ln x)^{-\frac{1}{2}}],$$

where  $[\cdot]$  denotes the integral part, and set

$$P_{lower}(x) = l\left(\frac{x}{\ln x}\right), \quad P_{upper}(x) := u\left(\frac{x}{\ln x} \left(1 + \frac{2(m+3 + (m-1)c_5)}{(\ln(x/\ln \ln x))^{\frac{1}{2}}}\right)\right). \quad (\mathcal{B})$$

Note that when  $x$  assumes values in a quickly increasing sequence of real numbers, the intervals  $[P_{lower}(x), P_{upper}(x)]$  are disjoint and tend to  $\infty$ .

**THEOREM 1.1.** — Assume that the  $F$ -series  $f_1(z) \equiv 1, f_2(z), \dots, f_m(z)$  are linearly independent over  $\mathbf{K}(z)$  and constitute a solution of  $(\mathbf{D})$ . Let  $\xi$  be a non-zero ordinary point of  $\mathbf{D}$ , and recall the notations  $(\mathcal{A}), (\mathcal{B})$  above. Let further

$$\Lambda(y_1, \dots, y_m) = h_1 y_1 + \dots + h_m y_m$$

be a nonzero linear form with coefficients  $h_i \in \mathbf{Z}_{\mathbf{K}}$  and height

$$H(\Lambda) = \max_{i=1, \dots, m} H(h_i).$$

Then for any  $H \geq \max(H(\Lambda), H_0)$ , there exists a prime number  $p$  with

$$P_{lower}(\ln H) < p < P_{upper}(\ln H)$$

and a valuation  $v|p$  on  $\mathbf{K}$  such that in  $\mathbf{K}_v$

$$\Lambda(\xi) = \Lambda(f_1(\xi), \dots, f_m(\xi)) \neq 0,$$

and more precisely such that in this  $\mathbf{K}_v$

$$|\Lambda(\xi)|_v \geq H^{-m - \frac{m+3+2mc_5}{\sqrt{\ln \ln H}}}.$$

Making these bounds on  $p$  and  $|\Lambda(\xi)|_v$  effective therefore reduces to the question of finding an upper bound for  $n_0(\mathbf{D})$  depending effectively on the data  $m, \kappa, \text{deg}(\mathbf{D}), H(\mathbf{D})$  of the differential system  $\mathbf{D}$ . This can be achieved

under certain conditions on the differential system  $\mathbf{D}$ , cf. [Br]. At about the same time, J. Yebbou (unpublished, but see [Be 1]) proved the existence of an effective upper bound for  $n_0(\mathbf{D})$  in the general case. Developing his method, we shall here show:

**THEOREM 1.2.** — *For any differential system  $(\mathbf{D})$  over  $\mathbf{K}(z)$ , there exist two positive real numbers  $C, c$ , effectively computable in terms of  $q = \text{deg}(\mathbf{D})$ ,  $m$ , and  $\kappa = [\mathbf{K} : \mathbf{Q}]$ , such that*

$$n_0(\mathbf{D}) \leq C(\kappa, m, q) H(\mathbf{D})^{c(\kappa, m, q)}.$$

More precisely, we can take

$$c(\kappa, m, q) = \text{Log}_2 C(\kappa, m, q) = (2\kappa(q+1)m)^{(2(q+1)m)^{8m}}.$$

*Remark 1.3.* — Since our main concern is with effectivity, we made no attempt to get sharp bounds in Theorem 1.2. It is likely that on using Corel's notion of exponents for differential systems [Co], the upper bound for  $c(\kappa, m, q)$  can be considerably reduced. A natural question is whether it can be reduced to  $c(\kappa, m, q) = 1$  (as shown by the differential equation  $y' = \frac{H}{z}y, H \in \mathbf{N}$ ,  $n_0(\mathbf{D})$  is at least linear in  $H(\mathbf{D})$ ).

*Remark 1.4.* — To handle *polynomial relations* of arbitrary degree  $K \geq 1$  instead of linear ones (as in the introduction of the paper, assuming in particular that the functions  $f_1, \dots, f_m$  are homogeneously algebraically independent over  $\mathbf{K}(z)$ ), we can apply Theorem 1.1 to the set of monomials of degree  $K$  in these series. Indeed, for  $k_1 + \dots + k_m = K$ , the series  $f_1^{k_1} \dots f_m^{k_m}$  belong to the  $F$ -class with parameters

$$c_1 + \ln 2, \quad c_2, \quad (c_3 + 1)K, \quad d^{1+[\ln K]},$$

and constitute a solution of the differential system  $\text{Sym}^K(\mathbf{D})$ , which has rank  $\binom{m+K-1}{K}$ , degree  $\leq \text{deg}(\mathbf{D})$  and height  $\leq (m+K)^{m+K} H(\mathbf{D})$ . As for Theorem 2, it is more efficient to replace it by the relation

$$n_0(\text{Sym}^K(\mathbf{D})) \leq (m+K)^{2K} n_0(\mathbf{D}),$$

which immediately follows from [BB], top of p. 191.

*Remark 1.5 (heights).* — In this paper, we use the following definitions on heights and degrees. Let  $\overline{\mathbf{Q}}$  be the algebraic closure of  $\mathbf{Q}$  in  $\mathbf{C}$ , and let  $\overline{\mathbf{Z}}$  be the ring of algebraic integers. We define:

- the height  $H(\alpha)$  of an algebraic number  $\alpha$  as the maximum of the denominator  $den(\alpha) \in \mathbf{N}$  and of the size  $size(\alpha)$  of  $\alpha$ , where as in [Sh],  $den(\alpha)$  is the minimal positive integer  $d \in \mathbf{Z}$  such that  $d\alpha$  lies in  $\overline{\mathbf{Z}}$ , and  $size(\alpha)$  is the maximum of all the archimedean absolute values of  $\alpha$  (the height of  $\alpha$  in the sense of [Sh], i.e. the height of its minimal polynomial over  $\mathbf{Z}$ , will not be used here).

Thus, a non-zero element  $\beta$  of a number field  $\mathbf{K}$  of degree  $\kappa$  over  $\mathbf{Q}$  satisfies  $H(\frac{1}{\beta}) \leq H(\beta)^{2\kappa-1}$ ; in particular,  $H(\beta)^{-2\kappa+1} \leq |\beta| \leq H(\beta)$ .

- the height  $H(P)$  of a collection  $P = (P_1, \dots, P_t)$  of polynomials  $P_i \in \overline{\mathbf{Q}}[X]$  as the maximum of the least common denominator of all the coefficients of all the  $P_i$ 's and of all the heights of these coefficients; the degree  $deg(P)$  of  $P$  as the maximum of the degrees of the  $P_i$ 's (note that the height  $H(P)$  of a polynomial  $P$  coincides with that of [Sh] only when  $P$  has integral coefficients).

