ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

BOELE BRAAKSMA

ROBERT KUIK

Resurgence relations for classes of differential and difference equations

Annales de la faculté des sciences de Toulouse 6^e série, tome 13, n° 4 (2004), p. 479-492

<http://www.numdam.org/item?id=AFST_2004_6_13_4_479_0>

Annales de la Faculté des Sciences de Toulouse Vol. XIII, n° 4, 2004
pp. 479–492

Resurgence relations for classes of differential and difference equations (*)

BOELE BRAAKSMA (1), ROBERT KUIK (1)

ABSTRACT. — We consider meromorphic differential and difference equations of level 1. Small solutions of these equations can be represented as exponential series with as coefficients Borel sums of certain formal series. We consider the Stokes transition in these representations when the Borel sums of the coefficients are changed from one sector to another sector. Further we derive resurgence relations for the coefficients in the exponential series by using one-sided Borel transforms. Similar results have been obtained by means of staircase distributions by Costin [Cos98] and Kuik [Kuik03]. Finally a sketch is given of the bridge equation of Ecalle and its use for the equations considered here.

RÉSUMÉ. — On considère, dans le contexte méromorphe, des équations différentielles et des équations aux différences de niveau 1. On peut représenter les petites solutions de ces équations par des séries exponentielles où les coefficients sont des sommes de Borel de certaines séries formelles. On considère la transition de Stokes dans ces représentations quand les sommes de Borel des coefficients changent d'un secteur à un autre. On dérive des relations de résurgence pour les coefficients dans ces séries en utilisant des transformations de Borel unilatérales. Costin[Cos98] et Kuik[Kuik03] ont obtenu des résultats similaires par le biais des distributions en escalier. Finalement, on esquisse l'équation du pont de J. Ecalle et son utilisation dans les équations considérées ici.

1. Introduction

We study the differential equation

$$y'(x) + L(x)y(x) = f(x^{-1}, y(x)) \tag{1.1}$$

(*) Reçu le 29 août 2003, accepté le 14 janvier 2004

(1) B.L.J. Braaksma and G.R. Kuik, Department of Mathematics IWI, P.O.Box 800, 9700 AV Groningen, Pays-Bas.
E-mail: braaksma@math.rug.nl, g.r.kuik@gasunie.nl

and the difference equation

$$y(x+1) = \Lambda(x)y(x) + f(x^{-1}, y(x)) \tag{1.2}$$

where $y \in \mathbb{C}^n, x \in \mathbb{C}^*$, L and Λ are $n \times n$ matrices,

$$L(x) = \operatorname{diag}\{\mu_j - a_j x^{-1}\}, \Lambda(x) = \operatorname{diag}\{e^{-\mu_j}(1+x^{-1})^{a_j}\}, \tag{1.3}$$

with μ_j and $a_j \in \mathbb{C}, j = 1, \ldots, n$, and $f(x,y)$ is a bounded analytic $\mathbb{C}^n$-valued function for $|x| < r_1$ and $|y| < r_2$ such that $f(x,y) = O(x^2) + O(|y|^2)$ as $x \to 0$ and $y \to 0$. Here and in the following we use $|y| := \max_{j=1,\ldots,n} |y_j|$. We also assume the nonresonance condition $\mu_j \neq \mathbf{k} \bullet \boldsymbol{\mu}$ in case of (1.1) and $\mu_j \not\equiv \mathbf{k} \bullet \boldsymbol{\mu} \bmod 2\pi i$ in case of (1.2) for all $\mathbf{k} \in \mathbb{N}^n \setminus \{\mathbf{e}_j\}$. Here $\mathbf{e}_j$ denotes the jth unit vector and $\boldsymbol{\mu} := \sum_{j=1}^n \mu_j \mathbf{e}_j$.

The differential equation (1.1) which we will denote by (D) has been studied in [Cos98] and [Kuik03] and the difference equation (1.2) which we will denote by (Δ) has been studied in [Bra01], [Kuik03] and [BK04]. In particular, solutions which tend to 0 as $x \to \infty$ in some sector have been investigated. These solutions can be represented by a special type of transseries. First we collect some of their results.

Both equations (D) and (Δ) have a unique formal solution $\hat{y}_0(x) = \sum_{m=2}^\infty c_m x^{-m}$ in $\mathbb{C}^n[[x^{-1}]]$. This solution has the property that if y is a solution such that $y(x) \to 0$ as $x \to \infty$ in some sector S then $y(x) \sim \hat{y}_0(x)$ as $x \to \infty$ in S.

There is a formal transformation

$$y = \hat{T}(x,z) := \sum_{\mathbf{k} \in \mathbb{N}^n} z^{\mathbf{k}} \hat{y}_{\mathbf{k}}(x) \tag{1.4}$$

which transforms (D) and (Δ) into

$$z'(x) + L(x)z(x) = 0 \tag{1.5}$$

and

$$z(x+1) = \Lambda(x)z(x) \tag{1.6}$$

respectively where $\hat{y}_{\mathbf{k}}(x) \in \mathbb{C}^n[[x^{-1}]]$, $\hat{y}_{\mathbf{0}} := \hat{y}_0$ and if $\mathbf{k} \in \mathbb{N}^n \setminus \{\mathbf{0}\}$ then $\hat{y}_{\mathbf{k}}$ is a formal solution of

$$y'_{\mathbf{k}}(x) + (\hat{L}_1(x) - \mathbf{k} \bullet \boldsymbol{\mu} + \mathbf{k} \bullet \mathbf{a} x^{-1}) y_{\mathbf{k}}(x) = \hat{t}_{\mathbf{k}}(x), \tag{1.7}$$

in case (D) and of

$$e^{-\mathbf{k} \bullet \boldsymbol{\mu}} (1 + x^{-1})^{\mathbf{k} \bullet \mathbf{a}} y_{\mathbf{k}}(x+1) = \hat{\Lambda}_1(x) y_{\mathbf{k}}(x) + \hat{t}_{\mathbf{k}}(x), \tag{1.8}$$

in case (Δ), where $\mathbf{k} \in \mathbb{N}^n, |\mathbf{k}| \geqslant 1, \mathbf{a} := \sum_{j=1}^n a_j \mathbf{e}_j, \hat{L}_1(x) := L(x) - \partial_y f(x^{-1}, \hat{y}_0(x))$, $\hat{\Lambda}_1(x) := \Lambda(x) + \partial_y f(x^{-1}, \hat{y}_0(x))$ and $\hat{t}_{\mathbf{k}}(x)$ is the coefficient of $z^{\mathbf{k}}$ in the Taylor expansion with respect to $z \in \mathbb{C}^n$ of $f(x^{-1}, \sum_{\mathbf{k}' \prec \mathbf{k}} z^{\mathbf{k}'} \hat{y}_{\mathbf{k}'})$ if $|\mathbf{k}| > 1$ and $\hat{t}_{\mathbf{k}}(x) = 0$ if $|\mathbf{k}| = 1$. Here $\prec$ corresponds to the usual lexicographical ordering. Choosing $\hat{y}_{\mathbf{e}_j}(\infty) = \mathbf{e}_j$ the formal solutions $\hat{y}_{\mathbf{k}}(x) \in \mathbb{C}^n[[x^{-1}]]$ of (1.7) and (1.8) can be constructed recursively and they are unique.