Thus, all roots  $\epsilon$  of a non-zero polynomial  $P \in \mathbf{K}[X]$  of degree  $q$  are algebraic numbers of degree  $\leq \kappa q$  and height  $H(\epsilon) \leq qH(P)^{2\kappa}$ .

- the height  $H(A)$  (resp. degree  $deg A$ ) of a collection  $A = (A_1, \dots, A_t)$  of rational functions  $A_i \in \overline{\mathbf{Q}}(X)$  as the height (resp. degree) of the collection of polynomials  $T, TA_1, \dots, TA_t$ , where  $T$  is the monic polynomial of minimal degree in  $\overline{\mathbf{Q}}[X]$  such that all the  $TA_i$ 's lie in  $\overline{\mathbf{Q}}[X]$ ; this applies in particular to the set of coefficients  $A = (A_{ij}; 1 \leq i, j \leq m)$  of the differential system  $\mathbf{D}$  considered in this paper, so that the notions of height  $H(\mathbf{D}) := H(A)$  and degree  $deg(\mathbf{D}) := deg(A)$  of  $\mathbf{D}$  are well-defined.

For a collection  $A = (A_1, \dots, A_t)$  of rational functions  $A_i \in \mathbf{K}(X)$ , we also say that  $A$  has “height and degree at most  $H_1(A), deg_1(A)$ ” if there exists a non-zero polynomial  $T_1$  such that  $T_1, T_1A_1, \dots, T_1A_t$  lie in  $\mathbf{Z}_{\mathbf{K}}[X]$  and form a collection of polynomials of height at most  $H_1(A)$  and degree at most  $deg_1(A)$ . One easily checks that one may take  $H_1(A) = H(A)^2, deg_1(A) = deg(A)$ . Conversely, a Gelfond-type lemma shows that for any choice of  $T_1$  and consequent  $H_1(A), deg_1(A) = q_1$ , we have:  $deg(A) \leq deg_1(A)$  and  $H(A) \leq (2q_1)^{q_1} H_1(A)^{2\kappa q_1}$ . These estimates will be used at the beginning of the proof of Theorem 1.1, and at the end of the proof of Theorem 1.2.

## 2. Proof of Theorem 1.1

We shall prove Theorem 1.1 under slightly less restrictive conditions on the constant  $H_0$  from Formula (A) and on the functions  $\ell, u, P_{lower}, P_{upper}$  from Formulae (B) of §1.

We start with the same hypotheses 1, 2, 3 as in §1 on the  $f_i$ 's and  $\mathbf{D}$ , and use the same notations  $c_1, c_2, c_3, d, c_4, c_5, n_0(\mathbf{D})$ . We further fix a non zero polynomial  $T_1(z) \in \mathbf{Z}_{\mathbf{K}}[z]$  such that  $T_1(z)A_{i,j}(z) \in \mathbf{Z}_{\mathbf{K}}[z]$  for all  $i, j$ , and denote by  $\deg_1(A)$  and  $H_1(A)$ , respectively, the maximum of the degrees (and, respectively, heights) of  $T_1(z)$  and of all  $T_1(z)A_{i,j}(z)$ . We assume that  $\xi = \frac{a}{b} \in K$  is not a zero of  $T_1$ . We further denote by  $c = c(\mathbf{K})$  the constant which comes from Siegel's lemma, see, e.g. [Sh], page 105. This constant is effectively computable in terms of the degree  $\kappa$  and the discriminant  $Disc(\mathbf{K})$  of  $\mathbf{K}$ , and indeed, one may take  $c(\mathbf{K}) = 2(Disc(\mathbf{K}))^{\frac{1}{2\kappa}}$ , in view of the well-known Bombieri-Vaaler version of Siegel's lemma.

Let  $[x]$  be the integral part of a real number  $x$ , and consider all positive integers  $N \in \mathbf{N}$  such that

$$N \geq n_0(\mathbf{D}) \tag{2.1}$$

$$N \geq 2m \ln N + (m + 2)(\ln N)^{\frac{1}{2}} \tag{2.2}$$

$$N \geq \exp(4(2(m + 3) + c_4(m - 1)))^2 \tag{2.3}$$

$$c_4[N(\ln N)^{-\frac{1}{2}}] > \ln(N + 1) + \ln c + \ln m$$

$$N > (\ln c)^2$$

$$\ln N > h(\xi)(\ln N)^{\frac{1}{2}} + \deg_1(A)(h(\xi) + 2) + \ln H_1(A). \tag{2.4}$$

Suppose further that a decreasing function  $\varepsilon(x)$  is given with  $\lim_{x \rightarrow +\infty} \varepsilon(x) = 0$ , and that either

$$\lim_{x \rightarrow +\infty} \varepsilon(x)(\ln x)^{\frac{1}{2}} = +\infty, \tag{2.5}$$

in which case we set  $\varepsilon_1(x) = (m + 3)\varepsilon(x)$ ,

or that

$$\varepsilon(x) = (\ln x)^{-\frac{1}{2}}, \tag{2.6}$$

in which case we set  $\varepsilon_1(x) = (m + 3 + (m - 1)c_5)(\ln x)^{-\frac{1}{2}}$  (this second case is the one considered in §1). We then require that  $N$  also satisfy

$$\varepsilon(N) \ln N > \ln b + \ln d. \tag{2.7}$$

Let  $N_0$  be the minimal natural number for which all these hypotheses about  $N$  hold, and set

$$H_0 = \exp(N_0 \ln N_0 (1 - \varepsilon_1(N_0))). \tag{A'}$$

Notice again that this real number  $H_0$  is effectively computable in terms of the data  $c_1, c_2, c_3, d, m, \kappa, Disc(\mathbf{K}), H(\xi), deg(\mathbf{D}), H(\mathbf{D})$ , the chosen function  $\varepsilon$  and the constant  $n_0(\mathbf{D})$ .



Finally, we consider the positive functions in the real variable  $x > 3$  defined by

$$l(x) := x^{\varepsilon(x)} \tag{2.8}$$

$$u(x) = m(x + 1) - [x(\ln x)^{-\frac{1}{2}}]. \tag{2.9}$$

and set

$$P_{lower}(x) = l\left(\frac{x}{\ln x}\right), \quad P_{upper}(x) := u\left(\frac{x}{\ln x}\left(1 + 2\varepsilon_1\left(\frac{x}{\ln x}\right)\right)\right). \tag{B'}$$

Note again that when  $x$  assumes values in a quickly increasing sequence of real numbers, the intervals  $[P_{lower}(x), P_{upper}(x)]$  are disjoint and tend to  $\infty$ .

We now claim that *Theorem 1.1 holds in the more general setting when Formulae (A) and (B) of §1 are respectively replaced by (A'), (B')*. Notice that on using the estimates on  $\deg_1(A), H_1(A)$  and  $c(\mathbf{K})$  given above, we do retrieve in our “second case” the statement of Theorem 1.1.

The arguments leading to this sharpened form of Theorem 1.1 are close to those of [Ch2], and we shall here only develop the new aspects of the proof.

LEMMA 2.1. — *Let  $f_i, i = 1, \dots, m$  and  $\xi$  be as in Theorem 1.1. Let  $N$  satisfy (1)–(4). Then there exist linearly independent forms*

$$\Lambda_k = \sum_{i=1}^m h_{k,i} y_i, \quad k = 1, \dots, m$$

with coefficients  $h_{k,i} \in \mathbf{Z}_{\mathbf{K}}$  such that

$$H(h_{k,i}) \leq \exp(N \ln N + c_5 N (\ln N)^{\frac{1}{2}})$$

and such that for any prime  $p$  with  $3c_3 < p \leq u(N)$  and any  $v|p$ , the  $v$ -adic numbers  $\Lambda_k(\xi) := \sum_{i=1}^m h_{k,i} f_i(\xi), k = 1, \dots, m$ , satisfy

$$|\Lambda_k(\xi)|_v \leq |u(N)!|_v \exp\left(\frac{\kappa_v}{\kappa} ((c_3 + 2) \log_p u(N) + c_3 u(N) p^{-2}) \ln p\right).$$

(Here  $\kappa_v = [\mathbf{K}_v : \mathbf{Q}_p], \kappa = [\mathbf{K} : \mathbf{Q}]$ )

The proof of this lemma is similar to those of Lemmas 14-16 from [Sh, pp. 107-114].

Now,  $m - 1$  among the forms  $\Lambda_1, \dots, \Lambda_m$ , together with  $\Lambda$ , are linearly independent over  $\mathbf{K}$ . Since the numeration of these forms is at our disposal, let  $\Lambda, \Lambda_2, \dots, \Lambda_m$  be linearly independent. Their determinant  $\Delta$  then is a nonzero element of  $\mathbf{Z}_{\mathbf{K}}$ . In any  $\mathbf{K}_v$  with  $p$  as in the Lemma, we proceed as follows. We take the products of  $f_i(\xi)$  and of the  $i$ -th column of  $\Delta$  and add them all to the first column, which now contains the elements  $\Lambda(\xi), \Lambda_2(\xi), \dots, \Lambda_m(\xi)$  in the local field  $\mathbf{K}_v$ . In this way, we get

$$\Delta = \Lambda(\xi)\Delta_1 + \sum_{i=2}^m \Lambda_i(\xi)\Delta_i \quad (2.10)$$

where  $\Delta_i$  is the determinant of the minor of the  $i$ -th element of the first column of  $\Delta$ . We shall evaluate  $\Delta$  by making a convenient choice of the natural number  $N$ .

In what follows,  $V_\infty$  and  $V_0$  are respectively the set of archimedean and non archimedean valuations on  $\mathbf{K}$ , and we put  $V_0 = V_1 \cup V_2 \cup V_3$ , where  $V_1$  consists of all  $v|p$  such that

$$l(N) < p \leq u(N); \quad (2.11)$$

$V_2$  consists of all  $v$ 's above primes  $p \leq l(N)$ ,  $V_3$  of all  $v$ 's above primes  $p > u(N)$ . We further use  $\Pi_i$  to denote a product over  $V_i$ ,  $\Pi_{i,j}$  for a product over  $V_i \cup V_j$ , and so on.

Let  $H$  be any positive real number such that  $H \geq \max(H(\Lambda), H_0)$ , and let  $N$  be the largest positive integer such that

$$(N - 1) \ln(N - 1)(1 - \varepsilon_1((N - 1))) \leq \ln H < N \ln N(1 - \varepsilon_1(N)). \quad (\mathcal{A}'')$$

By the definition ( $\mathcal{A}'$ ) of  $H_0$ , the number  $N$  is  $\geq N_0$ , and in particular, satisfies (2.1)-(2.4). Lemma 2.1 then gives

$$\Pi_\infty |\Delta|_v \leq m! H \exp((m - 1)N \ln N + (m - 1)c_5 N (\ln N)^{\frac{1}{2}}).$$

Since  $\Delta$  is a non-zero algebraic integer in  $\mathbf{K}$ ,  $\Pi_{2,3} |\Delta|_v \leq 1$  and the product formula  $\Pi_{1,2,3,\infty} |\Delta|_v = 1$  implies

$$\Pi_1 |\Delta|_v \geq (m! H \exp((m - 1)N \ln N + (m - 1)c_5 N (\ln N)^{\frac{1}{2}}))^{-1}. \quad (2.12)$$

We also deduce from Lemma 2.1 that

$$\begin{aligned} \Pi_1 \left| \sum_{i=2}^m \Delta_i \Lambda_i(\xi) \right|_v &\leq \Pi_1 \max |\Lambda_i(\xi)|_v \\ &\leq \Pi_{1,2} p^{\left(\frac{\kappa_v}{\kappa} \left( (c_3+2) \log_p u(N) + c_3 u(N) p^{-2} \right)\right)} \cdot \Pi_1 |u(N)!|_v. \end{aligned}$$

But clearly

$$\Pi_{1,2} \exp\left(\frac{\kappa_v}{\kappa} \ln p \left( (c_3+2) \log_p u(N) + c_3 u(N) p^{-2} \right)\right) \leq \exp\left((3c_3+2)mN\right),$$

while for  $N$  as above,

$$\Pi_2 |u(N)!|_v \geq \exp\left(-u(N) \ln l(N) - u(N)(2+4 \ln 2)\right).$$

Since  $\Pi_{1,2} |u(N)!|_v = (u(N)!)^{-1}$ , we finally get

$$\Pi_1 \left| \sum_{i=2}^m \Delta_i \Lambda_i(\xi) \right|_v \leq \exp\left(-mN \ln N + (m+2)N \ln N \varepsilon(N)\right). \quad (2.13)$$

Now if  $\Lambda(\xi) = 0$  in all  $\mathbf{K}_v$  with  $v|p$ ,  $l(N) < p \leq u(N)$ , then (2.10), (2.11), (2.12), (2.13) would imply

$$\begin{aligned} -\ln m! - \ln H - (m-1)N \ln N - (m-1)c_5 N (\ln N)^{\frac{1}{2}} \\ \leq -mN \ln N + (m+2)N \ln N \varepsilon(N). \end{aligned} \quad (2.14)$$

Then in both cases (2.5) or (2.6) we would get, with the corresponding function  $\varepsilon_1(N)$ , that

$$N \ln N (1 - \varepsilon_1(N)) - \ln H \leq 0, \quad (2.15)$$

and this contradicts  $(\mathcal{A}'')$ . We have thus established the non vanishing of the  $v$ -adic evaluation of  $\Lambda(\xi)$  for at least one place  $v \in V_1$ . Furthermore,  $(\mathcal{A}'')$  implies that

$$\frac{\ln H}{\ln \ln H} < N < \frac{\ln H}{\ln \ln H} \left(1 + 2\varepsilon_1\left(\frac{\ln H}{\ln \ln H}\right)\right), \quad (2.16)$$

so that in view of Formulae  $(\mathcal{B}')$  and (2.11), such a place  $v \in V_1$  lies above a prime  $p$  in the interval  $[P_{lower}(\ln H), P_{upper}(\ln H)]$ .