By substitution of the general solution of (1.5) and (1.6) in $\hat{T}(x, z)$ we obtain a so-called formal integral $\hat{y}(x, C)$ of (D) and (Δ) respectively

$$\hat{y}(x, C) = \sum_{\mathbf{k} \in \mathbb{N}^n} C^{\mathbf{k}} e^{-\mathbf{k} \bullet \boldsymbol{\mu} x} x^{\mathbf{k} \bullet \mathbf{a}} \hat{y}_{\mathbf{k}}(x) \tag{1.9}$$

where $C \in \mathbb{C}^n$ is an arbitrary constant in case of (D) and C is a one-periodic $\mathbb{C}^n$-valued function in case of (Δ).

The formal series $\hat{y}_{\mathbf{k}}$ with $\mathbf{k} \in \mathbb{N}^n$ are Borel summable in all directions except the singular directions $\arg x = \arg(\mu_j - \mathbf{k}' \bullet \boldsymbol{\mu} + 2l\pi i)$ where $j \in \{1, \ldots, n\}, \mathbf{0} \preceq \mathbf{k}' \preceq \mathbf{k}, \mathbf{k}' \neq \mathbf{e}_j$ and $l \in \mathbb{Z}$ in case $(\Delta), l = 0$ in case (D). The singular directions multiplied by -1 are Stokes directions, also called anti-Stokes directions in part of the literature. According to Ecalle [Eca85] $\hat{y}_{\mathbf{k}}$ is a resurgent function. This means that the Borel transform $Y_{\mathbf{k}}$ of $\hat{y}_{\mathbf{k}}$ is analytic in a neighborhood of 0 and can be analytically continued along paths which avoid a denumerable set of singularities (cf. [Eca85, Eca92, CNP93]). Here the singularities of Y_0 are in case of (D) the points of the set $\sum_{j=1}^n \mathbb{N}\mu_j \setminus \{\mathbf{0}\}$ and in case of (Δ) the same points with integral multiples of $2\pi i$ added.

One may give an analytic meaning to the formal integrals by restricting $\mathbf{k}$ suitably. We will do this for sectors containing the positive axis. In the case of (D) this is no loss of generality since by means of a rotation of the independent variable we may always achieve that the positive axis is included in the sector we consider. Suppose that *at least one of the numbers* μ_j *has positive real part.* Then we may rearrange the numbers μ_h such that $\Re\mu_j > 0$ if $j = 1, \ldots, p$ and $\Re\mu_j \leqslant 0$ if $j = p+1, \ldots, n$, where p is a positive integer, $p \leqslant n$. We now restrict $\mathbf{k}$ to $\mathbb{N}^p$ meaning that $\mathbf{k} \bullet \mathbf{e}_j = 0$ if $j > p$. Consider the singular directions associated with the first p eigenvalues μ_j: $\arg(\mu_j - \mathbf{k} \bullet \boldsymbol{\mu} + 2l\pi i)$ where $j \in \{1, \ldots, n\}, \mathbf{k} \in \mathbb{N}^p \setminus \{\mathbf{e}_j\}$ and $l = 0$ in case of (D) and $l \in \mathbb{Z}$ in case of (Δ). Let θ_- and θ_+ be two consecutive singular directions of this type with $-\pi/2 < \theta_+, \theta_- < \pi/2$. Let

$$S_1 := \{x \in \mathbb{C}^* | -\frac{\pi}{2} - \theta_+ < \arg x < \frac{\pi}{2} - \theta_-\}. \tag{1.10}$$

We now consider the Borel sum $y_{\mathbf{k},S_1}$ of $\hat{y}_{\mathbf{k}}$ on S_1 for $\mathbf{k} \in \mathbb{N}^p$. Let S_1' be a closed subsector of S_1. According to theorem 6 in [Bra01] there exist positive δ and ρ such that

$$T_p(S_1; x, z) := \sum_{\mathbf{k}\in\mathbb{N}^p} z^{\mathbf{k}} y_{\mathbf{k},S_1}(x) \tag{1.11}$$

converges for $x \in S_1', |x| \geqslant \rho$ and $z \in \mathbb{C}^p, |z| \leqslant \delta$. We define $\mathcal{M}$ as the manifold defined by $y = T_p(S_1; x, z)$ for x and z as above. Then we have

THEOREM 1.1. — *Let y be a solution of (D) or (Δ) in a sector $S := \{x \in \mathbb{C}^* | \varphi_- < \arg x < \varphi_+\}$ of opening at most π which contains the positive axis and such that $y(x) \sim 0$ as $x \to \infty$ in S. Then there exist θ_- and θ_+ as above such that the closure of S is contained in S_1 (cf.(1.10)). Let $y_{\mathbf{k},S_1}$ denote the Borel sum of $\hat{y}_{\mathbf{k}}$ on S_1 for $\mathbf{k} \in \mathbb{N}^p$ and let $\mathcal{M}$ be the manifold defined by $y = T_p(S_1; x, z)$ for $x \in S, |x| \geqslant \rho$ and $z \in \mathbb{C}^p, |z| \leqslant \delta$ as above.*

Then the given solution y belongs to $\mathcal{M}$ which therefore is a stable manifold. On this manifold (D) and (Δ) are transformed to (1.5) and (1.6) respectively with in both cases $z_j(x) = 0$ if $j = p+1, \ldots, n$.