We have just proved that for  $N$  as in (2.16) we have

$$\Pi_1 |\Delta|_v > \Pi_1 \left| \sum_{i=2}^m \Delta_i \Lambda_i(\xi) \right|_v$$

and therefore (2.10) implies that

$$\Pi_1|\Delta|_v = \Pi_1|\Lambda(\xi)\Delta_1|_v.$$

Hence for some place  $v \in V_1$  (above a prime  $p$  as above), we deduce from (2.12)

$$|\Lambda(\xi)|_v \geq (m!H \exp((m-1)N \ln N + (m-1)c_5N(\ln N)^{\frac{1}{2}}))^{-1}, \quad (2.17)$$

and (2.17) with (2.16) finally implies

$$|\Lambda(\xi)|_v \geq H^{-m - \frac{m+3+2mc_5}{\sqrt{\ln \ln H}}}. \quad (2.18)$$

This estimate shows that we are completely effective: to check the non-vanishing of  $\Lambda(f_1(\xi), \dots, f_m(\xi))$  in one of these fields  $\mathbf{K}_v$ , it suffices to truncate the series  $f_i$ 's at their initial terms, where the number  $\mathcal{N}$  of terms to be considered is effectively bounded from above in terms of  $H(\Lambda)$  and the parameters defining the constant  $H_0$  of Theorem 1.1. More precisely, assuming that  $H_0$  satisfies  $(\mathcal{A}')$ , that  $H_0 \geq \exp \exp(8m^2(m+3+(m-1)c_5)^2)$ , and that  $H = \max(H(\Lambda), H_0)$ , we may simply truncate the series  $f_1, \dots, f_m$  at any order

$$\mathcal{N} \geq 8\kappa(m+1)^2 \left( \frac{\ln H}{\ln \ln H} \right)^2.$$

### 3. Proof of Theorem 1.2

Let  $\mathbf{D}$  be any differential system as in §1, and let  $S$  be its set of singularity, i.e. the union of the zeroes of the denominator  $T$  of the  $A_{i,j}$ 's and (possibly) of the point  $\infty \in \mathbf{P}_1(\mathbf{C})$ . We have to bound the constant  $r_0$  appearing in Shidlovsky's lemma [Sh], p. 93, Lemma 8 (and we then obtain the bound  $n_0 \leq 2r_0$  by p. 99 (83)). By Theorem 1 of [BB] (applied to the linear case  $h = 1$ , and noticing that  $s$  is then bounded by the order  $m$  of  $\mathbf{D}$ ), Shidlovsky's lemma is valid with

$$r_0 = C_1m + C_2m^2$$

where  $C_1$  and  $C_2$  can be made explicit in terms of  $\mathbf{D}$  as follows:

– in view of [BB], Lemma 2bis,  $C_2$  is bounded from above by  $q := \deg A$ . Indeed, we are here considering only one point  $\beta$ , viz.  $\beta = 0$ , so  $\ell' = 0$  in the notations of this lemma;

– defining  $C_1$  requires a local analysis of  $\mathbf{D}$ : for each  $\alpha \in S$ , let  $r_\alpha$  be the largest non-positive real number such that all the solutions of  $\mathbf{D}$  near  $\alpha$  have generalized order  $\geq r_\alpha$ , in the sense of [BB], Prop. 1. Then, we may take  $C_1 = -\sum_{\alpha \in S} r_\alpha$ .

Denoting by  $\mathcal{R}(\mathbf{D})$  the maximum of the absolute values of the  $r_\alpha$ 's,  $\alpha \in S$ , we shall finally derive:

$$n_0(\mathbf{D}) \leq 2(q+1)m^2(\mathcal{R}(\mathbf{D}) + 1).$$

### 3.1 The case of differential equations

In order to bound  $\mathcal{R}(\mathbf{D})$  from above, we first assume that  $\mathbf{D} = \mathbf{D}_L$  is the companion form of a differential equation  $Ly = 0$  (i.e., with respect to Theorem 1.1, that the components of the  $F$ -series consist of a solution  $f$  of  $Ly = 0$  and of its derivatives  $f^{(k)}$  of order  $k \leq m-1$ ; the hypothesis of linear independence on the  $f_i$ 's then means that  $L$  is the annihilator of  $f$  in  $\mathbf{K}(z)[d/dz]$ ). Thus,

$$L = (d/dz)^m + a_1(z)(d/dz)^{m-1} + \dots + a_m(z) \in \mathbf{K}(z)[d/dz]$$

is a differential operator of order  $m$ , where  $a = (a_1, \dots, a_m)$  is a collection of elements of  $\mathbf{K}(z)$ . We let  $T \in \mathbf{K}[z]$  be their monic common denominator, and recall from §1, Remark 1.5, that the height  $H(a)$  and degree  $\deg(a)$  of  $a$  (which we also simply call the height  $H(L)$  and degree  $\deg(L)$  of  $L$  in what follows) is the height of the collection of polynomials  $T, Ta_1, \dots, Ta_m$ . We set  $q = \deg(L)$ .

In these conditions,  $L$  admits at  $\alpha \in S$  a set of  $m$  exponents  $e_1^\alpha, \dots, e_m^\alpha$  in the sense of [Be 2], Proposition 2. Comparing their definition to the generalized orders of  $\mathbf{D}_L$ , we obtain

$$r_\alpha \geq \min(0, \min_{i=1, \dots, m} \operatorname{Re}(e_i^\alpha)) - qm - (m-1),$$

where  $\operatorname{Re}(z)$  is the real part of the complex number  $z$ . Denoting by  $\mathcal{E}(L)$  the maximum of all the absolute values of the exponents of  $L$  at all singularities  $\alpha$  of  $L$ , we may then take

$$\mathcal{R}(\mathbf{D}_L) \leq \mathcal{E}(L) + (q+1)m.$$

Under the current assumption  $\mathbf{D} = \mathbf{D}_L$ , our task has thus turned into finding an effective upper bound for the absolute values of the exponents

of  $L$  at any given singularity  $\alpha$  in terms of the data  $\kappa, m, q = \deg(L), H(L)$  attached to the coefficients  $a_i(z)$  of  $L$ . We shall indeed prove:

LEMMA 3.1. — *Let  $L \in \mathbf{K}(z)[d/dz]$  be a monic differential operator of order  $m$ , degree  $\deg(L) = q$ , and height  $H(L)$ . The exponents of  $L$  at any of its singularities have absolute values at most*

$$\mathcal{E}(L) \leq 2^{(36(q+1)m\kappa)^{9(q+1)^2 m^{3m}}} H(L)^{(5\kappa(q+1)m)^{9(q+1)^2 m^{3m}}}.$$

The proof of Lemma 3.1 proceeds in three steps: we first study the fuchsian case, then the case where irregular singularities with unramified determining factors are allowed, and finally the general case.

### *The fuchsian case*

As a preparation for the general case, we first bound  $|r_\alpha|$  when  $\alpha$  is a regular singularity of  $L$ . In this case, the exponents are the usual exponents of Fuchs theory, i.e. the zeroes of the indicial polynomial  $P_{L,\alpha}(X) \in \mathbf{K}(\alpha)[X]$  of  $L$  at  $\alpha$ , so that their absolute values are bounded from above by  $m H(P_{L,\alpha})^{2[\mathbf{K}(\alpha):\mathbf{Q}]}$ . Let us first compute the height of this polynomial when  $\alpha = 0$ . Then,  $T(z)$  has a zero of order at most  $m$  at 0,  $z^i a_i(z)$  is holomorphic at 0 for all  $i$ 's, and the differential operator  $z^m L$  reads in terms of the derivation  $\theta = zd/dz$ :

$$z^m L = \theta^m + b_1(z)\theta^{m-1} + \cdots + b_m(z),$$

where for each  $i = 1, \dots, m, b_i$  is a linear combination of  $1, za_1, \dots, z^i a_i$  with constant integral coefficients of absolute value  $\leq (2m)^m$  (this is a rough upper bound for the coefficients of the matrix expressing the  $z^j (d/dz)^j$  in terms of the  $\theta^k$ ). Therefore, the indicial equation  $P_{L,0}(X) = X^m + b_1(0)X^{m-1} + \cdots + b_m(0)$  satisfies:

$$H(P_{L,0}) \leq (2m)^{m+1} H(L)^{2\kappa}.$$

Putting  $t = z^{-1}$ ,  $td/dt = -zd/dz$ , we obtain the same bound for  $H(P_{L,\infty})$ .

For a general finite  $\alpha$ , we set

$$t = z - \alpha, L = (d/dt)^m + a_1^\alpha(t)(d/dt)^{m-1} + \cdots + a_m^\alpha(t) \in \mathbf{K}(\alpha)(t)[d/dt],$$

where the  $a_i^\alpha(t) = a_i(t + \alpha)$  form a collection of rational functions in  $\mathbf{K}(\alpha)(X)$  of degree  $q$  and height

$$H(a^\alpha) \leq q^{2q} H(\alpha)^q H(a) \leq (2q)^q H(L)^{3\kappa q}$$

(recall that  $\alpha$  is a zero of  $T$ , hence has height  $\leq qH(L)^{2\kappa}$ ). Since  $[\mathbf{K}(\alpha) : \mathbf{Q}] \leq \kappa q$ , the indicial polynomial of  $L$  at  $\alpha$  finally has height

$$H(P_{L,\alpha}) \leq (2m)^{m+1} ((2q)^q H(L)^{3\kappa q})^{2\kappa q} \leq (4qm)^{2\kappa m q^2} H(L)^{6\kappa^2 q^2}.$$

Thus, Lemma 3.1 holds with

$$\mathcal{E}(L) \leq m((4qm)^{2\kappa m q^2} H(L)^{6\kappa^2 q^2})^{2\kappa q},$$

hence  $n_0(\mathbf{D}_L) \leq m(4mq)^{5\kappa^2 m q^3} H(L)^{12\kappa^3 q^3}$  whenever  $L$  is *fuchsian* over  $\mathbf{P}_1(\mathbf{C})$  -a situation which does not occur for  $F$ -series, but which would be automatic if dealing with  $G$ -functions.

*The irregular unramified case*

We now assume that  $\alpha$  is an irregular singularity of  $L$ . When  $\alpha$  is finite, this implies that  $q > 1$  (in fact,  $q > 1$  is automatic for  $F$ -series), and we shall henceforth suppose that  $q \geq 1$ . By the same method as above (taking  $t = 1/z$  when  $L$  is written in terms of  $\theta$ , replacing  $a(z)$  by  $a^\alpha(t) := a(t + \alpha)$  when  $\alpha$  is finite), we may restrict to the case  $\alpha = 0$ . In the final estimate for  $\mathcal{E}(L)$ , we'll just have to replace  $H(L)$  by  $(2q)^q H(L)^{3\kappa q}$  and  $\kappa$  by  $\kappa q$ .

Embedding  $\mathbf{K}(z)$  in the field  $\mathbf{K}((z))$  of meromorphic formal series, we view  $L$  as an element of  $\mathbf{K}((z))[d/dz]$ . Each exponent of  $L$  (at 0) is then a zero of the indicial polynomial  $P_{L,\alpha=0,\omega}$  attached to a ‘determining factor’  $\omega$  of  $L$ . We here bound their height under the assumption that all the determining factors of  $L$  at 0 are polynomials (rather than Puiseux polynomials) in  $\frac{1}{z}$ .