Let

$$F(S_1; x, C) = \sum_{\mathbf{k}\in\mathbb{N}^p} C^{\mathbf{k}} e^{-\mathbf{k}\bullet\boldsymbol{\mu}x} x^{\mathbf{k}\bullet\mathbf{a}} y_{\mathbf{k},S_1}(x). \tag{1.12}$$

In case (D) there exists a unique $C \in \mathbb{C}^p$ such that $y(x) = F(S_1; x, C)$ whereas in case (Δ) there exists a unique 1-periodic $\mathbb{C}^p$-valued trigonometric polynomial $C(x)$ such that $y(x) = F(S_1; x, C(x))$. In both cases these representations hold for all $x \in S$ with $|x| \geqslant \rho$ and $|C_j(x)e^{-\mu_j x}x^{a_j}| \leqslant \delta$ for $j \in \{1, \ldots, p\}$ where in case (D) we replace $C(x)$ by C. In particular, these representations hold in a neighborhood of ∞ of any closed subsector of S and $C_j(x)e^{-\mu_j x}, j = 1, \ldots, p$ and $y(x) - y_0(x)$ are exponentially small in this neighborhood.

Remark 1.2. — Note that in general C and $C(x)$ depend on the choice of S_1.

2. Stokes transition

Here we consider a small solution y on a sector S such that there are three consecutive singular directions θ_-, θ_0 and θ_+ associated with the first p eigenvalues such that $|\theta_0| < \pi/2, S \subset S_-$ and $S \subset S_+$ where

$$S_+ := \{x \in \mathbb{C}^* | -\theta_+ - \pi/2 < \arg x < -\theta_0 + \pi/2\}, \tag{2.1}$$

$$S_- := \{x \in \mathbb{C}^* | -\theta_0 - \pi/2 < \arg x < -\theta_- + \pi/2\}. \tag{2.2}$$

We now may apply the previous theorem with $S_1 = S_+$ and with $S_1 = S_-$. The associated coefficients $C(x)$ are now denoted by $C_-(x)$ and $C_+(x)$ respectively and where in case of (D) $C(x) = C$, a constant. They satisfy

THEOREM 2.1. — *Let $y, S, \theta_-, \theta_0, \theta_+, S_-$ and S_+ be as above. Then*

$$y(x) = F(S_-; x, C_-(x)) = F(S_+; x, C_+(x)) \tag{2.3}$$

for $x \in S$ with sufficiently large $|x|$. Suppose that $\arg \mu_j = \theta_0$ if and only if $j \in I$ where I is a subset of the first p positive integers.

Then there exist constants $\alpha_j \in \mathbb{C}, j \in I$, independent of y such that

$$C_+(x) - C_-(x) = \boldsymbol{\alpha} := \sum_{j \in I} \alpha_j \mathbf{e}_j \tag{2.4}$$

where $\mathbf{e}_j$ denotes the jth unit vector.

Proof. — We first choose $y(x) = y_{\mathbf{0},S_-}(x)$. Then $y(x) \sim \hat{y}_0$ for $-\theta_0 - \pi/2 < \arg x < -\theta_- + \pi/2$ and $y(x) = F(S_-; x, 0) = F(S_+; x, C_+(x))$, so $C_-(x) = 0$. Here $(C_+)_j(x)e^{-\mu_j x} \to 0$ for $-\theta_0 - \pi/2 < \arg x < -\theta_0 + \pi/2$. This sector is a halfplane. As moreover $\arg \mu_j = \theta_0$ only if $j \in I$ we see that $(C_+)_j(x) = 0$ if $j \notin I$ whereas $(C_+)_j(x)$ has to be a constant if $j \in I$ and we denote this constant by α_j. Hence

$$y_{\mathbf{0},S_-}(x) - y_{\mathbf{0},S_+}(x) = \sum_{\mathbf{k} \in \sum_{j \in I} \mathbb{N}\mathbf{e}_j, \mathbf{k} \neq \mathbf{0}} \boldsymbol{\alpha}^{\mathbf{k}} e^{-\boldsymbol{\mu} \bullet \mathbf{k} x} x^{\mathbf{k} \bullet \mathbf{a}} y_{\mathbf{k},S_+}(x). \tag{2.5}$$

Next consider the general case for y. From (2.3) and the definition of F we deduce

$$y_{\mathbf{0},S_-}(x) - y_{\mathbf{0},S_+}(x) = \tag{2.6}$$
$$\sum_{\mathbf{k} \in \sum_{j \in I} \mathbb{N}\mathbf{e}_j, \mathbf{k} \neq \mathbf{0}} e^{-\mathbf{k} \bullet \boldsymbol{\mu} x} x^{\mathbf{k} \bullet \mathbf{a}} \quad \{C_+(x)^{\mathbf{k}} y_{\mathbf{k},S_+}(x) - C_-(x)^{\mathbf{k}} y_{\mathbf{k},S_-}(x)\}.$$

Comparing terms with $|\mathbf{k}| = 1$ in the righthand sides of this formula and (2.5) we deduce (2.4). □

COROLLARY 2.2. — *If $\mathbf{k} \in \mathbb{N}^p$ then*

$$y_{\mathbf{k}}^-(x) = \sum_{\mathbf{m} \in \sum_{j \in I} \mathbb{N}\mathbf{e}_j, \mathbf{m} \neq \mathbf{0}} \binom{\mathbf{k} + \mathbf{m}}{\mathbf{k}} \boldsymbol{\alpha}^{\mathbf{m}} e^{-\boldsymbol{\mu} \bullet \mathbf{m} x} x^{\mathbf{m} \bullet \mathbf{a}} y_{\mathbf{k}+\mathbf{m}}^+(x), \tag{2.7}$$

where $y_{\mathbf{k}}^{\pm} := y_{\mathbf{k},S_\pm}$.

Proof. — In the previous theorem we choose $C \in \mathbb{C}^p$ independent of x with $|C|$ sufficiently small. Then

$$\sum_{\mathbf{k}\in\mathbb{N}^p} C^{\mathbf{k}} x^{\mathbf{k}\bullet\mathbf{a}} e^{-\boldsymbol{\mu}\bullet\mathbf{k}x} y_{\mathbf{k}}^-(x) = \sum_{\mathbf{k}\in\mathbb{N}^p} (C+\boldsymbol{\alpha})^{\mathbf{k}} x^{\mathbf{k}\bullet\mathbf{a}} e^{-\boldsymbol{\mu}\bullet\mathbf{k}x} y_{\mathbf{k}}^+(x).$$

We expand the righthand side in powers of C. By comparison of the coefficient of $C^{\mathbf{k}}$ on both sides we obtain (2.7). □

These results have been obtained by means of staircase distributions by Costin [Cos98] and Kuik [Kuik03].