We must first determine the determining factors themselves. As shown by J. Yebbou [Ye], this can be achieved by a direct formula when  $m \leq 3$ , but the general case requires an iterative computation, as follows. The first point in this computation is that the determining factors of  $L$  are invariant under truncation of the coefficients of  $L$  up to  $z$ -adic order  $m(q - 1)$ . This is shown in [BV], Theorem on p. 52 (for a more direct, though not explicit, proof of this fact, see [Ro] and the contemporary work of B. Malgrange, as quoted in Robba’s paper.) Thus, we can replace  $L$  by

$$\tilde{L} = (d/dz)^m + \tilde{a}_1(t)(d/dz)^{m-1} + \dots + \tilde{a}_m(z) \in \mathbf{K}[z, z^{-1}][d/dz],$$

where the  $\tilde{a}_i$ ’s are Laurent polynomials in  $z$  satisfying

$$\forall i = 1, \dots, m, \tilde{a}_i \equiv a_i \pmod{z^{m(q-1)}}.$$

The corresponding collection of rational functions  $\tilde{a}$  admits  $\tilde{T} = z^t$ , where  $t = \text{ord}_0(T) \leq q$ , as common denominator, and satisfies  $\deg(\tilde{a}) \leq m(q-1) + t \leq 2mq$ . Since the power series expansion of  $z^t/T$  is majorized by  $\sum_{i \geq 0} H(T)^{2\kappa(i+1)}(1+z)^{(q-1)^i} z^i$ , while the common denominator of its first  $m(q-1)$  terms is bounded by  $H(T)^{2\kappa(m(q-1)+1)}$ , we eventually get

$$H(\tilde{L}) := H(\tilde{a}) \leq 2mq \cdot mq^2 \cdot 2^{mq^2} H(L)^{2\kappa mq+1} \leq 8^{mq^2} H(L)^{3\kappa mq}.$$

View  $z^m \tilde{L} = \tilde{D}(\theta) = \theta^m + \tilde{b}_1(z)\theta^{m-1} + \dots$  as a polynomial in  $\theta = zd/dz$  (its coefficients  $\tilde{b}_i(z) = \sum_{-t < j \leq mq} \tilde{b}_{i,j} z^j$  therefore form a collection of Laurent polynomials of height  $H(\tilde{b}) \leq (2m)^m H(\tilde{L})$ ), and consider a positive slope  $s$  of its Newton polygon. By our current assumption in this paragraph, the slopes, which lie between 0 and  $q$ , are integers, so that the determining factors for this slope have the shape

$$\omega = \frac{\lambda_0}{z^s} + \frac{\lambda_1}{z^{s-1}} + \dots + \frac{\lambda_s}{z}.$$

They are characterized by the fact that the Newton polygon of  $\tilde{D}(\theta + \omega)$  has a horizontal side (i.e. that the minimal value of the  $z$ -adic valuations of its coefficients is reached by one of positive index). The terms  $\tilde{b}_{i,j}^\lambda$  of the operator  $\tilde{D}^{(\lambda)} := \tilde{D}(\theta + \frac{\lambda}{z^s}) = \sum_{i,j} \tilde{b}_{i,j}^\lambda z^j \theta^i$  are linear forms in the  $\tilde{b}_{i,j}$ 's, whose coefficients are polynomials in  $\lambda$  of degree  $\leq m$  and height  $\leq (2ms)^{ms}$  (note that  $(\theta + \frac{\lambda}{z^s})^i = \sum_{0 \leq j, k, \ell \leq i} d_{ijk\ell} z^{-ks} \lambda^\ell \theta^j$ , where the  $d_{ijk\ell}$  are integers of absolute value  $\leq (i+2)!s^i$ ), so that each  $\tilde{b}_{i,j}^\lambda$  can be viewed as an element of  $\mathbf{K}[\lambda]$  of degree  $\leq m$  in  $\lambda$  and height  $(2ms)^{2ms} H(\tilde{L})$ . Now,  $\lambda_0$  can occur as highest term of a determining factor  $\omega$  of degree  $s$  *only if* one of these polynomials  $\tilde{b}_{i,j}^\lambda$  vanishes at  $\lambda = \lambda_0$ : indeed, for a generic  $\lambda$ , the Newton polygon of  $\tilde{D}^{(\lambda)}$  has a side of minimal slope  $s$ , which can break down (providing a new slope  $\leq s-1$ ) only if one of its monomials of minimal valuation vanishes. Therefore,  $\lambda_0$  is an algebraic number of degree  $\leq \kappa m$  and height  $H(\lambda_0) \leq (2ms)^{5\kappa ms} H(\tilde{L})^{2\kappa}$ .

We can repeat this process to get  $\lambda_1$  from the now known  $\tilde{D}^{(\lambda_0)} = \theta^m + \tilde{b}_1^{\lambda_0}(z)\theta^{m-1} + \dots$ , whose coefficients  $\tilde{b}_i^{\lambda_0}(z) = \sum_{-t-m \leq j \leq mq} \tilde{b}_{i,j}^{\lambda_0} z^j$  form a collection of Laurent polynomials of height

$$H(\tilde{b}^{\lambda_0}) \leq (2ms)^{2ms+1} H(\tilde{L}) H(\lambda_0)^m \leq (2ms)^{8\kappa ms} H(\tilde{L})^{3\kappa m},$$

defined over a number field of degree  $\leq \kappa m$ , and whose Newton polygon does present a slope  $\leq s-1$ . In the end, we obtain that  $\omega$  is a element of  $\mathbf{K}_\omega[\frac{1}{z}]$  of degree  $s$ , where  $1 \leq s \leq q$ ,  $\mathbf{K}_\omega$  is a number field of degree  $\leq \kappa m^s$ , and

$$H(\omega) \leq (2ms)^{(8s\kappa)^s m^{s^2}} H(\tilde{L})^{(3\kappa)^s m^{s^2}}.$$



To compute the exponents corresponding to this determining factor, we must go back to  $L$  (truncating at  $\tilde{L}$  would not be enough). Writing  $z^m L = D(\theta)$ , we know by now that the Newton polygon of  $D(\theta + \omega)$  has a side of slope 0, hence an indicial polynomial  $P_{L,0,\omega} := P_{D^\omega}$  of positive degree (equal to the length of that side), and the exponents corresponding to  $\omega$  are precisely the roots of  $P_{D^\omega}$ , cf. [Be 2]. The coefficients of  $D^\omega$  are rational functions  $b_i^\omega(z) \in \mathbf{K}_\omega(z)$ , admitting  $z^{q'} T(z)$ , for some  $q' \leq mq$ , as common denominator. Expressing them as linear forms in those of  $D(\theta)$  (which have height  $\leq (2m)^{m+1} H(L)$ ) and recalling that  $H(\tilde{L}) \leq 8^{mq^2} H(L)^{3\kappa mq}$ , we deduce as above that their height is bounded by

$$H(b^\omega) \leq (2mq)^{(8q\kappa)^9 m^{q^2}} (8^{mq^2} H(L)^{3\kappa mq})^{(3\kappa)^9 m^{q^2}} \leq 2^{(32q\kappa m)^{2q^2}} H(L)^{(3\kappa m)^{3q^2}}.$$

An analysis similar to the Fuchsian case gives:

$$H(P_{L,0,\omega}) \leq (2m)^{m+1} (H(b^\omega))^{2\kappa m^q},$$

so that all the exponents of  $L$  at 0 have absolute value at most

$$mH(P_{L,0,\omega})^{2\kappa m^q} \leq 2^{(34q\kappa m)^{2q^2}} H(L)^{(5\kappa m)^{3q^2}}.$$