3. Some resurgence relations

In this section we will derive some relations between the Borel transforms of $\hat{y}_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^p$. Let

$$\tilde{Y}_{\mathbf{k}} = \hat{\mathcal{B}}(x^{\mathbf{k}\bullet\mathbf{a}}\hat{y}_{\mathbf{k}}).$$

So, if $\hat{y}_{\mathbf{k}}(x) = \sum_{m=0}^{\infty} y_{\mathbf{k},m} x^{-m}$ where $y_{\mathbf{k},m} \in \mathbb{C}^n$, then

$$\tilde{Y}_{\mathbf{k}}(t) = \sum_{m=0}^{\infty} y_{\mathbf{k},m} \frac{t^{m-\mathbf{k}\bullet\mathbf{a}-1}}{\Gamma(m-\mathbf{k}\bullet\mathbf{a})},$$

and this series converges for $|t|$ positive and sufficiently small. We denote the Borel transform of $\hat{y}_0$ also by Y_0. Let $\theta_-, \theta_0, \theta_+$ be consecutive singular directions associated with the first p eigenvalues as in the previous section and suppose $\arg \mu_j = \theta_0$ for $j \in I$ as in the previous section. We also use the sectors S_- and S_+ given by (2.1)and (2.2).

Now

$$x^{\mathbf{k}\bullet\mathbf{a}} y_{\mathbf{k},S_\pm} = \mathcal{L}_{\theta^\pm} \tilde{Y}_{\mathbf{k}},$$

where $\mathcal{L}_{\theta^+}$ denotes the Laplace transform in the direction $\theta^+ \in (\theta_0, \theta_+)$ and the righthand side also denotes the analytic continuation on S_+ of $\mathcal{L}_{\theta^+}\tilde{Y}_{\mathbf{k}}$ and similarly for $\mathcal{L}_{\theta^-}$ with $\theta^- \in (\theta_-, \theta_0)$ and S_+ replaced by S_-.

If $\Re(\mathbf{k}\bullet\mathbf{a}) \geqslant 0$ and $\mathbf{k}\bullet\mathbf{a}$ is not an integer, then this Laplace transform has to be interpreted as the sectorial Laplace transform in the sense of Ecalle (cf.[Eca85],[Mal85]) which is defined as follows: Let T denote the monodromy operator of analytic continuation along a positively oriented loop around the origin and let the variation operator be defined by var $= T-$identity. Let $\tilde{Y}_{\mathbf{k}}^\vee$ be such that var $\tilde{Y}_{\mathbf{k}}^\vee = \tilde{Y}_{\mathbf{k}}$ and $\eta = \varepsilon e^{i\theta}$ with $\varepsilon > 0$ sufficiently small. Then

$$\mathcal{L}_\theta \tilde{Y}_{\mathbf{k}}(x) := \int_\eta^{(0^+)} e^{-xs} \tilde{Y}_{\mathbf{k}}^\vee(s) ds + \int_\eta^{\infty:\theta} e^{-xs} \tilde{Y}_{\mathbf{k}}(s) ds,$$

where the first integral is over a positively oriented loop around the origin starting at η and the upper limit $\infty : \theta$ of the last integral means that the path of integration ends in ∞ in the direction θ. If $\mathbf{k} \bullet \mathbf{a}$ is not an integer then we may choose

$$\tilde{Y}_{\mathbf{k}}^{\vee}(s) = (e^{-2\pi i \mathbf{k}\bullet\mathbf{a}} - 1)^{-1}\tilde{Y}_{\mathbf{k}}(s).$$

Now (2.5) may be written as

$$\mathcal{L}_{\theta^-}Y_0 - \mathcal{L}_{\theta^+}Y_0 = \sum_{\mathbf{k}\in\sum_{j\in I}\mathbb{N}\mathbf{e}_j, \mathbf{k}\neq\mathbf{0}} \boldsymbol{\alpha}^{\mathbf{k}} e^{-\boldsymbol{\mu}\bullet\mathbf{k}x}\mathcal{L}_{\theta^+}\tilde{Y}_{\mathbf{k}}. \tag{3.1}$$

We want to apply some form of the inverse Laplace transform to this equation. As the usual Borel transform cannot be applied to the lefthand side since this is only defined in an open sector of opening π we use a one-sided Borel transform instead (cf. [Eca85]). This transform is defined as follows:

$$(\mathcal{B}_{\psi,b}f)(t) = \frac{1}{2\pi i}\int_b^{\infty:\psi} e^{tx}f(x)dx. \tag{3.2}$$

Here it is assumed that f is continuous on the path of integration and that it is of exponential type. So the integral converges for $|\arg(-t) + \psi| < \pi/2$ and $|t|$ sufficiently large.

Let $F_\pm := \mathcal{B}_{\psi,b}y_0^\pm$ with ψ the direction of a halfline in $S_\pm$ and $b = b_\pm \in S_\pm$ with $|b|$ sufficiently large. So

$$F_\pm(t) = \frac{1}{2\pi i}\int_b^{\infty:\psi} dx e^{tx}\int_0^{\infty:\theta^\pm} e^{-xs}Y_0(s)ds, \tag{3.3}$$

where $\theta^\pm$ is as above. Here the last integral converges for $\arg x = \psi$, $|\psi + \theta^\pm| < \frac{\pi}{2}$ and sufficiently large $|x|$, and we choose $b = b_\pm$ in this region. The repeated integral is absolutely convergent if $|\arg(s-t) + \psi| < \frac{\pi}{2}$ for $\arg s = \theta^\pm$. If s varies over the halfline $\arg s = \theta^\pm$ then $\arg(s-t)$ varies between $\arg(-t)$ and $\theta^\pm$. So there is absolute convergence if

$$|\psi + \theta^\pm| < \frac{\pi}{2}, |\arg(-t) + \psi| < \frac{\pi}{2}$$

and changing the order of integration we obtain the Cauchy integral

$$F_\pm(t) = \frac{1}{2\pi i}\int_0^{\infty:\theta^\pm} e^{-b(s-t)}\frac{Y_0(s)}{s-t}ds. \tag{3.4}$$

The righthand side is independent of ψ and is analytic for $\arg t \in (\theta^\pm, \theta^\pm + 2\pi)$ and $\Re(be^{i\theta^\pm})$ sufficiently large. By varying the path of integration we see that $F_+(t)$ can be analytically continued for $\arg t \in (\theta_0, 2\pi + \theta_+)$. In the same way $F_-(t)$ is analytic for $\arg t \in (\theta_-, \theta_0 + 2\pi)$.