### *The irregular ramified case*

In general, the slopes at an irregular singularity (say again at  $\alpha = 0$ ) are not integral, but the general theory (see, for instance, [BV]) implies that they all become integers after extension of scalars from  $\mathbf{C}((z))$  to  $\mathbf{C}((z^{1/m!}))$ . More precisely, setting  $t^{m!} = z$ , and  $\mathbf{K}_m = \mathbf{K}(e^{2\pi i/m!})$ , the differential operator  $L$  turns into a differential operator  $L_m \in \mathbf{K}_m(t)[d/dt]$  whose slopes at  $t = 0$  are integers. In the process, the exponents of  $L$  are multiplied by  $m!$  (but are unchanged when expressed in the parameter  $z$ ),  $\kappa$  becomes  $\kappa_m \leq \kappa m^m$ , the degree  $q$  of  $L$  is turned into  $\deg(L_m) \leq m^m q$ , while  $H(L_m) \leq m^{2m} H(L)$ . Applying the previous result to  $L_m$ , we derive that in the general case, the exponents of  $L$  at 0 have absolute value at most

$$\begin{aligned} & 2^{(34qm^m \kappa m^{m+1})^{2q^2 m^{2m}}} (m^{2m} H(L))^{(5\kappa m^{m+1})^{3q^2 m^{2m}}} \\ & \leq 2^{(35qm\kappa)^{6q^2 m^{3m}}} H(L)^{(5\kappa m)^{6q^2 m^{3m}}}. \end{aligned}$$

The same bounds holds when the singularity is at  $\infty$ . Replacing  $H(L)$  by  $(2q)^q H(L)^{3\kappa q}$  and  $\kappa$  by  $\kappa q$  to deal with singularities  $\alpha$  at finite distance, we finally obtain the upper bound

$$\mathcal{E}(L) \leq 2^{(36qm\kappa)^{9q^2 m^{3m}}} H(L)^{(5\kappa qm)^{9q^2 m^{3m}}},$$

and the proof of Lemma 3.1, hence of Theorem 1.2 (under the current assumption  $\mathbf{D} = \mathbf{D}_L$ ), is completed, with  $c(\kappa, m, q) = \text{Log}_2 C(\kappa, m, q) = (2\kappa q m)^{(2qm)^{4m}}$  (when  $q \geq 1$ ; the case  $q = 0$  is easily settled by a direct study).

### 3.2 From differential equations to general systems

Finally, we show how to reduce the estimates for a general differential system of rank  $m$  as in §1 :

$$Y' = A(z)Y \tag{D}$$

to the case, treated just above, of a differential equation  $Ly = 0$ . This is achieved by an effective version of the existence theorem for cyclic vectors, according to which there exists a matrix  $P \in GL_m(\mathbf{K}(z))$  such that  $A_{[P]} = P^{-1}AP - P^{-1}P'$  is the companion matrix of a differential equation  $Ly = 0$ , for some differential operator  $L \in \mathbf{K}(z)[d/dz]$  of order  $m$ . The gauge transformation  $V = P^{-1}Y$  then turns the solutions  $Y$  of  $Y' = AY$  into the solutions  $V$  of  $V' = A_{[P]}V$ , which, on setting  $V = {}^t(y, y', \dots, y^{(m-1)})$ , are in bijection with the solutions  $y$  of  $Ly = 0$ .

As in §1, Remark 1.5, we denote by  $H(\mathbf{D}) = H(A)$  (resp.  $\text{deg}(\mathbf{D}) = \text{deg}(A)$ ) the height (resp. degree) of the collection of rational functions  $A_{i,j} \in \mathbf{K}(z)$  formed by the entries of the matrix  $A$ , by  $T \in \mathbf{K}[z]$  their monic common denominator, and we set  $\text{deg}(\mathbf{D}) = q$ . Our task thus consists in constructing a differential operator  $L$  (i.e. a matrix  $P$ ) as above, and then

1) bounding from above the data  $H(L), \text{deg}(L)$  corresponding to  $L$  in terms of  $\kappa, m, q, H(\mathbf{D}), \text{deg}(\mathbf{D})$  on the one hand;

2) bounding from above the maximum  $\mathcal{R}(\mathbf{D})$  of the absolute values of the generalized orders  $r_\alpha \leq 0$  of  $\mathbf{D}$  (cf. beginning of §3) in terms of  $\mathcal{E}(L)$  on the other hand.

The first task is achieved by the following

LEMMA 3.2. — *Let  $\mathbf{K}$  be a number field of degree  $\kappa$  over  $\mathbf{Q}$ , and let  $\mathbf{D}$  be a differential system of rank  $m$  over  $\mathbf{K}(z)$ , with degree  $\text{deg}(\mathbf{D}) = q$  and height  $H(\mathbf{D})$ . There exists a differential operator  $L \in \mathbf{K}(z)[d/dz]$  of order  $m$ , degree  $\text{deg}(L) \leq 5(q+1)m^2$  and height*

$$H(L) \leq ((2m+q)H(\mathbf{D}))^{164\kappa(q+1)m^4}$$

*such that  $\mathbf{D}$  and the differential equation  $Ly = 0$  are equivalent over  $\mathbf{K}(z)$ .*

*Proof.* — Consider the differential operator  $U \mapsto \check{D}U := U + {}^tAU$  on  $(\mathbf{K}(z))^m$ , and denote by  $E_i = {}^t(0, \dots, 0, 1, 0, \dots, 0)$  the standard basis of  $(\mathbf{K}(z))^m$ . To a given element  $\gamma$  of  $\mathbf{K}$ , we associate the vector

$$U_\gamma = \sum_{j=0, \dots, m-1} (z - \gamma)^j (\sum_{\ell=0, \dots, j} (-1)^\ell \frac{(m-1)!}{\ell!(j-\ell)!} \check{D}^\ell (E_{j-\ell}))$$

of  $(\mathbf{K}(z))^m$ , and we let  $Q = Q_\gamma$  be the  $m \times m$  matrix whose columns are the  $m \times 1$  vectors

$$U_\gamma, \check{D}U_\gamma, \dots, \check{D}^{m-1}U_\gamma.$$

According to [K],  $\det(Q_\gamma)$  is a non-zero element of  $\mathbf{K}(z)$  apart from at most  $m(m-1)$  values of  $\gamma$ , or as soon as  $\gamma$  is not a pole of the matrix  ${}^tA$ . Under either assumption,  $\check{D}^m U_\gamma$  can be expressed as a linear combination

$$\check{D}^m U_\gamma = -\sum_{i=0, \dots, m-1} a_{m-i} \check{D}^i U_\gamma$$

with coefficients  $a_i \in \mathbf{K}(z)$  depending only on  $\gamma$  and  $A$ . We can then consider the matrix  $P = {}^tQ^{-1}$ , and a standard computation shows that