However, we may obtain the analytic continuation of $F_\pm$ by deforming in (3.4) the path of integration in such a way that the singularities of $Y_0(t)$ on the half line $\arg t = \theta_0$ are avoided. Let $\omega_1, \omega_2, \ldots$ denote these singularities with $|\omega_1| < |\omega_2| < \ldots$. Let $Y_0^{(-,r)}$ denote the analytic continuation of Y_0 from the sector $\theta_- < \arg t < \theta_0$ through the interval (ω_r, ω_{r+1}) to the sector $\theta_0 < \arg t < \theta_+$ and let $\gamma_{-,r}$ be a path from 0 to $\infty : \theta^-$ that only crosses the halfline $\arg s = \theta_0$ twice between ω_r and ω_{r+1}. Now we replace the path of integration in (3.4) with the lower sign by $\gamma_{-,r}$ and in this way we obtain the analytic continuation $F_{-,r}$ of F_- to the sector $\theta_0 + 2\pi < \theta_+ + 2\pi$. An analogous result holds with + and − signs interchanged.

From the definition of $\mathcal{B}_{\psi,b}(y_0^- - y_0^+)(t)$ we deduce that this expression exists if $|\psi + \theta_0| < \pi/2$, $|\arg(-t) + \psi| < \pi/2$ and $\Re(be^{i\theta_0})$ sufficiently large and that

$$\mathcal{B}_{\psi,b}(y_0^- - y_0^+)(t) = F_-(t) - F_+(t). \tag{3.5}$$

Using (2.5) we see that

$$F_- - F_+ = \mathcal{B}_{\psi,b}\Big(\sum_{\mathbf{k}\in\sum_{j\in I}\mathbb{N}\mathbf{e}_j, \mathbf{k}\neq\mathbf{0}} \boldsymbol{\alpha}^{\mathbf{k}} e^{-\boldsymbol{\mu}\bullet\mathbf{k}x} x^{\mathbf{k}\bullet\mathbf{a}} y_{\mathbf{k}}^+\Big). \tag{3.6}$$

Here the series in the righthand side is uniformly convergent for $|x|$ sufficiently large and $-\frac{\pi}{2} - \theta_+ + \varepsilon \leqslant \arg x \leqslant \frac{\pi}{2} - \theta_0 - \varepsilon$ and sufficiently small positive ε due to the convergence properties of $T_p(S_1; x, z)$ mentioned after (1.11). Hence we may apply $\mathcal{B}_{\psi,b}$ termwise to this series. Similarly as above we obtain

$$\mathcal{B}_{\psi,b}(x^{\mathbf{k}\bullet\mathbf{a}} y_{\mathbf{k}}^+)(t) =$$

$$\frac{1}{2\pi i}\Big[\int_\eta^{\infty:\theta^+} \tilde{Y}_{\mathbf{k}}(s)\frac{e^{-(s-t)b}}{s-t}ds + \int_\eta^{(0^+)} \tilde{Y}_{\mathbf{k}}^{\vee}(s)\frac{e^{-(s-t)b}}{s-t}ds\Big] =: F_{\mathbf{k}}(t)$$

if $|\psi + \theta^+| < \pi/2$ and $|\arg(-t) + \psi| < \pi/2$ and where we choose $0 < \eta e^{-i\theta^+} < |t|$. As before in the case of F_+ also $F_{\mathbf{k}}$ may be analytically continued for $\arg t \in (\theta_0, 2\pi + \theta_+)$. From (3.6) we deduce

$$F_-(t) - F_+(t) = \sum_{\mathbf{k}\in\sum_{j\in I}\mathbb{N}\mathbf{e}_j, \mathbf{k}\neq\mathbf{0}} \boldsymbol{\alpha}^{\mathbf{k}} F_{\mathbf{k}}(t - \mathbf{k}\bullet\boldsymbol{\mu}). \tag{3.7}$$

Now we take variations. From the definition of $F_{\mathbf{k}}$ it follows that if $\arg t \in (\theta_0, \theta_+)$ then $\operatorname{var} F_{\mathbf{k}}(t) = -\tilde{Y}_{\mathbf{k}}(t)$. For these values of $\arg t$ we consider the analytic continuation of the terms in (3.7) along a positively oriented loop around $[0, \omega_r]$ and close to it, which starts at t and intersects the half line $\arg s = \theta_0$ only once between ω_r and ω_{r+1}. Then consider the difference of these two values. If $\arg(\mathbf{k} \bullet \boldsymbol{\mu} - \omega_r) = \theta_0$ then $t - \mathbf{k} \bullet \boldsymbol{\mu}$ does not turn around 0 and so $F_{\mathbf{k}}(t - \mathbf{k} \bullet \boldsymbol{\mu})$ returns to its original value, whereas if $\arg(\omega_r - \mathbf{k} \bullet \boldsymbol{\mu}) = \theta_0$ then $t - \mathbf{k} \bullet \boldsymbol{\mu}$ turns once around 0 and $F_{\mathbf{k}}(t - \mathbf{k} \bullet \boldsymbol{\mu})$ is decreased by $\tilde{Y}_{\mathbf{k}}(t - \mathbf{k} \bullet \boldsymbol{\mu})$. Similarly $F_+(t)$ and $F_-(t)$ are decreased by $Y_0(t)$ and $Y_0^{(-,r)}(t)$, where in the last case we used the analytic continuation $F_{-,r}$ of F_-.

Hence

$$Y_0^{(-,r)}(t) - Y_0(t) = \sum_{\mathbf{k} \bullet \boldsymbol{\mu} \in (0, \omega_r]} \boldsymbol{\alpha}^{\mathbf{k}} \tilde{Y}_{\mathbf{k}}(t - \mathbf{k} \bullet \boldsymbol{\mu}) \text{ if } \theta_0 < \arg t < \theta_+. \tag{3.8}$$

By comparison of (3.8) and the same formula with r replaced by $r-1$ with $r \geqslant 1$ we deduce

$$Y_0^{(-,r)}(t) - Y_0^{(-,r-1)}(t) = \sum_{\mathbf{k} \bullet \boldsymbol{\mu} = \omega_r} \boldsymbol{\alpha}^{\mathbf{k}} \tilde{Y}_{\mathbf{k}}(t - \mathbf{k} \bullet \boldsymbol{\mu}) \text{ if } \theta_0 < \arg t < \theta_+. \tag{3.9}$$

Moreover, we see that $\omega_r \in \sum_{j \in I} \mathbb{N} \mu_j$.

Similar results hold with $+$ and $-$ signs interchanged. Instead of (2.5) we now use

$$y_0^+ - y_0^- = \sum_{\mathbf{k} \in \sum_{j \in I} \mathbb{N} \mathbf{e}_j, \mathbf{k} \neq \mathbf{0}} (-\boldsymbol{\alpha})^{\mathbf{k}} e^{-\boldsymbol{\mu} \bullet \mathbf{k} x} x^{\bullet \mathbf{a}} y_{\mathbf{k}}^-, \tag{3.10}$$

which follows from theorem 2.1 with $C_+ = 0$. Then we obtain in the same way as above

$$Y_0^{(+,r)}(t) - Y_0(t) = \sum_{\mathbf{k} \bullet \boldsymbol{\mu} \in (0, \omega_r]} (-\boldsymbol{\alpha})^{\mathbf{k}} \tilde{Y}_{\mathbf{k}}(t - \mathbf{k} \bullet \boldsymbol{\mu}) \text{ if } \theta_- < \arg t < \theta_0. \tag{3.11}$$

and

$$Y_0^{(+,r)}(t) - Y_0^{(+,r-1)}(t) = \sum_{\mathbf{k} \bullet \boldsymbol{\mu} = \omega_r} (-\boldsymbol{\alpha})^{\mathbf{k}} \tilde{Y}_{\mathbf{k}}(t - \mathbf{k} \bullet \boldsymbol{\mu}) \text{ if } \theta_- < \arg t < \theta_0. \tag{3.12}$$

We may extend these resurgence relations for Y_0 to $Y_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^p$, using (2.7). Applying the one-sided Borel transform to both sides of this formula we may deduce in a similar way as above

$$\tilde{Y}_{\mathbf{k}}^{(-,r)}(t) - \tilde{Y}_{\mathbf{k}}(t) = \sum_{\mathbf{m} \bullet \boldsymbol{\mu} \in (0, \omega_r]} \binom{\mathbf{k} + \mathbf{m}}{\mathbf{m}} \boldsymbol{\alpha}^{\mathbf{m}} \tilde{Y}_{\mathbf{k}+\mathbf{m}}(t - \mathbf{m} \bullet \boldsymbol{\mu}) \tag{3.13}$$

if $\mathbf{k} \in \mathbb{N}^p, \theta_0 < \arg t < \theta_+$, and similarly for $\tilde{Y}_{\mathbf{k}}^{(+,r)}$. Moreover,

$$\tilde{Y}_{\mathbf{k}}^{(-,r)}(t) - \tilde{Y}_{\mathbf{k}}^{(-,r-1)}(t) = \sum_{\mathbf{m}\bullet\boldsymbol{\mu}=\omega_r} \binom{\mathbf{k}+\mathbf{m}}{\mathbf{m}} \boldsymbol{\alpha}^{\mathbf{m}} \tilde{Y}_{\mathbf{k}+\mathbf{m}}(t - \mathbf{m}\bullet\boldsymbol{\mu}), \quad (3.14)$$

if $\mathbf{k} \in \mathbb{N}^p, \theta_0 < \arg t < \theta_+$, and

$$\tilde{Y}_{\mathbf{k}}^{(+,r)}(t) - \tilde{Y}_{\mathbf{k}}^{(+,r-1)}(t) = \sum_{\mathbf{m}\bullet\boldsymbol{\mu}=\omega_r} \binom{\mathbf{k}+\mathbf{m}}{\mathbf{m}} (-\boldsymbol{\alpha})^{\mathbf{m}} \tilde{Y}_{\mathbf{k}+\mathbf{m}}(t - \mathbf{m}\bullet\boldsymbol{\mu}) \quad (3.15)$$

if $\mathbf{k} \in \mathbb{N}^p, \theta_- < \arg t < \theta_0$.

In the resurgence relations above we may let t tend to a point on the halfline $\arg t = \theta_0$ different from the singularities $\omega_h, h \in \mathbb{N}$ and thus these relations remain valid on this halfline. Such resurgence relations have been studied by Costin and Kuik in [Cos98] and [Kuik03] by means of staircase distributions in case that only one eigenvalue μ_j has argument θ_0. Kuik also considered the case that $L(x)$ and $\Lambda(x)$ (cf.(1.3)) are not diagonalisable.

COROLLARY 3.1. — *If (D) or (Δ) is real, so L, Λ, f are real analytic, and $\theta_0 = 0$ in the previous theorem, then $\alpha_j, j \in I$ is purely imaginary and $F_\pm(S_\pm; x, C \pm \frac{1}{2}\boldsymbol{\alpha})$ is a real-analytic solution of (D) or (Δ) if $C \in \mathbb{R}^p$ or C is a $\mathbb{R}^p$-valued trigonometric polynomial with period 1 on $\mathbb{R}$ respectively. These solutions correspond to the median Laplace transform considered by Ecalle (cf. [Eca92]) and the balanced averages considered by Costin (cf. [Cos98]).*

Proof. — In case (D) $\mu_j \in \mathbb{R}$ and in case (Δ) we may choose $\mu_j \in \mathbb{R}$. Now the formal series $\hat{y}_{\mathbf{k}}$ have real coefficients. Hence $\tilde{Y}_{\mathbf{k}}(t) = \overline{\tilde{Y}_{\mathbf{k}}(\bar{t})}$ in a neighborhood of the origin. Now we use (3.8) with $r = 1$ where we let t tend to a value on (ω_1, ω_2). Then we conclude that $\alpha_j, j \in I$ is purely imaginary. The last statement of the corollary is a consequence of $\overline{C(x) + \frac{1}{2}\boldsymbol{\alpha}} = C(x) - \frac{1}{2}\boldsymbol{\alpha}$ and $\overline{y}_{\mathbf{k}}^{+}(x) = y_{\mathbf{k}}^{-}(x)$ for $x \in \mathbb{R}$ where the last equality follows from the definition of Borel sum. □

4. The bridge equation of Ecalle

In this section we give a very sketchy indication of how the bridge equation of Ecalle may be applied to the equations (D) and (Δ). For more details we refer to [Eca85] and [CNP93].