$$A_{[P]} := P^{-1}AP - P^{-1}P' = {}^t[Q^{-1}({}^tA)Q + Q^{-1}Q']$$

is the companion matrix to the differential equation  $Ly = 0$ , where

$$L = L_{\gamma, A} = (d/dz)^m + a_1(z)(d/dz)^{m-1} + \dots + a_m(z).$$

We now fix some natural number  $\gamma \in \mathbf{Z}, 0 \leq \gamma \leq m^2$  satisfying the above requirement  $\det(Q_\gamma) \neq 0$ . Denoting by  $\tau_1(z) \in \mathbf{Z}_{\mathbf{K}}[z]$  a suitable common denominator (relative to the ring  $\mathbf{Z}_{\mathbf{K}}[z]$ ) of the corresponding rational functions  $a_i$ , we shall compute upper bounds  $q_1, H_1$  for the degree and height of the collection of polynomials  $\tau_1, \tau_1 a_i \in \mathbf{Z}_{\mathbf{K}}[z]$  in terms of  $m, q, H(A)$ . Under these conditions,  $\deg(L) \leq q_1$ , while  $H(L) \leq (2q_1)^{q_1} H_1^{2\kappa q_1}$  by Gelfond's lemma, so that our first task will be completed. Let  $\delta$  be the least common denominator of the coefficients of the polynomials  $T, TA_{i,j}$ , and let  $T_1 = \delta T$ . Note that  $T_1, T_1 A_{i,j}$  form a collection of polynomials in  $\mathbf{Z}_{\mathbf{K}}[z]$  of degree  $q$  and height  $H_1(A) \leq H(A)^2$ , and that  $T_1 \check{D}$  maps the  $\mathbf{Z}_{\mathbf{K}}$ -module  $(\mathbf{Z}_{\mathbf{K}}[z])^m$  of integral polynomial vectors into itself.

By Cramer's rule,  $a_{m-i} = -\frac{\det(U_\gamma, \dots, \check{D}^m U_\gamma, \dots, \check{D}^{m-1} U_\gamma)}{\det(U_\gamma, \dots, \check{D}^i U_\gamma, \dots, \check{D}^{m-1} U_\gamma)}$  for all  $i = 0, \dots, m-1$ , which by Leibniz formula may be rewritten as

$$a_{m-i} = -T_1^{m^2+m(m-1)/2} \frac{\det(U_\gamma, \dots, \check{D}^m U_\gamma, \dots, \check{D}^{m-1} U_\gamma)}{\det(T_1^m U_\gamma, \dots, (T_1 \check{D})^i (T_1^m U_\gamma), \dots, (T_1 \check{D})^{m-1} (T_1^m U_\gamma))}.$$

Since  $T_1\check{D}$  maps polynomial vectors of degree  $d$  into polynomial vectors of degree  $\leq q + d$ , and since  $T_1^m U_\gamma \in (\mathbf{Z}_K[z])^m$  has degree  $\leq m + qm$ , the bottom term  $\Delta := \det(T_1^m U_\gamma, \dots, (T_1\check{D})^{m-1}(T_1^m U_\gamma))$  appearing in the last quotient is an integral polynomial of degree  $\leq 2(q+1)m^2$ . A rougher analysis of the upper term then implies that  $\tau_1 := T_1^{2m^2} \Delta$  is a common denominator for the  $a_i$ 's with respect to the ring  $\mathbf{Z}_K[z]$ . The degree of  $\tau_1$  is  $\leq 4(q+1)m^2$ , while its height is bounded from above by

$$H(\tau_1) \leq (2m+q)^{14m^2} H_1(A)^{4m^2}.$$

Indeed,  $T_1\check{D}$  maps polynomial vectors of height  $h$  and degree  $d$  in  $(\mathbf{Z}_K[z])^m$  into polynomial vectors of height  $\leq m(q+d+1)hH_1(A) + (q+d)dhH_1(A) \leq (m+d)(q+d+1)hH_1(A)$ , while the height of  $T_1^m U_\gamma$  is bounded from above by  $(m+q)^{8m} H_1(A)^m$  (recall that the height of  $\gamma$  is  $\leq m^2$ ).

A similar computation on the numerators  $\tau_1 a_i$  shows that their degrees are bounded by  $5(q+1)m^2 = q_1$ , while their heights are  $\leq (2m+q)^{16m^2} H_1(A)^{6m^2} = H_1$ . Our differential operator  $L = L_{\gamma,A}$  therefore satisfies

$$\tilde{q} := \deg(L) \leq 5(q+1)m^2; \quad H(L) \leq (2m+q)^{164(q+1)\kappa m^4} H(A)^{120\kappa(q+1)m^4}.$$

This concludes the proof of Lemma 3.2, and in conjunction with our previous step, implies that

$$\mathcal{E}(L) \leq 2^{(36\tilde{q}m\kappa)^{9\tilde{q}^2 m^{3m}}} H(L)^{(5\tilde{q}\tilde{q}m)^{9\tilde{q}^2 m^{3m}}} \leq (2H(A))^{(2\kappa(q+1)m)^{(2(q+1)m)^{7m}}}$$

Our second task is easier to achieve: since the  $\mathbf{C}$ -linear map  $y \mapsto Y = P^t(y, y', \dots, y^{(m-1)})$  yields an isomorphism between the spaces of solutions of  $Ly = 0$  and of  $Y' = AY$ , the generalized orders  $r_\alpha \leq 0$  of the system  $(\mathbf{D})$  at any of its singularities  $\alpha$  are bounded from below by  $v_\alpha(P) - \mathcal{E}_\alpha(L) - (m-1) - qm$ , where  $\mathcal{E}_\alpha(L)$  (resp.  $v_\alpha(P)$ ) denotes an upper bound for the absolute values of the exponents of  $L$  at  $\alpha$  (resp. for the order of  $P$  at  $\alpha$ ). Since  $P = {}^t Q_\gamma^{-1}$ , we infer from the analysis of the determinant  $\Delta$  above and of its minors that  $v_\alpha(P) \leq 4(q+1)m^2$  at all  $\alpha$ 's (including  $\infty$ ), so that

$$\mathcal{R}(\mathbf{D}) \leq \mathcal{E}(L) + 5(q+1)m^2,$$

and Shidlovsky's constant for a general system  $(\mathbf{D})$  may at long last be bounded from above by  $n_0(\mathbf{D}) \leq C(\kappa, m, q) H(A)^{c(\kappa, m, q)}$ , with

$$c(\kappa, m, q) = \text{Log}_2 C(\kappa, m, q) = (2\kappa(q+1)m)^{(2(q+1)m)^{8m}}.$$

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