Let θ be a direction and $\underline{\underline{\mathcal{R}}}_\theta$ be the set of resurgent functions ϕ in the direction θ. Here ϕ is resurgent in the direction θ if there is a function

Φ which is analytic in a neighborhood of 0 on $\mathbb{C}_\infty$ such that var $\Phi = \phi$ and ϕ can be analytically continued on paths along the halfline θ except for a discrete set of singularities $\Omega = \{\omega_1, \omega_2, \ldots\}$ on this halfline but one may analytically continue ϕ on both sides of the halfline around ω_j for all j. Moreover we assume that the function obtained by analytic continuation along the righthand side of the halfline is at most of exponential growth, and the same with righthand side replaced by lefthand side. Using an extension of the usual convolution $\underline{\underline{\mathcal{R}}}_\theta$ becomes a convolution algebra.

With this convolution algebra one may associate two models: the sectorial model obtained by application of the Laplace transform $\mathcal{L}_{\theta+}$ and $\mathcal{L}_{\theta-}$ with Laplace integrals along paths to the left or to the right of the halfline θ, and the formal model obtained by taking asymptotic expansions of these functions. From the formal model one returns to the convolution model by application of the formal Borel transform $\hat{\mathcal{B}}$. From the formal model one goes to the sectorial model by Borel summation $s_{\theta+}$ and $s_{\theta-}$ in the direction θ_+ and θ_- respectively. The elements in the sectorial model have subexponential growth in a sectorial neighborhood of ∞ with opening π and bisector $-\theta$.

Given a direction θ one considers an algebra of resurgent symbols $\dot{\phi} = \sum_{\tau\in T} \phi_\tau e^{-\tau x}$ with obvious product where T is a denumerable set in $\mathbb{C}$ and $\phi_\tau, \tau \in T$, belongs to the formal model associated with $\underline{\underline{\mathcal{R}}}_\theta$ and the discrete set Ω_τ of singularities on the half line $\arg t = \theta$ of the Borel transform of ϕ_τ satisfies $\Omega_\tau + \tau \subset T$ for all $\tau \in T$. Here T depends on $\dot{\phi}$ and the expression $\phi_\tau e^{-\tau x}$ with $\tau \in T$ is called a resurgent symbol with support τ. An example of a resurgent symbol is given by the formal integral $\hat{y}(x, C)$ in (1.9) of (D) and of (Δ).

Then one defines $\dot{s}_{\theta\pm}\dot{\phi}(x) := \sum_{\tau\in T}(s_{\theta\pm}\phi_\tau)(x)e^{-\tau x}$. Here $s_{\theta\pm}$ is an isomorphism of the formal model to the sectorial model. So

$$\mathfrak{S}_\theta := (\dot{s}_{\theta+})^{-1}\dot{s}_{\theta-} \tag{4.1}$$

is an automorphism of the algebra of resurgent symbols in direction θ which is called the Stokes automorphism. Here $\mathfrak{S}_\theta$ may be decomposed as $\mathfrak{S}_\theta = \sum_{\arg\omega=\theta} e^{-\omega x}\Delta_\omega^+$, where Δ_ω^+ transforms resurgent symbols of support τ into such symbols. Furthermore one defines the alien derivation $\underline{\Delta}_\theta :=$ Log $\mathfrak{S}_\theta$ which may be decomposed similarly as

$$\underline{\Delta}_\theta := \sum_{\arg\omega=\theta} \dot{\Delta}_\omega, \dot{\Delta}_\omega = e^{-\omega x}\Delta_\omega. \tag{4.2}$$

As Borel summation commutes with $\frac{\partial}{\partial x}$, also $\mathfrak{S}_\theta, \underline{\Delta}_\theta$ and $\dot{\Delta}_\omega$ commute with $\frac{\partial}{\partial x}$ and with the shift over 1.

We apply this to the formal integral $\hat{y}(x, C)$ of (Δ) mentioned above where now we choose $C \in \mathbb{C}^n$ independent of x. Let θ be a direction which contains singularities. If $\arg \omega = \theta$ then

$$\dot{\Delta}_\omega \hat{y}(x, C)\big|_{x \mapsto x+1} = \Lambda(x) \dot{\Delta}_\omega \hat{y}(x, C) + \dot{\Delta}_\omega f(x^{-1}, \hat{y}(x, C)),$$

where $\dot{\Delta}_\omega f(x^{-1}, \hat{y}(x, C)) = \frac{\partial f}{\partial y}(x^{-1}, y)\big|_{y=\hat{y}(x,C)} \dot{\Delta}_\omega \hat{y}(x, C)$ as follows for instance by using the Taylor expansion of $f(x^{-1}, y)$ with respect to y. Hence $\dot{\Delta}_\omega \hat{y}(x, C))$ is a solution of the linearized equation of (Δ) which admits $\frac{\partial}{\partial C} \hat{y}(x, C)$ as fundamental matrix. Therefore there exists a vector

$$P_\omega(x, C) = \sum_{j \in \mathbb{Z}} P_{\omega,j}(C) e^{-2\pi i j x}$$

where the coefficients $P_{\omega,j}(C) \in \mathbb{C}^n$ are formal power series in C, such that

$$\dot{\Delta}_\omega \hat{y}(x, C) = \frac{\partial}{\partial C} \hat{y}(x, C) P_\omega(x, C). \tag{4.3}$$

This is the bridge equation of Ecalle.

We choose for ω a singularity $\omega := \mathbf{l} \bullet \boldsymbol{\mu} + 2\pi i g$ with $\mathbf{l} \in \mathbb{Z}^n, g \in \mathbb{Z}$ with $\arg \omega = \theta$. Then the previous equation may be rewritten as

$$\sum_{\mathbf{k}} e^{-((\mathbf{k}+\mathbf{l})\bullet\boldsymbol{\mu}+2\pi i g)x} C^{\mathbf{k}} \Delta_\omega(x^{\mathbf{k}\bullet\mathbf{a}} \hat{y}_{\mathbf{k}}(x)) =$$

$$\sum_{\mathbf{h}} e^{-\mathbf{h}\bullet\boldsymbol{\mu}x} x^{\mathbf{h}\bullet\mathbf{a}} \hat{y}_{\mathbf{h}}(x) (\frac{\partial}{\partial C} C^{\mathbf{h}}) P_\omega(x, C).$$

Comparing coefficients of $e^{-((\mathbf{k}+\mathbf{l})\bullet\boldsymbol{\mu}+2\pi i g)x}$ we obtain

$$\Delta_\omega(x^{\mathbf{k}\bullet\mathbf{a}} \hat{y}_{\mathbf{k}}(x)) = \sum_J x^{\mathbf{h}\bullet\mathbf{a}} \hat{y}_{\mathbf{h}}(x) C^{-\mathbf{k}} (\frac{\partial}{\partial C} C^{\mathbf{h}}) P_{\omega,j}(C), \tag{4.4}$$

where J consists of the pairs $\mathbf{h}, j$ such that $\mathbf{h} \in \mathbb{N}^n, j \in \mathbb{Z}, \mathbf{h} \bullet \boldsymbol{\mu} + 2\pi i j = (\mathbf{k}+\mathbf{l}) \bullet \boldsymbol{\mu} + 2\pi i g$. Now assume that $\mu_1, \mu_2, \ldots, \mu_n$ are linearly independent over $\mathbb{Z}$ mod $2\pi i$. Then $\mathbf{h} \bullet \boldsymbol{\mu} + 2\pi i j = (\mathbf{k}+\mathbf{l}) \bullet \boldsymbol{\mu} + 2\pi i g$ only if $\mathbf{h} = \mathbf{k}+\mathbf{l}$ and $P_{\omega,j}(C) = 0$ if $j \neq g$. From (4.4) it now follows that $C^{-\mathbf{k}}(\frac{\partial}{\partial C} C^{\mathbf{k}+\mathbf{l}}) P_{\omega,g}(C)$ is independent of C. From this we may deduce in a straightforward manner that if $P_{\omega,g} \neq 0$ then at most one component of $\mathbf{l}$ is positive and in the latter case this component equals one. So only in these cases $\Delta_\omega(x^{\mathbf{k}\bullet\mathbf{a}} \hat{y}_{\mathbf{k}}(x))$ with $\omega = \mathbf{l} \bullet \boldsymbol{\mu} + 2\pi i g$ may be different from 0.

If $l_j \leqslant 0$ for all j then we may deduce that

$$P_\omega(x, C) = e^{-2\pi i g x} C^{-\mathbf{l}} \sum_{j=1}^{n} \alpha_j(\omega) C_j \mathbf{e}_j \tag{4.5}$$

for certain constants $\alpha_j(\omega) \in \mathbb{C}$ and we obtain from the bridge equation (4.3)

$$\dot{\Delta}_\omega \hat{y}(x, C) = e^{-2\pi i g x} C^{-\mathbf{l}} \sum_{j=1}^{n} \alpha_j(\omega) C_j \frac{\partial}{\partial C_j} \hat{y}(x, C). \tag{4.6}$$

Similarly, if $l_j = 1$ then $l_m \leqslant 0$ if $m \neq j$ and (4.5) holds with $\alpha_m(\omega) = 0$ if $m \neq j$.

From the bridge equation (4.6) we may derive $\underline{\Delta}_\theta$ via (4.2) and then by exponentiation also $\mathfrak{S}_\theta$. Using the decomposition of $\mathfrak{S}_\theta$ in homogeneous components we get with (4.1) that

$$\dot{s}_{\theta-} = \sum_{j=1}^{\infty} e^{-\omega_j x} s_{\theta+} \Delta^{+}_{\omega_j}. \tag{4.7}$$

For example if $\arg \mu_j = \theta$ and there are no other singularities with argument θ then we have (4.5) with $\omega = \mu_j, \mathbf{l} = \mathbf{e}_j, g = 0$ and $\underline{\Delta}_\theta = \dot{\Delta}_\omega$, so $\mathfrak{S}_\theta \hat{y}(x, C) = \exp(\alpha_j \frac{\partial}{\partial C_j}) \hat{y}(x, C) = \hat{y}(x, C + \alpha_j \mathbf{e}_j)$, which corresponds with (2.4) in (2.3) for the case $I = \{j\}$. From this we may deduce resurgence relations as before. Resurgence relations for general singular directions may be obtained similarly with some more effort.

Ecalle (cf.[Eca85]) has shown that the set of $\alpha_j(\omega)$ form a complete and free system of holomorphic invariants for (Δ) (invariance for analytic changes of the variables which do not change the formal normal form). Similarly for (D) where now we do not use the periodic factors $e^{2\pi i g x}$.

Remark 4.1. — The discussion in this paper may be extended to the case where in (1.1) $y'(x)$ is replaced by $x^{1-p} y'(x)$ and where in (1.2) now $\Lambda(x)$ and $f(x, y)$ are analytic in $x^{-1/p}$ with $\Lambda(\infty) = \mathrm{diag}\{e^{-\mu_j}\}$, $f(x, y) = O(x^{-2/p}) + O(|y|^2)$ and in both cases $p \in \mathbb{N}$.

Acknowledgement. — We want to thank Trudeke Immink for her comments on a first draft of this paper.

Bibliography

[BK04] BRAAKSMA (B.), KUIK (R.). — Asymptotics and singularities for a class of difference equation, *Contemporary Mathematics*, to appear (2004).

[Bra01] BRAAKSMA (B.L.J.). — Transseries for a class of nonlinear difference equations, *J. Diff. Equations Appl.*, **7(5)**, p. 717-750 (2001).

[CNP93] CANDELPERGHER (B.), NOSMAS (J.-C.), PHAM (F.). — *Approche de la résurgence*, Actualités Mathématiques, Herman, Paris (1993).

[Cos98] COSTIN (O.). — On Borel summation and Stokes phenomena for rank-1 nonlinear systems of ordinary differential equations, *Duke Math. J.*, **93(2)**, p. 289-344 (1998).

[Eca85] ECALLE (J.). — *Les fonctions résurgentes. Tome III*, Université de Paris-Sud Département de Mathématique, Orsay (1985).

[Eca92] ECALE (J.). — *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*, Actualités Mathématiques, Hermann, Paris (1992).

[Kui03] KUIK (R.) *Transseries in difference and differential equations*, Thesis, University of Groningen (2003).

[Mal85] MALGRANGE (B.). — Introduction aux travaux de J. Ecalle, *Enseign. Math. (2)* **31(3-4)**, p. 261-282 (1985).